On the Bernstein Constants of Polynomial Approximation

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Abstract

Let $\alpha > 0$ not be an integer. In papers published in 1913 and 1938, S. N. Bernstein established the limit

$$\Lambda^*_{\infty,\alpha} = \lim_{n\to\infty} n^\alpha E_n\left[|x|^\alpha ; L_\infty [-1,1]\right].$$

Here $E_n\left[|x|^\alpha ; L_\infty [-1,1]\right]$ denotes the error in best uniform approximation of $|x|^\alpha$ on $[-1,1]$ by polynomials of degree $\leq n$. Bernstein proved that $\Lambda^*_{\infty,\alpha}$ is itself the error in best uniform approximation of $|x|^\alpha$ by entire functions of exponential type at most 1, on the whole real line. We prove that the best approximating entire function is unique, and satisfies an alternation property. We show that the scaled polynomials of best approximation converge to this unique entire function. We derive a representation for $\Lambda^*_{\alpha,\infty}$, as well as its L_p analogue for $1 \leq p < \infty$.

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1 Introduction

Let $\alpha > 0$ be not an even integer. In papers published in 1913 and 1938, S. N. Bernstein [2], [3] established the limit

$$\Lambda^*_{\infty,\alpha} = \lim_{n\to\infty} n^\alpha E_n \left[|x|^\alpha ; L_\infty [-1,1] \right],$$

where

$$E_n [f; L_p [a,b]] = \inf \left\{ \|f - P\|_{L_p[a,b]} : \deg (P) \le n \right\}$$

denotes the error in best L_p approximation of a function f on $[a,b]$ by polynomials of degree $\le n$. The most studied case of this limit is $\alpha = 1$. Bernstein's first proof for this case was in the 1913 paper, and was long and difficult. Later he obtained a much simpler proof, for all α, involving dilations of the interval, making essential use of the homogeneity of $|x|^\alpha$, namely that for $\lambda > 0$,

$$|\lambda x|^\alpha = \lambda^\alpha |x|^\alpha .$$

This enabled Bernstein to relate the error in approximation on $[-\lambda, \lambda]$ to that on $[-1,1]$. It also yielded a formulation of the limit as the error in approximation on the whole real axis by entire functions of exponential type, namely

$$\Lambda^*_{\infty,\alpha} \\ = \inf \left\{ \| \, |x|^\alpha - f(x) \, \|_{L_\infty(\mathbb{R})} : f \text{ is entire of exponential type} \le 1 \right\}.$$

Recall here that f is of exponential type $A \ge 0$ means that for each $\varepsilon > 0$, and for $|z|$ large enough,

$$|f(z)| \le \exp \left(|z| (A + \varepsilon) \right).$$

Moreover, A is the smallest number with this property. This formula was extended to L_p by Raitsin [25]. We reproduce a variant of Bernstein's argument in Section 8 below. Bernstein also showed that [3], [5, p. 194]

$$|\sin \alpha\pi| \, \Gamma(2\alpha) \left(1 - \frac{1}{2\alpha - 1} \right) < \Lambda^*_{\infty,\alpha} < \frac{|\sin \alpha\pi|}{\pi} \Gamma(2\alpha).$$

However, Bernstein did not establish uniqueness of the entire function attaining the inf, nor characterize it. Bernstein did not determine the value of $\Lambda^*_{\alpha,\infty}$, but speculated that

$$\Lambda^*_{\infty,1} = \lim_{n\to\infty} n E_n \left[|x| ; L_\infty [-1,1] \right] = \frac{1}{2\sqrt{\pi}} = 0.28209\,47917\ldots.$$

Some 70 years later, this was disproved by Varga and Carpenter [34], [33] using high precision scientific computation. They showed that

$$\Lambda^*_{\infty,1} = 0.28016\,94990\ldots$$

They also showed numerically that the normalized error $2nE_{2n}[|x|; L_\infty[-1,1]]$ should admit an asymptotic expansion in negative powers of n. Further numerical explorations for approximation of $|x|^\alpha$ have been provided by Varga and Carpenter [5]. Their crucial numerical work pointed the direction for analytic investigations into a number of classical approximation problems.

Surprisingly, the much deeper analogous problem of rational approximation has already been solved, by H. Stahl in a series of seminal papers [27], [28], [29]. He proved, using sophisticated methods of potential theory and other complex analytic tools, that

$$\lim_{n\to\infty} e^{\pi\sqrt{n}} R_n\left[|x|\,; L_\infty[-1,1]\right] = 8,$$

where $R_n\left[|x|\,; L_\infty[-1,1]\right]$ denotes the error in best L_∞ approximation of $|x|$ on $[-1,1]$ by rational functions with numerator and denominator degree $\leq n$. Later [29], he extended this to $|x|^\alpha$, establishing

$$\lim_{n\to\infty} e^{\pi\sqrt{\alpha n}} R_n\left[|x|^\alpha\,; L_\infty[-1,1]\right] = 4^{1+\alpha/2}\left|\sin\frac{\pi\alpha}{2}\right|.$$

Although $\Lambda^*_{\alpha,\infty}$ is not known explicitly, the ideas of Bernstein have been refined, and greatly extended. M. Ganzburg has shown limit relations of this type for large classes of functions, in one and several variables, even when weighted norms are involved [9], [10]. He and others such as Nikolskii and Raitsin have considered not only uniform, but also L_p norms. It is known [10] that for $1 \leq p \leq \infty$, there exists

$$\Lambda^*_{p,\alpha} = \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} E_n[|x|^\alpha\,; L_p\,[-1,1]].$$

In particular, Nikolskii [23] proved that at least for odd integers α,

$$\Lambda^*_{1,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 8\Gamma\left(\alpha+1\right)\sum_{k=0}^{\infty}\left(-1\right)^k\left(2k+1\right)^{-\alpha-2}.$$

He also established an integral representation valid for all $\alpha > -1$, and Bernstein later noted that this implies the above series representation for all

$\alpha > -1$. Raitsin [26] proved that for $\alpha > -\frac{1}{2}$,

$$\Lambda^*_{2,\alpha} = \frac{\left|\sin \frac{\alpha\pi}{2}\right|}{\pi} 2\Gamma(\alpha+1)\sqrt{\pi/(2\alpha+1)}.$$

These are the only known explicit values of $\Lambda^*_{p,\alpha}$.

Vasiliev [32] extended Bernstein's results in another direction, replacing the interval $[-1,1]$ by fairly general compact sets. Totik [31] has put Vasiliev's results in final form, using sophisticated estimates for harmonic measures. For example, if K is a compact set containing 0 in its interior, then the Vasiliev-Totik result has the form

$$\lim_{n\to\infty} n^\alpha E_n\left[|x|^\alpha; L_\infty(K)\right] = (\pi\omega_K(0))^{-\alpha}\Lambda^*_{\infty,\alpha},$$

where ω_K denotes the equilibrium density of the set K (in the sense of classical potential theory). The Bernstein constant continues to attract attention: for example it is discussed in the recent book of Finch on mathematical constants [8, p. 257 ff.] in different branches of mathematics.

In this paper, we prove:

Theorem 1.1. *Let $1 \le p \le \infty$ and $\alpha > -\frac{1}{p}$, not an even integer. For $n \ge 1$, let P^*_n denote the unique polynomial of degree $\le n$ that best approximates $|x|^\alpha$ in the $L_p[-1,1]$ norm, so that*

$$\left\| |x|^\alpha - P^*_n \right\|_{L_p[-1,1]} = \inf_{\deg(P)\le n} \left\| |x|^\alpha - P \right\|_{L_p[-1,1]}. \tag{1.1}$$

Then uniformly in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} n^\alpha P^*_n(z/n) = H^*(z), \tag{1.2}$$

where H^* is an entire function of exponential type 1 satisfying

$$\left\| |x|^\alpha - H^*(x) \right\|_{L_p(\mathbb{R})} = \inf \left\| |x|^\alpha - f(x) \right\|_{L_p(\mathbb{R})}, \tag{1.3}$$

the inf *being taken over all functions f that are entire of exponential type ≤ 1.*

The scaled asymptotics for P^*_n are new for all α and all $1 \le p \le \infty$. Concerning H^*, we prove that it is the unique best approximant:

Theorem 1.2. *Let $1 \leq p \leq \infty$ and $\alpha > -\frac{1}{p}$, not an even integer. There is exactly one entire function H^*, of exponential type ≤ 1, satisfying* (1.3)*. Moreover, the function H^* is even.*
(a) If $1 \leq p < \infty$, H^* is characterized by the conditions that

$$\| \, |x|^{\alpha} - H^* (x) \, \|_{L_p(\mathbb{R})} < \infty \tag{1.4}$$

and

$$\int_{-\infty}^{\infty} ||x|^{\alpha} - H^* (x)|^{p-1} \operatorname{sign} (|x|^{\alpha} - H^* (x)) \, f(x) \, dx = 0 \tag{1.5}$$

for all entire functions f that are of exponential type ≤ 1, and that satisfy $f_{|\mathbb{R}} \in L_p(\mathbb{R})$.
(b) If $p = \infty$, there exist alternation points

$$0 = y_0^* < y_1^* < y_2^* < \cdots \tag{1.6}$$

with

$$y_j^* \in [(j-1)\pi, j\pi] \, , j \geq 1, \tag{1.7}$$

and for $j \geq 0$,

$$\left|y_j^*\right|^{\alpha} - H^* \left(\pm y_j^*\right) = (-1)^{j+\overline{\alpha/2}} \| \, |x|^{\alpha} - H(x) \, \|_{L_\infty(\mathbb{R})}, \tag{1.8}$$

where $\overline{\alpha/2}$ is the least integer exceeding $\alpha/2$.

Remarks
(a) The uniqueness in Theorem 1.2 is new for $p = \infty$, and is probably the most delicate result in this paper. There are very few results on uniqueness of best approximating entire functions of exponential type in the uniform norm. The most powerful is that due to Achieser [1], though it is in a more general setting. In our context, it requires that the alternation points are the zeros of an entire function $\Omega(z)$ such that

$$\lim_{y \to \infty} \frac{y e^{2y}}{\Omega(iy)} = 0.$$

This limit apparently fails in our case. Achieser also proved a characterization under this limit, but the author does not know of even a bounded function on the real line for which it applies. In Achieser's context, it was applied to a different type of extremal problem.

(b) Note that we do not have a complete characterization in the uniform norm at present – we established that the best approximant satisfies an alternation theorem, but did not prove that the alternation conditions characterize H^*. We believe that together with the condition

$$\| \, |x|^\alpha - H^*(x) \, \|_{L_\infty(\mathbb{R})} < \infty$$

they do. The only proper L_∞ characterization for approximation by entire functions that the author could find is due to Achieser [1], but again requires that limit that fails in our case.

(c) For $p < \infty$, the characterization above is known, and follows easily from results in Timan's book [30, p. 84]. The uniqueness for $p > 1$ is also then a simple consequence of the equality case of Hölder's inequality. The uniqueness for $p = 1$ is more delicate, and the author thanks M. Ganzburg for pointing out how to correct an oversight in the author's original proof of this.

We also prove a representation for the function H^* in Theorem 1.1, and a representation for $\Lambda^*_{\alpha,p}$:

Theorem 1.3. *Let* $1 \leq p \leq \infty$ *and* $\alpha > -\frac{1}{p}$*, not an even integer.*
(a) For $\operatorname{Im}(z) \neq 0$*, the function* H^* *admits the representation*

$$(z \operatorname{sign}(\operatorname{Re} z))^\alpha - H^*(z) = -\frac{\sin \frac{\alpha}{2}\pi}{\pi} F^*(z) \int_{-\infty}^{\infty} \frac{|s|^{\alpha+1}}{s^2+z^2} \frac{ds}{F^*(is)}, \qquad (1.9)$$

where

$$F^*(z) = \prod_{j=1}^{\infty} \left(1 - \left(z/x_j^*\right)^2\right), \qquad (1.10)$$

with

$$0 < x_1^* < x_2^* < x_3^* < \cdots,$$
$$x_j^* \in \left[\left(j - \frac{3}{2}\right)\pi, \left(j - \frac{1}{2}\right)\pi\right], \qquad j \geq 2. \qquad (1.11)$$

(b) If $p < \infty$*,*

$$\begin{aligned} \Lambda^*_{p,\alpha} &= \lim_{n \to \infty} n^{\alpha + \frac{1}{p}} E_n[|x|^\alpha ; L_p[-1,1]] \\ &= \frac{\left|\sin \frac{\alpha}{2}\pi\right|}{\pi} \left(\int_{-\infty}^{\infty} \left| F^*(x) \int_{-\infty}^{\infty} \frac{|s|^{\alpha+1}}{s^2+x^2} \frac{ds}{F^*(is)} \right|^p dx \right)^{1/p}. \qquad (1.12) \end{aligned}$$

(c) If $p = \infty$,

$$\Lambda^*_{\infty,\alpha} = \lim_{n\to\infty} n^\alpha E_n[|x|^\alpha ; L_\infty[-1,1]] = \frac{\left|\sin \frac{\alpha}{2}\pi\right|}{\pi} \int_{-\infty}^{\infty} \frac{|s|^{\alpha-1}}{F^*(is)} ds. \tag{1.13}$$

Remarks

(a) The representations for H^* and $\Lambda^*_{p,\alpha}$ are new, although as we noted, explicit formulae are available for $\Lambda^*_{p,\alpha}$ when $p = 1$ and $p = 2$. In the case $p = 1$, $F^*(z) = \cos(z)$, and [10, p. 197]

$$H^*(z) = \cos z \left\{ P(z) + 2z^\ell \sum_{j=1}^{\infty} \frac{(-1)^j \left(\left(j - \frac{1}{2}\right)\pi\right)^{\alpha-\ell+1}}{z^2 - \left(\left(j-\frac{1}{2}\right)\pi\right)^2} \right\},$$

where ℓ is the (unique) even integer in $(\alpha - 1, \alpha + 1]$, and P is an explicitly given polynomial of degree $\leq \ell - 2$. An interpolation series of this type can also be given for the case $p = \infty$, though it is not fully explicit [21]. In the case $p = 2$, M. Ganzburg informed the author that an explicit formula can be derived from the orthogonality (1.5) and Paley-Wiener theory:

$$H^*(z) = -\frac{2\sin\frac{\alpha\pi}{2}}{\pi}\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k-\alpha)(2k)!}.$$

An alternative representation involving the Bessel kernel is also possible for $p = 2$ [21].

(b) Some of these results were announced in [20].

(c) One can show that for fixed p, the jth zero x_j of F^* is a monotone increasing function of α. In the case $p = \infty$, one can show that F^* is different for different α.

(d) The representation (1.9) may seem strange. However, it reflects the fact that the right-hand side of (1.9) is an even function of z, as is H^*. For $\mathrm{Re}(z) > 0$, the left-hand side is $z^\alpha - H^*(z)$, and for $\mathrm{Re}(z) < 0$, the left-hand side is $(-z)^\alpha - H^*(z)$.

We shall prove the theorems above by the standard transformation $x \to x^2$ of $[0,1]$, approximating x^α on $[0,1]$ instead of $|x|^{2\alpha}$ on $[-1,1]$. The theorems above, in the transformed case, and with more detail, are given in Section 3, together with the organization of this paper. In the next section, we list our notation.

2 Notation

In this section, we record our notation. Some has already been given in the introduction, but we repeat it for the reader's convenience. In the sequel, $C, C_1, C_2, \ldots$ denote constants independent of n, x, z. The same symbol does not necessarily denote the same constant, even in successive occurrences. Given sequences of real numbers $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exists $C > 1$ such that

$$C^{-1} \leq c_n/d_n \leq C$$

for the relevant range of n (usually for all $n \geq 1$ or n large enough). Similar notation is used for functions and sequences of functions.

(I) Given $\alpha > -1$ that is not an integer, $\{\alpha\} \in (0,1)$ denotes the fractional part of α. (If $\alpha < 0$, we take the fractional part of α to be $1+\alpha$.) Let

$$\overline{\alpha} = \alpha - \{\alpha\} + 1, \tag{2.1}$$

so that

$$\overline{\alpha} = \text{least integer } > \alpha. \tag{2.2}$$

Also let

$$f_\alpha(x) = x^\alpha. \tag{2.3}$$

(II) Given $1 \leq p \leq \infty, n \geq 1$, and $f \in L_p[a,b]$, $E_n[f; L_p[a,b]]$ denotes the error in approximation of f by polynomials of degree $\leq n$ on $[a,b]$ in the L_p norm. That is,

$$E_n[f; L_p[a,b]] = \inf\left\{\|f - P\|_{L_p[a,b]} : \deg(P) \leq n\right\}. \tag{2.4}$$

For $[a,b] \subset [0,\infty)$, let

$$E_n[f; L_{p,\surd}[a,b]] = \inf\left\{\|f - P\|_{L_{p,\surd}[a,b]} : \deg(P) \leq n\right\}, \tag{2.5}$$

where for $p < \infty$,

$$\|f\|_{L_{p,\surd}[a,b]} = \left(\int_a^b |f(x)|^p \frac{dx}{\sqrt{x}}\right)^{1/p}, \tag{2.6}$$

and for $p = \infty$,

$$\|f\|_{L_{p,\surd}[a,b]} = \|f\|_{L_\infty[a,b]}.$$

We make frequent use of the fact that for $0 < r \leq \infty$, and with a mild abuse of notation,

$$\|f\left(x^2\right)\|_{L_p\left[-\sqrt{r},\sqrt{r}\right]} = \|f\|_{L_{p,\surd}[0,r]}. \tag{2.7}$$

Because of symmetry considerations,

$$E_n\left[x^\alpha; L_{p,\surd}[0,1]\right] = E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right]. \tag{2.8}$$

(III) For $\alpha > -\frac{1}{2p}$ and not an integer, the unique best polynomial approximation of degree $\leq n$ to $f(x) = x^\alpha$ in the norm $L_{p,\surd}[0,1]$ is denoted by $P_{n,p,\alpha}$. Thus

$$\|x^\alpha - P_{n,p,\alpha}(x)\|_{L_{p,\surd}[a,b]} = \inf\left\{\|x^\alpha - P(x)\|_{L_{p,\surd}[a,b]} : \deg(P) \leq n\right\}. \tag{2.9}$$

Let

$$R_{n,p,\alpha}(z) = z^\alpha - P_{n,p,\alpha}(z) \tag{2.10}$$

denote the remainder function in $\mathbb{C}\backslash(-\infty,0]$. Here the branch of z^α is the principal one. It is known that this residual has $n+1$ distinct zeros in $[0,1]$, which we denote by

$$0 < x_{1,n,p,\alpha} < x_{2,n,p,\alpha} < \cdots < x_{n+1,n,p,\alpha} < 1. \tag{2.11}$$

(We shall expand on this in Section 5.) We shall sometimes drop the subscripts p and α.
(IV) Let

$$X_{n,p,\alpha}(z) = \prod_{j=1}^{n+1}(z - x_{j,n,p,\alpha}) \tag{2.12}$$

denote the monic polynomial whose zeros are the zeros of the remainder function $R_{n,p,\alpha}$ in $[0,1]$. We also let

$$0 = y_{0n,\alpha} < y_{1n,\alpha} < \cdots < y_{nn,\alpha} < y_{n+1,n,\alpha} = 1$$

denote the alternation points of $R_{n,\infty,\alpha}$ in $[0,1]$, so that

$$R_{n,\infty,\alpha}(y_{jn,\alpha}) = (-1)^j R_{n,\infty,\alpha}(0) = \pm E_n\left[x^\alpha; L_{\infty,\surd}[0,1]\right]. \tag{2.13}$$

Set

$$Y_{n,\alpha}(z) = \prod_{j=1}^{n} (z - y_{jn,\alpha}).$$

(We shall prove that 0 and 1 are alternation points in Section 5.)
(V) In Section 12, where we prove the results of Section 1, we shall use P_n^* to denote the polynomial of degree $\leq n$ of best L_p approximation to $|x|^\alpha$ on $[-1,1]$. Throughout the paper the superscript * is reserved for approximation on a symmetric interval containing 0, typically $[-1,1]$ or $\mathbb{R}$.
(VI) Let

$$\Lambda_{p,\alpha}^* = \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} E_n[|x|^\alpha; L_p[-1,1]]; \tag{2.14}$$

$$\Lambda_{p,\alpha} = \lim_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_n[x^\alpha; L_{p,\surd}[0,1]]; \tag{2.15}$$

$$H_{p,\alpha}(z) = \lim_{n\to\infty} n^{2\alpha} P_{n,p,\alpha}\left(z/n^2\right); \tag{2.16}$$

and

$$F_{p,\alpha}(z) = \lim_{n\to\infty} X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0). \tag{2.17}$$

Of course, the proof that these limits exist is a major part of this paper, and is contained in Sections 7–11. Starting in Section 7, we shall use $H_{p,\alpha}$ and $F_{p,\alpha}$ to denote a subsequential limit, rather than a limit through all positive integers.
(VII) We let $T_{n,p}$ denote the monic polynomial of degree n and minimal L_p norm with weight $x^{-1/2p}$ on $[0,1]$. Thus

$$\|T_{n,p}\|_{L_{p,\surd}[0,1]} = \min_{\deg(P)<n} \|x^n - P(x)\|_{L_{p,\surd}[0,1]}.$$

A substitution $x = t^2$ gives

$$\|T_{n,p}\left(t^2\right)\|_{L_p[-1,1]} = \min_{\deg(P)<2n} \left(\int_{-1}^{1} \left|t^{2n} - P(t)\right|^p dt\right)^{1/p},$$

as the extremal polynomials are known to be even on $[-1,1]$ when the degree (namely $2n$) is even. Thus $T_{n,p}(t^2)$ is also the monic L_p extremal polynomial of degree $2n$ with the weight 1 on $[-1,1]$. We denote the zeros of $T_{n,p}$ by

$$0 < t_{1n,p} < t_{2n,p} < \cdots < t_{nn,p} < 1$$

and its extrema in $(0,1)$ by

$$0 < s_{1n,p} < s_{2n,p} < \cdots < s_{nn,p} < 1.$$

We also set

$$s_{0n,p} = 0 \text{ and } s_{n+1,n,p} = 1.$$

In the special case $p = \infty$, recall that $T_{n,\infty}$ may be expressed in terms of the classical Chebyshev polynomials T_n : for $n \geq 1$,

$$T_{n,\infty}(x) = 2^{-2n+1} T_n(2x-1)$$

and

$$T_{n,\infty}\left(x^2\right) = 2^{-2n+1} T_{2n}(x)$$

and its zeros and alternation points are respectively

$$t_{jn,\infty} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n}\right) \text{ and } s_{jn,\infty} = \sin^2\left(\frac{j\pi}{n}\right). \tag{2.18}$$

In the special case $p = 1$, the above shows that $T_{n,1}\left(x^2\right) = 2^{-2n} U_{2n}(x)$, where U_{2n} is the clasical Chebyshev polynomial of the second kind, and its zeros are

$$t_{jn,1} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n+1}\right). \tag{2.19}$$

(VIII) Given $\phi : \mathbb{R} \to \mathbb{R}$, and $\sigma > 0$, let

$$A_\sigma\left[\phi; L_p(\mathbb{R})\right] = \inf\left\{\|\phi - f\|_{L_p(\mathbb{R})} : f \text{ entire of exponential type } \leq \sigma\right\}.$$

That is, $A_\sigma\left[\phi; L_p(\mathbb{R})\right]$ denotes the error in best L_p approximation of ϕ by entire functions of exponential type $\leq \sigma$. We shall show in Lemma 8.7 that

$$A_\sigma\left[|x|^\alpha; L_p(\mathbb{R})\right] = \sigma^{-p-1/\alpha} A_1\left[|x|^\alpha; L_p(\mathbb{R})\right]. \tag{2.20}$$

Similarly, let

$$\begin{aligned} &A_\sigma\left[\phi; L_{p,\surd}[0,\infty)\right] \\ &= \inf\left\{\|\phi - f\|_{L_{p,\surd}[0,\infty)} : f\left(x^2\right) \text{ entire of exponential type } \leq \sigma\right\}. \end{aligned}$$

Note that if $\phi(x) \equiv \psi\left(x^2\right)$, then

$$A_\sigma\left[\phi; L_p(\mathbb{R})\right] = A_\sigma\left[\psi; L_{p,\surd}[0,\infty)\right]. \tag{2.21}$$

L_p^σ denotes the set of all entire functions f such that $f(x^2)$ is entire of exponential type $\leq \sigma$ and such that

$$\|f\|_{L_{p,\sqrt{}}[0,\infty)} < \infty.$$

(IX) Given increasing sequences $\{a_j\}$ and $\{b_j\}$ of real numbers, we say they *interlace strictly* if

$$a_1 < b_1 < a_2 < b_2 < \cdots$$

or

$$b_1 < a_1 < b_2 < a_2 < \cdots$$

We say they *interlace weakly* if we have $\leq$ rather than $<$ above.

3 Limits in Approximation on $[0,1]$

We now present our results in the setting of best approximation of x^α on $[0,1]$, with a more detailed statement than given in Section 1. Recall that given $1 \leq p \leq \infty$ and $\alpha > -\frac{1}{2p}$ not an integer, $P_{n,\alpha,p}$ denotes the best polynomial approximation of degree $\leq n$ to x^α in the $L_{p,\sqrt{}}[0,1]$ norm, and $\{x_{jn,p,\alpha}\}_{j=1}^{n+1}$ denote the zeros of the remainder function

$$R_{n,p,\alpha}(x) = x^\alpha - P_{n,p,\alpha}(x)$$

in increasing order. Moreover,

$$X_{n,p,\alpha}(z) = \prod_{j=1}^{n+1} (z - x_{jn,p,\alpha}).$$

Theorem 3.1. *Let $1 \leq p \leq \infty$ and $\alpha > -\frac{1}{2p}$, with α not an integer.*
(I) For $j \geq 1$, there exists

$$x_j = \lim_{n\to\infty} n^2 x_{jn,p,\alpha} > 0. \tag{3.1}$$

Moreover, for $j \geq 2$,

$$x_j \in \left[\left(\left(j - \frac{3}{2}\right)\frac{\pi}{2}\right)^2, \left(\left(j - \frac{1}{2}\right)\frac{\pi}{2}\right)^2\right]. \tag{3.2}$$

(II) Uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0) = \prod_{j=1}^{\infty}\left(1-\frac{z}{x_j}\right) =: F_{p,\alpha}(z). \tag{3.3}$$

(III) Uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} n^{2\alpha} P_{n,p,\alpha}\left(z/n^2\right) = H_{p,\alpha}(z), \tag{3.4}$$

where $H_{p,\alpha}(z^2)$ is an entire function of exponential type 2 satisfying

$$\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\surd}[0,\infty)} = A_2\left[x^\alpha; L_{p,\surd}[0,\infty)\right]. \tag{3.5}$$

(IV) Uniformly for z in compact subsets of $\mathbb{C}\backslash(-\infty,0]$,

$$\lim_{n\to\infty} n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right) = z^\alpha - H_{p,\alpha}(z). \tag{3.6}$$

Theorem 3.2. *(I) There is exactly one function $H_{p,\alpha}$ such that $H_{p,\alpha}(x^2)$ is entire of exponential type ≤ 2 and satisfies (3.5).*
(II) $H_{p,\alpha}$ admits in $\mathbb{C}\backslash(-\infty,0]$ the representation

$$z^\alpha - H_{p,\alpha}(z) = -\frac{\sin\alpha\pi}{\pi} F_{p,\alpha}(z)\int_0^\infty \frac{t^\alpha}{t+z}\frac{dt}{F_{p,\alpha}(-t)}. \tag{3.7}$$

(III) Let $p<\infty$. Then $H_{p,\alpha}$ is characterized by the conditions that $\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\surd}[0,\infty)}$ is finite and

$$\int_0^\infty |x^\alpha - H_{p,\alpha}(x)|^{p-1}\operatorname{sign}(x^\alpha - H_{p,\alpha}(x))\, f(x)\frac{dx}{\sqrt{x}} = 0, \tag{3.8}$$

for all $f\in L_p^2$. Moreover

$$\begin{aligned}\Lambda_{p,\alpha} &= \lim_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_n[x^\alpha; L_{p,\surd}[0,1]]\\ &= \frac{|\sin\alpha\pi|}{\pi}\left(\int_0^\infty \left|F_{p,\alpha}(u)\int_0^\infty \frac{t^\alpha}{t+u}\frac{dt}{F_{p,\alpha}(-t)}\right|^p \frac{du}{\sqrt{u}}\right)^{1/p}.\end{aligned} \tag{3.9}$$

(IV) Let $p=\infty$. There exist alternation points

$$0 = y_0 < y_1 < y_2 < \cdots$$

with

$$y_j \in \left[\left((j-1) \frac{\pi}{2} \right)^2, \left(j \frac{\pi}{2} \right)^2 \right], \qquad j \geq 1,$$

and for $j \geq 0$,

$$y_j^\alpha - H_{\infty,\alpha}(y_j) = (-1)^{j+\overline{\alpha}} \|x^\alpha - H_{\infty,\alpha}(x)\|_{L_\infty[0,\infty)}, \tag{3.10}$$

where $\overline{\alpha}$ is the least integer exceeding α. Moreover,

$$\begin{aligned} \Lambda_{\infty,\alpha} &= \lim_{n\to\infty} n^{2\alpha} E_n[x^\alpha; L_{\infty,\surd}[0,1]] \\ &= \frac{|\sin \alpha\pi|}{\pi} \int_0^\infty \frac{t^{\alpha-1}}{F_{\infty,\alpha}(-t)} dt. \end{aligned} \tag{3.11}$$

We believe the conditions in (IV) characterize $H_{\infty,\alpha}$ and "almost" have a proof. Concerning the alternation points for the case $p = \infty$, we prove:

Theorem 3.3. *Let $\alpha > 0$, and not be an integer.*
(I) Fix $j \geq 1$. There exists

$$y_j = \lim_{n\to\infty} n^2 y_{jn,\alpha} > 0. \tag{3.12}$$

Moreover, for $j \geq 1$,

$$y_j \in \left[\left((j-1) \frac{\pi}{2} \right)^2, \left(j \frac{\pi}{2} \right)^2 \right], \tag{3.13}$$

and uniformly in j,

$$y_j - x_j \sim x_j - y_{j-1} \sim \sqrt{y_j}. \tag{3.14}$$

(II) Let

$$G_\alpha(z) = \prod_{j=1}^{\infty} (1 - z/y_j). \tag{3.15}$$

Then G_α is an entire function such that $G_\alpha(z^2)$ is of exponential type 2, and uniformly for z in compact subsets of $\mathbb{C}\backslash(-\infty, 0]$,

$$\lim_{n\to\infty} n^{2\alpha-2} R'_{n,\infty,\alpha}\left(z/n^2\right) = \frac{\alpha \sin \alpha\pi}{\pi} G_\alpha(z) \int_0^\infty \frac{t^{\alpha-1}}{t+z} \frac{dt}{G_\alpha(-t)}. \tag{3.16}$$

Moreover uniformly for z in compact subsets of $\mathbb{C}$

$$\lim_{n\to\infty} Y_{n,\alpha}\left(z/n^2\right) / Y_{n,\alpha}(0) = G_\alpha(z). \tag{3.17}$$

Our plan of proof of Theorems 3.1 to 3.3 is as follows: in the next section, we establish our basic representation for the remainder $R_{n,p,\alpha}(x) = x^\alpha - P_{n,p,\alpha}(x)$. Then in Section 5, we examine the interlacing properties of the zeros of $R_{n,p,\alpha}$. In Section 6, we use the latter and other tools to show that for $n \geq 1$,

$$x_{1n,p,\alpha} \geq Cn^{-2}.$$

In Section 7, we show that $\{X_{n,p,\alpha}(z)/X_{n,p,\alpha}(0)\}_{n=1}^\infty$ is a normal family in the plane, and pass to a subsequence converging locally uniformly to an entire function $F_{p,\alpha}$ – as well as related limits for the relevant scaled subsequence of $P_{n,p,\alpha}$. In Section 8, we use Bernstein's dilation argument to show that the limit $\Lambda_{p,\alpha}$ exists, and relate best approximation on $\mathbb{R}$ and $[0,\infty)$. In Section 9, we establish the uniqueness and characterization of the best entire approximants $H_{p,\alpha}$ in $L_{p,\surd}[0,\infty)$ for $p < \infty$, and deduce Theorems 3.1 and 3.2 for the case $p < \infty$. In Section 10, we establish a Lagrange interpolation series at the zeros of $F_{\infty,\alpha}$, and in Section 11, use this to prove uniqueness and alternation properties of $H_{\infty,\alpha}$. We prove Theorems 3.1 to 3.3 for $p = \infty$ at the end of Section 11. Finally in Section 12, we prove Theorems 1.1 to 1.3 for all $1 \leq p \leq \infty$.

4 Interpolation Identities

We begin by noting that $R_{n,p,\alpha}$ really does have exactly $n+1$ zeros in $(0,1)$. If $\alpha > 0$, this follows from the fact that $\{1, x, \ldots, x^n, x^\alpha\}$ is a Chebyshev system. If $-\frac{1}{2p} < \alpha < 0$, we need a little more care. We defer the proof of this to the next section, where we deal with the zeros and their interlacing properties.

The basic idea of this section is to interpolate, for fixed a, the function

$$h_a(x) = \frac{1}{1+ax}; \tag{4.1}$$

to multiply by $a^{-\{\alpha\}}$, where $\{\alpha\} \in (0,1)$ denotes the fractional part of α; and then integrate with respect to a, using the elementary identity

$$\int_0^\infty \frac{a^{-\{\alpha\}}}{1+ax} da = x^{\{\alpha\}-1} \int_0^\infty \frac{s^{-\{\alpha\}}}{1+s} ds = \frac{\pi}{|\sin \alpha\pi|} x^{\{\alpha\}-1} \tag{4.2}$$

[12, p. 285, no. 3.194.4]. Similar ideas were used in [16], [19]. Recall that $\overline{\alpha}$ is the least integer exceeding α.

Theorem 4.1. *Let $\alpha > -\frac{1}{2p}$ and $n \geq \overline{\alpha} - 1$. Then for $x \in [0, \infty)$,*
(a)

$$R_{n,p,\alpha}(x) = x^\alpha - P_{n,p,\alpha}(x) = -\frac{\sin \alpha\pi}{\pi} X_{n,p,\alpha}(x) \int_0^\infty \frac{s^\alpha}{X_{n,p,\alpha}(-s)} \frac{ds}{s+x}. \quad (4.3)$$

(b)

$$R'_{n,p,\alpha}(x) = \alpha x^{\alpha-1} - P'_{n,p,\alpha}(x) = \alpha \frac{\sin \alpha\pi}{\pi} Y_{n,p,\alpha}(x) \int_0^\infty \frac{s^{\alpha-1}}{Y_{n,p,\alpha}(-s)} \frac{ds}{s+x}. \quad (4.4)$$

Proof. (a) Let $U(x) = x^{\overline{\alpha}}$, let $a \geq 0$, and let $L_n[Uh_a]$ denote the Lagrange interpolation polynomial of degree $\leq n$ to Uh_a at the $n+1$ zeros of $X_{n,p,\alpha}$ in $(0,1)$. Then

$$L_n[Uh_a]/h_a - U = (L_n[Uh_a] - Uh_a)/h_a,$$

is a polynomial of degree $\leq n+1$ that vanishes at the zeros of $X_{n,p,\alpha}$. Since $X_{n,p,\alpha}$ is a polynomial of degree $n+1$, it follows that for some constant c,

$$L_n[Uh_a]/h_a - U = cX_{n,p,\alpha}.$$

Setting $x = -1/a$ gives

$$-U(-1/a) = cX_{n,p,\alpha}(-1/a),$$

so

$$L_n[Uh_a]/h_a - U = X_{n,p,\alpha}(-1)^{\overline{\alpha}+1} a^{-\overline{\alpha}} / X_{n,p,\alpha}(-1/a).$$

So for all real x,

$$L_n[Uh_a](x) - \frac{x^{\overline{\alpha}}}{1+ax} = X_{n,p,\alpha}(x)(-1)^{\overline{\alpha}+1} \frac{a^{-\overline{\alpha}}}{1+ax} \frac{1}{X_{n,p,\alpha}(-1/a)}.$$

Now multiply by $a^{-\{\alpha\}}$, and integrate over $a \in [0, \infty)$, and use (4.2) to obtain

$$\frac{\pi}{|\sin \alpha\pi|} L_n\left[Ux^{\{\alpha\}-1}\right](x) - \frac{\pi}{|\sin \alpha\pi|} x^{\overline{\alpha}+\{\alpha\}-1}$$
$$= X_{n,p,\alpha}(x)(-1)^{\overline{\alpha}+1} \int_0^\infty \frac{a^{-\overline{\alpha}-\{\alpha\}}}{1+ax} \frac{da}{X_{n,p,\alpha}(-1/a)}.$$

Recalling (2.1) and (2.3) this gives

$$\begin{aligned}L_n[f_\alpha](x) - x^\alpha &= (-1)^{\overline{\alpha}+1}\frac{|\sin\alpha\pi|}{\pi}X_{n,p,\alpha}(x)\int_0^\infty \frac{a^{-\alpha-1}}{1+ax}\frac{da}{X_{n,p,\alpha}(-1/a)} \\ &= (-1)^{\overline{\alpha}+1}\frac{|\sin\alpha\pi|}{\pi}X_{n,p,\alpha}(x)\int_0^\infty \frac{s^\alpha}{x+s}\frac{ds}{X_{n,p,\alpha}(-s)},\end{aligned}$$

by the substitution $s = 1/a$. Here

$$(-1)^{\overline{\alpha}+1}|\sin\alpha\pi| = \sin\alpha\pi. \tag{4.5}$$

Finally, note that $P_{n,p,\alpha}(x)$ is a polynomial of degree $\leq n$ that interpolates to f_α at the $n+1$ zeros of $X_{n,p,\alpha}$, so by uniqueness of Lagrange interpolation,

$$P_{n,p,\alpha}(x) = L_n[f_\alpha](x).$$

(b) This is similar. Let $V(x) = x^{\overline{\alpha}-1}$ and $L_n^{\#}[Vh_a]$ denote the Lagrange interpolation polynomial to Vh_a at the n zeros of $Y_{n,p,\alpha}$. Proceeding as above, we obtain

$$L_n^{\#}[Vh_a]/h_a - V = Y_{n,p,\alpha}(-1)^{\overline{\alpha}}a^{-\overline{\alpha}+1}/Y_{n,p,\alpha}(-1/a). \tag{4.6}$$

So for all real x,

$$L_n^{\#}[Vh_a](x) - \frac{x^{\overline{\alpha}-1}}{1+ax} = Y_{n,p,\alpha}(x)(-1)^{\overline{\alpha}}\frac{a^{-\overline{\alpha}+1}}{1+ax}\frac{1}{Y_{n,p,\alpha}(-1/a)}.$$

Now multiply by $a^{-\{\alpha\}}$, and integrate over $a \in [0,\infty)$, and use (4.2) to obtain

$$\begin{aligned}&\frac{\pi}{|\sin\alpha\pi|}L_n^{\#}\left[Vx^{\{\alpha\}-1}\right](x) - \frac{\pi}{|\sin\alpha\pi|}x^{\overline{\alpha}+\{\alpha\}-1} \\ &= Y_{n,p,\alpha}(x)(-1)^{\overline{\alpha}}\int_0^\infty \frac{a^{-\overline{\alpha}-\{\alpha\}+1}}{1+ax}\frac{da}{Y_{n,p,\alpha}(-1/a)},\end{aligned}$$

and hence

$$\begin{aligned}&\alpha L_n^{\#}[f_{\alpha-1}](x) - \alpha x^{\alpha-1} \\ &= (-1)^{\overline{\alpha}}\alpha\frac{|\sin\alpha\pi|}{\pi}Y_{n,p,\alpha}(x)\int_0^\infty \frac{a^{-\alpha}}{1+ax}\frac{da}{Y_{n,p,\alpha}(-1/a)} \\ &= (-1)^{\overline{\alpha}}\alpha\frac{|\sin\alpha\pi|}{\pi}Y_{n,p,\alpha}(x)\int_0^\infty \frac{s^{\alpha-1}}{x+s}\frac{ds}{Y_{n,p,\alpha}(-s)}.\end{aligned}$$

As $P'_{n,p,\alpha}$ is also a polynomial of degree $\leq n-1$ interpolating to $\alpha x^{\alpha-1}$ at the n zeros of $Y_{n,p,\alpha}$,

$$P'_{n,p,\alpha}(x) = \alpha L_n^{\#}[f_{\alpha-1}](x),$$

and (4.4) follows. □

There is another way to derive Theorem 4.1, without using (4.2). One can apply Cauchy's integral formula to the function $R_{n,p,\alpha}(z)/X_{n,p,\alpha}(z)$, with the contour of integration taken as a circle centre 0, but cut above and below the negative real axis to avoid the branchpoint of z^α at 0. One deforms the circular part into ∞, giving 0. The jump of z^α across the negative real axis gives the representation. An idea similar to this was used in one form by Bernstein, and by M. Ganzburg [11] to deduce a representation similar to (4.3).

5 Interlacing Properties

In this section, we list a number of interlacing properties of the zeros $\{x_{jn,p,\alpha}\}_{j=1}^{n+1}$ and extrema $\{y_{jn,\alpha}\}_{j=0}^{n+1}$. Most of these follow from classical results of Pinkus and Ziegler [24]. We also use results of Kroo and Peherstorfer [15]. Recall that $T_{n,p}$ is the extremal monic polynomial of degree n and minimal L_p norm with weight $x^{-1/2p}$ on $[0,1]$. Recall that we denote the zeros of $T_{n,p}$ by

$$0 < t_{1n,p} < t_{2n,p} < \cdots < t_{nn,p} < 1$$

and its extrema in $(0,1)$ by

$$0 < s_{1n,p} < s_{2n,p} < \cdots < s_{nn,p} < 1.$$

We also set

$$s_{0n,p} = 0 \quad \text{and} \quad s_{n+1,n,p} = 1.$$

In the special case $p = \infty$, recall that the zeros and alternation points of $T_{n,\infty}$ are

$$t_{jn,\infty} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n}\right) \text{ and } s_{jn,\infty} = \sin^2\left(\frac{j\pi}{2n}\right), \quad 1 \leq j \leq n. \quad (5.1)$$

In the special case $p = 1$, recall that the zeros of $T_{n,1}$ are

$$t_{jn,1} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n+1}\right), \quad 1 \leq j \leq n. \quad (5.2)$$

Theorem 5.1. *Let $1 \leq p \leq \infty$, and $\alpha > -\frac{1}{2p}$ not be an integer.*
(a) The zeros of $R_{n,p,\alpha}$ and $T_{n,p}$ interlace weakly, that is,

$$0 < x_{1n,p,\alpha} \leq t_{1n,p} \leq x_{2n,p,\alpha} \leq t_{2n,p} \leq \cdots \leq t_{nn,p} \leq x_{n+1,n,p,\alpha} \leq 1. \qquad (5.3)$$

If $\alpha > 0$, the interlacing is strict.
(b) Let $n > \beta > \alpha > 0$ with β not an integer. Then for $1 \leq j \leq n+1$,

$$x_{jn,\infty,\alpha} \leq x_{jn,\infty,\beta}. \qquad (5.4)$$

(c) For $p = 1$, and for $1 \leq j \leq n$,

$$x_{j,n-1,1,\alpha} = t_{jn,1} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n+1}\right). \qquad (5.5)$$

We note that (5.4) should hold also for $p < \infty$. Pinkus and Ziegler proved an L_p version of this, but with Lebesgue measure, while we have a weight $1/\sqrt{x}$. Concerning the case $p = \infty$ and the alternation points $\{y_{jn,\alpha}\}_{j=0}^{n+1}$, we need:

Theorem 5.2. *Let $\alpha > 0$.*
(a)

$$0 = y_{0n,\alpha} < x_{1n,\infty,\alpha} < y_{1n,\alpha} < x_{2n,\infty,\alpha} < \cdots < x_{n+1,n,\infty,\alpha} < y_{n+1,n,\alpha} = 1. \qquad (5.6)$$

(b) The alternation points of $R_{n,\infty,\alpha}(x)$and $T_{n,\infty}$ interlace weakly, that is,

$$0 < y_{1n,\alpha} \leq s_{1n,\infty} \leq y_{2n,\alpha} \leq s_{2n,\infty} \leq \cdots \leq s_{nn,\infty} \leq y_{n+1,n,\alpha} < 1. \qquad (5.7)$$

Corollary 5.3. *For $1 \leq p \leq \infty$, $\alpha > -\frac{1}{2p}$ and $2 \leq j \leq n+1$,*

$$\sin^2\left(\left(j - \frac{3}{2}\right)\frac{\pi}{2n+1}\right) \leq x_{jn,p,\alpha} \leq \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n}\right). \qquad (5.8)$$

The upper bound remains valid for $j = 1$, and the lower bound remains valid for $j = n+1$.

To use the results of [24], we need some background on Chebyshev systems. Let $I = [0,1]$ and $\phi_1, \phi_2, \ldots, \phi_m$ be real valued functions on I. We say that $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is a *Chebyshev system* if whenever

$$0 \leq x_1 < x_2 < \cdots < x_m \leq 1, \qquad (5.9)$$

we have

$$\det\left(\phi_i\left(x_j\right)\right)_{1\leq i,j\leq m}\neq 0. \tag{5.10}$$

Note that this determinant is of one sign as long as the $\{x_j\}$ are strictly increasing. We say that $\{\phi_1,\phi_2,\ldots,\phi_m\}$ is a *positive determinant Chebyshev system* if whenever (5.9) holds, we have

$$\det\left(\phi_i\left(x_j\right)\right)_{1\leq i,j\leq m}>0. \tag{5.11}$$

For the purposes of the next two lemmas, let $w\in L_p\left(I\right)$ be a function positive a.e. in $[0,1]$. For $1\leq p\leq\infty$, the associated L_p Chebyshev polynomial is

$$T_{m-1,p}\left\{\phi_1,\phi_2,\ldots,\phi_{m-1};\phi_m\right\}=\phi_m-\sum_{j=0}^{m-1}c_j^*\phi_j$$

satisfying

$$\begin{aligned}&\left\|T_{m-1,p}\left\{\phi_1,\phi_2,\ldots,\phi_{m-1};\phi_m\right\}w\right\|_{L_p(I)}\\&=\inf_{c_1,c_2,\ldots,c_{m-1}}\left\|\left(\phi_m-\sum_{j=0}^{m-1}c_j\phi_j\right)w\right\|_{L_p(I)}.\end{aligned}$$

The terms in $\{\}$ after $T_{m-1,p}$ distinguish this Chebyshev polynomial from the specialized polynomial $T_{m-1,p}\left(x\right)$.

We record some of the interlacing properties, proved by Pinkus and Ziegler [24]. We warn the reader that in some places those authors use the convention that the determinant in (5.10) is positive, with differing values of m, and essentially they then assume they are working with what we call here a positive determinant Chebyshev system. This is why we introduced that term.

Lemma 5.4 (Successive Chebyshev Polynomials). *Assume that for* $k=m,m+1,m+2$, $\{\phi_1,\phi_2,\ldots,\phi_k\}$ *are sets of continuous functions that are Chebyshev systems on* $[0,1]$. *Then for* $1\leq p\leq\infty$, *the zeros of* $T_{m,p}\left\{\phi_1,\phi_2,\ldots,\phi_m;\phi_{m+1}\right\}$ *and* $T_{m+1,p}\left\{\phi_1,\phi_2,\ldots,\phi_{m+1};\phi_{m+2}\right\}$ *strictly interlace. Moreover the alternation points of* $T_{m,\infty}\left\{\phi_1,\phi_2,\ldots,\phi_m;\phi_{m+1}\right\}$ *and* $T_{m+1,\infty}\left\{\phi_1,\phi_2,\ldots,\phi_{m+1};\phi_{m+2}\right\}$ *interlace weakly.*

Proof. See [24, Corollary 1.1, p. 2]. $\square$

We also need the notion of a *Descartes system*. We say that $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is a Descartes system on $[a, b]$ if for any $k \geq 1$ and any

$$1 \leq i_1 < i_2 < \cdots < i_k \leq m,$$

$\{\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_k}\}$ is a positive determinant Chebyshev system on $[a, b]$.

Lemma 5.5 (Lexicographic Property). *Assume that $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is a Descartes system of continuous functions on $[0, 1]$. Let $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ be increasing sets of indices in $\{1, 2, \ldots, m-1\}$ with*

$$i_s \leq j_s, 1 \leq s \leq k$$

and strict inequality for at least one s. Then the zeros of $T_{k,\infty}\{\phi_{j_1}, \phi_{j_2}, \ldots, \phi_{j_k}; \phi_m\}$ lie strictly to the right of the zeros of $T_{k,\infty}\{\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_k}; \phi_m\}$.

Proof. See [24, Thm. 5.3, p. 14] or [4, p. 116ff.]. $\square$.

Pinkus and Ziegler also proved an L_p lexicographic property, in the unweighted case $w(x) \equiv 1$. We expect that their proof goes through for our case involving the weight $x^{-1/2p}$, but we do not need the extension.

Now we turn to

The Proof of Theorem 5.1(a) for $\alpha > 0$. When $\alpha > 0$, all the functions involved are continuous, and we can directly apply the results of Pinkus and Ziegler. We may apply the property of successive Chebyshev polynomials, Lemma 5.4 with $w(x) = x^{-1/2p}$, to

$$R_{n,p,\alpha} = T_{n+1,p}\left\{1, x, x^2, \ldots, x^n; x^\alpha\right\}$$

and

$$T_{n,p} = T_{n,p}\left\{1, x, x^2, \ldots, x^{n-1}; x^n\right\}.$$

The positivity of $x_{1n,p,\alpha}$ also follows from Lemma 5.4, since it lies strictly to the right of the smallest (non-negative) zero of $T_{n+2,p}\{1, x, x^2, \ldots, x^{n+1}; x^\alpha\}$. $\square$

The Proof of Theorem 5.1(a) for $\alpha < 0$. Let $\varepsilon \in (0, 1)$. Note that $\{x^\alpha, 1, x, x^2, \ldots, x^n\}$ is a Chebyshev system on $[\varepsilon, 1]$. In fact, it is even a positive determinant Chebyshev system there. Let $R^\varepsilon_{n,p,\alpha}$ denote the error function in best approximation of x^α by polynomials of degree $\leq n$ in the $L_{p,\surd}$ norm on $[\varepsilon, 1]$. Similarly, let $T^\varepsilon_{n,p}$ denote the L_p extremal polynomial of degree n in the $L_{p,\surd}$ norm on $[\varepsilon, 1]$. By the property of successive Chebyshev polynomials, Lemma 5.4 (translated to $[\varepsilon, 1]$) the zeros of $R^{\varepsilon}_{n,p,\alpha}$ and $T^{\varepsilon}_{n,p}$ strictly interlace. Next, as $\varepsilon \to 0+$, $R^{\varepsilon}_{n,p,\alpha}$ and $T^{\varepsilon}_{n,p}$ necessarily converge to $R_{n,p,\alpha}$ and $T_{n,p}$ respectively. This follows because the error of best approximation on $[\varepsilon, 1]$ decreases on $[\varepsilon, 1]$ as ε increases, and because in our context, the best approximations are unique. Then the zeros of $R_{n,p,\alpha}$ and $T_{n,p}$ weakly interlace. The positivity of $x_{1n,p,\alpha}$ follows from the unboundedness of $R_{n,p,\alpha}$ at 0. □

The Proof of Theorem 5.1(b). When we place $0, 1, \ldots, n, \alpha, \beta$ in increasing order, it is well known [4, p. 130] that the corresponding powers $1, x, \ldots, x^n, x^\alpha, x^\beta$ (ordered correspondingly) form a Descartes system on $[\varepsilon, 1]$ for any $\varepsilon \in (0, 1)$. Let $R^{\varepsilon}_{n,\infty,\alpha}$ denote the error function in best approximation of x^α by polynomials of degree $\leq n$ in the L_∞ norm on $[\varepsilon, 1]$, with similar notation for $R^{\varepsilon}_{n,\infty,\beta}$. Apart from a constant multiple, $R^{\varepsilon}_{n,\infty,\alpha}$ is also the error function in best approximation of x^n by $1, x, x^2 \ldots, x^{n-1}, x^\alpha$ in the L_∞ norm on $[\varepsilon, 1]$. A similar remark applies to $R^{\varepsilon}_{n,\infty,\beta}$. Since x^α precedes x^β in this ordering, the lexicographic property Lemma 5.5 shows that the zeros of $R^{\varepsilon}_{n,\infty\,\alpha}$ lie strictly to the left of the zeros of $R^{\varepsilon}_{n,\infty,\beta}$. Next, because of uniqueness of best approximations in our context, and with these functions, as $\varepsilon \to 0+$, $R^{\varepsilon}_{n,\infty,\alpha}$ and $R^{\varepsilon}_{n,\infty,\beta}$ necessarily converge to $R_{n,\infty,\alpha}$ and $R_{n,\infty,\beta}$ respectively. □

The Proof of Theorem 5.1(c). We must prove (5.5), namely that

$$x_{j,n-1,1,\alpha} = t_{jn,1} = \sin^2 \left(\left(j - \frac{1}{2} \right) \frac{\pi}{2n+1} \right).$$

This follows from well known results that best L_1 approximations are interpolation polynomials at "canonical points." That in turn implies that the zeros $\{x_{j,n-1,\alpha}\}_{j=1}^{n}$ of $R_{n-1,p,\alpha}$ are exactly the canonical points. Unfortunately we could not find a text giving immediately all that we need. We use [6, pp. 82–88] and start with the unweighted case on $[-1, 1]$. Let $a_0 = 1$, $a_{m+1} = 1$ and

$$a_k = \cos\left(\frac{m+1-k}{m+1}\pi\right), \qquad 1 \leq k \leq m \tag{5.12}$$

and define the signature

$$\sigma(x) = \operatorname{sign} U_m(x),$$

which changes sign exactly at the $\{a_k\}_{k=1}^{m}$. It is known [6, p. 87] that $\{a_k\}_{k=1}^{n}$ is a set of "canonical points" for $[-1,1]$ so that

$$0 = \int_{-1}^{1} \sigma(x) P(x)\, dx = \int_{-1}^{1} (\text{sign } U_m(x)) P(x)\, dx = 0$$

whenever P is a polynomial of degree $\leq m-1$. Now we set $m = 2n$ and deduce that for P of degree $\leq n-1$,

$$\begin{aligned} 0 &= \int_{-1}^{1} (\text{sign } U_{2n}(x)) P\left(x^2\right) dx \\ &= 2\int_{0}^{1} (\text{sign } U_{2n}(x)) P\left(x^2\right) dx = \int_0^1 \left(\text{sign } U_{2n}\left(\sqrt{t}\right)\right) P(t) \frac{dt}{\sqrt{t}}. \end{aligned}$$

Here sign $U_{2n}\left(\sqrt{t}\right)$ is a signature with sign changes in $(0,1)$ exactly at the points

$$t_{jn,1} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n+1}\right), \qquad 1 \leq j \leq n.$$

(To derive this, set $m = 2n$ and $k = n + j$ in (5.12), and observe that a_k^2 is a zero of $U_{2n}\left(\sqrt{t}\right)$.) Theorem 10.5 in [6, p. 84] asserts that the best L_1 approximant to a continuous function f on $[0,1]$ with weight $1/\sqrt{t}$ from the polynomials of degree $\leq n-1$ is just the Lagrange interpolation polynomial, P_0 say, to f at $\{t_{jn,1}\}_{j=1}^{n}$, provided $f - P_0$ changes sign at each $t_{jn,1}$ and nowhere else in $[0,1]$. For $f_\alpha(x) = x^\alpha$, where $\alpha > 0, \alpha \notin \{0,1,2,\ldots,n-1\}$, this sign changing condition follows from the fact that $\{1, x, \ldots, x^{n-1}, x^\alpha\}$ is a Chebyshev system on $[0,1]$. When $\alpha < 0$, we cannot apply Theorem 10.5 there because f_α is not continuous at 0. However, we can apply Theorem 10.4 there. The Lagrange interpolation polynomial P_0 to $f(x) = x^\alpha$ still has these properties, so

$$\int_0^1 (\text{sign } (f - P_0)(t)) P(t) \frac{dt}{\sqrt{t}} = \int_0^1 \left(\text{sign } U_{2n}\left(\sqrt{t}\right)\right) P(t) \frac{dt}{\sqrt{t}} = 0$$

for all polynomials P of degree $\leq n-1$. Then Theorem 10.4 in [6, p. 84] shows that P_0 is still the best $L_{1,\sqrt{}}$ approximant on $[0,1]$. □

The Proof of Theorem 5.2. (a) The interlacing of extrema and zeros of the remainder $R_{n,\infty,\alpha}$ is obvious. The only thing to be proved is that both 0 and 1 are indeed alternation points, so that

$$0 = y_{0n,\alpha} < y_{n+1,n,\alpha} = 1.$$

Suppose for example that $0 < y_{0n,\alpha}$, so that $y_{0n,\alpha}$ is a local extremum for $R_{n,\infty,\alpha}$ in $(0,1)$. Then $R_{n,\infty,\alpha}$ has $n+1$ extrema inside $(0,1)$, so $R'_{n,\infty,\alpha}$ has $n+1$ zeros there. But $R'_{n,\infty,\alpha}$ is a linear combination of the $n+1$ functions $1, x, x^2, \ldots, x^{n-1}, x^{\alpha-1}$, which form a Chebyshev system in $[\varepsilon, 1]$ for each $\varepsilon > 0$. So it cannot have $n+1$ zeros in $\left(\frac{1}{2}y_{0n}, 1\right)$, and we have a contradiction. Similarly if $y_{n+1,n} < 1$.
(b) This follows from Lemma 5.4 much as above. □

The Proof of Corollary 5.3. Let $T^{\#}_{n,p}$ denote the monic L_p extremal polynomial of degree n for the unweighted case on $[-1,1]$, with zeros

$$-1 < t^{\#}_{1n,p} < t^{\#}_{2n,p} < \cdots < t^{\#}_{nn,p} < 1.$$

Kroo and Peherstorfer [15, Theorem 2, p. 655] (with different notation) proved that for $1 < p < \infty$, and each $j > n/2$,

$$t^{\#}_{j,n,1} < t^{\#}_{jn,p} < t^{\#}_{jn,\infty}.$$

Since, as in Section 2(VII),

$$T_{n,p}\left(x^2\right) = T^{\#}_{2n,p}\left(x\right)$$

we obtain for $1 \leq j \leq n$,

$$t_{jn,1} < t_{jn,p} < t_{jn,\infty}.$$

For $p = 1$ or $p = \infty$, these inequalities remain valid with $\leq$ replacing $<$. Theorem 5.1(a) shows that for $1 \leq j \leq n$,

$$x_{jn,p,\alpha} \leq t_{jn,p} \leq t_{jn,\infty} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n}\right)$$

while for $2 \leq j \leq n+1$,

$$x_{jn,p,\alpha} \geq t_{j-1,n,p} \geq t_{j-1,n,1} = \sin^2\left(\left(j - \frac{3}{2}\right)\frac{\pi}{2n+1}\right). \qquad \square$$

6 Smallest Zero of the Error

In Corollary 5.3, we showed that

$$x_{1n,p,\alpha} \leq \sin^2\left(\frac{\pi}{4n}\right) \leq \left(\frac{\pi}{4n}\right)^2. \tag{6.1}$$

In this section we establish a lower bound for $x_{1n,p,\alpha}$:

Theorem 6.1. *Let $1 \leq p \leq \infty$ and $\alpha > -\frac{1}{2p}$, not an integer. Then there exists $C_0 > 0$ such that for large enough n,*

$$x_{1n,p,\alpha} \geq C_0 n^{-2}. \tag{6.2}$$

Proof of Theorem 6.1 for $p = \infty$. Suppose first $\alpha < \frac{1}{2}$. By Jackson's Theorems, and Bernstein's Theorems, [7, p. 110, no. 3],

$$E_n\left[x^\alpha; L_\infty[0,1]\right] = \inf_{\deg(P) \leq n} \|f_\alpha - P\|_{L_\infty[0,1]} \sim n^{-2\alpha}.$$

We use an estimate of Ditzian and Totik for derivatives of polynomials of best approximation: [7, p. 98]

$$\sup_{x\in[-1,1]} \sqrt{x(1-x)} \left|P^{*\prime}_{n,\infty,\alpha}(x)\right| \leq C \sum_{k=0}^{n} E_k\left[x^\alpha; L_\infty[0,1]\right] \leq C n^{1-2\alpha},$$

since $\alpha < \frac{1}{2}$. Then at least for large enough n,

$$\begin{aligned}
\left|P^*_{n,\infty,\alpha}(x_{1n,\infty,\alpha}) - P^*_{n,\infty,\alpha}(0)\right| &= \left|\int_0^{x_{1n,\infty,\alpha}} P^{*\prime}_{n,\infty,\alpha}(t)\,dt\right| \\
&\leq C n^{1-2\alpha} \int_0^{x_{1n,\infty,\alpha}} \frac{dt}{\sqrt{t}} \\
&\leq C n^{1-2\alpha} \sqrt{x_{1n,\infty,\alpha}}.
\end{aligned}$$

Since 0 is an alternation point,

$$\begin{aligned}
C_1 n^{-2\alpha} &\leq E_n\left[x^{2\alpha}; L_\infty[0,1]\right] = \left|R_{n,\infty,\alpha}(0)\right| \\
&= \left|R_{n,\infty,\alpha}(0) - R_{n,\infty,\alpha}(x_{1n,\infty,\alpha})\right| \\
&= \left|-P^*_{n,\infty,\alpha}(0) - \left(x^\alpha_{1n,\infty,\alpha} - P^*_{n,\infty,\alpha}(x_{1n,\infty,\alpha})\right)\right| \\
&\leq C n^{1-2\alpha} \sqrt{x_{1n,\infty,\alpha}} + x^\alpha_{1n,\infty,\alpha}.
\end{aligned}$$

Then

$$C_1 \leq C \sqrt{n^2 x_{1n,\infty,\alpha}} + \left(n^2 x_{1n,\infty,\alpha}\right)^\alpha.$$

It follows that

$$\liminf_{n\to\infty} n^2 x_{1n,\infty,\alpha} > 0.$$

If $\alpha \geq \frac{1}{2}$, we can use the monotonicity of $x_{1n,\infty,\alpha}$ in α (Theorem 5.1(b)). $\square$

Proof of Theorem 6.1 for $p = 1$. This follows immediately from the identity (5.5), namely

$$x_{1,n-1,1,\alpha} = \sin^2\left(\frac{1}{2}\frac{\pi}{2n+1}\right). \qquad \square$$

The Proof of Theorem 6.1 for $1 < p < \infty$ is more difficult. We first need a Schur type inequality:

Lemma 6.2. *Let $1 < p < \infty$. There exist $B_0, n_0 > 0$ such that for $n \geq n_0$ and polynomials S of degree $\leq n$,*

$$\int_0^{B_0 n^{-2}} |S(x)|^{p-1} \frac{dx}{\sqrt{x}} \leq \int_{B_0 n^{-2}}^1 |S(x)|^{p-1} \frac{dx}{\sqrt{x}}.$$

Proof. This follows, for example, from Theorem 14 in [22, p. 113]. $\square$

Next, recall from Theorem 4.1 that

$$|R_{n,p,\alpha}|(x) = \frac{|\sin \alpha\pi|}{\pi} |X_{n,p,\alpha}(x)| \int_0^\infty \frac{s^\alpha}{|X_{n,p,\alpha}(-s)|} \frac{ds}{s+x}.$$

(We can take absolute values inside the integral as all zeros of $X_{n,p,\alpha}$ are positive.) We shall use the abbreviation

$$\chi = x_{1n,p,\alpha},$$

so suppressing, for notational simplicity, the dependence on n, p, α. Let

$$X_n^{\#}(x) = \prod_{j=2}^{n+1} (x - x_{jn,p,\alpha})$$

so that

$$X_{n,p,\alpha}(x) = (x - \chi) X_n^{\#}(x).$$

Moreover, let

$$w_n(x) = \int_0^\infty \frac{s^\alpha}{\left|X_n^{\#}(-s)\right|} \frac{ds}{(s+\chi)(s+x)}. \tag{6.3}$$

We see that then

$$|R_{n,p,\alpha}(x)| = \frac{|\sin \alpha\pi|}{\pi} |x - \chi| \left|X_n^{\#}(x)\right| w_n(x). \tag{6.4}$$

In the rest of this section, B_0 denotes the constant from Lemma 6.2.

We shall need upper and lower bounds for w_n in certain ranges:

Lemma 6.3. *(a) For $x \in [0,1]$,*

$$w_n(x)\left|X_n^{\#}(0)\right| \le C\left\{\chi^{\alpha-1} + \left(n^{-2}\right)^{\alpha-1}\right\}, \tag{6.5}$$

where C is independent of n, x and χ.
(b) For $x \in [B_0 n^{-2}, 1]$,

$$w_n(x)\left|X_n^{\#}(0)\right| \ge C\frac{n^{-2\alpha}}{x}, \tag{6.6}$$

where C is independent of n, x and χ.

Proof. (a) We see that

$$\begin{aligned} w_n(x) &\le \frac{1}{\chi}\int_0^{\chi}\frac{s^{\alpha-1}}{\left|X_n^{\#}(-s)\right|}ds + \int_{\chi}^{\infty}\frac{s^{\alpha-2}}{\left|X_n^{\#}(-s)\right|}ds \\ &= \frac{n^{-2\alpha}}{\chi}\int_0^{n^2\chi}\frac{t^{\alpha-1}}{\left|X_n^{\#}(-n^{-2}t)\right|}dt + n^{-2(\alpha-1)}\int_{n^2\chi}^{\infty}\frac{t^{\alpha-2}}{\left|X_n^{\#}(-n^{-2}t)\right|}dt. \end{aligned}$$

Recall here from Corollary 5.3 that for $j \ge 1$,

$$n^2 x_{jn,p,\alpha} \le n^2\sin^2\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2n}\right) \le \left(\frac{j\pi}{2}\right)^2,$$

so for $t \ge 0$,

$$\begin{aligned} \frac{X_n^{\#}(0)}{X_n^{\#}(-n^{-2}t)} &= \prod_{j=2}^{n+1}\left(1+\frac{t}{n^2 x_{jn,p,\alpha}}\right)^{-1} \\ &\le \prod_{j=2}^{n+1}\left(1+\frac{t}{Cj^2}\right)^{-1} \le 1. \end{aligned}$$

Then if $L > \alpha$ is a fixed positive integer and $n \ge L$,

$$\begin{aligned} w_n(x)\left|X_n^{\#}(0)\right| &\le \frac{n^{-2\alpha}}{\chi}\int_0^{n^2\chi}t^{\alpha-1}dt + n^{-2(\alpha-1)}\int_{n^2\chi}^{\infty}\frac{t^{\alpha-2}}{\prod_{j=2}^{L}\left(1+\frac{t}{Cj^2}\right)}dt \\ &\le \left\{\frac{1}{\alpha}\chi^{\alpha-1} + n^{-2(\alpha-1)}\left[\int_{n^2\chi}^{1}t^{\alpha-2}dt + C\int_1^{\infty}t^{\alpha-1-L}dt\right]\right\} \\ &\le C\left\{\chi^{\alpha-1} + \left(n^{-2}\right)^{\alpha-1}\right\}. \end{aligned}$$

Here we have also used the bound (6.1),

$$n^2\chi \leq \left(\frac{\pi}{4}\right)^2 \leq 1; \tag{6.7}$$

the fact that $L > \alpha$, and have separately considered the cases $\alpha > 1$ or $\alpha < 1$.
(b) From (6.3),

$$\begin{aligned} w_n(x) &\geq \int_{\chi}^{\frac{2}{B_0}x} \frac{s^\alpha}{\left|X_n^{\#}(-s)\right|} \frac{ds}{(2s)\left(\left(1+\frac{2}{B_0}\right)x\right)} \\ &= \frac{n^{-2\alpha}}{2\left(1+\frac{2}{B_0}\right)x} \int_{n^2\chi}^{n^2\frac{2}{B_0}x} \frac{t^{\alpha-1}}{\left|X_n^{\#}(-n^{-2}t)\right|} dt \\ &\geq \frac{Cn^{-2\alpha}}{x\left|X_n^{\#}(0)\right|} \int_1^2 t^{\alpha-1} \left|\frac{X_n^{\#}(0)}{X_n^{\#}(-n^{-2}t)}\right| dt, \end{aligned}$$

recall (6.7). Here using Corollary 5.3, we easily see that for $t \in [1,2]$,

$$\left|\frac{X_n^{\#}(0)}{X_n^{\#}(-n^{-2}t)}\right| \geq \prod_{j=2}^{\infty}\left(1+\frac{t}{Cj^2}\right)^{-1} \geq C_1,$$

and then (6.6) follows. □

Proof of Theorem 6.1 for $1 < p < \infty$. We use the characterization of best L_p approximants, $1 < p \leq \infty$ [6, Cor. 10.2, p. 83]:

$$\int_0^1 |R_{n,p,\alpha}(x)|^{p-2} R_{n,p,\alpha}(x) S(x) \frac{dx}{\sqrt{x}} = 0, \tag{6.8}$$

for all polynomials S of degree $\leq n$. Letting

$$S(x) = X_n^{\#}(x) = \frac{X_{n,p,\alpha}(x)}{x-\chi},$$

and recalling that $R_{n,p,\alpha}$ changes sign exactly where $X_{n,p,\alpha}$ does, gives

$$\int_0^1 |R_{n,p,\alpha}(x)|^{p-1} \frac{|X_{n,p,\alpha}(x)|}{x-\chi} \frac{dx}{\sqrt{x}} = 0. \tag{6.9}$$

In particular, this gives

$$\int_0^\chi |R_{n,p,\alpha}(x)|^{p-1} \left|X_n^\#(x)\right| \frac{dx}{\sqrt{x}} = \int_\chi^1 |R_{n,p,\alpha}(x)|^{p-1} \left|X_n^\#(x)\right| \frac{dx}{\sqrt{x}}.$$

We now assume that for infinitely many n, say for $n \in \mathcal{S}$,

$$\chi \leq \frac{B_0}{2} n^{-2}.$$

(If no such $\mathcal{S}$ exists, then we already have the desired result.) Then recalling (6.4), we have

$$\begin{aligned}
1 &= \frac{\int_0^\chi |x-\chi|^{p-1} \left|X_n^\#(x)\right|^p w_n(x)^{p-1} \frac{dx}{\sqrt{x}}}{\int_\chi^1 |x-\chi|^{p-1} \left|X_n^\#(x)\right|^p w_n(x)^{p-1} \frac{dx}{\sqrt{x}}} \\
&\leq (2\chi)^{p-1} \frac{\int_0^\chi \left|X_n^\#(x)\right|^p w_n(x)^{p-1} \frac{dx}{\sqrt{x}}}{\int_{2\chi}^1 x^{p-1} \left|X_n^\#(x)\right|^p w_n(x)^{p-1} \frac{dx}{\sqrt{x}}} \\
&\leq (2\chi)^{p-1} C \left[\frac{\chi^{\alpha-1} + \left(n^{-2}\right)^{\alpha-1}}{n^{-2\alpha}}\right]^{p-1} \frac{\int_0^\chi \left|X_n^\#(x)\right|^p \frac{dx}{\sqrt{x}}}{\int_{B_0 n^{-2}}^1 \left|X_n^\#(x)\right|^p \frac{dx}{\sqrt{x}}} \\
&\leq C\left[\left(n^2\chi\right)^\alpha + n^2\chi\right]^{p-1}.
\end{aligned}$$

In the second last line, we used the bounds of Lemma 6.3 on w_n, and in the last line, we used the Schur inequality Lemma 6.2. It is of course crucial that B_0 and C are independent of n and χ. We deduce that for $n \in \mathcal{S}$,

$$n^2\chi = n^2 x_{1n,p,\alpha} \geq C_1,$$

so (6.2) follows. □

7 Subsequential Limits

Let us fix p, α and $j \geq 2$. It follows from Corollary 5.3 that

$$\left[\left(j-\frac{3}{2}\right)\frac{\pi}{2}\right]^2 \leq \liminf_{n\to\infty} n^2 x_{jn,p,\alpha} \leq \limsup_{n\to\infty} n^2 x_{jn,p,\alpha} \leq \left[\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right]^2.$$

For $j = 1$, we instead have from Theorem 6.1 that the last $\liminf$ is positive, and from Corollary 5.3 that the upper bound persists. By a diagonal choice argument, (for example, as in the proof of the Arzela-Ascoli Theorem), we can extract an infinite sequence of positive integers $\mathcal{S}$ such that for each fixed $j \geq 1$, there exists

$$\lim_{n\to\infty, n\in\mathcal{S}} n^2 x_{jn,p,\alpha} = x_j. \tag{7.1}$$

In fact, given any infinite sequence of positive integers, we can extract a further subsequence $\mathcal{S}$ with this property. Moreover, we see that for $j \geq 2$,

$$\left[\left(j - \frac{3}{2}\right)\frac{\pi}{2}\right]^2 \leq x_j \leq \left[\left(j - \frac{1}{2}\right)\frac{\pi}{2}\right]^2, \tag{7.2}$$

while

$$0 < C_0 \leq x_1 \leq \left[\frac{\pi}{4}\right]^2, \tag{7.3}$$

where C_0 is as in Theorem 6.1. We emphasize that C_0 is independent of the subsequence $\mathcal{S}$. Throughout this section, we assume that $\mathcal{S}$ is the sequence above, and we define

$$F_{p,\alpha}(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{x_j}\right). \tag{7.4}$$

In our first result, we establish subsequential asymptotics for the remainder of approximation, and associated quantities:

Theorem 7.1. *(a) The family* $\left\{X_{n,p,\alpha}(z/n^2)/X_{n,p,\alpha}(0)\right\}_{n=1}^{\infty}$ *is a normal family in* $\mathbb{C}$*. Uniformly in compact subsets of* $\mathbb{C}$*,*

$$\lim_{n\to\infty, n\in\mathcal{S}} X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0) = F_{p,\alpha}(z). \tag{7.5}$$

(b) Uniformly in compact subsets of $\mathbb{C}\backslash(-\infty, 0]$*,*

$$\lim_{n\to\infty, n\in\mathcal{S}} n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right) = -\frac{\sin\alpha\pi}{\pi} F_{p,\alpha}(z) \int_0^{\infty} \frac{t^{\alpha}}{t+z} \frac{dt}{F_{p,\alpha}(-t)}. \tag{7.6}$$

(c) Uniformly in compact subsets of $\mathbb{C}$*, there exists*

$$H_{p,\alpha}(z) = \lim_{n\to\infty, n\in\mathcal{S}} n^{2\alpha} P_{n,p,\alpha}\left(z/n^2\right). \tag{7.7}$$

The function $H_{p,\alpha}(z)$ is entire and in $\mathbb{C}\backslash(-\infty,0]$,

$$H_{p,\alpha}(z) = z^{\alpha} + \frac{\sin\alpha\pi}{\pi} F_{p,\alpha}(z) \int_0^{\infty} \frac{t^{\alpha}}{t+z} \frac{dt}{F_{p,\alpha}(-t)}. \tag{7.8}$$

Moreover, for all $z \in \mathbb{C}$,

$$H_{p,\alpha}(z) = \frac{\sin\alpha\pi}{\pi} \int_0^{\infty} \frac{t^{\alpha}}{t+z} \left\{ \frac{F_{p,\alpha}(z)}{F_{p,\alpha}(-t)} - \left(\frac{-z}{t}\right)^{\overline{\alpha}} \right\} dt. \tag{7.9}$$

Proof. (a) Now

$$X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0) = \prod_{j=1}^{n+1} \left(1 - \frac{z}{n^2 x_{jn,p,\alpha}}\right)$$

so

$$\left|X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0)\right| \leq \prod_{j=1}^{n+1} \left(1 + \frac{|z|}{n^2 x_{jn,p,\alpha}}\right).$$

Recall from Corollary 5.3, that for $j \geq 2$,

$$n^2 x_{jn,p,\alpha} \geq \left(\left(j - \frac{3}{2}\right)\frac{2n}{2n+1}\right)^2 \geq \frac{4}{9}\left(j - \frac{3}{2}\right)^2$$

and

$$n^2 x_{1n,p,\alpha} \geq C_0,$$

so for all z,

$$\begin{aligned}\left|X_{n,p,\alpha}\left(z/n^2\right)/X_{n,p,\alpha}(0)\right| &\leq \left(1 + \frac{|z|}{C_0}\right) \prod_{j=2}^{\infty} \left(1 + \frac{9|z|}{4\left(j - \frac{3}{2}\right)^2}\right) \\ &=: P(|z|).\end{aligned}$$

Since the bound on the right-hand side is independent of n, and the infinite product converges for all z, we see that $\{X_{n,p,\alpha}(z/n^2)/X_{n,p,\alpha}(0)\}_{n=1}^{\infty}$ is uniformly bounded in compact subsets of the plane. Thus it is a normal family there. Let $\varepsilon, R > 0$. By a similar argument, and the convergence of the bounding product, it follows that we can choose N such that for $|z| \leq R$, and $n \geq N$,

$$1 - \varepsilon \leq \prod_{j=N}^{n+1} \left|1 - \frac{z}{n^2 x_{jn,p,\alpha}}\right| \leq 1 + \varepsilon,$$

while uniformly for $|z| \leq R$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \prod_{j=1}^{N-1} \left(1 - \frac{z}{n^2 x_{jn,p,\alpha}}\right) = \prod_{j=1}^{N-1} \left(1 - \frac{z}{x_j}\right).$$

Then the uniform convergence to $F_{p,\alpha}$ follows.
(b) Making a substitution in Theorem 4.1(a) gives, for $z \in [0, \infty)$,

$$n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right) = -\frac{\sin\alpha\pi}{\pi} \frac{X_{n,p,\alpha}\left(z/n^2\right)}{X_{n,p,\alpha}(0)} \int_0^\infty \frac{t^\alpha}{\frac{X_{n,p,\alpha}(-t/n^2)}{X_{n,p,\alpha}(0)}} \frac{dt}{t+z}.$$

As both sides are analytic in $\mathbb{C}\backslash(-\infty, 0]$, the identity persists in that region. Then

$$\left|n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right)\right| \leq \frac{|\sin\alpha\pi|}{\pi} \left|\frac{X_{n,p,\alpha}\left(z/n^2\right)}{X_{n,p,\alpha}(0)}\right| \int_0^\infty \frac{t^\alpha}{\left|\frac{X_{n,p,\alpha}(-t/n^2)}{X_{n,p,\alpha}(0)}\right|} \frac{dt}{|t+z|}. \tag{7.10}$$

(Recall that $X_{n,p,\alpha}(0)$ and $X_{n,p,\alpha}(-t/n^2)$ both have sign $(-1)^{n+1}$.) Let $N \geq 1$. Here for $t \geq 0$ and $n \geq N$,

$$\begin{aligned} \frac{X_{n,p,\alpha}\left(-t/n^2\right)}{X_{n,p,\alpha}(0)} &= \prod_{j=1}^{n+1} \left(1 + \frac{t}{n^2 x_{jn,p,\alpha}}\right) \\ &\geq \prod_{j=1}^{N} \left(1 + \frac{t}{\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2}\right)^2}\right), \end{aligned} \tag{7.11}$$

by Corollary 5.3, with a similar upper bound. Since $\left\{\frac{X_{n,p,\alpha}\left(z/n^2\right)}{X_{n,p,\alpha}(0)}\right\}_{n=1}^\infty$ is uniformly bounded in compact sets, the normality of $\left\{n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right)\right\}_{n=1}^\infty$ follows. Next, for each fixed t, we have

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{X_{n,p,\alpha}\left(-t/n^2\right)}{X_{n,p,\alpha}(0)} = F_{p,\alpha}(-t).$$

This, the bound (7.11) (with N large enough) and Lebesgue's Dominated Convergence Theorem give

$$\lim_{n\to\infty, n\in\mathcal{S}} \int_0^\infty \frac{t^\alpha}{\frac{X_{n,p,\alpha}(-t/n^2)}{X_{n,p,\alpha}(0)}} \frac{dt}{t+z} = \int_0^\infty \frac{t^\alpha}{F_{p,\alpha}(-t)} \frac{dt}{t+z}.$$

Then we obtain (7.6).

(c) Now

$$n^{2\alpha}P_{n,p,\alpha}\left(z/n^2\right)=z^\alpha-n^{2\alpha}R_{n,p,\alpha}\left(z/n^2\right),$$

so uniformly in compact subsets of $\mathbb{C}\backslash(-\infty,0]$,

$$\begin{aligned}H_{p,\alpha}(z) &:= \lim_{n\to\infty,n\in\mathcal{S}} n^{2\alpha}P_{n,p,\alpha}\left(z/n^2\right)\\ &= z^\alpha+\frac{\sin\alpha\pi}{\pi}F_{p,\alpha}(z)\int_0^\infty \frac{t^\alpha}{t+z}\frac{dt}{F_{p,\alpha}(-t)}.\end{aligned}$$

So $H_{p,\alpha}$ is defined and analytic in $\mathbb{C}\backslash(-\infty,0]$. To prove that it is entire, we recall that if $\overline{\alpha}$ is the least integer exceeding α, (2.1), (4.2) and (4.5) give for $z\in\mathbb{C}\backslash(-\infty,0]$,

$$z^\alpha=z^{\overline{\alpha}+\{\alpha\}-1}=z^{\overline{\alpha}}(-1)^{\overline{\alpha}+1}\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{t^{\{\alpha\}-1}}{t+z}dt.$$

Then at least for $z\in\mathbb{C}\backslash(-\infty,0]$,

$$H_{p,\alpha}(z)=\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{t^\alpha}{t+z}\left\{\frac{F_{p,\alpha}(z)}{F_{p,\alpha}(-t)}-\left(\frac{-z}{t}\right)^{\overline{\alpha}}\right\}dt.$$

Moreover, we see that the integrand is continuous as a function of (z,t) provided $t\in(0,\infty)$ and z lies in a compact subset of the plane, even one intersecting the negative real axis. Hence $H_{p,\alpha}$ is well defined and continuous on the negative real axis, and is also real valued there. Finally we see that the limits (7.6) and (7.7) remain valid at $z=0$ also. By Morera's theorem, $H_{p,\alpha}$ is analytic on the negative real axis too. So $H_{p,\alpha}$ is entire. We must still show that $n^{2\alpha}P_{n,p,\alpha}(z/n^2)$ converges uniformly on compact sets intersecting the negative real axis. To do this, we proceed as above and establish the representation

$$\begin{aligned}&n^{2\alpha}P_{n,p,\alpha}\left(z/n^2\right)\\ &=\frac{\sin\alpha\pi}{\pi}\int_0^\infty\frac{t^\alpha}{t+z}\left\{\frac{X_{n,p,\alpha}(z/n^2)/X_{n,p,\alpha}(0)}{X_{n,p,\alpha}(-t/n^2)/X_{n,p,\alpha}(0)}-\left(\frac{-z}{t}\right)^{\overline{\alpha}}\right\}dt.\end{aligned}$$

The integrand converges as $n\to\infty$, uniformly for z in a compact subset of $\mathbb{C}$ (even one intersecting the negative real axis), and t in a compact subset of $(0,\infty)$, to

$$\frac{t^\alpha}{t+z}\left\{\left(\frac{-z}{t}\right)^{\overline{\alpha}}-\frac{F_{p,\alpha}(z)}{F_{p,\alpha}(-t)}\right\}.$$

Of course for $z=-t$, the quotient is replaced by a derivative. Hence the desired convergence. □

Now we start our investigation of the growth in the plane of the functions F and H. We begin with F.

Theorem 7.2. *(a) For some $C_0>0$, and all complex z,*

$$\left|F_{p,\alpha}(z)\right| \leq \left(1+\frac{|z|}{C_0}\right)\cosh\left(2\sqrt{|z|}\right). \tag{7.12}$$

(b) For $\operatorname{Re}(z)\leq 0$,

$$\left|F_{p,\alpha}(z)\right| \geq \left|\cos\left(2\sqrt{z}\right)\right|.$$

(c) $F_{p,\alpha}(z^2)$ is entire of exponential type 2.
(d) There exists $C>0$ such that for all z,

$$\left|\log\left|F_{p,\alpha}(z)\right| - \log\left|\cos\left(2\sqrt{z}\right)\right|\right| \leq \left|\log|z-\zeta|\right| + \log^+|z| + C, \tag{7.13}$$

where ζ is the closest zero of either $F_{p,\alpha}$ or $\cos\left(2\sqrt{\cdot}\right)$ to z.
(e) There exists $C>0$ such that for all z,

$$\left|F_{p,\alpha}\left(z^2\right)\right| \leq C\left(1+|z|\right)^2 e^{2|\operatorname{Im} z|}. \tag{7.14}$$

Proof. (a) From (7.2) to (7.4),

$$\begin{aligned}\left|F_{p,\alpha}(z)\right| &\leq \left(1+\frac{|z|}{x_1}\right)\prod_{j=2}^{\infty}\left(1+\frac{|z|}{\left[\left(j-\frac{3}{2}\right)\frac{\pi}{2}\right]^2}\right)\\ &= \left(1+\frac{|z|}{C_0}\right)\cosh\left(2\sqrt{|z|}\right).\end{aligned}$$

(b) If $\operatorname{Re}(z)\leq 0$, our upper bound for x_j in (7.2) gives

$$\begin{aligned}\left|1-\frac{z}{x_j}\right|^2 &= 1+\frac{2\left|\operatorname{Re}(z)\right|}{x_j}+\frac{|z|^2}{x_j^2}\\ &\geq 1+\frac{2\left|\operatorname{Re}(z)\right|}{\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right)^2}+\frac{|z|^2}{\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right)^4}\\ &= \left|1-\frac{z}{\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right)^2}\right|^2.\end{aligned}$$

Then

$$|F_{p,\alpha}(z)| \geq \left|\prod_{j=1}^{\infty}\left(1-\frac{z}{\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right)^2}\right)\right| = \left|\cos\left(2\sqrt{z}\right)\right|.$$

(c) We see that as all the x_j are positive,

$$\log\left(\max_{|z|=r}\left|F_{p,\alpha}\left(z^2\right)\right|\right) = \log F_{p,\alpha}\left(-r^2\right)$$
$$\begin{cases} \leq \log\left(1+\frac{r^2}{C_0}\right)+\log\left(\cosh\left(2r\right)\right) \\ \geq \log\left(\cosh\left(2r\right)\right) \end{cases},$$

by (a), (b), so

$$\lim_{r\to\infty}\frac{\log\left(\max_{|z|=r}\left|F_{p,\alpha}\left(z^2\right)\right|\right)}{r} = \lim_{r\to\infty}\frac{\log F_{p,\alpha}\left(-r^2\right)}{r} = 2.$$

(d) Let $t_k = \left(\left(k-\frac{1}{2}\right)\frac{\pi}{2}\right)^2$, $k \geq 1$ and $t_0 < \min\{C_0, t_1\}$, where C_0 is as in (6.2). Recall that

$$x_k \in \left[t_{k-1}, t_k\right], \qquad k \geq 1.$$

Fix $z = x + iy$, and assume that for some $j \geq 2$,

$$x \in [t_{j-1}, t_j).$$

Then

$$k \geq j+1 \Rightarrow |z-x_k| \leq |z-t_k|;$$
$$k \leq j-1 \Rightarrow |z-x_k| \leq |z-t_{k-1}|.$$

(Draw a diagram!). We split

$$\log|F_{p,\alpha}(z)| = \left(\sum_{k=1}^{j-1}+\sum_{k=j}^{j}+\sum_{k=j+1}^{\infty}\right)\left[\log|z-x_k|-\log x_k\right]$$
$$=: T_1+T_2+T_3.$$

Firstly,

$$T_1 \leq \sum_{k=1}^{j-1}\left[\log|z-t_{k-1}|-\log t_{k-1}\right] = \sum_{k=0}^{j-2}\log\left|1-\frac{z}{t_k}\right|.$$

Next,

$$T_3 \leq \sum_{k=j+1}^{\infty} [\log |z - t_k| - \log t_{k-1}] = \left(\sum_{k=j+1}^{\infty} \log \left| 1 - \frac{z}{t_k} \right| \right) - \log t_j.$$

Since

$$\log \left| \cos \left(2\sqrt{z} \right) \right| = \sum_{k=1}^{\infty} \log \left| 1 - \frac{z}{t_k} \right|,$$

we obtain on combining the above estimates, that

$$\begin{aligned} & \log |F_{p,\alpha}(z)| - \log \left| \cos \left(2\sqrt{z} \right) \right| \\ & \leq \log \left| 1 - \frac{z}{t_0} \right| - \sum_{k=j-1}^{j} \log \left| 1 - \frac{z}{t_k} \right| + \log \left| 1 - \frac{z}{x_j} \right| - \log t_j \\ & \leq |\log |z - \zeta|| + \log^+ |z| + C. \end{aligned}$$

Here we have used $x_j \in [t_{j-1}, t_j]$ and $t_j / t_{j-1} \leq 9$. The lower bound is similar. The case $x < t_1$ is similar, but easier.

(e) We use the bound on $F_{p,\alpha}$ from (d) and the maximum-modulus principle. Let $T > 0$, $j \geq 2$, and consider the interval $[\sqrt{t_{j-1}}, \sqrt{t_j}] = \left[\left(j - \frac{3}{2} \right) \frac{\pi}{2}, \left(j - \frac{1}{2} \right) \frac{\pi}{2} \right]$. It contains at most one zero $\sqrt{x_j}$ of $F_{p,\alpha}(z^2)$. Then we can find a point $r \in [\sqrt{t_{j-1}}, \sqrt{t_j}]$ such that

$$\left| r - \sqrt{t_j} \right|, \left| r - \sqrt{t_{j-1}} \right|, \left| r - \sqrt{x_j} \right| \geq \left(\sqrt{t_j} - \sqrt{t_{j-1}} \right) / 4 = \frac{\pi}{8}.$$

Consider the rectangular contour Γ with sides parallel to the x, y axes intersecting the $x-$axis at $\pm r$ and the $y-$axis at $\pm iT$. For $z \in \Gamma$, the above considerations and the estimate of (d) give

$$\begin{aligned} \log \left| F_{p,\alpha}\left(z^2 \right) \right| & \leq \log |\cos (2z)| + \left| \log \frac{\pi}{8} \right| + \log \left(r^2 + T^2 \right) + C \\ & \leq 2T + \log \left(r^2 + T^2 \right) + C. \end{aligned}$$

By the maximum-modulus principle, for $|\mathrm{Re}\,(z)| \leq \sqrt{t_{j-1}}$ and $|\mathrm{Im}\, z| \leq T$,

$$\left| F_{p,\alpha}\left(z^2 \right) \right| \leq C \left(r^2 + T^2 \right) e^{2T} \leq C_1 (t_{j-1}^2 + T^2) e^{2T},$$

since $r^2 / t_{j-1} \leq t_j / t_{j-1} \leq 9$. Finally given $S \geq \sqrt{t_2}$, choose j such that $S \in [\sqrt{t_{j-1}}, \sqrt{t_j}]$. The above and the maximum-modulus principle give for $|\mathrm{Re}\,(z)| \leq S$, $|\mathrm{Im}\,(z)| \leq T$,

$$\left| F_{p,\alpha}\left(z^2 \right) \right| \leq C \left(t_j^2 + T^2 \right) e^{2T} \leq C_1 (S^2 + T^2) e^{2T}$$

as $t_j/S^2 \leq t_j/t_{j-1} \leq 9$. Then with a change of notation, the stated assertion follows. □

For future use, we record a generalization of Theorem 7.2(d):

Lemma 7.3. *Let*

$$f_1(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{\alpha_j}\right) \quad \textit{and} \quad f_2(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{\beta_j}\right)$$

be entire functions with non-negative zeros that weakly interlace, in the sense that

$$0 < \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots.$$

Assume moreover, that

$$\beta_1 < \beta_2 < \beta_3 < \cdots$$

and

$$\sup_j \frac{\beta_{j+1}}{\beta_j} < \infty.$$

Then there exists $C > 0$ such that for all z,

$$|\log|f_1(z)| - \log|f_2(z)|| \leq |\log|z - \zeta|| + \log^+|z| + C, \tag{7.15}$$

where ζ is the closest zero of either f_1 or f_2 to z.

Proof. This is the same as that of Theorem 7.2(d), with the relevant notational changes. □

Next, we consider the properties of $H_{p,\alpha}$, but first need a lemma on (what are essentially) Hilbert transforms. The conclusion is fairly standard, but we could not find the form we need in the literature:

Lemma 7.4. *Let $g : [0,\infty) \to \mathbb{R}$ be continuous, with g' continuous in $(0,\infty)$. Assume moreover, that $g \in L_1[0,\infty)$ and $\frac{g(t)}{t} \in L_1[0,\infty)$, and as $x \to \infty$,*

$$\left|g^{(j)}(x)\right| = O\left(\frac{1}{x^{1+j}}\right), \qquad j = 0, 1. \tag{7.16}$$

Let

$$T(z) = \int_0^\infty \frac{g(t)}{t+z}dt, \qquad z \in \mathbb{C}\backslash(-\infty, 0], \tag{7.17}$$

and define T on $(-\infty,0]$ as a boundary value from the upper-half plane, using the Sokhotkii-Plemelj formulas. Then as $r\to\infty$,

$$\sup_{|z|=r}|T(z)| = O\left(\frac{1}{r}\right). \tag{7.18}$$

Proof. We write $z=x+iy$, and consider two ranges of z:
Case 1: $|y|\geq \frac{1}{4}|z|$.
Then

$$|T(z)|\leq \frac{1}{|y|}\int_0^\infty |g(t)|\,dt \leq \frac{4}{|z|}\int_0^\infty |g(t)|\,dt.$$

Case 2: $|y|<\frac{1}{4}|z|$
In this case, $|x|>\frac{\sqrt{15}}{4}|z|>\sqrt{15}\,|y|$. If x is positive, then

$$|T(z)|\leq \frac{1}{|x|}\int_0^\infty |g(t)|\,dt \leq \frac{4}{\sqrt{15}\,|z|}\int_0^\infty |g(t)|\,dt.$$

So assume $x<0$. We split

$$\begin{aligned} T(z) &= \left(\int_0^{|x|-\frac{1}{2}|z|} + \int_{|x|-\frac{1}{2}|z|}^{|x|+\frac{1}{2}|z|} + \int_{|x|+\frac{1}{2}|z|}^{\infty}\right)\frac{g(t)}{t+z}dt \\ &=: I_1+I_2+I_3. \end{aligned}$$

In I_1 and I_3,

$$|t+z|\geq |t+x|\geq \frac{1}{2}|z|,$$

so

$$|I_1|+|I_3|\leq \frac{2}{|z|}\int_0^\infty |g(t)|\,dt.$$

Next, write

$$\begin{aligned} I_2 &= \int_{|x|-\frac{1}{2}|z|}^{|x|+\frac{1}{2}|z|}\frac{g(t)-g(|x|)}{t+z}dt + g(|x|)\int_{|x|-\frac{1}{2}|z|}^{|x|+\frac{1}{2}|z|}\frac{dt}{t+z} \\ &=: I_{21}+I_{22}. \end{aligned}$$

In I_{21},

$$\left|\frac{g(t)-g(|x|)}{t+z}\right| \leq \left|\frac{g(t)-g(|x|)}{t-|x|}\right|$$
$$\leq \sup\left\{g'(s) : s \in \left[|x|-\frac{1}{2}|z|, |x|+\frac{1}{2}|z|\right]\right\}$$
$$\leq \sup\left\{g'(s) : s \geq \left(\frac{\sqrt{15}}{4}-\frac{1}{2}\right)|z|\right\} = O\left(\frac{1}{|z|^2}\right),$$

by (7.16). So

$$|I_{21}| = O\left(\frac{1}{|z|}\right).$$

Next, the substitution $t = |x| + \frac{u}{2}|z|$ gives

$$I_{22} = g(|x|)\int_{-1}^{-1}\frac{du}{u+\frac{2}{|z|}iy}$$
$$= g(|x|)\, S\left(\frac{2}{|z|}iy\right),$$

where

$$S(w) = \int_{-1}^{1}\frac{du}{u+w}, \qquad |w| < 1.$$

Note that S has well defined boundary values on $(-1,1)$ from the upper and lower half plane. We interpret $S(w)$ as its boundary value from the upper half-plane for $w \in (-1,1)$ and use the fact that $|S|$ is bounded in compact subsets of the open unit ball. Since for this range of z, $\left|\frac{2}{|z|}iy\right| < \frac{1}{2}$, we have

$$|I_{22}| \leq |g(|x|)| \max\left\{|S(w)| : |w| \leq \frac{1}{2}\right\}$$
$$= O\left(\frac{1}{|x|}\right) = O\left(\frac{1}{|z|}\right).$$

The constant in the order term is independent of z (and in particular $|y|$). Then the above estimates for I_1, I_{21} and I_{22} gives the result for $y = \operatorname{Im} z \neq 0$. The Sokotkii-Plemelj formulas show that the estimate for T persists on $(-\infty, 0)$ when we interpret T as a boundary value from the upper half plane. □

Theorem 7.5. *(a) There exists $C > 0$ such that for $|z| \geq 1$,*

$$|H_{p,\alpha}(z)| \leq |z|^{\alpha} + C\frac{|F_{p,\alpha}(z)|}{|z|}. \tag{7.19}$$

Moreover for $r \geq 1$,

$$\max_{|z|=r} |H_{p,\alpha}(z)| \leq C \max_{|z|=r} |F_{p,\alpha}(z)| / r. \tag{7.20}$$

(b) As $|z| \to \infty$,

$$H_{p,\alpha}(z) = z^{\alpha} + \frac{F_{p,\alpha}(z)}{z}\left[\frac{\sin \alpha\pi}{\pi}\int_0^{\infty} \frac{t^{\alpha}}{F_{p,\alpha}(-t)} dt\right]\left(1 + O\left(\frac{1}{z}\right)\right). \tag{7.21}$$

(c) $H_{p,\alpha}(z^2)$ is entire of exponential type 2.

Proof. (a) We know that for $z \in \mathbb{C}\backslash(-\infty, 0]$,

$$H_{p,\alpha}(z) = z^{\alpha} + \frac{\sin \alpha\pi}{\pi} F_{p,\alpha}(z) \int_0^{\infty} \frac{g(t)}{t+z} dt,$$

where

$$g(t) = \frac{t^{\alpha}}{F_{p,\alpha}(-t)}, \qquad t \in (0, \infty)$$

satisfies all the hypotheses of Lemma 7.4, since $|F_{p,\alpha}(-t)| \geq \cosh\left(2\sqrt{t}\right)$, $t \geq 0$, with a similar growth for the derivative. The lemma then gives (7.19). Moreover, (7.20) follows as on the circle $|z| = r$, $\max_{|z|=r} |F_{p,\alpha}(z)|$ grows at least as fast as $\cosh(2\sqrt{r})$.
(b) With g as in (a), we have

$$\begin{aligned} H_{p,\alpha}(z) &= z^{\alpha} + \frac{\sin \alpha\pi}{\pi} F_{p,\alpha}(z)\left[\frac{1}{z}\int_0^{\infty} g(t)\, dt - \frac{1}{z}\int_0^{\infty} \frac{t}{t+z} g(t)\, dt\right] \\ &= z^{\alpha} + \frac{\sin \alpha\pi}{\pi} F_{p,\alpha}(z)\left[\frac{1}{z}\int_0^{\infty} g(t)\, dt + O\left(\frac{1}{z^2}\right)\right], \end{aligned}$$

by the lemma, applied to $tg(t)$.
(c) This follows from (a), (b) and Theorem 7.2(c). □

8 A Proof of Bernstein's Limit Theorem

In this section, we prove that the limit

$$\Lambda^*_{p,\alpha} = \lim_{n\to\infty} n^{\alpha+1/p} E_n \left[|x|^\alpha ; L_p\left[-1,1\right]\right]$$

exists, as well as the fact that any subsequential limit $H_{p,\alpha}$ is a best approximant. The ideas of the proof go back to Bernstein (at least for the case $p=\infty$) and are available in one form in the book of Timan [30, Section 2.6, p. 45. ff; Section 5.4.5, p. 285 ff.], and in more general form in the work of M. Ganzburg (see the survey [9] for references), but we present a proof for the specific context that we need. Recall that

$$\begin{aligned} &A_\sigma \left[|x|^\alpha ; L_p\left(\mathbb{R}\right)\right] \\ &= \inf\left\{\| \, |x|^\alpha - f\left(x\right) \|_{L_p(\mathbb{R})} : f \text{ is entire of exponential type } \le \sigma\right\} \end{aligned}$$

and

$$\begin{aligned} &A_\sigma \left[x^\alpha ; L_{p,\surd}[0,\infty)\right] \\ &= \inf\left\{\|x^\alpha - f\left(x\right)\|_{L_{p,\surd}[0,\infty)} : f\left(x^2\right) \text{ is entire of exponential type } \le \sigma\right\}. \end{aligned}$$

Theorem 8.1. *Let $1 \le p \le \infty$ and $\alpha > -\frac{1}{2p}$, not an integer. Then*

$$\Lambda^*_{p,2\alpha} = \lim_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_n \left[|x|^{2\alpha} ; L_p\left[-1,1\right]\right] \tag{8.1}$$

exists, and

$$\Lambda^*_{p,2\alpha} = 2^{2\alpha+1/p} A_2 \left[|x|^{2\alpha} ; L_p\left(\mathbb{R}\right)\right] = A_1 \left[|x|^{2\alpha} ; L_p\left(\mathbb{R}\right)\right]. \tag{8.2}$$

Moreover,

$$\Lambda_{p,\alpha} = \lim_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_n \left[x^\alpha ; L_{p,\surd}\left[0,1\right]\right] \tag{8.3}$$

exists, and

$$\Lambda_{p,\alpha} = A_2 \left[|x|^{2\alpha} ; L_p\left(\mathbb{R}\right)\right] = A_2 \left[x^\alpha ; L_{p,\surd}[0,\infty)\right]. \tag{8.4}$$

We begin the proof with

Lemma 8.2. *Let $a, b > 0$.*
(a)

$$E_n\left[|x|^{2\alpha}; L_p[-b,b]\right] = \left(\frac{b}{a}\right)^{2\alpha+\frac{1}{p}} E_n\left[|x|^{2\alpha}; L_p[-a,a]\right]. \tag{8.5}$$

(b)

$$n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] = n^{2\alpha+\frac{1}{p}} E_n\left[x^{\alpha}; \sqrt{}; L_p[0,1]\right]. \tag{8.6}$$

Proof. (a) Let $P_{n,a}^*(x)$ denote the best polynomial approximation of degree $\leq n$ to $|x|^{2\alpha}$ in the L_p norm on $[-a,a]$. Then

$$\begin{aligned}
& E_n\left[|x|^{2\alpha}; L_p[-b,b]\right] \\
\leq & \left(\int_{-b}^{b}\left||x|^{2\alpha} - \left(\frac{b}{a}\right)^{2\alpha} P_{n,a}^*\left(\frac{ax}{b}\right)\right|^p dx\right)^{1/p} \\
= & \left(\frac{b}{a}\right)^{2\alpha+\frac{1}{p}}\left(\int_{-a}^{a}\left||t|^{2\alpha} - P_{n,a}^*(t)\right|^p dt\right)^{1/p} \\
= & \left(\frac{b}{a}\right)^{2\alpha+\frac{1}{p}} E_n\left[|x|^{2\alpha}; L_p[-a,a]\right].
\end{aligned}$$

Then (8.5) follows on swopping the roles of a and b.
(b) Since $|x|^{2\alpha}$ is even, its (unique) best polynomial approximations on $[-1,1]$ are also even. Then a substitution $x^2 = t$ easily gives the result. □

Lemma 8.3.

$$\infty > \liminf_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] \geq A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right]. \tag{8.7}$$

Proof. Let us denote the lim inf in the statement by A, and choose a subsequence $\mathcal{S}$ of integers such that

$$A = \lim_{n\to\infty, n\in\mathcal{S}} n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right].$$

Recall that this is finite by the Bernstein-Jackson approximation theorems [7, p. 110, no. 3]. By passing to a further subsequence, we may assume that (7.1) and the associated limits (7.5), (7.7) from Section 7 hold. Then for each $r > 0$, Lemma 8.2(b) followed by a substitution gives

$$\begin{aligned} A &\geq \lim_{n\to\infty, n\in\mathcal{S}} n^{2\alpha+\frac{1}{p}} E_n \left[x^\alpha; \surd; L_p[0,1]\right] \\ &= \lim_{n\to\infty, n\in\mathcal{S}} \left(\int_0^{n^2} \left| x^\alpha - n^{2\alpha} P_{n,p,\alpha}\left(x/n^2\right)\right|^p \frac{dx}{\sqrt{x}} \right)^{1/p} \\ &\geq \lim_{n\to\infty, n\in\mathcal{S}} \left(\int_0^{r} \left| x^\alpha - n^{2\alpha} P_{n,p,\alpha}\left(x/n^2\right)\right|^p \frac{dx}{\sqrt{x}} \right)^{1/p} \\ &= \left(\int_0^{r} \left| x^\alpha - H_{p,\alpha}(x)\right|^p \frac{dx}{\sqrt{x}} \right)^{1/p}. \end{aligned}$$

As this is true for each $r > 0$, we obtain

$$\begin{aligned} A &\geq \left(\int_0^{\infty} \left| x^\alpha - H_{p,\alpha}(x)\right|^p \frac{dx}{\sqrt{x}} \right)^{1/p} \\ &= \| \, |t|^{2\alpha} - H_{p,\alpha}\left(t^2\right) \|_{L_p(\mathbb{R})} \\ &\geq A_2 \left[|x|^{2\alpha}; L_p(\mathbb{R}) \right], \end{aligned} \tag{8.8}$$

since $H_{p,\alpha}(t^2)$ is entire of exponential type ≤ 2 (Theorem 7.5(c)). □

Lemma 8.4. *There exists f^*, entire of exponential type ≤ 2, such that*

$$\| \, |x|^{2\alpha} - f^*(x) \|_{L_p(\mathbb{R})} = A_2 \left[|x|^{2\alpha}; L_p(\mathbb{R}) \right]. \tag{8.9}$$

Proof. We have already shown in the proof above that $A_2 := A_2 \left[|x|^{2\alpha}; L_p(\mathbb{R}) \right]$ is finite. Choose a sequence (f_n) of entire functions of exponential type ≤ 2 such that

$$\| \, |x|^{2\alpha} - f_n(x) \|_{L_p(\mathbb{R})} \leq A_2 + \frac{1}{n}, \qquad n \geq 1.$$

For $n \geq 1$,

$$\|f_n - f_1\|_{L_p(\mathbb{R})} \leq 2A_2 + 2.$$

It then follows that for some $C > 0$ independent of n and z,

$$|f_n - f_1|(z) \leq C \exp(2 |\mathrm{Im}\, z|)$$

(See [17, Theorem 3, p. 38] for $p = \infty$, and [17, eqn. (3), p. 149] for $p < \infty$.) Hence $\{f_n - f_1\}_{n=1}^{\infty}$ is uniformly bounded in each compact set. We can then extract a subsequence converging uniformly in compact sets to some entire function $f^* - f_1$ of exponential type ≤ 2, and bounded on the real axis. For notational simplicity, we assume the full sequence converges. We see that for $r > 0, n \geq 1$,

$$\begin{aligned} &\| \, |x|^{2\alpha} - f^* (x) \, \|_{L_p(-r,r)} \\ &\leq \| \, |x|^{2\alpha} - f_n (x) \, \|_{L_p(\mathbb{R})} + \| f_n (x) - f^* (x) \, \|_{L_p(-r,r)}. \end{aligned}$$

Letting $n \to \infty$ gives

$$\| \, |x|^{2\alpha} - f^* (x) \, \|_{L_p(-r,r)} \leq A_2.$$

Since $r > 0$ is arbitrary, we have

$$\| \, |x|^{2\alpha} - f^* (x) \, \|_{L_p(\mathbb{R})} \leq A_2.$$

The converse inequality is immediate, and (8.9) follows. □

Lemma 8.5. *Let* f^* *be as in* Lemma 8.4. *Then there exists* $C_2 > 0$ *such that*

$$|f^* (z)| \leq C_2 (1 + |z|)^{2\alpha} e^{2|\text{Im}\, z|}, \qquad z \in \mathbb{C}. \tag{8.10}$$

Proof. First note that

$$\| \, |x|^{2\alpha} - H_{p,\alpha} \left(x^2 \right) \|_{L_p(\mathbb{R})} = \| x^\alpha - H_{p,\alpha} (x) \, \|_{L_{p,\sqrt{}}[0,\infty)} < \infty$$

so

$$\| f^* (x) - H_{p,\alpha} \left(x^2 \right) \|_{L_p(\mathbb{R})} < \infty.$$

As $f^* (z) - H_{p,\alpha} (z^2)$ is entire of exponential type at most 2, we obtain the bound throughout the complex plane [17, p. 149, eqn. (3)],

$$\left| f^* (z) - H_{p,\alpha} \left(z^2 \right) \right| \leq C e^{2|y|},$$

with C independent of $z = x + iy$. Next, from (7.14) and (7.19), for $|z| \geq 1$,

$$\left| H_{p,\alpha} \left(z^2 \right) \right| \leq |z|^{2\alpha} + C e^{2|\text{Im}\, z|} \tag{8.11}$$

so

$$|f^* (z)| \leq |z|^{2\alpha} + C e^{2|\text{Im}\, z|}.$$

Then the result follows. □

Lemma 8.6. *Let* f^* *be as in* Lemma 8.4. *Let* $\lambda \in (0,1)$. *Then*

$$\lim_{n\to\infty} E_{2n}\left[f^*; L_p\left[-\lambda n, \lambda n\right]\right] = 0. \tag{8.12}$$

Proof. Fix $\lambda \in (0,1)$. Let $L_{2n}(x)$ denote the Lagrange interpolation polynomial to $f^*(\lambda n x)$ at the zeros of the Chebyshev polynomial $T_{2n}(x)$. Let also $\sigma > 1$. The standard contour integral estimate for the error of interpolation gives

$$\max_{x\in[-1,1]} \left|f^*(\lambda n x) - L_{2n}(x)\right| \leq C \max\left\{\left|\frac{f^*(\lambda n z)}{T_{2n}(z)}\right| : z \in \mathcal{E}_\sigma\right\},$$

where C is independent of f^*, n, λ and $\mathcal{E}_\sigma$ denotes the ellipse with foci at $-1, 1$, intersecting the x and y axis respectively at $\pm\frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)$ and $\pm\frac{1}{2}\left(\sigma - \frac{1}{\sigma}\right)$. To estimate the max, we use the fact that each $z = x + iy \in \mathcal{E}_\sigma$ can be represented in the form

$$z = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)\cos\theta + \frac{i}{2}\left(\sigma - \frac{1}{\sigma}\right)\sin\theta,$$

some $\theta \in [0, 2\pi]$. For such a z, the bound (8.10) gives

$$\left|f^*(\lambda n z)\right| \leq C (\lambda n \sigma)^{2\alpha} \exp\left(n\lambda\left(\sigma - \frac{1}{\sigma}\right)\right)$$

while

$$\left|T_{2n}(z)\right| = \frac{1}{2}\left(\sigma^{2n} + \sigma^{-2n}\right).$$

Hence, at least for large enough n,

$$\max_{x\in[-1,1]} \left|f^*(\lambda n x) - L_{2n}(x)\right| \leq C_5 n^{2\alpha} \exp\left(n\left[\lambda\left(\sigma - \frac{1}{\sigma}\right) - 2\log\sigma\right]\right).$$

As $\lambda < 1$, we can choose $\sigma > 1$ so close to 1 that $\lambda\left(\sigma - \frac{1}{\sigma}\right) - 2\log\sigma < 0$. Thus we obtain for large enough n,

$$\max_{x\in[-1,1]} \left|f^*(\lambda n x) - L_{2n}(x)\right| \leq C_8 \exp\left(-C_9 n\right).$$

Then (8.12) follows. □

Finally for the proof of Theorem 8.1 and later use, we need:

Lemma 8.7. *(a)*

$$\sigma^{\alpha+1/p} A_\sigma \left[|x|^\alpha ; L_p(\mathbb{R}) \right] = A_1 \left[|x|^\alpha ; L_p(\mathbb{R}) \right]. \tag{8.13}$$

Moreover, if $H^{\#}$ is a best approximation in $L_p(\mathbb{R})$ to $|x|^\alpha$ from entire functions of exponential type ≤ 1, then $\sigma^{-\alpha} H^{\#}(\sigma z)$ is a best approximation in $L_p(\mathbb{R})$ to $|x|^\alpha$ from entire functions of exponential type $\leq \sigma$, and conversely.
(b)

$$A_\sigma \left[|x|^\alpha ; L_p(\mathbb{R}) \right] = A_\sigma \left[x^{\alpha/2} ; L_{p,\surd}[0,\infty) \right]. \tag{8.14}$$

Moreover, if $H^{\#}$ is even and a best approximation in $L_p(\mathbb{R})$ to $|x|^\alpha$ from entire functions of exponential type $\leq \sigma$, then $H^{\#}(\sqrt{z})$ is a best approximation in $L_{p,\surd}[0,\infty)$ to $x^{\alpha/2}$ and conversely.

Proof. (a) Suppose that $p < \infty$. Then

$$\begin{aligned} A_1 \left[|x|^\alpha ; L_p(\mathbb{R}) \right] &= \left(\int_{-\infty}^{\infty} \left| |x|^\alpha - H^{\#}(x) \right|^p dx \right)^{1/p} \\ &= \sigma^{\alpha+\frac{1}{p}} \left(\int_{-\infty}^{\infty} \left| |t|^\alpha - \sigma^{-\alpha} H^{\#}(\sigma t) \right|^p dt \right)^{1/p}, \end{aligned}$$

by the substitution $x = \sigma t$. As $\sigma^{-\alpha} H^{\#}(\sigma t)$ is entire of exponential type $\leq \sigma$, this leads to the inequality

$$A_1 \left[|x|^\alpha ; L_p(\mathbb{R}) \right] \geq \sigma^{\alpha+1/p} A_\sigma \left[|x|^\alpha ; L_p(\mathbb{R}) \right].$$

The converse inequality follows similarly.
(b) This follows easily by a substitution in the definition of $A_\sigma \left[|x|^\alpha ; L_p(\mathbb{R}) \right]$. $\square$

Proof of Theorem 8.1. Let f^* be as in Lemma 8.4. Let $\lambda \in (0,1)$. By Lemma 8.2(a),

$$\begin{aligned} &(\lambda n)^{2\alpha+\frac{1}{p}} E_{2n} \left[|x|^{2\alpha} ; L_p[-1,1] \right] \\ &\quad = E_{2n} \left[|x|^{2\alpha} ; L_p[-\lambda n, \lambda n] \right] \\ &\quad \leq E_{2n} \left[|x|^{2\alpha} - f^* ; L_p[-\lambda n, \lambda n] \right] + E_{2n} \left[f^* ; L_p[-\lambda n, \lambda n] \right] \\ &\quad \leq \| \, |x|^{2\alpha} - f^*(x) \, \|_{L_p(\mathbb{R})} + E_{2n} \left[f^* ; L_p[-\lambda n, \lambda n] \right] \\ &\quad = A_2 \left[|x|^{2\alpha} ; L_p(\mathbb{R}) \right] + E_{2n} \left[f^* ; L_p[-\lambda n, \lambda n] \right]. \end{aligned}$$

Using Lemma 8.6, we obtain

$$\limsup_{n\to\infty} (\lambda n)^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] \le A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right].$$

Letting $\lambda \to 1-$ gives

$$\limsup_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] \le A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right].$$

Together with Lemma 8.3, this gives

$$\lim_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] = A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right] \tag{8.15}$$

and hence

$$\begin{aligned}\lim_{n\to\infty} (2n)^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] &= 2^{2\alpha+\frac{1}{p}} A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right] \\ &= A_1\left[|x|^{2\alpha}; L_p(\mathbb{R})\right],\end{aligned}$$

by (8.13). The monotonicity of the errors of approximation ($E_{2n} \le E_{2n-1}$) then gives both (8.1) and (8.2). The remaining assertions follow from (8.6), (8.14) and (8.15). □

For future use, we recall that in the proof of Lemma 8.3 (see (8.8)), we proved

$$\begin{aligned}&\liminf_{n\to\infty} n^{2\alpha+\frac{1}{p}} E_{2n}\left[|x|^{2\alpha}; L_p[-1,1]\right] \\ &\quad \ge \|\, |x|^{2\alpha} - H_{p,\alpha}\left(x^2\right)\|_{L_p(\mathbb{R})} \ge A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right].\end{aligned}$$

We then obtain from (8.15),

$$\|\, |x|^{2\alpha} - H_{p,\alpha}\left(x^2\right)\|_{L_p(\mathbb{R})} = A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right], \tag{8.16}$$

and hence

$$\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\sqrt{}}[0,\infty)} = A_2\left[x^\alpha; L_{p,\sqrt{}}[0,\infty)\right]. \tag{8.17}$$

9 Uniqueness in L_p Approximation, $1 \le p < \infty$

Recall that L_p^σ denotes the set of all entire functions f such that $f\left(x^2\right)$ is entire of exponential type $\le \sigma$, and such that

$$\|f\|_{L_{p,\sqrt{}}[0,\infty)} < \infty.$$

Theorem 9.1. *Let $1 \leq p < \infty$ and $\alpha > -\frac{1}{2p}$, not an integer.*
(i) There exists a unique entire function $H_{p,\alpha}$ such that $H_{p,\alpha}(x^2)$ is entire of exponential type ≤ 2 and

$$\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\surd}[0,\infty)} = A_2\left[x^\alpha; L_{p,\surd}[0,\infty)\right]. \tag{9.1}$$

(ii) $H_{p,\alpha}$ is characterized by the condition that $\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\surd}[0,\infty)}$ is finite and

$$\int_0^\infty |x^\alpha - H_{p,\alpha}(x)|^{p-1} \operatorname{sign}(x^\alpha - H_{p,\alpha}(x)) f(x) \frac{dx}{\sqrt{x}} = 0 \tag{9.2}$$

for all $f \in L_p^2$.

We deduce:

Corollary 9.2. *Let $1 \leq p < \infty$ and $\alpha > -\frac{1}{2p}$, α not an integer. Then all the conclusions of* Theorems 3.1 *and* 3.2 *are true.*

Proof of Theorem 9.1

Step 1 Characterization for $1 < p < \infty$. Let $\mathcal{S}$ be an infinite subsequence of integers as in Section 7, so that (7.1) holds and the conclusion of Theorem 7.1 holds. In Section 8 – see (8.17) – we proved that

$$\left\||x|^{2\alpha} - H_{p,\alpha}\left(x^2\right)\right\|_{L_p(\mathbb{R})} = A_2\left[|x|^{2\alpha}; L_p(\mathbb{R})\right]$$

and

$$\|x^\alpha - H_{p,\alpha}(x)\|_{L_{p,\surd[0,\infty)}} = A_2\left[x^\alpha; L_p[0,\infty)\right].$$

That is, $H_{p,\alpha}(x^2)$ is a b.a. to $|x|^{2\alpha}$ in $L_p(\mathbb{R})$ from all entire functions of exponential type ≤ 2. Equivalently 0 is a b.a. to $|x|^{2\alpha} - H_{p,\alpha}(x^2)$ in $L_p(\mathbb{R})$ from all entire functions of exponential type ≤ 2. By Theorem 2.12.6 in [30, p. 84],

$$\int_{\mathbb{R}} \left||x|^{2\alpha} - H_{p,\alpha}\left(x^2\right)\right|^{p-1} \operatorname{sign}\left(|x|^{2\alpha} - H_{p,\alpha}\left(x^2\right)\right) g(x)\, dx = 0 \tag{9.3}$$

for all $g \in L_p^2$, and this relation characterizes 0 and hence $H_{p,\alpha}(x^2)$. A substitution then gives (9.2). We already proved (9.1) in Section 8, see (8.17). (Note that (9.3) remains valid for $p = 1$ as a necessary condition for best approximation.)

Step 2 Uniqueness. We have already shown that $H_{p,\alpha}$ is a best approximant that satisfies the orthogonality condition (9.2). Suppose that $H^{\#}$ is another best approximant. Then

$$\|H^{\#} - H_{p,\alpha}\|_{L_{p,\sqrt{}}[0,\infty)} \leq \|H^{\#}(x) - x^{\alpha}\|_{L_{p,\sqrt{}}[0,\infty)} + \|x^{\alpha} - H_{p,\alpha}(x)\|_{L_{p,\sqrt{}}[0,\infty)},$$

so $H^{\#} - H_{p,\alpha} \in L_p^2$. Our orthogonality condition (9.2) gives

$$0 = \int_0^{\infty} |x^{\alpha} - H_{p,\alpha}(x)|^{p-1} \operatorname{sign}(x^{\alpha} - H_{p,\alpha}(x)) \left(H^{\#} - H_{p,\alpha}\right)(x) \frac{dx}{\sqrt{x}}$$

and hence

$$\begin{aligned} &\int_0^{\infty} |x^{\alpha} - H_{p,\alpha}(x)|^{p} \frac{dx}{\sqrt{x}} \\ &= \int_0^{\infty} |x^{\alpha} - H_{p,\alpha}(x)|^{p-1} \operatorname{sign}(x^{\alpha} - H_{p,\alpha}(x)) \left(x^{\alpha} - H^{\#}(x)\right) \frac{dx}{\sqrt{x}}. \end{aligned}$$

This can be re-expressed as the case of equality in Holder's inequality. Indeed, letting q be the dual parameter of p (that is $\frac{1}{p} + \frac{1}{q} = 1$),

$$\begin{aligned} \psi(x) &= |x^{\alpha} - H_{p,\alpha}(x)|^{p-1} \operatorname{sign}(x^{\alpha} - H_{p,\alpha}(x)); \\ \phi(x) &= x^{\alpha} - H^{\#}(x), \end{aligned}$$

and using that $x^{\alpha} - H_{p,\alpha}$ and $x^{\alpha} - H^{\#}$ have equal norm in $L_{p,\sqrt{}}[0,\infty)$, we obtain

$$\begin{aligned} \|\psi\|_{L_{q,\sqrt{}}[0,\infty)} \|\phi\|_{L_{p,\sqrt{}}[0,\infty)} &= \int_0^{\infty} |x^{\alpha} - H_{p,\alpha}(x)|^{p} \frac{dx}{\sqrt{x}} \\ &= \int_0^{\infty} \psi(x)\,\phi(x) \frac{dx}{\sqrt{x}}. \end{aligned}$$

This forces [14, p. 410], [18, p. 46] for some positive constant c,

$$\psi(x) = c\,|\phi(x)|^{p-1} \operatorname{sign}(\phi(x)),$$

a.e. in $[0,\infty)$. That is, we have a.e.

$$\begin{aligned} &|x^{\alpha} - H_{p,\alpha}(x)|^{p-1} \operatorname{sign}(x^{\alpha} - H_{p,\alpha}(x)) \\ &\quad = c\left|x^{\alpha} - H^{\#}(x)\right|^{p-1}(x) \operatorname{sign}\left(x^{\alpha} - H^{\#}(x)\right) \\ &\quad \Rightarrow \operatorname{sign}\left(x^{\alpha} - H^{\#}(x)\right) = \operatorname{sign}(x^{\alpha} - H_{p,\alpha}(x)) \text{ a.e.} \qquad (9.4) \end{aligned}$$

Then also, we see that $c = 1$, and if $p > 1$,

$$x^\alpha - H_{p,\alpha}(x) = x^\alpha - H^{\#}(x) \text{ a.e.,}$$

whence $H^{\#} = H_{p,\alpha}$ because of analyticity. This completes the proof of Theorem 9.1 for $p > 1$.

Step 3 Uniqueness and Characterization for $p = 1$. Here we still know that $H_{p,\alpha}$ satisfies (9.3), and hence (9.4) is true, but this does not imply the required uniqueness. So we need an extra argument. Recall from Theorem 5.1(c) that the zeros of $x^\alpha - P_{n,1,\alpha}(x)$ are

$$x_{jn,1,\alpha} = \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n+3}\right)$$

and hence

$$x_j = \lim_{n\to\infty, n\in\mathcal{S}} n^2 x_{jn,1,\alpha} = \left(\left(j - \frac{1}{2}\right)\frac{\pi}{2}\right)^2,$$

so

$$F_{1,\alpha}(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{\left(\left(j-\frac{1}{2}\right)\frac{\pi}{2}\right)^2}\right) = \cos\left(2\sqrt{z}\right).$$

Theorem 7.1 shows that $x^\alpha - H_{1,\alpha}(x)$ changes sign in $(0,\infty)$ exactly at the zeros of $\cos(2\sqrt{z})$. Thus $x^\alpha - H^{\#}(x)$ also changes sign exactly at the zeros of $F_{1,\alpha}(z) = \cos(2\sqrt{z})$. Let

$$f(x) = H_{1,\alpha}\left(x^2\right) - H^{\#}\left(x^2\right).$$

This is entire of exponential type at most 2, has $\|f\|_{L_1(\mathbb{R})} < \infty$ and has zeros at the zeros of $\cos(2z)$ on the real line. Then [17, p. 149, eqn. (3)] there exists $C > 0$ such that for all complex z,

$$|f(z)| \le Ce^{2|\operatorname{Im} z|}.$$

In particular, f is bounded on the real axis, and satisfies 4.3(5), 4.3(6) and 4.3(7) in [30, pp. 180-181]. We then have the convergent interpolation formula [30, p. 183, eqn. 4.3(13)]

$$f(z) = \frac{\sin\left(2\left(z - \frac{\pi}{4}\right)\right)}{2} f'\left(\frac{\pi}{4}\right) + \sum_{k=-\infty}^{\infty} f\left(k\frac{\pi}{2} + \frac{\pi}{4}\right) g_k(z),$$

where the $\{g_k\}$ are explicitly given functions. In our case $f\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)=\cos\left(2\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)\right)=0$ for all k, so

$$f(z)=-\frac{\cos(2z)}{2}f'\left(\frac{\pi}{4}\right).$$

Integrability of f over the real line then gives $f'\left(\frac{\pi}{4}\right)=0$ and hence $f=0$, as desired. Note that we actually proved the following: any entire function $H^{\#}$ with $H^{\#}(x^2)$ of exponential type ≤ 2 that satisfies the orthogonality condition (9.2) equals $H_{1,\alpha}$. Hence (9.2) characterizes the b.a. even for $p=1$. □

We note that one can avoid use of the interpolation formula, instead applying Lemma 10.5 below.

Proof of Corollary 9.2
Proof of Theorem 3.1 for $1\leq p<\infty$. As any subsequential limit $H_{p,\alpha}$ of $\{n^{2\alpha}P_{n,p,\alpha}(z/n^2)\}_{n=1}^{\infty}$ is a b.a. (recall (8.17)) and b.a.'s are unique, we obtain

$$\lim_{n\to\infty}n^{2\alpha}P_{n,p,\alpha}\left(z/n^2\right)=H_{p,\alpha}(z)$$

uniformly in compact subsets of the plane. Then all the assertions of Theorem 3.1 follow from Theorem 7.1 and (7.1).

Proof of Theorem 3.2 for $1\leq p<\infty$. The uniqueness of $H_{p,\alpha}$ has just been established, as has the orthogonality (3.8). The representation (3.7) was established in Theorem 7.1. It remains to prove (3.9). From (8.3) and (8.4) of Theorem 8.1, and then (8.17),

$$\begin{aligned}\Lambda_{p,\alpha}&=\lim_{n\to\infty}n^{2\alpha+\frac{1}{p}}E_n[x^\alpha;L_{p,\surd}[0,1]]\\&=A_2\left[x^\alpha;L_{p,\surd}[0,\infty)\right]\\&=\|x^\alpha-H_{p,\alpha}(x)\|_{L_{p,\surd}[0,\infty)}.\end{aligned}$$

Now all we need to do is to substitute (3.7) into this last formula. □

10 An Interpolation Series for $p=\infty$

In this section, we establish representations by interpolation series at the zeros of $F_{\infty,\alpha}$ and also establish some estimates involving $F_{\infty,\alpha}$. Our main result summarizes some of these. Throughout we assume that we have a subsequence $\mathcal{S}$ as in Section 7, so that

$$x_j = \lim_{n\to\infty, n\in\mathcal{S}} n^2 x_{jn}, \qquad j \geq 1,$$

exists, and uniformly in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} X_{n,\infty,\alpha}\left(z/n^2\right)/X_{n,\infty,\alpha}(0) = F_{\infty,\alpha}(z).$$

Then as $n \to \infty$ through the same subsequence, the scaled alternation points also converge (because the differentiated sequence $\left\{\frac{d}{dz} n^{2\alpha} R_{n,\infty,\alpha}\left(z/n^2\right)\right\}_{n\in\mathcal{S}}$ converges), so that

$$y_j = \lim_{n\to\infty, n\in\mathcal{S}} n^2 y_{jn,\alpha} \tag{10.1}$$

exists for $j \geq 0$. In view of the interlacing properties in Theorem 5.2(b),

$$\sin^2 \frac{(j-1)\pi}{2n} = s_{j-1,n,\infty} \leq y_{jn,\alpha} \leq s_{jn,\infty} = \sin^2 \frac{j\pi}{2n} \tag{10.2}$$

and hence, if

$$s_j = \left(\frac{j\pi}{2}\right)^2, \qquad j \geq 0, \tag{10.3}$$

then

$$s_{j-1} \leq y_j \leq s_j, \qquad j \geq 1. \tag{10.4}$$

The main results in this section are:

Theorem 10.1. *Let $\alpha > 0$ and f be an entire function such that both*
(i) for some $C > 0$ and large enough $|z|$,

$$\left|f\left(z^2\right)\right| \leq e^{2|\mathrm{Im}\, z|} |z|^C. \tag{10.5}$$

(ii) for some $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$|f(x)| = O\left(x^{1-\varepsilon}\right), x \to \infty. \tag{10.6}$$

Then in the plane,

$$f(z) = F_{\infty,\alpha}(z) \sum_{j=1}^{\infty} \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(z - x_j)}. \tag{10.7}$$

Theorem 10.2. *Let $\alpha > 0$. Uniformly for $j \geq 1$,*
(a)

$$(-1)^j F'_{\infty,\alpha}(x_j) \geq C\sqrt{x_j}. \tag{10.8}$$

(b)

$$F_{\infty,\alpha}(y_j) \sim (-1)^j y_j. \tag{10.9}$$

(c)

$$y_j - x_j \sim x_j - y_{j-1} \sim \sqrt{y_j} \sim y_j - y_{j-1}. \tag{10.10}$$

We shall first establish some estimates involving the polynomials $X_{n,\infty,\alpha}$, making essential use of the equioscillation points $\{y_{jn,\alpha}\}_{j=0}^{n+1}$. In this section, since α is fixed, and we treat only the L_∞ case, we use the abbreviations

$$X_n(z) = X_{n,\infty,\alpha}(z)\,;\; R_n(z) = R_{n,\infty,\alpha}(z)\,;$$
$$x_{jn} = x_{jn,\infty,\alpha};\; y_{jn} = y_{jn,\alpha};\; s_{jn} = s_{jn,\infty}.$$

Lemma 10.3. *Let $\beta > 0$ and $\varepsilon > 0$. Then for $n \geq 1$,*

$$\int_0^\infty \frac{X_n(0)}{X_n(-s)} s^{\beta-1} ds \sim \int_0^\infty \frac{X_n(0)}{X_n(-s)} \frac{s^{\beta-1}}{1+s} ds \sim \int_{\varepsilon n^{-2}}^\infty \frac{X_n(0)}{X_n(-s)} s^{\beta-1} ds$$
$$\sim \int_0^{\varepsilon n^{-2}} \frac{X_n(0)}{X_n(-s)} s^{\beta-1} ds \sim n^{-2\beta}. \tag{10.11}$$

Proof. Let J_n denote the first integral in (10.11) and I_n denote the last. We see that

$$\begin{aligned} I_n &= \int_0^{\varepsilon n^{-2}} \frac{X_n(0)}{X_n(-s)} s^{\beta-1} ds \\ &= n^{-2\beta} \int_0^\varepsilon \left[\prod_{j=1}^{n+1} \left(1 + \frac{u}{n^2 x_{jn}}\right)^{-1}\right] u^{\beta-1} du. \end{aligned} \tag{10.12}$$

Here by Fatou's lemma,

$$\begin{aligned} \liminf_{n\to\infty} n^{2\beta} I_n &\geq \int_0^\varepsilon \liminf_{n\to\infty} \left\{\prod_{j=1}^{n+1} \left(1 + \frac{u}{n^2 x_{jn}}\right)^{-1} u^{\beta-1}\right\} du \\ &\geq \int_0^\varepsilon \left[\prod_{j=1}^{\infty} \left(1 + \frac{u}{\liminf_{n\to\infty} n^2 x_{jn}}\right)^{-1}\right] u^{\beta-1} du \\ &\geq \int_0^\epsilon \left[\prod_{j=1}^{\infty} \left(1 + \frac{u}{\max\left\{C_0, \left[\left(j - \frac{3}{2}\right)\frac{\pi}{2}\right]^2\right\}}\right)^{-1}\right] u^{\beta-1} du, \end{aligned}$$

by (7.1) to (7.3). (Recall that C_0 there is independent of the subsequence $\mathcal{S}$.) Moreover, by Corollary 5.3,

$$\begin{aligned} n^{2\beta} J_n &= \int_0^\infty \left[\prod_{j=1}^{n+1} \left(1 + \frac{u}{n^2 x_{jn}}\right)^{-1} \right] u^{\beta-1} du \\ &\leq \int_0^\infty \left[\prod_{j=1}^{n+1} \left(1 + \frac{u}{\left(\left(j - \frac{1}{2}\right) \frac{\pi}{2}\right)^2}\right)^{-1} \right] u^{\beta-1} du \\ &\to \int_0^\infty \left[\prod_{j=1}^{\infty} \left(1 + \frac{u}{\left(\left(j - \frac{1}{2}\right) \frac{\pi}{2}\right)^2}\right)^{-1} \right] u^{\beta-1} du, \end{aligned}$$

$n \to \infty$, by Lebesgue's Monotone Convergence Theorem. As $I_n \leq J_n$, we obtain

$$n^{2\beta} I_n \sim n^{2\beta} J_n \sim 1.$$

This gives the first and last $\sim$ relations in (10.11). The other two are similar. □

Now we can prove:

Lemma 10.4. *(a)*

$$X_n(1) \sim n^2 X_n(0) (-1)^{n+1}. \tag{10.13}$$

(b) Uniformly for $n \geq 1$ and $1 \leq j \leq n$,

$$(-1)^j X_n(y_{jn}) \sim X_n(0)\, n^2 y_{jn}. \tag{10.14}$$

(c) There exists $C > 0$ such that for $n \geq 1$ and $x \in [0,1]$,

$$\left| \frac{X_n(x)}{X_n(1)} \right| \leq x + (1-x) \frac{C}{n^2}. \tag{10.15}$$

(d) Uniformly in j and n,

$$\left| \frac{X_n'(x_{jn})}{n^2 X_n(0)} \right| \geq C n \sqrt{x_{jn}}. \tag{10.16}$$

(e) Uniformly for $n \geq 1$ and j such that $y_{jn} \leq \frac{1}{2}$, we have

$$y_{jn} - x_{jn} \sim x_{jn} - y_{j-1,n} \sim \sqrt{y_{jn}}/n. \tag{10.17}$$

Proof. (a) Now from (4.3) and as 0 and 1 are alternation points,

$$1=\left|\frac{R_n(0)}{R_n(1)}\right|=\left|\frac{X_n(0)}{X_n(1)}\right|\frac{\int_0^\infty \frac{s^{\alpha-1}}{X_n(-s)}ds}{\int_0^\infty \frac{s^\alpha}{X_n(-s)}\frac{ds}{s+1}}\sim\left|\frac{X_n(0)}{X_n(1)}\right|n^2,$$

by Lemma 10.3. Finally, $X_n(0)$ has sign $(-1)^{n+1}$.

(b)

$$\begin{aligned}1&=(-1)^j\frac{R_n(y_{jn})}{R_n(0)}=(-1)^j\frac{X_n(y_{jn})}{X_n(0)}\frac{\int_0^\infty \frac{s^\alpha}{X_n(-s)}\frac{ds}{s+y_{jn}}}{\int_0^\infty \frac{s^{\alpha-1}}{X_n(-s)}ds}\\&\le\frac{1}{y_{jn}}(-1)^j\frac{X_n(y_{jn})}{X_n(0)}\frac{\int_0^\infty \frac{s^\alpha}{X_n(-s)}ds}{\int_0^\infty \frac{s^{\alpha-1}}{X_n(-s)}ds}\le\frac{C(-1)^j}{y_{jn}}\frac{X_n(y_{jn})}{n^2X_n(0)},\end{aligned}$$

by Lemma 10.3. Also for $j\ge1$, $y_{jn}\ge x_{1n}\ge C_0n^{-2}$, so

$$\begin{aligned}1&\ge(-1)^j\frac{X_n(y_{jn})}{X_n(0)}\frac{\int_0^{C_0n^{-2}}\frac{s^\alpha}{X_n(-s)}\frac{ds}{2y_{jn}}}{\int_0^\infty \frac{s^{\alpha-1}}{X_n(-s)}ds}\\&\ge\frac{C(-1)^j}{y_{jn}}\frac{X_n(y_{jn})}{n^2X_n(0)},\end{aligned}$$

by Lemma 10.3 again. Thus

$$(-1)^jX_n(y_{jn})\sim n^2X_n(0)\,y_{jn}.$$

(c) Now

$$\begin{aligned}\frac{X_n(x)}{X_n(1)}-x\frac{R_n(x)}{R_n(1)}&=\frac{X_n(x)}{X_n(1)}\left\{1-\frac{\int_0^\infty\frac{s^\alpha}{X_n(-s)}\frac{x}{s+x}ds}{\int_0^\infty\frac{s^\alpha}{X_n(-s)}\frac{1}{s+1}ds}\right\}\\&=\frac{X_n(x)}{X_n(1)}\frac{\int_0^\infty\frac{s^\alpha}{X_n(-s)}\frac{s(1-x)}{(s+1)(s+x)}ds}{\int_0^\infty\frac{s^\alpha}{X_n(-s)}\frac{1}{s+1}ds}.\end{aligned}$$

Here for $x\in[0,1]$,

$$\begin{aligned}&\int_0^\infty\frac{s^\alpha}{|X_n(-s)|}\frac{s(1-x)}{(s+1)(s+x)}ds\\&\le(1-x)\min\left\{\int_0^\infty\frac{s^\alpha}{|X_n(-s)|}\frac{ds}{1+s},\frac{1}{x}\int_0^\infty\frac{s^{\alpha+1}}{X_n(-s)}\frac{ds}{1+s}\right\},\end{aligned}$$

so using Lemma 10.3,

$$\left|\frac{X_n(x)}{X_n(1)} - x\frac{R_n(x)}{R_n(1)}\right| \leq \left|\frac{X_n(x)}{X_n(1)}\right|(1-x)\min\left\{1, \frac{C}{n^2 x}\right\}.$$

Then,

$$\left|\frac{X_n(x)}{X_n(1)}\right|\left[1-(1-x)\min\left\{1, \frac{C}{n^2 x}\right\}\right] \leq x\left|\frac{R_n(x)}{R_n(1)}\right| \leq x.$$

(Recall that 1 is an alternation point.) We deduce that for $x \geq \frac{2C}{n^2}$,

$$\left|\frac{X_n(x)}{X_n(1)}\right| \leq \frac{x}{1-(1-x)\frac{C}{n^2 x}} \leq x + (1-x)\frac{2C}{n^2},$$

by the inequality $\frac{1}{1-u} \leq 1+2u, u \in \left[0, \frac{1}{2}\right]$. For $0 \leq x \leq \frac{2C}{n^2}$, we instead use

$$\begin{aligned}1 \geq \left|\frac{R_n(x)}{R_n(0)}\right| &= \left|\frac{X_n(x)}{X_n(0)}\right| \int_0^\infty \frac{t^\alpha}{X_n(-t)}\frac{dt}{t+x} / \int_0^\infty \frac{t^{\alpha-1}}{X_n(-t)}dt \\ &\geq \left|\frac{X_n(x)}{X_n(0)}\right| \frac{1}{2}\int_x^\infty \frac{t^{\alpha-1}}{X_n(-t)}dt / \int_0^\infty \frac{t^{\alpha-1}}{X_n(-t)}dt \\ &\geq C_1\left|\frac{X_n(x)}{X_n(0)}\right|,\end{aligned}$$

by Lemma 10.3. Thus for such x,

$$\begin{aligned}|X_n(x)| &\leq C_2 |X_n(0)| \\ &\leq C_3 X_n(1) n^{-2} \leq C_4 X_n(1)\frac{1-x}{n^2},\end{aligned}$$

by (a) of this lemma. So we again obtain (10.15).
(d) As X_n has $n+1$ simple zeros, X_n' has n simple zeros that interlace those of X_n. Let us denote these zeros by

$$x_{jn}' \in (x_{jn}, x_{j+1,n}), \qquad 1 \leq j \leq n.$$

Similarly X_n'' has $n-1$ simple zeros that interlace those of X_n'. Now let us fix $j \geq 2$, and consider x_{jn}. As $\left(x_{j-1,n}', x_{jn}'\right)$ contains exactly one simple zero of X_n'', either the interval $(x_{j-1,n}', x_{jn})$ or the interval (x_{jn}, x_{jn}') does not contain a zero of X_n''. Suppose $(x_{j-1,n}', x_{jn})$ does not contain a zero (the other case is similar). Then X_n is concave up or concave down in $(x'_{j-1,n}, x_{jn})$. Suppose, for example, it is concave down (the other case is similar). Then necessarily,

$$X'_n(x_{jn}) < \frac{X_n(x_{jn}) - X_n\left(x'_{j-1,n}\right)}{x_{jn} - x'_{j-1,n}} < 0,$$

so

$$|X'_n(x_{jn})| > \frac{\left|X_n\left(x'_{j-1,n}\right)\right|}{x_{jn} - x'_{j-1,n}}.$$

But $X'_n\left(x'_{j-1,n}\right) = 0$, so $|X_n|$ has its unique maximum in $(x_{j-1,n}, x_{jn})$ at $x'_{j-1,n}$. In particular, as the alternation point $y_{j-1,n}$ lies in this interval,

$$\left|X_n\left(x'_{j-1,n}\right)\right| \geq |X_n(y_{j-1,n})| \geq C y_{j-1,n} n^2 |X_n(0)| \geq C x_{jn} n^2 |X_n(0)|,$$

by (10.14) and as Corollary 5.3 shows that $x_{jn} \geq C x_{j-1,n}$. Hence

$$\frac{|X'_n(x_{jn})|}{n^2 |X_n(0)|} \geq \frac{C x_{jn}}{x_{jn} - x_{j-1,n}}.$$

Now we know from Corollary 5.3 that

$$\begin{aligned} x_{jn} - x_{j-1,n} &\leq \sin^2\left(\left(j - \frac{1}{2}\right)\frac{\pi}{2n}\right) - \sin^2\left(\left(j - \frac{5}{2}\right)\frac{\pi}{2n}\right) \\ &\leq \frac{C}{n}\sqrt{x_{jn}}, \end{aligned}$$

uniformly in $n \geq 1$ and $2 \leq j \leq n$. (If $j = 2$, we take the lower bound for x_{1n} as 0.) Then

$$\frac{|X'_n(x_{jn})|}{n^2 |X_n(0)|} \geq Cn\sqrt{x_{jn}}.$$

The case $j = 1$ is not included above. For this case, we note that as the smallest zero of X''_n lies in (x'_{1n}, x'_{2n}), we shall have that X''_n is of one sign in (x_{1n}, x'_{1n}), so is concave up or concave down there. Then we can proceed as before.

(e) From (c), for all $x \in [0,1]$,

$$\begin{aligned} \left|\frac{(1-x)X_n(x)}{n^2 X_n(1)}\right| &\leq C_1\left\{\left[\frac{\sqrt{x(1-x)}}{n}\right]^2 + \left[\frac{1}{n^2}\right]^2\right\} \\ &\leq C_2\left\{\frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right\}^2. \end{aligned}$$

By an inequality of Brudnyi and Lebed [6, p. 241, Theorem 2.3],

$$\left|\frac{d}{dx}\left\{\frac{(1-x)X_n(x)}{n^2X_n(1)}\right\}\right| \leq C\left\{\frac{\sqrt{x(1-x)}}{n}+\frac{1}{n^2}\right\}.$$

From this, and using (10.15), we derive the bound

$$\left|\frac{X_n'(x)}{X_n(1)}\right| \leq C\left(n\sqrt{\frac{x}{1-x}}+\frac{1}{1-x}\right), \qquad x \in [0,1].$$

Then

$$\begin{aligned}\left|\frac{X_n(y_{jn})}{X_n(1)}\right| &= \left|\int_{x_{jn}}^{y_{jn}} \frac{X_n'(x)}{X_n(1)}dx\right| \\ &\leq C\left[n\sqrt{\frac{y_{jn}}{1-y_{jn}}}+\frac{1}{1-y_{jn}}\right](y_{jn}-x_{jn}).\end{aligned}$$

Using (a) and (b), we obtain if $y_{jn} \leq \frac{1}{2}$, and since $y_{1n} > x_{1n} \geq C_0n^{-2}$,

$$y_{jn} \leq Cn\sqrt{y_{jn}}\,(y_{jn}-x_{jn})$$

so

$$y_{jn}-x_{jn} \geq C\sqrt{y_{jn}}/n.$$

The corresponding upper bound follows easily from (5.6) and (5.8). The first $\sim$ relation in (10.17) follows. The remaining $\sim$ relation in (10.17) is similar. □

Proof of Theorem 10.2. (a) From the locally uniform limit

$$\lim_{n\to\infty, n\in\mathcal{S}} X_n\left(z/n^2\right)/X_n(0) = F_{\infty,\alpha}(z)$$

we deduce that locally uniformly

$$\lim_{n\to\infty, n\in\mathcal{S}} n^{-2}X_n'\left(z/n^2\right)/X_n(0) = F_{\infty,\alpha}'(z).$$

Then because of the uniform convergence

$$\left|F_{\infty,\alpha}'(x_j)\right| = \lim_{n\to\infty, n\in\mathcal{S}} n^{-2}\left|X_n'\left(n^2x_{jn}/n^2\right)/X_n(0)\right| \geq C\sqrt{x_j},$$

by Lemma 10.4(d). As $F_{\infty,\alpha}(0) = 1$, and $F_{\infty,\alpha}$ has sign changes at each x_j, we see that $F'_{\infty,\alpha}(x_j)$ has sign $(-1)^j$.

(b) Recall from Lemma 10.4(b) that

$$(-1)^j X_n(y_{jn}) / X_n(0) \sim n^2 y_{jn}.$$

Then because of locally uniform convergence,

$$F_{\infty,\alpha}(y_j) = \lim_{n\to\infty, n\in\mathcal{S}} X_n\left(n^2 y_{jn}/n^2\right)/X_n(0) \sim (-1)^j y_j.$$

(c) This follows from Lemma 10.4(e):

$$n^2 y_{jn} - n^2 x_{jn} \sim n^2 x_{jn} - n^2 y_{j-1,n} \sim \sqrt{n^2 y_{jn}},$$

giving as $n \to \infty$ through $\mathcal{S}$,

$$y_j - x_j \sim x_j - y_{j-1} \sim \sqrt{y_j}.$$

Finally, we see from (10.4) that.

$$\sqrt{y_j} \geq Cj \geq C_1(y_j - y_{j-1}).$$

Then the last $\sim$ relation in (10.10) follows. □

We need one more growth lemma:

Lemma 10.5. *Let g be entire of exponential type, and assume that for some C_1, C_2, C_3 and $|\mathrm{Im}\, z| \geq C_3$,*

$$|g(z)| \leq C_1 (1 + |z|)^{C_2}. \tag{10.18}$$

Then g is a polynomial.

Proof. We shall show that the estimate in (10.18) holds even for $|\mathrm{Im}\, z| \leq C_3$. Then Liouville's Theorem gives the desired result. We use the indicator function

$$h_g(\theta) := \limsup_{r\to\infty} \frac{\log\left|g\left(re^{i\theta}\right)\right|}{r}, \qquad \theta \in [0, 2\pi].$$

Our bound above shows that

$$h_g(\theta) \leq 0, \qquad \theta \in (0,\pi) \cup (\pi, 2\pi).$$

As g is entire, it is known that h_g is continuous [17, p. 55], [13, p. 465], and hence

$$h_g \leq 0 \text{ in } [0, 2\pi].$$

By Theorem 2 in [17, p. 56], given $\varepsilon > 0$, there exists r_ε such that for $|z| \geq r_\varepsilon$, and all θ

$$\left|g\left(re^{i\theta}\right)\right| \leq e^{r(h_g(\theta)+\varepsilon)} \leq e^{\varepsilon r}.$$

We use this estimate, (10.18) and subharmonicity of $\log|g|$ to prove (10.18) for $|\mathrm{Im}\, z| \leq C_3$. Suppose that $z = x + iy$ with $|y| \leq C_3 \leq \frac{1}{4}|x|$. Then

$$\begin{aligned}
\log|g(z)| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|g\left(z + |x| e^{it}\right)\right| dt \\
&\leq \frac{1}{2\pi}\int_{\{t\in[-\pi,\pi]:|x\sin t|\geq 2C_3\}} \log\left[C_1\left(1 + \left|z + |x| e^{it}\right|\right)^{C_2}\right] dt \\
&\quad + \frac{1}{2\pi}\int_{\{t\in[-\pi,\pi]:|x\sin t|< 2C_3\}} \varepsilon\left|z + |x| e^{it}\right| dt.
\end{aligned}$$

In the second integral, the range of integration is contained in intervals with endpoints $0, \pm\pi$, and length $O\left(\frac{1}{|x|}\right)$. Hence,

$$\log|g(z)| \leq C_3 + C_4 \log|z| + C_5\frac{|z|}{|x|} \leq C\log(1 + |z|).$$

Then (10.18) follows for $|\mathrm{Im}\, z| \leq C_3$ with $|z|$ large enough. (For small $|z|$, we can just use continuity of g.) $\square$

Proof of Theorem 10.1. We break the proof into three steps.

Step 1. The Interpolation Series

Let

$$L(z) = F_{\infty,\alpha}(z)\sum_{j=1}^{\infty} \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(z - x_j)}.$$

We first show that this series converges uniformly in compact subsets of the plane. Suppose that $|z| \leq r$. For j such that $x_j \geq 2r$,

$$\left|\frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(z - x_j)}\right| \leq Cx_j^{-\left(\frac{1}{2}+\varepsilon\right)},$$

by (10.6) and (10.8). As $x_j \geq Cj^2$, we have

$$\sum_{j:x_j \geq 2r} \left| \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(z - x_j)} \right| \leq C \sum_{j:x_j \geq 2r} j^{-1-2\varepsilon} \leq Cr^{-\varepsilon}. \tag{10.19}$$

Then the uniform convergence follows (for example, via the Weierstrass $M-$test).

Step 2. The function g
Let

$$g(z) = \left(f\left(z^2\right) - L\left(z^2\right) \right) / F_{\infty,\alpha}\left(z^2\right).$$

This is entire, and we shall apply Lemma 10.5 to show that g is a polynomial. Firstly, our hypotheses show that $f(z^2)$ is entire of exponential type ≤ 2. We claim the same is true of $L(z^2)$. To see this, let $r \geq 1$ and suppose that $|z|^2 \leq r$, but $|z^2 - x_j| \geq 1$ for all j. We see that

$$\begin{aligned} \sum_{j:x_j < 2r} \left| \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(z^2 - x_j)} \right| &\leq C \sum_{j:x_j < 2r} x_j^{\frac{1}{2}-\varepsilon} \\ &\leq C \sum_{j:j < C_1\sqrt{2r}} j^{1-2\varepsilon} \leq Cr. \end{aligned}$$

Replacing z by z^2 in (10.19) and combining with this gives

$$\left| L\left(z^2\right) \right| \leq Cr \left| F_{\infty,\alpha}\left(z^2\right) \right|, \tag{10.20}$$

for all $|z| \leq \sqrt{r}$ such that $|z^2 - x_j| \geq 1$ for all $j \geq 1$. Since $x_j - x_{j-1} \sim j$, we can use the maximum modulus principle as in Theorem 7.2(e) to show that

$$\sup_{|z| \leq \sqrt{r}} \left| L\left(z^2\right) \right| \leq Cr \sup_{|z| \leq \sqrt{r}} \left| F_{\infty,\alpha}\left(z^2\right) \right|.$$

As $F_{\infty,\alpha}(z^2)$ is of exponential type 2 (Theorem 7.2(c)), the same is then true of $L(z^2)$. As $f(z^2) - L(z^2)$ and $F_{\infty,\alpha}(z^2)$ are of exponential type, their quotient g is also [17, p. 13, Theorem 1]. We next show that g satisfies (10.18). For $|\mathrm{Im}\, z| \geq 1$, we have $|z^2 - x_j| \geq 1$, so by (10.20),

$$\left| L\left(z^2\right) \right| \leq C|z| \left| F_{\infty,\alpha}\left(z^2\right) \right|. \tag{10.21}$$

Moreover from our hypothesis (10.5) and Theorem 7.2(d),

$$\begin{aligned} \log \left| f\left(z^2\right) / F_{\infty,\alpha}\left(z^2\right) \right| &\leq 2 |\mathrm{Im}\, z| \\ &+ C \log|z| - \log|\cos(2z)| + \log^+ |z| + \left| \log\left| z^2 - \zeta \right| \right| + C, \end{aligned}$$

where ζ is the closest zero of $F_{\infty,\alpha}$ or $\cos\left(2\sqrt{\cdot}\right)$ to z^2. By an elementary calculation, for $|\mathrm{Im}\, z| \geq 1$, we have

$$2\,|\mathrm{Im}\, z| - \log|\cos(2z)| \leq C.$$

Then for $|\mathrm{Im}\, z| \geq 1$,

$$\log\left|f\left(z^2\right)/F_{\infty,\alpha}\left(z^2\right)\right| \leq C\left(1 + \log|z|\right).$$

Combining this with (10.21), we obtain

$$|g(z)| \leq C\left(1 + |z|\right)^C, \qquad |\mathrm{Im}\, z| \geq 1.$$

By Lemma 10.5, g is a polynomial.

Step 3. We show $g \equiv 0$.

To do this observe that from (10.6), (10.9),

$$|f(y_k)/F_{\infty,\alpha}(y_k)| = O\left(y_k^{-\varepsilon}\right) = o(1), \qquad k \to \infty.$$

We claim that also

$$|L(y_k)/F_{\infty,\alpha}(y_k)| = \left|\sum_{j=1}^{\infty} \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(y_k - x_j)}\right| = o(1), \qquad k \to \infty. \quad (10.22)$$

Once we have this relation, we will have

$$\lim_{k\to\infty} g(y_k) = 0,$$

which will give the desired conclusion that g is identically 0, as g is a polynomial. Firstly, as at (10.19),

$$\sum_{j:x_j \geq 2y_k} \left|\frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(y_k - x_j)}\right| \leq C \sum_{j:x_j \geq 2y_k} j^{-1-2\varepsilon} \leq C y_k^{-\varepsilon}. \quad (10.23)$$

Next,

$$\begin{aligned}\sum_{j:x_j \leq y_k/2} \left|\frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(y_k - x_j)}\right| &\leq \frac{C}{y_k} \sum_{j:x_j \leq y_k/2} x_j^{\frac{1}{2}-\varepsilon} \\ &\leq \frac{C}{y_k} \sum_{j:j \leq C\sqrt{y_k/2}} j^{1-2\varepsilon} \\ &\leq C y_k^{-\varepsilon}. \quad (10.24)\end{aligned}$$

Finally,

$$\sum_{j:y_k/2<x_j<2y_k} \left| \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(y_k - x_j)} \right| \leq C y_k^{\frac{1}{2}-\varepsilon} \sum_{j:y_k/2<x_j<2y_k} \frac{1}{|y_k - x_j|}.$$

Here from (10.10), the closest x_j to y_k will be at a distance $\sim \sqrt{y_k}$ away. Moreover, we know that for indices j in this last sum,

$$x_j - x_{j-1} \sim j \sim \sqrt{y_k}$$

Then

$$\begin{aligned} \sum_{j:y_k/2<x_j<2y_k} \left| \frac{f(x_j)}{F'_{\infty,\alpha}(x_j)(y_k - x_j)} \right| &\leq C y_k^{-\varepsilon} \sum_{j:y_k/2<x_j<2y_k} \frac{x_j - x_{j-1}}{|y_k - x_j|} \\ &\leq C y_k^{-\varepsilon} \int_{[y_k/2,2y_k]\setminus[y_k - C\sqrt{y_k}, y_k + C\sqrt{y_k}]} \frac{dx}{|y_k - x|} \\ &\leq C y_k^{-\varepsilon} \log y_k. \end{aligned}$$

This and the estimates (10.23), (10.24) give (10.22) and the result. □

11 Uniqueness in L_∞ Approximation

In this section, we prove uniqueness of the best L_∞ approximants:

Theorem 11.1. *Let $\alpha > 0$ and $\overline{\alpha}$ be the least integer $> \alpha$.*
(a) Then there exists a unique entire function H^* such that $H^*(x^2)$ is entire of exponential type ≤ 2 and

$$\|x^\alpha - H^*(x)\|_{L_\infty[0,\infty)} = A_2[x^\alpha; L_\infty[0,\infty)]. \tag{11.1}$$

Moreover, $H^* = H_{\infty,\alpha}$, where $H_{\infty,\alpha}$ is the locally uniform limit of some subsequence of $\{n^{2\alpha} P_{n,\infty,\alpha}(z/n^2)\}_{n=1}^{\infty}$.
(b) There exist alternation points

$$0 = y_0 < y_1 < y_2 < \cdots \tag{11.2}$$

with

$$y_j^\alpha - H^*(y_j) = (-1)^{j+\overline{\alpha}} A_2[x^\alpha; L_\infty[0,\infty)], \qquad j \geq 0, \tag{11.3}$$

and

$$y_j \in \left[\left((j-1)\frac{\pi}{2} \right)^2, \left(j\frac{\pi}{2} \right)^2 \right], \qquad j \geq 1. \tag{11.4}$$

We already know from Lemma 8.4 that a best approximant H^* exists. We shall show that $H^* = H_{\infty,\alpha}$, where $H_{\infty,\alpha}$ denotes the locally uniform limit of some fixed subsequence of $\left\{n^{2\alpha} P^*_{n,\infty,\alpha}\left(z/n^2\right)\right\}$, that is

$$\lim_{n\to\infty, n\in\mathcal{S}} n^{2\alpha} P^*_{n,\infty,\alpha}\left(z/n^2\right) = H_{\infty,\alpha}(z)$$

uniformly in compact subsets of $\mathbb{C}$. We proved that $H_{\infty,\alpha}$ is a best approximant in the sense outlined above, recall (8.17). In the sequel, let

$$R(z) = z^\alpha - H_{\infty,\alpha}(z) = \lim_{n\to\infty} n^{2\alpha} R_{n,p,\alpha}\left(z/n^2\right) \text{ and } R^*(z) = z^\alpha - H^*(z). \tag{11.5}$$

As $n \to \infty$ through the same subsequence $\mathcal{S}$, the scaled alternation points also converge, so that

$$\lim_{n\to\infty, n\in\mathcal{S}} n^2 y_{jn,\alpha} = y_j, \qquad j \geq 0, \tag{11.6}$$

and (10.4) holds. The alternation property (11.3) for $H_{\infty,\alpha}$ then follows from that for $R_{n,p,\alpha}$ and (11.5), except that the specific sign $(-1)^{j+\overline{\alpha}}$ has not been established. In this section, we let

$$A = A_2\left[x^\alpha; L_{\infty,\surd}[0,\infty)\right]. \tag{11.7}$$

Lemma 11.2. *(a) For $j \geq 0$,*

$$(H^* - H_{\infty,\alpha})(y_j)(-1)^{j+\overline{\alpha}} \geq 0. \tag{11.8}$$

(b) We can choose a sequence $\{z_j\}_{j=1}^\infty$ of zeros of $H_{\infty,\alpha} - H^*$ satisfying for $j \geq 1$,

$$z_j \in [y_{j-1}, y_j] \tag{11.9}$$

and

$$z_j < z_{j+2}, \qquad j \geq 1. \tag{11.10}$$

If for some j, $z_j = z_{j+1}$ then $z_j = y_j$ and y_j is a double zero of $R - R^*$.

Proof. (a) From (4.3) and (4.5),

$$\operatorname{sign}(R_{n,\infty,\alpha}(0)) = -\operatorname{sign}(\sin\alpha\pi) = (-1)^{\overline{\alpha}}$$

and hence also

$$\operatorname{sign}(R(0)) = -\operatorname{sign}(\sin\alpha\pi) = (-1)^{\overline{\alpha}}.$$

Then for $j \geq 0$,
$$R(y_j) = (-1)^{j+\overline{\alpha}} A.$$
Since
$$\|R^*\|_{L_\infty[0,\infty)} = A,$$
we obtain
$$(H^* - H_{\infty,\alpha})(y_j)(-1)^{j+\overline{\alpha}} = (R - R^*)(y_j)(-1)^{j+\overline{\alpha}} \geq 0.$$
(b) From (a), $H^* - H_{\infty,\alpha}$ has at least one zero in $[y_{j-1}, y_j]$ for $j \geq 1$. For $j \geq 1$, let z_j denote the smallest zero of $H^* - H_{\infty,\alpha}$ in $[y_{j-1}, y_j]$. Note that it is possible that $z_{j+1} = z_j$, but this occurs iff both equal y_j, which will then be a double zero of $H^* - H_{\infty,\alpha} = R - R^*$. (For if y_j is a zero of $R - R^*$, then both R and R^* will have an alternation point at y_j.) Finally, observe that
$$z_{j+2} \geq y_{j+1} > y_j \geq z_j. \qquad \square$$

If $z_1 > 0$, we let
$$\Psi(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right). \tag{11.11}$$
If $z_1 = 0$, we instead let
$$\Psi(z) = 2z \prod_{k=2}^{\infty} \left(1 - \frac{z}{z_k}\right).$$
We also use the notation (10.3) and let
$$G_\alpha(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{y_k}\right).$$
Lemma 11.3. *(a) For all $z \in \mathbb{C}$,*
$$\begin{aligned} &\left| \log|\Psi(z)| - \log\left|\frac{\sin(2\sqrt{z})}{2\sqrt{z}}\right| \right| \\ &\leq 2\left(|\log|z - \zeta|| + \log^+|z| + C\right), \end{aligned} \tag{11.12}$$
where ζ is the closest zero of Ψ, G_α or $\sin(2\sqrt{z})$ to z.
(b) Suppose $z_1 > 0$. Then for $\mathrm{Re}(z) \leq 0$,
$$|\Psi(z)| \geq \left|\frac{\sin(2\sqrt{z})}{2\sqrt{z}}\right|. \tag{11.13}$$

If $z_1 = 0$, *this holds for* $\mathrm{Re}\,(z) \leq 0$ *with* $|z| \geq 1$.
(c) $\Psi\left(z^2\right)$ *is entire of exponential type* 2.

Proof. (a) From (11.4), the zeros of $G_\alpha(z)$ and those of $\frac{\sin\left(2\sqrt{z}\right)}{2\sqrt{z}}$ weakly interlace. Then Lemma 7.3 gives

$$\left|\log|G_\alpha(z)| - \log\left|\frac{\sin\left(2\sqrt{z}\right)}{2\sqrt{z}}\right|\right| \leq |\log|z-\zeta|| + \log^+|z| + C,$$

where ζ is the closest zero of G_α or $\frac{\sin\left(2\sqrt{\cdot}\right)}{2\sqrt{\cdot}}$ to z. Moreover, from (11.9), the zeros of Ψ and G_α weakly interlace. Again Lemma 7.3 gives

$$|\log|\Psi(z)| - \log|G_\alpha(z)|| \leq |\log|z-\zeta|| + \log^+|z| + C,$$

where ζ is the closest zero of G_α or Ψ to z. Combining these inequalities gives the result.
(b) Suppose first that $z_1 > 0$. We have for $j \geq 1$, (recall (10.4), (11.9))

$$z_j \leq y_j \leq s_j = \left(\frac{j\pi}{2}\right)^2.$$

Then for $\mathrm{Re}\,(z) \leq 0$,

$$\begin{aligned}\left|1 - \frac{z}{z_j}\right|^2 &= 1 + \frac{|z|^2}{z_j^2} + 2\frac{|\mathrm{Re}\,z|}{z_j} \\ &\geq 1 + \frac{|z|^2}{s_j^2} + 2\frac{|\mathrm{Re}\,z|}{s_j} = \left|1 - \frac{z}{s_j}\right|^2.\end{aligned}$$

So

$$|\Psi(z)| = \prod_{j=1}^{\infty}\left|1 - \frac{z}{z_j}\right| \geq \prod_{j=1}^{\infty}\left|1 - \frac{z}{s_j}\right| = \left|\frac{\sin\left(2\sqrt{z}\right)}{2\sqrt{z}}\right|.$$

If instead $z_1 = 0$, the only difference is the term for $j = 1$. We use instead the estimate

$$\left|1 - \frac{z}{s_1}\right|^2 \leq (1 + |z|)^2 \leq (2|z|)^2,$$

provided $|z| \geq 1$.
(c) This follows easily from (a) and the maximum-modulus theorem. □

We can now prove:

Lemma 11.4. *Let*

$$\Phi\left(z\right)=\frac{H^{*}\left(z\right)-H_{\infty,\alpha}\left(z\right)}{\Psi\left(z\right)}. \tag{11.14}$$

Then Φ is a polynomial.

Proof. Now

$$C_{0}=\|\left(H^{*}-H_{\infty,\alpha}\right)\left(x^{2}\right)\|_{L_{\infty}(\mathbb{R})}<\infty. \tag{11.15}$$

Moreover, $\left(H^{*}-H_{\infty,\alpha}\right)\left(z^{2}\right)$ is entire of exponential type ≤ 2. Then as per usual [17, p. 38, Theorem 3]

$$\left|\left(H^{*}-H_{\infty,\alpha}\right)\left(z^{2}\right)\right|\leq C_{0}e^{2|\operatorname{Im}z|},\qquad z\in\mathbb{C}. \tag{11.16}$$

This and Lemma 11.3(a) give

$$\begin{aligned}\log\left|\Phi\left(z^{2}\right)\right| &\leq \log C_{0}+2\left|\operatorname{Im}z\right|-\log\left|\frac{\sin\left(2z\right)}{2z}\right|\\ &\quad+2\left(\left|\log\left|z^{2}-\zeta\right|\right|+\log^{+}\left|z\right|+C\right),\end{aligned}$$

where ζ is the closest zero of Ψ, G_{α} or $\frac{\sin\left(2\sqrt{\cdot}\right)}{2\sqrt{\cdot}}$ to z^{2}. Next, if $z=x+iy$, some elementary manipulations give

$$2\left|\operatorname{Im}z\right|-\log\left|\sin\left(2z\right)\right|\leq C,\qquad\left|\operatorname{Im}z\right|\geq 1.$$

Hence for $\left|\operatorname{Im}z\right|\geq 1$,

$$\log\left|\Phi\left(z^{2}\right)\right|\leq C\left(1+\left|\log\left|z\right|\right|\right).$$

Moreover, as Φ is the ratio of entire functions of exponential type, it is also of exponential type, [17, p. 13]. Then Lemma 10.5 shows that $\Phi\left(z^{2}\right)$ and hence $\Phi\left(z\right)$, is a polynomial. □

Now we can give

Proof of Theorem 11.1(a). We claim first that

$$H^{*}\left(z\right)-H_{\infty,\alpha}\left(z\right)=F_{\infty,\alpha}\left(z\right)\sum_{j=1}^{\infty}\frac{\left(H^{*}-H_{\infty,\alpha}\right)\left(x_{j}\right)}{F_{\infty,\alpha}^{\prime}\left(x_{j}\right)\left(z-x_{j}\right)}. \tag{11.17}$$

This follows from Theorem 10.1: the bounds (10.5) and (10.6) for $f=H^{*}-H_{\infty,\alpha}$ were established in stronger form, in the proof of Lemma 11.4. So all the hypotheses of Theorem 10.1 are satisfied, and we have (11.17). Next for $x > 2$, (10.8) and (11.15), (11.17) give

$$\begin{aligned}
|H^* - H_{\infty,\alpha}|(-x) &\leq C\,|F_{\infty,\alpha}(-x)| \sum_{j=1}^{\infty} \frac{1}{\sqrt{x_j}\,(x+x_j)} \\
&\leq C\,|F_{\infty,\alpha}(-x)| \left\{ \frac{1}{x} \sum_{j:x_j<x} \frac{1}{\sqrt{x_j}} + \sum_{j:x_j\geq x} \frac{1}{x_j^{3/2}} \right\} \\
&\leq C\,|F_{\infty,\alpha}(-x)| \left\{ \frac{1}{x} \sum_{j:j<C_1\sqrt{x}} \frac{1}{j} + \sum_{j:j\geq C_2\sqrt{x}} \frac{1}{j^3} \right\} \\
&\leq C\,|F_{\infty,\alpha}(-x)| \frac{\log x}{x}.
\end{aligned}$$

Then Φ of (11.14) satisfies

$$|\Phi(-x)| = \left| \frac{H^* - H_{\infty,\alpha}}{\Psi} \right| (-x) \leq C \left| \frac{F_{\infty,\alpha}}{\Psi}(-x) \right| \frac{\log x}{x}. \tag{11.18}$$

We shall show that the right-hand side tends to 0 as $x \to \infty$. Then in as much as Φ is a polynomial, it is identically zero, and the result $H^* = H_{\infty,\alpha}$ follows. We assume $z_1 > 0$ (the case $z_1 = 0$ requires trivial modifications) and write

$$\left| \frac{F_{\infty,\alpha}}{\Psi}(-x) \right| = \prod_{j=1}^{\infty} \frac{1 + \frac{x}{x_j}}{1 + \frac{x}{z_j}}. \tag{11.19}$$

Recall (from (10.10) and (11.9)) that both

$$x_j, z_j \in [y_{j-1}, y_j]. \tag{11.20}$$

Next,

$$\left| \frac{1 + \frac{x}{x_j}}{1 + \frac{x}{z_j}} - 1 \right| = \frac{x\,|z_j - x_j|}{x_j\,(z_j + x)}. \tag{11.21}$$

If $z_j \geq x$, we estimate this above by

$$\frac{x\,(y_j - y_{j-1})}{x_j z_j} \leq \frac{Cx}{j^3},$$

recall (11.4). Then

$$\prod_{j:z_j\geq x}\left|\frac{1+\frac{x}{x_j}}{1+\frac{x}{z_j}}\right| \leq \prod_{j:z_j\geq x}\left(1+\frac{Cx}{j^3}\right)$$
$$\leq \exp\left(Cx\sum_{j:z_j\geq x}\frac{1}{j^3}\right)$$
$$\leq \exp\left(Cx\sum_{j:j\geq C\sqrt{x}}\frac{1}{j^3}\right)\leq C. \tag{11.22}$$

Next if $z_j \leq x$, we estimate (recall (11.20))

$$\frac{x\,|z_j-x_j|}{x_j\,(z_j+x)} \leq \frac{\max\{y_j-x_j, x_j-y_{j-1}\}}{x_j}.$$

Using the crucial estimate (10.10), we see that for some $\Lambda\in(0,1)$ independent of j,

$$y_j-x_j = y_j-y_{j-1}-(x_j-y_{j-1})$$
$$\leq (y_j-y_{j-1})\,(1-\Lambda)\,.$$

A similar upper bound holds for x_j-y_{j-1}. Also for some C_0, (10.10) shows that

$$x_j \geq y_j - C_0\sqrt{y_j},$$

and the function $t\to\frac{1}{t-C_0\sqrt{t}}$ is decreasing for $t\geq (C_0/2)^2$. Hence if j_0 is such that $y_{j_0-1}\geq (C_0/2)^2$, then for $j\geq j_0$,

$$\frac{x\,|z_j-x_j|}{x_j\,(z_j+x)} \leq (1-\Lambda)\,\frac{y_j-y_{j-1}}{y_j-C_0\sqrt{y_j}}$$
$$\leq (1-\Lambda)\int_{y_{j-1}}^{y_j}\frac{dt}{t-C_0\sqrt{t}}.$$

Then recalling (11.21),

$$\prod_{j\geq j_0:z_j<x}\left|\frac{1+\frac{x}{x_j}}{1+\frac{x}{z_j}}\right| \leq \prod_{j\geq j_0:z_j<x}\left(1+(1-\Lambda)\int_{y_{j-1}}^{y_j}\frac{dt}{t-C_0\sqrt{t}}\right)$$
$$\leq \exp\left((1-\Lambda)\int_{y_{j_o-1}}^{2x}\frac{dt}{t-C_0\sqrt{t}}\right)$$
$$\leq C_1 x^{1-\Lambda}, \tag{11.23}$$

since

$$\int_{y_{j_o-1}}^{2x}\frac{dt}{t-C_0\sqrt{t}}-\int_{y_{j_o-1}}^{2x}\frac{dt}{t}=O\left(\int_{y_{j_o-1}}^{2x}\frac{dt}{t^{3/2}}\right)=O\left(1\right).$$

Finally, as $x\to\infty$,

$$\prod_{j=1}^{j_0-1}\left|\frac{1+\frac{x}{x_j}}{1+\frac{x}{z_j}}\right|\leq C_1.$$

Substituting this and the estimates (11.22), (11.23) into (11.19) gives

$$\left|\frac{F_{\infty,\alpha}}{\Psi}\left(-x\right)\right|\leq Cx^{1-\Lambda},\qquad x\to\infty$$

and hence (11.18) gives

$$\left|\Phi\left(-x\right)\right|\leq Cx^{-\Lambda}\log x\to 0,\qquad x\to\infty.$$

So Φ is identically zero, and $H^*=H_{\infty,\alpha}$ identically. Because of uniqueness of the best approximant, $H_{\infty,\alpha}$ is independent of the subsequence. So it is the unique best approximant. □

Proof of Theorem 11.1(b). Because $H_{\infty,\alpha}$ is independent of the subsequence $\mathcal{S}$, and is the unique best approximation to x^α, Theorem 7.1 gives, uniformly in compact subsets of $\mathbb{C}$,

$$z^\alpha-H_{\infty,\alpha}\left(z\right)=R\left(z\right)=\lim_{n\to\infty}n^{2\alpha}R_{n,\infty,\alpha}\left(z/n^2\right).$$

Because of the uniform convergence, this relation can be differentiated. As y_{jn} is the jth alternation point of $R_{n,\infty,\alpha}\left(x\right)=x^\alpha-P_{n,\infty,\alpha}\left(x\right)$, we obtain the existence of

$$y_j=\lim_{n\to\infty}n^2y_{jn},\qquad j\geq 1.$$

The relation (11.3) was proved in Lemma 11.2, while (11.4) follows from (10.4). □

Corollary 11.5. *Let $p = \infty$ and $\alpha > 0$. Then all the conclusions of* Theorems 3.1 *to* 3.3 *are true.*

Proof.

Proof of Theorem 3.1 for $p = \infty$. As any subsequential limit $H_{\infty,\alpha}$ of $\{n^{2\alpha} P_{n,\infty p,\alpha}(z/n^2)\}_{n=1}^{\infty}$ is a b.a. and b.a.'s are unique, we obtain

$$\lim_{n\to\infty} n^{2\alpha} P_{n,\infty,\alpha}\left(z/n^2\right) = H_{\infty,\alpha}(z)$$

uniformly in compact subsets of the plane. Then all the assertions of Theorem 3.1 follow from Theorem 7.1 and (7.1).

Proof of Theorem 3.2 for $p = \infty$. The uniqueness of $H_{\infty,\alpha}$ has just been established, as has the alternation property (3.10). The representation (3.7) was established in Theorem 7.1. It remains to prove (3.11). From (8.3) and (8.4) of Theorem 8.1,

$$\begin{aligned}\Lambda_{\infty,\alpha} &= \lim_{n\to\infty} n^{2\alpha} E_n[x^\alpha; L_{\infty,\surd}[0,1]] \\ &= A_2\left[x^\alpha; L_{\infty,\surd}[0,\infty)\right] \\ &= \|x^\alpha - H_{\infty,\alpha}(x)\|_{L_{\infty,\surd}[0,\infty)} = |H_{\infty,\alpha}(0)|.\end{aligned}$$

Now all we need to do is to substitute (3.7) into this last formula.

Proof of Theorem 3.3. We established (3.12) and (3.12) already – see (11.4) and (11.6), and recall that all the limits are independent of the subsequence $\mathcal{S}$. The relation (3.14) is Theorem 10.2(c). The remaining relations (3.16) to (3.17) follow by a scale change in Theorem 4.1(b) and taking limits there. □

12 Proofs of Theorems 1.1 to 1.3

Proof of Theorem 1.1. Let P_n^* denote the polynomial of degree $\leq n$ that best approximates $|x|^\alpha$ in $L_p[-1,1]$, as in Theorem 1.1. Then

$$P_{2n}^*(z) = P_{n,p,\frac{\alpha}{2}}\left(z^2\right)$$

as follows from (2.8). So

$$\begin{aligned}\lim_{n\to\infty}(2n)^\alpha P_{2n}^*(z/(2n)) &= 2^\alpha \lim_{n\to\infty}\left(n^2\right)^{\alpha/2} P_{n,p,\frac{\alpha}{2}}\left((z/2)^2/n^2\right) \\ &= 2^\alpha H_{p,\frac{\alpha}{2}}\left((z/2)^2\right),\end{aligned}$$

by (3.4) in Theorem 3.1. Because $|x|^\alpha$ is even, $P^*_{2n} = P^*_{2n-1}$ so the uniform convergence allows us to deduce

$$\lim_{n\to\infty} n^\alpha P^*_n(z/n) = 2^\alpha H_{p,\frac{\alpha}{2}}\left((z/2)^2\right).$$

Now at (8.16), we noted that $H_{p,\frac{\alpha}{2}}(z^2)$ is a best approximation to $|x|^\alpha$ from entire functions of exponential type ≤ 2. By Lemma 8.7, $H^*(z) = 2^\alpha H_{p,\frac{\alpha}{2}}\left((z/2)^2\right)$ is a best approximation to $|x|^\alpha$ from entire functions of exponential type ≤ 1. So we have (1.3). □

Proof of Theorem 1.2 for $1 \leq p < \infty$. We already know that $H^*(z) = 2^\alpha H_{p,\frac{\alpha}{2}}\left((z/2)^2\right)$ satisfies (1.3) and (1.4). It also satisfies (1.5) for all f of exponential type ≤ 1 such that $f \in L_p(\mathbb{R})$ and f is even. Indeed this follows from (3.8) by a substitution. Of course if f is odd, (1.5) is also immediate. Since every function is the sum of its even and odd parts, we obtain (1.5) in full generality. Once we have the characterization (1.5), the uniqueness follows as in the proof of Theorem 9.1. □

Proof of Theorem 1.2 for $p = \infty$. Let $\{y_j\}_{j=1}^\infty$ be as in Theorem 3.2(IV) and let

$$y^*_j = 2\sqrt{y_j}, j \geq 1.$$

Then $H^*(z) = 2^\alpha H_{\infty,\frac{\alpha}{2}}\left((z/2)^2\right)$ satisfies (1.7) to (1.8) as follows from (3.10). Moreover, we know that H^* is a best approximant in $L_\infty(\mathbb{R})$ to $|x|^\alpha$ from the entire functions of exponential type ≤ 1. Suppose $H^\#$ is another best approximant. We decompose it as the sum of its even and odd components:

$$H^\# = H^\#_e + H^\#_o.$$

As $|x|^\alpha$ is even, $H^\#_e$ is also a best approximant to $|x|^\alpha$, and then via Lemma 8.7, $2^{-\alpha} H^\#_e(2\sqrt{z})$ is a best approximant in the setting of Theorem 3.2. The uniqueness part of Theorem 3.2 gives

$$2^{-\alpha} H^\#_e\left(2\sqrt{z}\right) = H_{\infty,\frac{\alpha}{2}}(z).$$

So

$$H^\#(z) = 2^\alpha H_{\infty,\frac{\alpha}{2}}\left((z/2)^2\right) + H^\#_o(z).$$

We now proceed to show that

$$H^\#_o \equiv 0,$$

which will give the result. First observe that (1.8) applied to $2^\alpha H_{\infty,\frac{\alpha}{2}}\left((z/2)^2\right)$ gives

$$\begin{aligned}&(-1)^{j+\overline{\frac{\alpha}{2}}}H_o^{\#}\left(\pm y_j^*\right)\\&=(-1)^{j+\overline{\frac{\alpha}{2}}}\left[\left|y_j^*\right|^\alpha-2^\alpha H_{\infty,\frac{\alpha}{2}}\left(\left(\pm y_j^*/2\right)^2\right)\right]-(-1)^{j+\overline{\frac{\alpha}{2}}}\left[\left|y_j^*\right|^\alpha-H^{\#}\left(\pm y_j^*\right)\right]\\&=A_1\left[|x|^\alpha;L_\infty(\mathbb{R})\right]-(-1)^{j+\overline{\frac{\alpha}{2}}}\left[\left|y_j^*\right|^\alpha-H^{\#}\left(\pm y_j^*\right)\right]\\&\geq 0.\end{aligned}$$

Since $H_o^{\#}$ is odd, this relation forces

$$H_o^{\#}\left(\pm y_j^*\right)=0,$$

and hence also

$$(-1)^{j+\overline{\frac{\alpha}{2}}}\left[\left|y_j^*\right|^\alpha-H^{\#}\left(\pm y_j^*\right)\right]=A_1\left[|x|^\alpha;L_\infty(\mathbb{R})\right].$$

Thus $\pm y_j^*$ are extrema of $|x|^\alpha-H^{\#}(x)$, and we know they are extrema of $|x|^\alpha-2^\alpha H_{\infty,\frac{\alpha}{2}}\left((x/2)^2\right)$. Then they are also critical points of $H_o^{\#}$, that is,

$$H_o^{\#\prime}\left(\pm y_j^*\right)=0.$$

Thus $H_o^{\#}$ has double zeros at $\pm y_j^*$. But $H_o^{\#}(z)=H^{\#}(z)-2^\alpha H_{\infty,\frac{\alpha}{2}}\left((z/2)^2\right)$ is also entire of exponential type ≤ 1. We show that $H_o^{\#}$ has too many zeros for a function of its growth and hence is the zero function. Let $n(r)$ denote the total multiplicity of zeros of $H_o^{\#}$ in the ball $|z|\leq r$. Since $y_j^*=2\sqrt{y_j}\in[(j-1)\pi,j\pi]$, we see that $n(j\pi)\geq 4j$, which easily implies

$$\liminf_{r\to\infty}\frac{n(r)}{r}\geq\liminf_{j\to\infty}\frac{4j}{(j+1)\pi}=\frac{4}{\pi}>1.$$

Since $H_o^{\#}$ is at most of order 1 and type 1, this implies [17, Theorem 3, p. 19] that $H_o^{\#}\equiv 0$. So $H^{\#}(z)\equiv 2^\alpha H_{\infty,\frac{\alpha}{2}}\left((z/2)^2\right)$. □

Proof of Theorem 1.3. We use (3.7) with α replaced by $\alpha/2$, and z replaced by $(z/2)^2$, and multiply by 2^α :

$$\begin{aligned}&z^\alpha-2^\alpha H_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right)\\&\quad=-2^\alpha\frac{\sin\frac{\alpha}{2}\pi}{\pi}F_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right)\int_0^\infty\frac{t^{\alpha/2}}{t+(z/2)^2}\frac{dt}{F_{p,\alpha}(-t)}.\end{aligned}$$

We make the substitution $t = (s/2)^2$, giving

$$= -\frac{\sin \frac{\alpha}{2}\pi}{\pi} F_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right) \int_{-\infty}^{\infty} \frac{|s|^{\alpha+1}}{s^2+z^2} \frac{dt}{F_{p,\alpha}\left(-\left(\frac{s}{2}\right)^2\right)}.$$

We set

$$F^*(z) := F_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right) = \prod_{j=1}^{\infty}\left(1-\frac{z^2}{4x_j}\right) = \prod_{j=1}^{\infty}\left(1-\left(\frac{z}{x_j^*}\right)^2\right),$$

with

$$x_j^* := 2\sqrt{x_j} \in \left[\left(j-\frac{3}{2}\right)\pi, \left(j-\frac{1}{2}\right)\pi\right], \qquad j \geq 1.$$

This gives the result for $\mathrm{Re}\,(z) > 0$. For $\mathrm{Re}\,(z) < 0$, we use the fact that all the terms other than z^α in (1.9) are even.

Finally in Theorem 8.1, we showed that

$$\Lambda_{p,\alpha}^* = A_1[|x|^\alpha; L_p(\mathbb{R})]$$

and using (1.3), (1.9), we continue this, if $p < \infty$, as

$$\begin{aligned} &= \left|\left||x|^\alpha - H^*(x)\right|\right|_{L_p(\mathbb{R})} \\ &= \frac{\left|\sin \frac{\alpha}{2}\pi\right|}{\pi}\left(\int_{-\infty}^{\infty}\left|F^*(x)\int_{-\infty}^{\infty}\frac{|s|^{\alpha+1}}{s^2+x^2}\frac{ds}{F^*(is)}\right|^p dx\right)^{1/p}. \end{aligned}$$

If $p = \infty$, we use instead that

$$\begin{aligned} \Lambda_{\infty,\alpha}^* &= A_1[|x|^\alpha; L_\infty(\mathbb{R})] \\ &= \left|\left||x|^\alpha - H^*(x)\right|\right|_{L_\infty(\mathbb{R})} = |H^*(0)| \\ &= \frac{\left|\sin \frac{\alpha}{2}\pi\right|}{\pi}\int_{-\infty}^{\infty}\frac{|s|^{\alpha-1}}{F^*(is)}ds. \end{aligned}$$

□

For future use, we state part of what is proved above:

Lemma 12.1. *Let H^* denote the b.a. to $|x|^\alpha$ in $L_p(\mathbb{R})$ from the entire functions of exponential type ≤ 1. Then*

$$H^*(z) = 2^\alpha H_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right)$$

and

$$F^*(z) = F_{p,\frac{\alpha}{2}}\left(\left(\frac{z}{2}\right)^2\right)$$

Moreover, for $j \geq 1$,

$$x_j^* = 2\sqrt{x_j} \quad \textit{and} \quad y_j^* = 2\sqrt{y_j}.$$

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