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# ON THE BERNSTEIN-GELFAND-GELFAND RESOLUTION AND THE DUFLO SUM FORMULA 

O. Gabber and A. Joseph


#### Abstract

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In ([8], Prop. 12) Duflo gave a remarkable sum formula interrelating induced ideals. The main result of this paper provides a natural generalization of this formula and more precisely gives a resolution for certain primitive quotients of the enveloping algebra $U(\mathfrak{g})$. The proof has three distinct steps. One, the extension of the Bernstein-Gelfand-Gelfand (in short, B.G.G.) resolution of a simple finite dimensional $U(\mathfrak{g})$ module to certain simple highest weight modules. Two, the description of the so-called $\mathfrak{l}$-finite part of the space of homomorphisms of any one Verma module to any other. Three, the proof of exactness of a certain functor. The last can be viewed as a non-trivial generalization of the fact that a Verma module with dominant highest weight is projective in the so-called $\mathbb{O}$ category. A by-product gives some results on a problem of Kostant relating $U(\mathfrak{g})$ to the f -finite part of the space of endomorphisms of a simple highest weight module.


## 1. Preliminaries

1.1: Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{b}$ a Cartan subalgebra for $\mathfrak{g}, R$ the set of non-zero roots, $R^{+} \subset R$ a system of positive roots, $B \subset R^{+}$the set of simple roots, $\rho$ the half sum of the positive roots, $s_{\alpha} \in \operatorname{Aut}\left(\mathfrak{h}^{*}\right)$ the reflection corresponding to the root $\alpha \in R$, and $W$ the group generated by the $s_{\alpha}: \alpha \in B$. Let $X_{\alpha}$ be the element of a

Chevalley basis for $\mathfrak{g}$ corresponding to the root $\alpha$ and set

$$
\mathfrak{n}^{+}=\sum_{a \in \mathbb{R}^{+}} \mathbb{C} X_{\alpha}, \mathfrak{n}^{-}=\sum_{\alpha \in \mathbb{R}^{+}} \mathbb{C} X_{-\alpha}, \mathfrak{b}=\mathfrak{h} \oplus \mathfrak{n}^{+}
$$

1.2: For each $\lambda \in \mathfrak{b}^{*}$, set $R_{\lambda}=\{\alpha \in R: 2(\lambda, \alpha) /(\alpha, \alpha) \in \mathbb{Z}\}$ (which is itself a root system) and $R_{\lambda}^{+}=R_{\lambda} \cap R^{+}$, with $B_{\lambda} \subset R_{\lambda}^{+}$the corresponding set of simple roots. Call $\lambda$ regular (resp. dominant) if $(\lambda, \alpha) \neq 0$ (resp. $(\lambda, \alpha) \geq 0$ ) for all $\alpha \in R^{+}$. For each $B^{\prime} \subset B_{\lambda}$, let $W_{B^{\prime}}$ be the subgroup of $W$ generated by the $s_{\alpha}: \alpha \in B^{\prime}$ and $w_{B^{\prime}}$ the largest element of $W_{B^{\prime}}$ with respect to its Bruhat order $\leq$ (as defined in [7], 7.7.3). If $B^{\prime}=B_{\lambda}$ we write $W_{B^{\prime}}=W_{\lambda}, w_{B^{\prime}}=w_{\lambda}$. Let $M(\lambda)$ denote the Verma module with highest weight $\lambda-\rho$ associated to the quadruplet $\mathfrak{g}$, $\mathfrak{h}, B, \lambda$ (see [7], 7.1.4), $\overline{M(\lambda)}$ the unique maximal submodule of $M(\lambda)$, and set $L(\lambda)=M(\lambda) / \overline{M(\lambda)}, J(\lambda)=$ Ann $L(\lambda)$. For each $\mathfrak{b}$ module $V$ we set $V_{\lambda}=\{v \in V: H v=(\lambda, H) v$, for all $H \in \mathfrak{h}\}$. Let $e_{\lambda}$ denote the canonical generator of $M(\lambda)$ (which has weight $\lambda-\rho$ ). Set $R_{\lambda}^{0}=$ $\{\alpha \in R:(\alpha, \lambda)=0\}$.
1.3: Let $u \mapsto$ ǔ (resp. $u \mapsto{ }^{\prime} u$ ) denote the involutory antiautomorphism of $U(\mathfrak{g})$ defined by $\check{X}=-X: X \in \mathfrak{g}$ (resp. ${ }^{t} X_{\alpha}=X_{-\alpha}: \alpha \in R$, $\left.{ }^{\mathrm{t}} H=H: H \in \mathfrak{h}\right)$. Identify $U:=U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \times \mathfrak{g})$. Define $j: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ through $j(X)=\left(X,-^{\mathfrak{t}} X\right)$, set $\mathfrak{k}=j(\mathfrak{g})$, so $U(\mathfrak{f})$ may be regarded as a subalgebra of $U$. Let $\mathfrak{f}^{\wedge}$ denote the set of equivalence classes of finite dimensional irreducible representations of $\mathfrak{f}$. For each locally finite $\mathfrak{f}$ module $L$ and each $\sigma \in \mathfrak{f}^{\wedge}$, we let $L_{\sigma}$ denote the isotypical component of type $\sigma$ of L. Let $\iota: U(\mathfrak{F}) \rightarrow U(\mathfrak{g})$ be the $\mathbb{C}$ algebra isomorphism sending $\left(-^{t} X, X\right)$ to $X$ for every $X \in \mathfrak{g}$. If $R \rightarrow \underset{\varphi}{ } S$ is a ring homomorphism and $M$ is a left $S$ module, we let $M^{\varphi}$ denote the left $R$ module which consists of the underlying abelian group $|M|$ of $M$ together with the operation $(r, m) \mapsto \varphi(r) \cdot m$ of $R$ on $|M|$.
1.4: Let $\mathcal{O}$ denote the category of finitely generated $U(\mathfrak{g})$ modules which are $\mathfrak{b}$ semisimple and $\mathfrak{b}$ locally finite (see $[1-3,6]$ ). Each $M \in$ $0 b \odot$ has finite length [2]. This category has enough projectives and so the extension groups $\operatorname{Ext}^{k}(\cdot, \cdot)$ relative to $\mathbb{O}$ are thereby defined. Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$. Then $\operatorname{Max} Z(\mathfrak{g})$ is isomorphic to $\mathfrak{b}^{*} / W$ such that for each $\lambda \in \mathfrak{b}^{*}, \hat{\lambda}:=W \lambda$ corresponds to the element $Z(\mathfrak{g}) \cap$ $J(\lambda)$ of Max $Z(\mathfrak{g})$. Let $\mathscr{O}_{\hat{\lambda}}$ denote the subcategory of $\mathcal{O}$ of all modules annihilated by a power of this maximal ideal. Each $M \in 0 b \odot$ admits a primary decomposition and we denote by $p_{i}: 0 b \odot \rightarrow 0 b \mathcal{O}_{\dot{\lambda}}$ the projection onto the primary component defined by $\hat{\lambda}$. It is an exact functor on 0 .
1.5: Given $M, N \in O b \mathcal{O}$, consider $\operatorname{Hom}_{\mathbf{C}}(M, N)\left(\operatorname{resp} .(M \otimes N)^{*}\right)$ as a $U$ module through $((a \otimes b) \cdot x) m=\left({ }^{\prime} a \check{a} x \check{b}\right) m \quad$ (resp. $\quad((a \otimes b) \cdot y$,
$m \otimes n)=(y, \quad \check{a} m \otimes \check{b} n))$ where $a, b \in U(g), \quad m \in M, \quad n \in N, \quad x \in$ $\operatorname{Hom}(M, N), y \in(M \otimes N)^{*}$. We remark that $(M(-\lambda) \otimes M(-\mu))^{*}$ is isomorphic to the $\mathfrak{g} \times \mathfrak{g}$ module co-induced from the $\mathfrak{b} \times \mathfrak{b}$ module $\mathbb{C}_{\lambda+\rho, \mu+\rho}$. Let $L(M, N)\left(\right.$ resp. $\left.L(M \otimes N)^{*}\right)$ denote the set of all f-finite elements of $\operatorname{Hom}(M, N)\left(\right.$ resp. $\left.(M \otimes N)^{*}\right)$ which we remark is again a $U$ module. For $\lambda, \mu \in \mathfrak{b}^{*}$, we set $L(\lambda, \mu)=L(M(-\lambda) \otimes M(-\mu))^{*}$.
1.6: Let $E$ be a finite dimensional $U(\mathfrak{g})$ module and given $M \in 0 b 0$, consider $E \otimes M$ as a $U(\mathfrak{g})$ module through the diagonal action. One has $E \otimes M \in 0 b \mathcal{O}$ and the functor $M \mapsto E \otimes M$ is exact. Again one has the natural isomorphisms

$$
\operatorname{Hom}_{\mathfrak{q}}\left(E, \operatorname{Hom}_{\mathrm{C}}(M, N)\right) \simeq \operatorname{Hom}_{9}(E \otimes M, N) \leftrightarrows \operatorname{Hom}_{9}\left(M, E^{*} \otimes N\right) .
$$

The latter gives on taking projective resolutions natural isomorphisms

$$
\operatorname{Ext}^{k}(E \otimes M, N) \xrightarrow{\sim} \operatorname{Ext}^{k}\left(M, E^{*} \otimes N\right): k \in N
$$

1.7: Let $\mathscr{H}$ denote the category of all $U$ modules which satisfy the following properties. One, each $L \in 0 b \mathscr{H}$ is locally finite as a $\mathfrak{k}$ module. Two, $\operatorname{dim} L_{\sigma}<\infty$ for each $\sigma \in \mathbb{P}^{\wedge}$. Three, each $L \in 0 b \mathscr{H}$ admits a finite filtration such that the centre of $U$ acts by scalars on each subquotient. Clearly $\mathscr{H}$ is stable under tensoring with finite dimensional $U$ modules. It follows from the classification ([21], I, Sect. 4) of the simple modules in $\mathscr{H}$ that each $L \in 0 b \mathscr{H}$ has finite length (for example, as shown in ([2], 4.2)). For each $M, N \in 0 b \mathcal{O}$, one has $L(M, N) \in \mathscr{H}$. Indeed, the first property holds by construction. The second obtains from the isomorphism valid for any simple finite dimensional $U(\mathfrak{g})$ module $E$, namely $\operatorname{Hom}_{\mathfrak{l}}\left(E^{\iota}, L(M, N)\right) \approx$ $\operatorname{Hom}_{q}(E \otimes M, N)$, the last space being finite dimensional (since $E \otimes M, N$ have finite length). The third obtains by taking composition series for $M, N$. We have shown that

Lemma: For each $M, N \in 0 b \mathcal{O}$, the $U$ module $L(M, N)$ has finite length.
1.8: Observe that $\tau: a \mapsto^{t} \check{a}$ is an involutory automorphism of $U(\mathfrak{g})$. Given $M \in 0 b \mathcal{O}$, we let $\delta(M)$ denote the submodule of $\left(M^{*}\right)^{\top}$ of all $\mathfrak{b}$ finite elements. Through the existence of a non-degenerate contravariant form on $L(\lambda)$ (see [11], 1.6), one has $L(\lambda) \cong \delta(L(\lambda)$ ). In particular $E^{*} \cong E^{\top}$ for any finite dimensional module $E$. Again each $M \in 0 b \odot$ has finite length, so $\delta(M) \in 0 b \odot$ and $\delta(M)$ has the same composition factors as $M$ (with the same multiplicities).
1.9: For each $M, N \in 0 b \mathcal{O}$, define $\sigma: \operatorname{Hom}_{\mathrm{c}}\left(M,\left(N^{*}\right)^{\tau}\right) \rightarrow(N \otimes M)^{*}$ through $\quad(\sigma(x), m \otimes n)=(x m, n)$. From $\quad(\sigma((a \otimes b) \cdot x), m \otimes n)=$ $(((a \otimes b) \cdot x) m, n)=\left({ }^{( } \mathfrak{a} x \check{b} m, n\right)=(x \check{b r} m$, ăn $)=(\sigma(x)$, ăn $\otimes \check{b} m)=$ $((a \otimes b) \cdot \sigma(x), n \otimes m)$, it follows that $\sigma$ is a $U$ module homomorphism. Again $\sigma$ is obviously injective. Given $y \in(N \otimes M)^{*}$, then for each $m \in M$ the map $g(y, m): n \mapsto(y, n \otimes m)$ of $N$ to $\mathbb{C}$ is $\mathbb{C}$-linear. It follows that the map $\eta(y): m \mapsto g(y, m)$ of $M$ to $\left(N^{*}\right)^{\tau}$ is $\mathbb{C}$-linear and the map $\eta: y \mapsto \eta(y)$ is inverse to $\sigma$.

Lemma: The map $\sigma$ restricts to a $U$ module isomorphism of $L(M, \delta(N))$ onto $L(N \otimes M)^{*}$. In particular $L(N \otimes M)^{*}$ has finite length as a $U$ module.

If $x \in L(M, \delta(N))$, then $\sigma(x)$ is obviously f -finite. Conversely for each $y \in L(N \otimes M)^{*}, \quad m \in M, \quad X \in \mathfrak{g}$, we have $X(\eta(y) m)=$ $\eta\left(j\left({ }^{( } \dot{X}\right) y\right) m+\eta(y) X m$, and so the local finiteness of $\mathfrak{b}$ on $M$ implies that $\eta(y) m \in \delta(N)$. Hence the surjectivity of the restriction of $\sigma$. The last part follows from 1.7.
1.10: Define an ordering on $\mathbb{Z} B$ through $\mu \geq \nu$ if $\mu-\nu \in \mathbb{N} B$. Given $M \in 0 b \subset$, set $\Omega(M)=\left\{\lambda \in \mathfrak{h}^{*}: M_{\lambda} \neq 0\right\}$. If $M \neq 0$, then $\Omega(M)$ admits at least one maximal element. Note that $H_{0}\left(\mathrm{n}^{-}, M\right)=M / \mathrm{n}^{-} M$ is a locally finite semisimple $\mathfrak{b}$ module.

Lemma: Suppose $M, N \in 0 b \mathcal{O}$ with $N$ a submodule of $M$. If $H_{0}\left(\mathfrak{n}^{-}, M\right), H_{0}\left(\mathfrak{n}^{-}, N\right)$ are isomorphic as $\mathfrak{b}$ modules, then $M=N$.

Assume $Q:=M / N \neq 0$. Let $\mu \in \Omega(Q)$ be maximal. Through the maximality of $\mu$ one has $\left(\mathfrak{n}^{-} M\right)_{\mu}=\Sigma X_{-\alpha} M_{\mu+\alpha}=\Sigma X_{-\alpha} N_{\mu+\alpha}=\left(\mathfrak{n}^{-} N\right)_{\mu}$. Yet $\operatorname{dim} N_{\mu} /\left(\mathfrak{n}^{-} N\right)_{\mu}=\operatorname{dim} M_{\mu} /\left(\mathfrak{n}^{-} M\right)_{\mu}$, by hypothesis. This gives $M_{\mu}=$ $N_{\mu}$, which is a contradiction.
1.11: For each $M \in 0 b \mathcal{O}$, let [ $M$ ] denote the corresponding element in the Grothendieck group $\mathscr{G}$ of $\mathfrak{O}$. For each $\hat{\lambda} \in \mathfrak{b}^{*} / W$, let $\mathscr{G}_{\hat{\lambda}}$ denote the subgroup of $\mathscr{G}$ corresponding to $\mathcal{O}_{\hat{\lambda}}$. It is well-known that $\{[L(\mu)]: \mu \in \hat{\lambda}\}$ is a basis for $\mathscr{G}_{\hat{\lambda}}$. Again each $M(\lambda): \lambda \in \mathfrak{b}^{*}$ has finite length with simple factors amongst the $L(\mu): \mu \in \hat{\lambda}$ and we denote by [ $M(\lambda): L(\mu)]$ the number of times $L(\mu)$ occurs in $M(\lambda)$. The resulting matrix is invertible (by [7], 7.6.23) and (by [7], 7.6.14) one has

$$
[E \otimes M(\lambda)]=\sum_{\nu \in \Omega(E)}[M(\lambda+\nu)] \operatorname{dim} E_{v}
$$

for any finite dimensional $U(\mathfrak{g})$ module $E$.
1.12: Let $P(R)$ denote the lattice of integral weights. Let $P(R)^{+}$ (resp. $P(R)^{++}$) denote the dominant (resp. dominant and regular) elements of $P(R)$. For each $\nu \in P(R)$, let $E(\nu)$ denote a (unique up to isomorphism) simple finite dimensional $U(\mathfrak{g})$ module with extreme weight $\nu$. The map $\nu \mapsto E(\nu)^{\imath}$ identifies the $\mathfrak{l}^{\wedge}$ of classes of finite dimensional simple $U(\xi)$ modules with $P(R) / W$ and hence with $P(R)^{+}$. Frobenius reciprocity gives $\operatorname{dim} \operatorname{Hom}_{l}\left(E(\nu)^{\imath}, \quad L(\lambda, \mu)\right)=$ $\operatorname{dim} E(\nu)_{\mu-\lambda}$ for all $\lambda, \mu \in \mathfrak{b}^{*}, \nu \in P(R)$. In particular, $L(\lambda, \mu) \neq 0$ if and only if $\lambda-\mu \in P(R)$. Now assume $\lambda-\mu \in P(R)$. Then by 1.9, $L(\lambda, \mu)$ has finite length. Since $\operatorname{dim} E(\lambda-\mu)_{\lambda-\mu}=1$, it follows that $L(\lambda, \mu)$ admits a unique simple subquotient, which we denote by $V(\lambda, \mu)$, satisfying $\operatorname{dim} \operatorname{Hom}_{\mathfrak{f}}(E(\lambda-\mu), V(\lambda, \mu))=1$. We shall need the following

## Theorem:

(i) Every simple module in $\mathscr{H}$ is isomorphic to some $V(\lambda, \mu)$.
(ii) $V(\lambda, \mu)$ is isomorphic to $V\left(\lambda^{\prime}, \mu^{\prime}\right)$ if and only if there exists $w \in W$ such that $\lambda^{\prime}=w \lambda, \mu^{\prime}=w \mu$.
(iii) Suppose $\lambda \in \mathfrak{b}^{*}$ is dominant. Then if $L(M(\lambda), L(\mu)) \neq 0$ (which holds in particular if $\lambda$ is regular), it is isomorphic to $V(-\mu,-\lambda)$. Furthermore every simple $V \in 0 b \mathscr{H}$ is so obtained.
(i), (ii) are just ([9], I, 4.1, 4.5) and (iii) follows from ([14], 4.7) and (i), (ii).
1.13: Given $-\lambda \in \mathfrak{b}^{*}$ dominant, then for each $-\mu \in-\lambda+P(R)$ dominant we define following Jantzen ([11], Sect. 2) a translation operator $T_{\lambda}^{\mu}: \mathcal{O} \rightarrow \mathbb{O}$ through $T_{\lambda}^{\mu} M=p_{\hat{\mu}}\left(E(\mu-\lambda) \otimes p_{\hat{\lambda}}(M)\right.$. If $R_{\lambda}^{0} \subset$ $R_{\mu}^{0}$, then for all $w \in W_{\lambda}$ we have $T_{\lambda}^{\mu} M(w \lambda) \cong M(w \mu)$ (see [10], 2.10). Let $E$ be a finite dimensional $U(\mathfrak{g})$ module. Through the natural $U$ module isomorphisms $L(M, N) \otimes(\mathbb{C} \otimes E) \leftrightarrows L\left(M \otimes E^{*}, N\right)$, $L(M, N) \otimes(E \otimes \mathbb{C}) \widetilde{\rightarrow} L\left(M, N \otimes E^{\tau}\right)$, it is obvious how to define exact functors on $\mathscr{H}$ satisfying $R_{\lambda}^{\mu} L(M, N) \cong L\left(T_{\lambda}^{\mu} M, N\right), S_{\lambda}^{\theta} L(M, N) \cong$ $L\left(M, T_{\lambda}^{\mu} N\right)$ for $M, N \in 0 b \mathcal{O}$. Again by 1.6, $T_{\lambda}^{\mu}$ is both left and right adjoint to $T_{\mu}^{\lambda}$.
1.14: For each $j \in \mathbb{N}, \mu \in \mathfrak{b}^{*}, N \in 0 b \mathbb{C}$, one has $\operatorname{Ext}^{j}(M(\mu), N) \cong$ $H^{j}\left(\mathrm{n}^{+}, N\right)_{\mu-\rho} \cong\left(H_{j}\left(\mathrm{n}^{-}, \delta(N)\right)_{\mu-\rho}\right)^{*}$, the first isomorphism being due to Delorme ([6], Thm. 2), the second a formal consequence of the appropriate standard complexes.
1.15: Take $\lambda, \mu \in \mathfrak{b}^{*}$ and let us note the almost obvious fact that $L(M(\lambda), M(\mu))=0$ unless $\lambda-\mu \in P(R)$. This latter condition further implies that $W_{\lambda}=W_{\mu}$.

Lemma: Fix $-\lambda,-\mu \in \mathfrak{b}^{*}$ dominant with $\lambda-\mu \in P(R)$. Then for each $w \in W_{\lambda}$ and each finite dimensional $U(\mathfrak{g})$ module $E$ one has

$$
\begin{aligned}
& \operatorname{dim}_{\operatorname{Hom}_{\mathfrak{t}}}\left(E^{\iota}, L\left(M\left(w_{\lambda} \lambda\right), M(w \mu)\right)\right) \\
&=\operatorname{dim} \operatorname{Hom}_{\mathfrak{t}}\left(E^{\iota}, L\left(M\left(w^{-1} w_{\lambda} \lambda\right), M(\mu)\right)\right)
\end{aligned}
$$

We show that both sides equal $\operatorname{dim} E_{w \mu-w_{\lambda} \lambda}$. For the right hand side this follows from the fact that $M(\mu)$ is simple (and so isomorphic to $\delta M(\mu)), 1.9$ and 1.12 , noting that $\Omega(E)$ is $W$ stable. The left hand side equals (by 1.7) $\operatorname{dim} \operatorname{Hom}_{q}\left(M\left(w_{\lambda} \lambda\right), E^{*} \otimes M(w \mu)\right)$; since $M\left(w_{\lambda} \lambda\right)$ is projective in 0 , we have by 1.11 that the latter equals $\operatorname{dim}\left(E^{*}\right)_{w_{\lambda} \lambda-w \mu}=$ $\operatorname{dim} E_{w \mu-w_{\lambda} \lambda}$.

Remarks: Although this also follows from ([14], 4.10) the above proof is much simpler. It is not difficult to extend the above to a further proof of ([14], 4.3) and hence of Duflo's theorem ([8], Thm. 1); but then this becomes essentially the proof given in ([3], 4.4).
1.16: Take $\lambda \in \mathfrak{b}^{*}$ dominant. $Z(\mathfrak{g})$ acts on $M(\lambda)$ by a homomorphism $\chi_{\lambda}: Z(\mathfrak{g}) \rightarrow \mathbb{C}$. Let $C=\lambda+P(R)$, and let $O_{C}$ be the full subcategory of $\mathcal{O}$ consisting of those modules $M$ that satisfy $\Omega(M) \subset C$. Define a functor $T: O_{C} \rightarrow \mathscr{H}$ by $T(N)=L(M(\lambda), N)$ (cf. 1.7). $T$ is exact since any $M(\lambda) \otimes E$ ( $E$ being a finite dimensional $U(\mathfrak{g})$ module) is projective in $\mathcal{O}$. Let $\mathscr{H}$ consisting of those $M \in 0 b(\mathscr{H})$ on which $1 \otimes Z(\mathfrak{g})$ acts through $1 \otimes z \mapsto \chi_{\lambda}(\check{z})$. The image of $T$ lies in $\mathscr{H}_{\hat{\lambda}}$, and in the following theorem we view $\mathscr{H}_{\hat{\lambda}}$ as the target category of $T$.

## Theorem:

(i) $T$ has a left adjoint $T^{\prime}$.
(ii) The unit map $\eta: I d_{\mathscr{r}} \rightarrow T T^{\prime}$ is an isomorphism of functors.
(iii) If $\lambda$ is regular, then $T$ is an equivalence of categories.

We indicate a proof for the theorem, which has also been proved by Bernstein and Gelfand ([3], 6.3, 6.1 (ii), 5.9 (i)).
(i). If $M \in 0 b\left(\mathscr{H}_{\dot{\lambda}}\right)$, we make $M$ into a two-sided $U(\mathfrak{g})$ module by $\mathrm{amb}=\left({ }^{t} \check{a} \otimes \check{b}\right) \cdot m$ for all $m \in M, \quad a, b \in U(g)$. Define $T^{\prime}(M)=$ $M \otimes_{A} M(\lambda)$, where $A=U(\mathfrak{g}) / U(\mathfrak{g}) \operatorname{ker}\left(\chi_{\lambda}\right)$. Now $T^{\prime}(M) \in 0 b\left(O_{C}\right)$ because if $E \subset M$ is a finite dimensional $\mathfrak{F}$ stable generating subspace (so $M=E U(\mathfrak{g})$ ), then we get a surjective $\mathfrak{g}$ linear map $E^{t-1} \otimes M(\lambda) \rightarrow$ $T^{\prime}(M)$. If $M \in 0 b\left(\mathscr{H}_{\hat{\lambda}}\right)$ and $N \in 0 b\left(\mathcal{O}_{C}\right)$, one defines an isomorphism $\zeta(M, N):\left(\operatorname{Hom}_{q}\left(M \otimes_{A} M(\lambda), N\right) \rightarrow \operatorname{Hom}_{U}(M, L(M(\lambda), N))\right.$ by $\zeta(\varphi)=$ ( $m \mapsto \varphi(m \otimes u)$ ). This makes $T^{\prime}$ a left adjoint to $T$.
(ii). We have to show that for any $M \in 0 b\left(\mathcal{H}_{\hat{\lambda}}\right)$ the map $\eta(M): M \rightarrow$
$L\left(M(\lambda), M \otimes{ }_{A} M(\lambda)\right)$ (given by $m \mapsto(n \mapsto m \otimes n)$ )) is bijective. We make $A$ into a $U$ module by $(a \otimes b) \cdot x=^{t} a ̆ x \check{b}$, for all $x \in A, a$, $b \in U(g)$. Then $\eta(A)$ is an isomorphism by ([13], 6.4).

If $E$ is a finite dimensional $U(\mathfrak{a})$ module we have natural isomorphisms

$$
\begin{gathered}
T(E \otimes N) \longleftarrow\left(E^{\tau} \otimes \mathbb{C}\right) \otimes T(N), N \in 0 b\left(O_{C}\right) \\
T^{\prime}\left(\left(E^{\tau} \otimes \mathbb{C}\right) \otimes M\right) \leftarrow E \otimes T^{\prime}(M), M \in 0 b\left(\mathscr{H}_{\hat{\lambda}}\right) .
\end{gathered}
$$

Using these isomorphisms, one shows that if $\eta(M)$ is an isomorphism then so is $\eta\left(\left(E^{\tau} \otimes \mathbb{C}\right) \otimes M\right)$. In particular, $\eta\left(\left(E^{\tau} \otimes \mathbb{C}\right) \otimes A\right)$ is an isomorphism. This implies that $\eta(M)$ is an isomorphism for any $M \in 0 b\left(\mathscr{H}_{\hat{\lambda}}\right)$, by observing that $T T^{\prime}$ is right exact and that for suitable finite dimensional $U(\mathfrak{a})$ modules $E_{1}, E_{2}$ there exists an exact sequence $\left(E_{1} \otimes \mathbb{C}\right) \otimes A \rightarrow\left(E_{2} \otimes \mathbb{C}\right) \otimes A \rightarrow M \rightarrow 0$ in $\mathscr{H}_{\hat{\lambda}}$.
(iii). We have to show that the counit map $\epsilon: T^{\prime} T \rightarrow I d_{\theta_{C}}$ is also an isomorphism of functors. The composition $T \xrightarrow{\eta^{T}} T T^{\prime} T \xrightarrow{T \epsilon} T$ is $I d_{T}$, so by (ii) $T \epsilon$ is an isomorphism. Thus, as $T$ is exact, $0=$ $T(\operatorname{ker}(\epsilon(N)))=T(\operatorname{coker}(\epsilon(N)))$ for any $N \in 0 b\left(\mathcal{O}_{C}\right)$. So it remains to show that if $N \in 0 b\left(O_{C}\right)$ and $T N=0$ then $N=0$. Indeed, if $N \neq 0$, then $N$ contains a simple submodule $L(\mu): \mu \in C$, so $T N \supset T L(\mu)$; but by ([14], 4.7) $T L(\mu) \neq 0$, and we get a contradiction.

## 2. The generalized B.G.G. resolution

Throughout this section we fix $-\lambda \in \mathfrak{b}^{*}$ dominant and regular.
2.1: Given $\alpha \in B_{\lambda}$, one can choose $\nu_{\alpha} \in P(R)$ such that $-\lambda_{\alpha}:=$ $-\lambda+\nu_{\alpha}$ is dominant and $\left(\beta, \lambda_{\alpha}\right)=0: \beta \in R^{+}$is equivalent to $\beta=\alpha$. Following Vogan ([22]) we set $\theta_{\alpha}=T_{\lambda_{\alpha}}^{\lambda} \circ T_{\lambda}^{\lambda_{\alpha}}: 0 \rightarrow 0$. Using 1.13, $\theta_{\alpha}$ is left adjoint to $\theta_{\alpha}$. So we obtain natural isomorphisms

$$
\operatorname{Ext}^{j}\left(\theta_{\alpha} M, N\right) \xrightarrow{\sim} \operatorname{Ext}^{j}\left(M, \theta_{\alpha} N\right): j \in \mathbb{N}, M, N \in 0 b \mathbb{O} .
$$

2.2: For each $w \in W_{\lambda}$, let $l(w)$ denote the reduced length of $w$ with respect to $B_{\lambda}$. For each $w, w^{\prime} \in W_{\lambda}$, we define an expression $P_{w, w^{\prime}}$ in the indeterminate $q$ through

$$
P_{w, w}(q)=\sum_{k=0}^{\infty} q^{\left(I\left(w^{\prime}\right)-l(w)-k\right) / 2} \operatorname{dim} \operatorname{Ext}^{k}\left(M(w \lambda), L\left(w^{\prime} \lambda\right)\right)
$$

A result of Casselman and Schmid (proved also in [6], Thm. 4)
implies that $P_{w, w^{\prime}}(q)$ is polynomial in $q^{1 / 2}$. Kazhdan and Lusztig ([19], Conj. 1.5) have further conjectured that $P_{w, w}(q)$ is polynomial in $q$ and that this polynomial is determined by a particular purely combinatorial procedure which uses only the description of $W_{\lambda}$ as a Coxeter group. This has been shown to follow from certain other conjectures ([10], [23]); but for the moment remains an open problem. Here we just establish one of the identities which would follow from the Kazhdan-Lusztig conjecture.

Lemma: For each $w, w^{\prime} \in W_{\lambda}, \alpha \in B_{\lambda}$, such that $w^{\prime} s_{\alpha}<w^{\prime}$, one has that $P_{w_{s_{q}},}(q)=P_{w, w}(q)$. In particular, for each $B^{\prime} \subset B_{\lambda}, w \in W_{B^{\prime}}$, one has that $P_{w, w_{B}}(q)=1$.

We can assume $w s_{\alpha}<w$, without loss of generality. Then the conclusion of the lemma is equivalent to the identity
(*) $\left.\quad \operatorname{dim} \operatorname{Ext}^{j+1}\left(M\left(w s_{\alpha} \lambda\right), L\left(w^{\prime} \lambda\right)\right)=\operatorname{dim} \operatorname{Ext}^{j}(M(w \lambda)), L\left(w^{\prime} \lambda\right)\right), j \in \mathbb{N}$.
Under the hypothesis $w^{\prime} s_{\alpha}<w^{\prime}$, it follows that $M\left(w^{\prime} s_{\alpha} \lambda\right)$ is a submodule of $M\left(w^{\prime} \lambda\right)$ and by ([10], 2.10a) that $\theta_{\alpha} M\left(w^{\prime} \lambda\right) \rightrightarrows$ $\theta_{\alpha} M\left(w^{\prime} s_{\alpha} \lambda\right)$. Hence $\theta_{\alpha}\left(M\left(w^{\prime} \lambda\right) / M\left(w^{\prime} s_{\alpha} \lambda\right)\right)=0$. Since $L\left(w^{\prime} \lambda\right)$ is a quotient of $M\left(w^{\prime} \lambda\right) / M\left(w^{\prime} s_{\alpha} \lambda\right)$, it follows that $\theta_{\alpha} L\left(w^{\prime} \lambda\right)=0$, and so $\operatorname{Ext}^{i}\left(\theta_{\alpha} M(w \lambda), L\left(w^{\prime} \lambda\right)\right)=0$ by 2.1. In particular $L(w \lambda)$ is not a quotient of $\theta_{a} M(w \lambda)$. Then from ([11],2.17) we obtain an exact sequence

$$
0 \rightarrow M(w \lambda) \rightarrow \theta_{\alpha} M(w \lambda) \rightarrow M\left(w s_{\alpha} \lambda\right) \rightarrow 0
$$

from which the corresponding long exact sequence for $\operatorname{Ext}^{*}\left(\cdot, L\left(w^{\prime} \lambda\right)\right)$ gives (*).
2.3: From 1.8, 1.14 and 2.2 we obtain

Corollary: For each $B^{\prime} \subset B_{\lambda}, w \in W_{\lambda}$, one has

$$
\operatorname{dim} H_{i}\left(n^{-}, L\left(w_{B} \lambda\right)\right)_{w \lambda-\rho}=\left\{\begin{array}{l}
1: w \in W_{B^{\prime}}, j=l\left(w_{B}\right)-l(w), \\
0: \text { otherwise. }
\end{array}\right.
$$

Remarks. As is well-known the remaining weight spaces of $H_{j}\left(n^{-}, L\left(w_{B}, \lambda\right)\right)$ are null. This follows from the action of $Z(g)$ and the fact that $w \lambda-\lambda \in \mathbb{Z} B$ implies $w \in W_{\lambda}$. This result then generalizes the Bott-Kostant formula established for finite dimensional simple modules (i.e. when $-\lambda \in P(R)^{++}$and $B^{\prime}=B$ ).
2.4: Fix $B^{\prime} \subset B_{\lambda}$ and set $s=l\left(w_{B}\right)$. Then for each $j \in \mathbb{N}$, set $W_{B^{\prime}}^{j}=$
$\left\{w \in W_{B^{\prime}}: l(w)=j\right\}$, and

$$
C_{j}=\underset{w \in W_{B^{\prime}}^{\prime}}{\bigoplus_{i}} M(w \lambda) .
$$

As $M(w \lambda)$ is $U\left(\mathrm{n}^{-}\right)$free, we have for each $y \in W_{B^{\prime}}$ that

$$
\operatorname{dim} H_{t}\left(\mathfrak{n}^{-}, C_{j}\right)_{y \lambda-\rho}=\left\{\begin{array}{l}
1: t=0, j=l(y) \\
0: \text { otherwise }
\end{array}\right.
$$

2.5: For each $w \in W_{B^{\prime}}$, fix a $U(\mathfrak{g})$ module embedding $i_{w}: M(w \lambda) \hookrightarrow$ $M\left(w_{B} \lambda\right)$. For $w, w^{\prime} \in W_{B^{\prime}}$ such that $w \leq w^{\prime}$, let $i_{w, w^{\prime}}: M(w \lambda) \rightarrow M\left(w^{\prime} \lambda\right)$ be the embedding such that $i_{w^{\prime}} \circ i_{w, w^{\prime}}=i_{w}$.

Fix $\mathrm{j} \in\{1,2, \ldots, s\}$ and consider a $U(\mathfrak{g})$ module map $\partial_{\mathrm{j}}: C_{\mathrm{j}-1} \rightarrow C_{\mathrm{j}}$ defined by $\left(x_{w}\right)_{w \in W_{B}^{j-1}} \stackrel{\partial_{j}}{\mapsto}\left(y_{w^{\prime}}\right)_{w^{\prime} \in W_{B}^{j}}$, when

$$
y_{w^{\prime}}=\sum_{w \leq w^{\prime}} c_{w, w^{w}}^{j} i_{w, w^{\prime}}\left(x_{w}\right), \quad x_{w} \in M(w \lambda)
$$

where $c_{w, w^{\prime}}^{j} \in \mathbb{Z}$ is non-zero and defined whenever $w \leq w^{\prime}, w \in W_{B^{\prime}}^{i-1}$, $w^{\prime} \in W_{B^{\prime}}^{j}$.

Lemma: The natural surjection $H_{0}\left(\mathrm{n}^{-}, C_{j-1}\right) \rightarrow H_{0}\left(\mathrm{n}^{-}, \operatorname{Im} \partial_{j}\right)$ is bijective.

Set $K=\operatorname{ker} \partial_{j}, \quad V=W_{B^{\prime}}^{i-1}$. We have an exact sequence $0 \rightarrow$ $K / K \cap n^{-} C_{j-1} \rightarrow C_{j-1} / n^{-} C_{j-1} \rightarrow \partial_{j} C_{j-1} / n^{-}\left(\partial_{j} C_{j-1}\right) \rightarrow 0$, so the lemma is equivalent to $K \subset \underset{w \in V}{\oplus} n^{-} M(w \lambda)$, or to $K \subset \bigoplus_{w \in V} \overline{M(w \lambda)}$, that is to

$$
\bar{K}:=\operatorname{Im}(K \rightarrow \underset{w \in V}{\bigoplus} L(w \lambda))=0
$$

If $\bar{K} \neq 0$, there exists $w \in V$ such that $[\bar{K}: L(w \lambda)]>0$, and so [ $K: L(w \lambda)]>0$. Yet equality of lengths in $V$ implies through ([7], 7.6.23) that $\left[C_{j}: L(w \lambda)\right]=1$, so $0=\left[C_{j} / K: L(w \lambda)\right]=\left[\partial_{j} C_{j-1}: L(w \lambda)\right]$. On the other hand since there exists $w^{\prime} \in W_{B^{\prime}}^{\prime}$ such that $w \leq w^{\prime}$ and by hypothesis we then have $c_{w, w^{\prime}}^{j} \neq 0$, it follows that $\partial_{j}$ is injective on the summand $M(w \lambda)$ of $C_{j-1}$. Thus $\partial_{j} C_{j-1}$ contains a copy of $M(w \lambda)$, which implies $\left[\partial_{j} C_{j-1}: L(w \lambda)\right] \geqslant 1$. This contradiction proves the lemma.
2.6: An appropriate combinatorial property of the Bruhat ordering enables one to choose the $c_{w, w^{\prime}}^{j}$ of 2.5 such that $\partial_{j} \partial_{j-1}=0$, for all $j=2, \ldots, s$. (See [2], Sect. 11 or [7], 7.8.14). Furthermore

Proposition: The sequence

$$
0 \rightarrow C_{0} \rightarrow C_{1} \rightarrow \cdots C_{j-1} \xrightarrow{d_{j}} C_{j} \rightarrow \cdots \rightarrow C_{s} \rightarrow L\left(w_{B}, \lambda\right) \rightarrow 0
$$

is exact.
Set $X_{s+1}=Y_{s+1}=L\left(w_{B}, \lambda\right), \quad Y_{s}=\operatorname{ker}\left(C_{s} \rightarrow X_{s+1}\right)$, and for each $j \in$ $\{1,2, \ldots, s\}$, set $X_{j}=\operatorname{Im} \partial_{j}, Y_{j-1}=\operatorname{ker} \partial_{j}$. For each $j \in\{1,2, \ldots, s+1\}$, $X_{j}$ is a submodule of $Y_{j}$ and we show that $X_{j}=Y_{j}$. Fix $r \geq 1$ and assume that this has been established for all $j>r$. This means that we have the short exact sequences

$$
0 \rightarrow X_{j} \rightarrow C_{j} \rightarrow X_{i+1} \rightarrow 0: r<j \leqslant s
$$

By 2.4, the associated long exact sequence for homology implies for all $\mu \in \mathfrak{b}^{*}$ and $r<j \leq s$ that

$$
\operatorname{dim} H_{t}\left(\mathrm{n}^{-}, X_{j}\right)_{\mu}=\left\{\begin{array}{cl}
\operatorname{dim} H_{t+1}\left(\mathrm{n}^{-}, X_{j+1}\right)_{\mu} & : t>0 \\
\operatorname{dim} H_{1}\left(\mathrm{n}^{-}, X_{j+1}\right)_{\mu}-\operatorname{dim} H_{0}\left(\mathrm{n}^{-}, X_{j+1}\right)_{\mu} \\
+\operatorname{dim} H_{0}\left(\mathrm{n}^{-}, C_{j}\right)_{\mu} & : t=0 .
\end{array}\right.
$$

Then from 2.3 and 2.4 we obtain

$$
\operatorname{dim} H_{t}\left(\mathfrak{n}^{-}, X_{j}\right)_{\mu}=\left\{\begin{array}{l}
1: \mu=w \lambda, w \in W_{B^{\prime}}, l(w)=j-t-1 \\
0: \text { otherwise }
\end{array}\right.
$$

for all $j>r$ and in particular for $j=r+1$.
Finally from the long exact sequence associated to $0 \rightarrow Y_{r} \rightarrow C_{r} \rightarrow$ $X_{r+1} \rightarrow 0,2.4$ and the above we eventually obtain

$$
\operatorname{dim} H_{0}\left(\mathrm{n}^{-}, Y_{r}\right)_{\mu}=\operatorname{dim} H_{0}\left(\mathfrak{n}^{-}, C_{r-1}\right)_{\mu}
$$

for all $\mu \in \mathfrak{b}^{*}$. Then by $2.5, H_{0}\left(\mathfrak{n}^{-}, Y_{r}\right)$ and $H_{0}\left(n^{-}, X_{r}\right)$ are isomorphic as $\mathfrak{b}$-modules and so $X_{r}=Y_{r}$ by 1.10 . Noting that $\partial_{1}$ is injective completes the proof of the proposition.

Remark. This generalizes the B.G.G. resolution originally established [2] for the case $-\lambda \in P(R)^{++}, B^{\prime}=B$. The original proof is different to ours and can only be generalized to the case when $B^{\prime} \subset B$ (see [20] for this). The present proof was found following conversations with M. Duflo and P. Delorme.

## 3. Mappings of Verma modules

3.1: Take $-\lambda \in \mathfrak{h}^{*}$ dominant. Then $M(\lambda)$ is a simple module and so isomorphic to $\delta(M(\lambda))$. Then by 1.9 one has for all $\mu \in \mathfrak{b}^{*}$ that

$$
L(M(\mu), M(\lambda))=L(M(\mu), \delta M(\lambda))=L(-\lambda,-\mu)
$$

up to isomorphisms. This relationship of mappings of Verma modules to the principal series has been known for some time. Here we consider the most general form this takes when $-\lambda$ is not necessarily dominant. Some results in this direction were already obtained in ([5], $5.5)$ and in ([14], 4.10).
3.2: Fix $-\lambda,-\mu \in \mathfrak{b}^{*}$ dominant with $\lambda-\mu \in P(R)$ (recall 1.15). Choose $w_{1}, w_{2} \in W_{\lambda}$ and $\alpha \in B_{\lambda}$ such that $s_{\alpha} w_{1}>w_{1}, s_{\alpha} w_{2}<w_{2}$. The second relation implies that $M\left(s_{\alpha} w_{2} \lambda\right)$ is a submodule of $M\left(w_{2} \lambda\right)$.

Lemma: Under the above hypotheses, one has $L\left(M\left(w_{1} \mu\right)\right.$, $\left.M\left(w_{2} \lambda\right) / M\left(s_{a} w_{2} \lambda\right)\right)=0$.

Equivalently for any finite dimensional $U(\mathfrak{g})$ module $E$ one has $\operatorname{Hom}_{a}\left(M\left(w_{1} \mu\right),\left(E \otimes\left(M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right)\right)\right)=0$. To establish this it is enough to show that $L\left(w_{1} \mu\right)$ is not a subquotient of $p_{\hat{\mu}}\left(E \otimes M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right)$. Now by 1.11 and the invariance of $\Omega(E)$ under $W$ one has
$\left[E \otimes\left(M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right)\right]=\sum_{\nu \in \Omega(E)}\left(\left[M\left(w_{2}(\lambda+\nu)\right)\right]\right.$

$$
\left.-\left[M\left(s_{\alpha} w_{2}(\lambda+\nu)\right)\right]\right)\left(\operatorname{dim} E_{\nu}\right),
$$

and so

$$
\begin{gather*}
{\left[p_{\dot{\mu}}\left(E \otimes M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right)\right]=}  \tag{*}\\
=\sum_{\substack{w \in w_{i} \\
w \in \in \lambda+\hat{\beta}(E)}}\left(\operatorname{dim} E_{w_{\mu}-\lambda}\right)\left(\left[M\left(w_{2} w \mu\right)\right]-\left[M\left(s_{\alpha} w_{2} w \mu\right)\right]\right) .
\end{gather*}
$$

Through the hypothesis $s_{a} w_{1}>w_{1}$, one has by $([10], 5.19)$ that

$$
\begin{equation*}
\left[M\left(w_{2} w \mu\right): L\left(w_{1} \mu\right)\right]=\left[M\left(s_{\alpha} w_{2} w \mu\right): L\left(w_{1} \mu\right)\right] . \tag{**}
\end{equation*}
$$

Combined with (*) this establishes the assertion of the lemma.
Remarks. A technically easier proof of (**) follows from ([11],
2.16) and ([3], 4.5 (6)). Again the analysis of ([14], 5.4) can be combined with the operators of coherent continuation to give an alternative proof of the fact that $L\left(w_{1} \mu\right)$ is not a subquotient of $E \otimes\left(M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right)$.
3.3: Let $W$ be a Coxeter group with $S$ the corresponding set of simple reflections and length function $l(\cdot)$. It is well-known that there exists an associative product $*$ on $W$ uniquely defined through

$$
\begin{gathered}
w_{*} w^{\prime}=w w^{\prime} \quad \text { if } \quad l(w)+l\left(w^{\prime}\right)=l\left(w w^{\prime}\right), \\
s_{*} s=s \quad \text { if } \quad s \in S
\end{gathered}
$$

(Up to a sign, these are the defining relations for the generators of the "singular Hecke algebra" obtained say from ([19], Sect. 1) by putting $q=0$.)

Lemma: For all $w, y \in W, s \in S$, one has

$$
s_{*} w=\left\{\begin{array}{l}
s w: s w>w  \tag{i}\\
w: s w<w
\end{array}\right.
$$

(ii)

$$
w_{*} s=\left\{\begin{array}{l}
w s: w s>w, \\
w: w s<w .
\end{array}\right.
$$

$$
\begin{equation*}
\left(w_{*} y\right)^{-1}=y^{-1} * w^{-1} \tag{iii}
\end{equation*}
$$

$$
\begin{equation*}
w_{*} w^{\prime} \geq w^{\prime}, w . \tag{iv}
\end{equation*}
$$

The top lines of (i), (ii) are immediate from the definition of $*$. For the bottom line in say (ii), set $w^{\prime}=s w$. Then $s w^{\prime}>w^{\prime}$ and so $s_{*} w=$ $s_{*}\left(s_{*} w^{\prime}\right)=\left(s_{*} s\right)_{*} w^{\prime}=s_{*} w^{\prime}=w$.

We prove (iii) by induction on $l(w)$. For $l(w)=0,1$, it follows from (i), (ii). Otherwise write $w=s_{*} z: l(z)<l(w)$. Then $\left(w_{*} y\right)^{-1}=$ $\left(\left(s_{*} z\right)_{*} y\right)^{-1}=\left(s_{*}\left(z_{*} y\right)\right)^{-1}=\left(\left(z_{*} y\right)^{-1}{ }_{*} s\right)=\left(y^{-1}{ }_{*} z^{-1}\right)_{*} s=y^{-1} *\left(z^{-1} * s\right)=$ $y^{-1} * w^{-1}$. (iv) follows from (i), (ii).
3.4: Fix $-\lambda \in \mathfrak{b}^{*}$ dominant. For all $w_{1}, w_{2} \in W_{\lambda}$, one has from 3.3 (iv) that $w_{2}^{-1} * w_{1} w_{\lambda} \geq w_{1} w_{\lambda}$ and so $w_{3}:=\left(w_{2}^{-1} * w_{1} w_{\lambda}\right) w_{\lambda} \leq w_{1}$.

Proposition: Assume $-\lambda,-\mu \in \mathfrak{b}^{*}$ dominant with $\lambda-\mu \in P(R)$. Given $w_{1}, w_{2} \in W_{\lambda}$, define $w_{3} \in W_{\lambda}$ as above. Then the $U$-module
homomorphism of $L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ into $L\left(M\left(w_{3} \lambda\right), M\left(w_{2} \mu\right)\right)$ defined by restriction is injective with image $L\left(M\left(w_{3} \lambda\right), M(\mu)\right)$.

The assertion is clear for $w_{2}=1$. If $w_{2} \neq 1$, choose $\alpha \in B_{\lambda}$ such that $s_{\alpha} w_{2}<w_{2}$. If $s_{\alpha} w_{1}>w_{1}$, then by 3.2 the natural embedding $L\left(M\left(w_{1} \lambda\right)\right.$, $\left.M\left(s_{\alpha} w_{2} \mu\right)\right) \hookrightarrow L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ is surjective. If $s_{\alpha} w_{1}<w_{1}$, then by ([13], 6.1) the map of $L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ into $L\left(M\left(s_{\alpha} w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ defined by restriction is injective and so, by 3.2 again, we obtain an embedding of $L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ into $L\left(M\left(s_{\alpha} w_{1} \lambda\right), M\left(s_{\alpha} w_{2} \mu\right)\right)$. In either case we obtain an embedding of $L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$ into $L\left(M\left(\left(s_{\alpha *} w_{1} w_{\lambda}\right) w_{\lambda} \lambda\right), M\left(s_{\alpha} w_{2} \mu\right)\right)$, and so by induction an embedding into $L\left(M\left(w_{3} \lambda\right), M(\mu)\right)$. On the other hand we can take $\alpha \in B_{\lambda}$ such that $s_{\alpha} w_{1}<w_{1}$. Then a similar argument gives an embedding of $L\left(M\left(s_{\alpha} w_{1} \lambda\right), M\left(\left(s_{\alpha} * w_{2}\right) \mu\right)\right)$ into $L\left(M\left(w_{1} \lambda\right), M\left(w_{2} \mu\right)\right)$. By induction this gives an embedding of $L\left(M\left(w_{\lambda} \lambda\right), M\left(\left(w_{\lambda} w_{1}^{-1} * w_{2}\right) \mu\right)\right)$ into $L\left(M\left(w_{1} \lambda\right), \quad M\left(w_{2} \mu\right)\right)$ which we saw above further embeds in $L\left(M\left(w_{3} \lambda\right), M(\mu)\right)$, both maps having been defined by restriction. Now by 3.3, we have $\left(w_{2}^{-1} * w_{1} w_{\lambda}\right)^{-1}=w_{\lambda} w_{1}^{-1} * w_{2}$ and so by 1.15 the combined map is surjective. Consequently the second map must also be surjective, proving the assertion.
3.5: Assume $-\lambda,-\mu \in \mathfrak{h}^{*}$ dominant with $\lambda-\mu \in P(R)$ and fix $B^{\prime} \subset B_{\lambda}$.

Corollary: For each $w \in W_{B^{\prime}}$ and each finite dimensional $U(\mathfrak{g})$ module E, one has

$$
\operatorname{dim} \operatorname{Hom}_{a}(Q, M(w \mu))=\operatorname{dim} \operatorname{Hom}_{8}(Q, \delta M(w \mu)),
$$

where $Q=E \otimes M\left(w_{B}, \lambda\right)$.
From $l\left(w w_{\lambda}\right)=l\left(w_{\lambda}\right)-l(w)$ for all $w \in W_{\lambda}$ and an analogous assertion for $W_{B}$, we obtain $l\left(w^{-1} w_{B}, w_{\lambda}\right)=l\left(w^{-1}\right)+l\left(w_{B}, w_{\lambda}\right)$. Since $w_{\lambda}^{2}=1$, it follows from the definition of $*$ that $\left(w^{-1} * w_{B} w_{\lambda}\right) w_{\lambda}=w^{-1} w_{B^{\prime}}$, so by 3.4, 3.1 one has the isomorphisms $L\left(M\left(w_{B} \lambda\right), \quad M(w \mu)\right) \leftrightharpoons$ $L\left(M\left(w^{-1} w_{B^{\prime}} \lambda\right), M(\mu)\right) \leftrightarrows L\left(-\mu,-w^{-1} w_{B^{\prime}} \lambda\right)$. On the other hand, by 1.9 we have $L\left(M\left(w_{B} \cdot \lambda\right), \delta(M(w \mu))\right) \cong L\left(-w \mu,-w_{B} \cdot \lambda\right)$. Combined with 1.7 and 1.12 , these isomorphisms imply the assertion of the corollary.
3.6: $\operatorname{Tak} \in \lambda, \mu, w_{1}, w_{2}, \alpha$ as in 3.2.

## Lemma:

(i) $L\left(M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right), \delta M\left(w_{1} \mu\right)\right)=0$.
(ii) $L\left(L\left(w_{2} \lambda\right), L\left(w_{1} \mu\right)\right)=0$.
(iii) The map of $L\left(-w_{1} \mu,-w_{2} \lambda\right)$ into $L\left(-w_{1} \mu,-s_{\alpha} w_{2} \lambda\right)$ defined by restriction is injective.

For (i), observe that $L\left(w_{1} \mu\right)$ is the unique simple submodule of $\delta M\left(w_{1} \mu\right)$, so it suffices to show for any finite dimensional module $E$ that

$$
\begin{equation*}
\left[E \otimes\left(M\left(w_{2} \lambda\right) / M\left(s_{\alpha} w_{2} \lambda\right)\right): L\left(w_{1} \mu\right)\right]=0 . \tag{*}
\end{equation*}
$$

This obtains by an argument parallel to 3.2. Hence (i). Through the embedding $\operatorname{Hom}_{9}\left(E \otimes L\left(w_{2} \lambda\right), L\left(w_{1} \mu\right)\right) C \operatorname{Hom}_{9}\left(E \otimes\left(M\left(w_{2} \lambda\right) /\right.\right.$ $\left.M\left(s_{\alpha} w_{2} \lambda\right)\right), L\left(w_{1} \mu\right)$ ) and (*) we obtain (ii). Recalling 1.9, (i) gives (iii).

Remark. When $\alpha \in B$, the result in (iii) is due to Zelobenko (see [8], Lemmes 4, 5).
3.7: We conclude this section with a result of obvious importance which by virtue of ([4], 2.14) is a far reaching generalization of 3.6 (ii). We start with the following

Lemma: For all $\lambda, \mu, \nu \in \mathfrak{b}^{*}$ one has
(i) $L(L(\mu), L(\lambda)) \neq 0 \Leftrightarrow L(L(\lambda), L(\mu)) \neq 0$.
(ii) $L(L(\mu), L(\lambda)) L(L(\nu), L(\mu))=0$ implies that one of these modules must vanish.
(i) follows from the isomorphism $\delta(L \mu) \leftleftarrows L(\mu)$. (ii) follows from the simplicity of $L(\mu)$.
3.8: Proposition: Let $\lambda \in \mathfrak{h}^{*}$ be dominant and regular. Then for each $w, y \in W_{\lambda}$, one has

$$
L(L(w \lambda), L(y \lambda)) \neq 0 \Leftrightarrow J\left(w^{-1} \lambda\right)=J\left(y^{-1} \lambda\right) .
$$

Suppose $L(L(w \lambda), L(y \lambda)) \neq 0$. Then there exists a finite dimensional $U(\mathfrak{g})$ module $E$ such that $\operatorname{Hom}_{\mathfrak{q}}(L(w \lambda), L(y \lambda) \otimes E) \neq 0$ and so $L(w \lambda)$ is a submodule of $L(y \lambda) \otimes E$. It follows that $L(M(\lambda), L(w \lambda))$ is a submodule of $L(M(\lambda), L(y \lambda) \otimes E)$. Hence the right annihilator of $L(M(\lambda), L(w \lambda))$ contains the right annihilator $J$ of $L:=L(M(\lambda)$, $L(y \lambda) \otimes E)$. Since $L$ is isomorphic to $L(M(\lambda), L(y \lambda)) \otimes\left(E^{r} \otimes \mathbb{C}\right)$, it
follows that $J$ coincides with the right annihilator of $L(M(\lambda), L(y \lambda))$. By ([14], 4.7, 4.12) this gives $J\left(w^{-1} \lambda\right) \supset J\left(y^{-1} \lambda\right)$. By 3.7 (i), interchange of $w, y$ gives the reverse inclusion.

Suppose $J\left(w^{-1} \lambda\right)=J\left(y^{-1} \lambda\right)$. By ([8], Prop. 8) $U(\mathfrak{g}) / J\left(w^{-1} \lambda\right)$ has a unique $U$ submodule which is furthermore isomorphic to some $V(-\sigma \lambda,-\lambda)$ with $\sigma$ an involution of $W_{\lambda}$. By ([14], 4.12) it is clear that $J(\sigma \lambda)=J\left(w^{-1} \lambda\right)$. After Vogan ([24], 3.5) there exists a finite dimensional $U(\mathfrak{g})$ module $E$ such that $U(\mathfrak{g}) / J\left(w^{-1} \lambda\right)$ (and hence $V(-\sigma \lambda,-\lambda))$ is a submodule of $V\left(-w^{-1} \lambda,-\lambda\right) \otimes(\mathbb{C} \otimes E)$. From 1.12(i), we have $V\left(-w^{-1} \lambda,-\lambda\right) \cong V(-\lambda,-w \lambda)$, and so $V(-\sigma \lambda,-\lambda)$ is a submodule of $V(-w \lambda,-\lambda) \otimes(E \otimes \mathbb{C})$. Then by 1.12 (iii), $L(M(\lambda)$, $L(\sigma \lambda))$ is a submodule of $L(M(\lambda), L(w \lambda)) \otimes(E \otimes \mathbb{C})$ ) which is isomorphic to $L\left(M(\lambda), L(w \lambda) \otimes E^{\top}\right)$. The resulting injection $i: L(M(\lambda), L(\sigma \lambda)) \rightarrow L\left(M(\lambda), L(w \lambda) \otimes E^{\tau}\right)$ must come by $1.16($ iii $)$ by applying $T$ to an injection $L(\sigma \lambda) \rightarrow L(w \lambda) \otimes E^{\tau}$. Hence $L(L(\sigma \lambda)$, $L(w \lambda)) \neq 0$. Interchanging $w, y$ and using 3.7 gives $L(L(w \lambda)$, $L(y \lambda)) \neq 0$, as required.

## 4. Exactness of the functor $L\left(M\left(w_{B}, \lambda\right), \cdot\right)$.

In this section we fix $-\lambda \in \mathfrak{b}^{*}$ dominant and $B^{\prime} \subset B_{\lambda}$. Set $\Lambda=$ $\{\mu \in \lambda+P(R):-\mu$ is dominant $\}$.
4.1: Let $\mathscr{O}_{A}^{B^{\prime}}$ denote the subcategory of $\mathcal{O}$ consisting of all those modules (necessarily of finite length) whose simple factors are amongst the $L(w \mu): \mu \in \Lambda, w \in W_{B^{\prime}}$. By ([6], Thm. 4(iv)) it follows that the $M\left(w_{B}, \mu\right): \mu \in \Lambda$ are projective in $\mathcal{O}_{A}^{B^{\prime}}$. On the other hand $\mathcal{O}_{A}^{B^{\prime}}$ is not closed under tensoring with finite dimensional $U(\mathfrak{g})$ modules. Nevertheless we have the

Proposition: Suppose $M_{1}, M_{2}, M_{3} \in 0 b \odot_{1}^{B^{\prime}}$, with

$$
0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0
$$

exact. Then

$$
0 \rightarrow L\left(M\left(w_{B} \cdot \lambda\right), M_{1}\right) \rightarrow L\left(M\left(w_{B} \cdot \lambda\right), M_{2}\right) \rightarrow L\left(M\left(w_{B} \cdot \lambda\right), M_{3}\right) \rightarrow 0
$$

is exact.
This is proved in sections 4.2, 4.3.
4.2: A module $M \in 0 b \mathcal{O}$ is said to admit a $p$-filtration if it has a finite filtration with factors isomorphic to Verma modules. For
example, by $([7], 7.6 .14) E \otimes M(\mu)\left(E\right.$ finite dimensional, $\left.\mu \in \mathfrak{b}^{*}\right)$ has a p-filtration.

Lemma: Suppose $Q \in 0 b \bigcirc$ admits a $p$-filtration. Then for all $\mu \in$ $\mathfrak{b}^{*}, k>0$, one has

$$
\operatorname{Ext}^{k}(Q, \delta M(\mu))=0
$$

It is enough to prove the assertion for $Q$ a Verma module, say $M(\nu): \nu \in \mathfrak{b}^{*}$. By 1.14, $\operatorname{Ext}^{k}(M(\nu), \delta M(\mu))=\left(H_{k}\left(\mathfrak{n}^{-}, M(\mu)\right)_{\nu-\rho}\right)^{*}$, up to isomorphism, so the assertion follows from the fact that $M(\mu)$ is $U\left(\mathrm{n}^{-}\right)$free.
4.3: Let $E$ be a finite dimensional $U(g)$ module and set $Q=$ $E \otimes M\left(w_{B} \lambda\right)$ and fix $\mu \in \Lambda$. We show that $\operatorname{Ext}^{1}(Q, L(y \mu))=0: y \in$ $W_{B^{\prime}}$ by induction on $l(y)$. This will establish 4.1. When $l(y)=0$, that is $y=1$, we have $L(\mu) \cong M(\mu) \cong \delta M(\mu)$ and so the assertion follows from 4.2. Now fix $w \in W_{B^{\prime}}$ and suppose the assertion proved for all $y \in W_{B^{\prime}}$ such that $l(y)<l(w)$. In particular this gives

$$
\begin{equation*}
\operatorname{Ext}^{1}(Q, \overline{M(w \mu)})=0 \tag{1}
\end{equation*}
$$

From the exact sequence

$$
0 \rightarrow L(w \mu) \rightarrow \delta M(w \mu) \rightarrow \delta \overline{M(w \mu)} \rightarrow 0
$$

and 4.2 we obtain an exact sequence
(2) $0 \rightarrow \operatorname{Hom}(Q, L(w \mu)) \rightarrow \operatorname{Hom}(Q, \delta M(w \mu)) \rightarrow \operatorname{Hom}(Q, \delta \overline{M(w \mu)}) \rightarrow$

$$
\rightarrow \operatorname{Ext}^{1}(Q, L(w \mu)) \rightarrow 0 .
$$

From the exact sequence

$$
0 \rightarrow \overline{M(w \mu)} \rightarrow M(w \mu) \rightarrow L(w \mu) \rightarrow 0
$$

and (1) we obtain an exact sequence
(3) $0 \rightarrow \operatorname{Hom}(Q, \overline{M(w \mu)}) \rightarrow \operatorname{Hom}(Q, M(w \mu)) \rightarrow \operatorname{Hom}(Q, L(w \mu)) \rightarrow 0$.

$$
\begin{aligned}
\operatorname{dim} \operatorname{Ext}^{1}(Q, L(w \mu))= & \{\operatorname{dim} \operatorname{Hom}(Q, \delta \overline{M(w \mu)})- \\
& -\operatorname{dim} \operatorname{Hom}(Q, \overline{M(w \mu)})\} \\
& -\{\operatorname{dim} \operatorname{Hom}(Q, \delta M(w \mu)) \\
& -\operatorname{dim} \operatorname{Hom}(Q, M(w \mu))\}
\end{aligned}
$$

The first term in curly brackets vanishes by the induction hypothesis and the fact that $\delta \overline{M(w \mu)}$ and $\overline{M(w \mu)}$ have the same composition factors which are amongst the $L(y \mu): y<w$. The second term vanishes by 3.5 .
4.4: Let $M$ be a simple $U(\mathfrak{g})$ module. The natural action of $U(\mathfrak{g})$ in $M$ defines an embedding of $U(\mathfrak{g}) / \operatorname{Ann} M$ into $\operatorname{Hom}(M, M)$ and in fact the image lies in the $f$-finite part $L(M, M)$. Kostant has asked if the image is exactly $L(M, M)$. This is generally false ([5], 6.5; [13], 9.3, 9.4); yet it is quite important to ascertain when it does hold, especially for highest weight modules.

Theorem: For each $-\lambda \in \mathfrak{h}^{*}$ dominant and $\mathrm{B}^{\prime} \subset B$, one has

$$
U(g) / J\left(w_{B} \cdot \lambda\right)=L\left(L\left(w_{B^{\prime}} \lambda\right), L\left(w_{B^{\prime}} \cdot \lambda\right)\right)
$$

By 4.1, $L\left(M\left(w_{B} \cdot \lambda\right), L\left(w_{B^{\prime}} \lambda\right)\right)$ is a quotient of $L\left(M\left(w_{B}, \lambda\right), M\left(w_{B}, \lambda\right)\right)$ and the latter by ([14], 3.6) identifies with $U(\mathfrak{g}) / A n n M\left(w_{B} \lambda\right)$. Since $L\left(L\left(w_{B} \cdot \lambda\right), L\left(w_{B^{\prime}} \lambda\right)\right)$ is a submodule of $L\left(M\left(w_{B^{\prime}} \lambda\right), L\left(w_{B^{\prime}} \lambda\right)\right)$, this proves the theorem.

Remark. In the special case when $B^{\prime} \subset B$ the above result is due to Conze-Berline and Duflo ([5], 2.12, 6.3). Their proof does not admit further generalization since it uses induction from the parabolic subalgebra defined by $B^{\prime}$. When $B^{\prime}=B_{\lambda}$ with $\lambda$ regular, the result is noted in ([12], 5.7).
4.5: For $\mu \in \mathfrak{b}^{*}$, we write $A_{\mu}:=U(\mathfrak{g}) / J(\mu), A_{\mu}^{\prime}:=L(L(\mu), L(\mu))$. The embedding of $A_{\mu}$ into $A_{\mu}^{\prime}$ extends ([13], 4.3) to an embedding of Fract $\boldsymbol{A}_{\mu}$ into Fract $\boldsymbol{A}_{\mu}^{\prime}$. In order to compute the scale factors in the Goldie polynomial defined by the Goldie rank of $A_{\mu}$ (see [15], 5.12) it is useful to know when Fract $A_{\mu}=$ Fract $A_{\mu}^{\prime}$.

Since $J(\mu)$ is a prime ideal, $A_{\mu}$ admits a unique simple submodule $V_{\mu}$ which furthermore ([8], Prop. 4) has annihilator $J_{\mu}:=$ $\check{J}(\mu) \otimes U(g)+U(g) \otimes \check{J}(\mu)$. We let $l_{0}\left(A_{\mu}^{\prime}\right)$ denote the number of factors in a $U$ composition series of $A_{\mu}^{\prime}$ having annihilator $J_{\mu}$.

Lemma: $l_{0}\left(A_{\mu}^{\prime}\right)=1 \Leftrightarrow$ Fract $A_{\mu}=$ Fract $A_{\mu}^{\prime}$.

If $M$ is a finitely generated left $U(\mathfrak{g})$ module, let $\operatorname{Dim} M$ denote its Gelfand-Kirillov dimension over $U(\mathfrak{g})$ as defined in ([17], 2.1). Now let $M$ be a simple $U$ subquotient of $A_{\mu}^{\prime}$, which by $k$-finiteness is a finitely generated left $U(\mathfrak{g})$ module. By ([17], 1.4, 3.1 and 3.3 Remark) we have $\operatorname{Dim} M=\operatorname{Dim}\left(U(\mathfrak{g}) / \operatorname{Ann}_{U(\mathfrak{q})} M\right)$. Since $\operatorname{Ann}_{U(\mathfrak{q})} M \supset \check{J}(\mu)$, it follows from the primeness of $\breve{J}(\mu)$ that $\mathrm{Ann}_{U())} M=\check{J}(\mu)$ if and only if $\operatorname{Dim} M=\operatorname{Dim} V_{\mu}$. A similar argument on the right, taking account of ([8], Prop. 4), shows that $\operatorname{Ann} M=J_{\mu}$ if and only if $\operatorname{Dim} M=\operatorname{dim} V_{\mu}$. Let $S$ denote the set of regular elements of $A_{\mu}$. Since $A_{\mu}^{\prime}$ is f-finite and has finite length as a $U$ module, it follows from ([18], 3.7) that $S$ is an Ore subset of the regular elements of $A_{\mu}^{\prime}$ and $S^{-1} A_{\mu}^{\prime}=$ Fract $A_{\mu}^{\prime}$. Hence it remains to show that $S^{-1} M=0$ if and only if $\operatorname{Dim} M<$ $\operatorname{Dim} V_{\mu}=\operatorname{Dim} U(\mathfrak{g}) / J(\mu)$. This follows from ([16], 5.1, 5.2(i)).
4.6: Retain the above notation and take $\nu \in \mu+P(R)$ in the upper closure of the $W_{\mu}$ facette containing $\mu$ (for this see [11], 2.6).

Lemma: Set $H_{\mu}^{\nu}=R_{\mu}^{\nu} S_{\mu}^{\nu}$ (notation 1.13). Then
(i) $H_{\mu}^{\nu} A_{\mu}^{\prime}=A_{\mu}^{\prime}$.
(ii) $H_{\mu}^{\nu} A_{\nu}=A_{\nu}$.
(iii) $l_{0}\left(A_{\mu}^{\prime}\right)=l_{0}\left(A_{v}^{\prime}\right)$.
(iv) $H_{\mu}^{\nu} V_{\mu}=V_{\nu}$.
(v) Fract $A_{\mu}=$ Fract $A_{\mu}^{\prime} \Leftrightarrow$ Fract $A_{\nu}=$ Fract $A_{\nu}^{\prime}$.

By ([11], 2.10, 2.11) we have under the hypothesis of the lemma the isomorphisms $T_{\mu}^{\nu} L(\mu) \cong L(\nu)$ (resp. $T_{\mu}^{\nu} M(\mu) \cong M(\nu)$ ) and so by 1.13 the isomorphisms $H_{\mu}^{\nu} A_{\mu}^{\prime}=A_{\nu}^{\prime}$ (resp. $H_{\mu}^{\nu} L(M(\mu), M(\mu))=L(M(\nu)$, $M(\nu))$ ). Hence (i). Since $L(M(\mu), M(\mu)) \leftleftarrows U(\mathfrak{g}) /$ Ann $M(\mu)$ by ([13], 6.4) and $A_{\mu}$ is the image of $U(\mathfrak{g}) / \operatorname{Ann} M(\mu)$ in $A_{\mu}^{\prime}$, exactness of $H_{\mu}^{\nu}$ gives (ii). Now let $K$ be a simple $U$ subquotient of $A_{\mu}^{\prime}$. Then by $1.12, K$ is isomorphic to some $L\left(M\left(\lambda_{1}\right), L\left(\lambda_{2}\right)\right): \lambda_{1}, \lambda_{2} \in \mathfrak{b}^{*}$ with $\lambda_{1}$ dominant. Furthermore from the action of the centre of $U$ it easily follows that $\lambda_{1}, \lambda_{2} \in W \mu$. Then from ([11], 2.10, 2.11) and 1.12, 1.13, it follows that either $H_{\mu}^{\nu} K=0$ or is a simple subquotient of $A_{\nu}$; then, by an argument similar to that given in ([4], 2.11), $H_{\mu}^{\nu} K$ has the same Gelfand-Kirillov dimension as $K$. Moreover by a trivial extension of ([4], 2.4), whether or not $H_{\mu}^{\nu} K=0$ depends only on Ann $K$. Hence (iii), (iv). Finally (v) follows from (iii).
4.7: Corollary: Fix $-\lambda \in \mathfrak{h}^{*}$ dominant, regular and take $B^{\prime} \subset B_{\lambda}$. Then for each $\alpha \in B^{\prime}$, one has

Fract $U(g) / J\left(w_{B^{\prime}} s_{\alpha} \lambda\right)=$ Fract $L\left(L\left(w_{B} s_{\alpha} \lambda\right), L\left(w_{B} s_{\alpha} \lambda\right)\right)$.

With respect to $\lambda, \alpha$ define $\nu_{\alpha}$ as in 2.1. Then apply $4.6(v)$ to 4.4 with $\mu=w_{B^{\prime}} s_{\alpha} \lambda, \nu=w_{B^{\prime}} s_{\alpha}\left(\lambda-\nu_{\alpha}\right)=w_{B}\left(\lambda-\nu_{\alpha}\right)$.
4.8: For each $w \in W_{\lambda}$, set $S(w)=\left\{\alpha \in R_{\lambda}^{+}: w \alpha \in R_{\lambda}^{-}\right\}$. Define an ordering $\subseteq$ on $W_{\lambda}$ through $y \subseteq w$ given $S\left(y^{-1}\right) \subseteq S\left(w^{-1}\right)$. One checks that $y \subseteq w$ implies $y \leq w$ and that $y \subseteq w \Leftrightarrow\left(y_{*}^{-1} w w_{\lambda}\right) w_{\lambda}=y^{-1} w$. Thus the obvious generalization of 4.1 shows that $L(M(w \lambda), \cdot)$ is exact when restricted to the subcategory of $0_{\hat{\imath}}$ of all modules with simple factors $L(y \lambda): y \in W_{\lambda}$ where $y$ satisfies $y^{\prime} \leq y \Rightarrow y^{\prime} \subseteq w$. Since $s_{\alpha} \leq y$, $\forall \alpha \in \operatorname{supp} y$ it follows that supp $y \subset S\left(w^{-1}\right)$, that is $y \in W_{B^{\prime}}$ where $B^{\prime}=B_{\lambda} \cap S\left(w^{-1}\right)$. Though this rather weak generalization is probably not the best the corresponding assertion with $\subseteq$ replaced by $\leq$ is false for it implies that Kostant's problem has always a positive answer (which is false by ([5], 6.5)) for simple highest weight modules. This is in spite of the fact that $\operatorname{Ext}^{1}(M(w \lambda), L(y \lambda))=0$ if $w \geq y$.

## 5. Main theorem

5.1: Fix $-\lambda,-\mu \in \mathfrak{h}^{*}$ dominant, $\mu$ regular, with $\lambda-\mu \in P(R)$. Take $B^{\prime} \subset B_{\lambda}$. Let $s=l\left(w_{B}\right)$, and for each $j \in\{0,1,2, \ldots, s\}$ set $D_{j}=$ $\bigoplus_{w \in W_{B},} L\left(M\left(w_{B}, \lambda\right), M(w \mu)\right)$. Finally put

$$
L=L\left(M\left(w_{B^{\prime}} \lambda\right), L\left(w_{B^{\prime}} \mu\right)\right)
$$

Theorem: There is a long exact sequence

$$
0 \rightarrow D_{0} \rightarrow D_{1} \rightarrow \cdots \rightarrow D_{s} \rightarrow L \rightarrow 0
$$

Apply 4.1 to 2.6 .
5.2: When $\lambda=\mu$ in 5.1 , we have that $L=U(\mathfrak{g}) / J\left(w_{B}, \lambda\right)$ by 4.4. Again by 3.4 one has that

$$
\begin{gathered}
D_{s}=L\left(M\left(w_{B^{\prime}} \lambda\right), M\left(w_{B^{\prime}} \lambda\right)\right)=U(\mathfrak{g}) / J(\lambda) \\
D_{s-1}=\bigoplus_{\alpha \in B^{\prime}} L\left(M\left(w_{B^{\prime}} \lambda\right), M\left(w_{B^{\prime}} s_{\alpha} \lambda\right)\right)=\bigoplus_{\alpha \in B^{\prime}} L\left(M\left(s_{\alpha} \lambda\right), M(\lambda)\right) \\
=\bigoplus_{\alpha \in B^{\prime}} J\left(s_{\alpha} \lambda\right) .
\end{gathered}
$$

In view of the definition of the maps in 5.1 this gives the

Corollary: For each $B^{\prime} \subset B$, one has

$$
\sum_{\alpha \in B^{\prime}} J\left(s_{\alpha} \lambda\right)=J\left(w_{B} \lambda\right)
$$

Remark. When $B^{\prime} \subset B$, this result is due to Duflo ([8], Prop. 12). When $B^{\prime}=B_{\lambda}$, it is just ([12], 4.4, 4.5). By ([12], 4.5) it implies that $J\left(w_{B}, \lambda\right) / J(\lambda)$ is an idempotent ideal and has exactly card $B^{\prime}$ distinct maximal submodules.
5.3: Again take $\lambda=\mu$ in 5.1. Then by 3.1, 3.4

$$
D_{j}=\bigoplus_{w \in w_{b^{\prime}},} L\left(M\left(w_{B^{\prime}} \lambda\right), M(w \lambda)\right)=\bigoplus_{w \in W^{\prime} \cdot} L\left(-\lambda,-w^{-1} w_{B^{\prime}} \lambda\right) .
$$

Combined with 1.12 this gives the following multiplicity formula for simple $\ddagger$ submodules of $U(\mathfrak{g}) / J\left(w_{B} \cdot \lambda\right)$.

Corollary: Fix $-\lambda \in \mathfrak{h}^{*}$ dominant and regular. Then for each $\nu \in P(R)$ one has

$$
\operatorname{dim} \operatorname{Hom}_{\mathfrak{f}}\left(E(\nu), U(\mathfrak{g}) / J\left(w_{B^{\prime}} \cdot \lambda\right)\right)=\sum_{w \in W_{B^{\prime}}}(\operatorname{det} w) \operatorname{dim} E(\nu)_{\lambda-w \lambda} .
$$

Remarks. When $B^{\prime} \subset B$, Conze-Berline and Duflo ([5], 2.12, 6.3) gave a formula for the left hand side above. Their formula obtains from 4.4 and Frobenius reciprocity with respect to induction from the parabolic subalgebra defined by $B^{\prime}$. The equivalence of these two formulae imply a combinatorial statement concerning weight subspaces of finite dimensional $U(\mathfrak{g})$ modules.

## 6. Duality

6.1: Some of our results can be given a dual form with the help of the following. Fix $\lambda, \mu \in \mathfrak{b}^{*}$ with $\lambda-\mu \in P(R)$. Then (see 6.3) $L(\lambda, \mu) \times L(-\lambda,-\mu)$ admits a bilinear form 〈,〉 satisfying $\langle(a \otimes b) x, y\rangle=\langle x,(\check{a} \otimes \check{b}) y\rangle, \quad$ for $\quad$ all $\quad x \in L(\lambda, \mu), \quad y \in L(-\lambda,-\mu)$, $a, b \in U(\mathfrak{g})$. For each $\sigma, \tau \in k^{\wedge},\langle$,$\rangle restricts to a \mathfrak{l}$-invariant bilinear form on $L(\lambda, \mu)_{\sigma} \times L(-\lambda,-\mu)_{\tau}$ which is non-degenerate if $\tau$ is contragradient to $\sigma$ and zero otherwise.
6.2: To apply 6.1 to the comparison of mappings of principal series modules we start with the following observation. Suppose $\lambda, \lambda^{\prime} \in \mathfrak{b}^{*}$ are chosen so that we have an embedding of $M\left(\lambda^{\prime}\right)$ into $M(\lambda)$. Then there exists $a \in U\left(\mathrm{n}^{-}\right)_{\lambda^{\prime}-\lambda}$ such that $a e_{\lambda}=e_{\lambda^{\prime}}$. (Furthermore $a$ is unique up to a non-zero scalar which can be fixed canonically as follows. First, under the above hypothesis, $\lambda-\lambda^{\prime}$ is a non-negative integral linear combination of the $\alpha \in B$ (with say coefficients $k_{\alpha}$ ) and second, with respect to the canonical filtration of $U\left(\mathrm{n}^{-}\right)$, the leading term of $a$ is just

$$
\prod_{\alpha \in B} X_{-\alpha}^{k_{\alpha}}
$$

up to a non-zero scalar ([21], Lemma 1). Fix this scalar to be one.)
Lemma: There exists an embedding of $M(-\lambda)$ into $M\left(-\lambda^{\prime}\right)$ and ǎe $e_{\lambda^{\prime}}=s e_{-\lambda}$, with $s= \pm 1$.

Fix $\quad \alpha \in B$. Then $\left[X_{\alpha}, a\right] e_{\lambda}=0 \quad$ and $\quad$ so $\quad\left[X_{\alpha}, a\right] \in$ Ann $e_{\lambda}=$ $U(\mathfrak{g}) \mathrm{n}^{+}+\Sigma_{\beta \in B} U(g)\left(H_{\beta}-\left(\lambda-\rho, H_{\beta}\right)\right)$. Since $a \in U\left(\mathrm{n}^{-}\right)_{\lambda^{\prime}-\lambda}$ and $\alpha$ is simple, we have in fact the more precise result, namely

$$
\left[X_{\alpha}, a\right] \in U(\mathfrak{g})_{\eta}\left(H_{\alpha}-\left(\lambda-\rho, H_{\alpha}\right)\right)
$$

where $\eta=\lambda^{\prime}-\lambda+\alpha$. Hence

$$
\left[X_{\alpha}, \check{a}\right] \in\left(H_{\alpha}+\left(\lambda-\rho, H_{\alpha}\right)\right) U(\mathfrak{g})_{\eta}=U(\mathfrak{g})_{\eta}\left(H_{\alpha}+\left(\lambda+\eta-\rho, H_{\alpha}\right)\right) .
$$

Yet $-\left(\lambda+\eta-\rho, H_{\alpha}\right)=-\left(\lambda^{\prime}+\rho, H_{\alpha}\right)$, and so $X_{\alpha} \check{a} e_{-\lambda^{\prime}}=\left[X_{\alpha}, \check{a}\right] e_{-\lambda^{\prime}}=0$. Since $\alpha$ was arbitrary, it follows that $\check{a} e_{-^{\prime}}$ is a highest weight vector (necessarily non-zero) of weight $\left(\lambda^{\prime}-\lambda\right)-\left(\lambda^{\prime}+\rho\right)=-(\lambda+\rho)$ and hence proportional to the canonical generator $e_{-\lambda}$ of $M(-\lambda)$ embedded in $M(-\lambda$ ') "canonically" as above. Comparison of leading terms shows that the constant of proportionality is just $(-1)^{\Sigma \mathrm{k}_{\alpha}}$.

Remark. Of course the first part also obtains from ([7], 7.6.23). When $B_{\lambda} \subset B$, the second part can also be derived from ([7], 7.8.8).
6.3: The bilinear form referred to in 6.1 has been defined purely algebraically in ([7], 9.6.9) for the case $\lambda=\mu$. We describe the modifications needed in the general case. In this we denote by $t, u, v$ elements of $U(\mathfrak{f}), a, b$ elements of $U(\mathfrak{g}), \theta$ an element of $U(\mathfrak{f})^{*}, f$ an element of $L:=L(M(\lambda) \otimes M(\mu))^{*}$.

Define an action of $U(\mathfrak{l}) \otimes U(\mathfrak{k})$ on $U(\mathfrak{k})^{*}$ through $((u \otimes v) \cdot \theta)(t)=$ $\theta(\check{u} t v)$ and set

$$
U(\mathfrak{k})_{i}=U(\mathfrak{k}) \otimes \mathbb{C}, \quad U(\mathfrak{k})_{r}=\mathbb{C} \otimes U(\mathfrak{k}) .
$$

By ([7], 2.7.12) the sum of the simple finite dimensional $U(1))_{l}$ submodules of $U(\mathfrak{f})^{*}$ coincides with the sum of the simple finite dimensional $U(\mathbb{F})_{r}$ submodules of $U(\mathfrak{F})^{*}$, and we denote this subspace by $L\left(U(\mathfrak{f})^{*}\right)$. Let $\epsilon: U(\mathfrak{F}) \rightarrow \mathbb{C}$ be the augmentation. $\mathbb{C} \epsilon$ occurs as the unique one dimensional subrepresentation of $L\left(U(\mathfrak{F})^{*}\right)$. Let $\varphi_{0}: L\left(U(\mathfrak{F})^{*}\right) \rightarrow \mathbb{C}$ be the linear form on $L\left(U(\mathfrak{F})^{*}\right)$ which takes the value 1 on $\epsilon$ and zero on the $U(\mathcal{F}) \otimes U(\mathfrak{F})$ stable complement of $\mathbb{C} \epsilon$ in $L\left(U()^{*}\right)$.

Now for each $\nu \in \mathfrak{h}^{*}$, define $T_{\nu}=\left\{\theta \in U(\mathfrak{f})^{*}: \theta(u j(H))=(\nu, H) \theta(u)\right.$, for all $H \in \mathfrak{h}, u \in U(\mathfrak{F})\}$ which is a $U(\mathfrak{F})_{i}$ module. With $\nu=\lambda-\mu$, $f \in L$, we define $\theta_{f} \in T_{v}$ through $\theta_{f}(u)=f\left(u\left(e_{\lambda} \otimes e_{u}\right)\right)$. Then the map $f \mapsto \theta_{j}$ is a $U(\mathfrak{F})$ module isomorphism of $L$ onto the $U(\mathfrak{F})_{i}$ finite part $L\left(T_{\nu}\right)$ of $T_{\nu .}$ (For this see [7], 5.5 .8 or [8], Sect. I,2). Now take $\lambda^{\prime} \in \mathfrak{h}^{*}$ such that $M\left(\lambda^{\prime}\right) \subset M(\lambda)$ and $a \in U\left(n^{-}\right)_{\lambda^{\prime}-\lambda}$ as in 6.2. Then for all $f \in L, \quad$ we have $\left((1 \otimes j(a)) \cdot \theta_{f}\right)(u)=\theta_{f}(u j(a))=f\left(u j(a)\left(e_{\lambda} \otimes e_{\mu}\right)\right)=$ $f\left(u\left(a e_{\lambda} \otimes e_{\mu}\right)\right)$, since $a \in U\left(n^{-}\right)$and ${ }^{t} X e_{\mu}=0$ for all $X \in n^{-}$.

Let $\psi: L \rightarrow L^{\prime}:=L\left(M\left(\lambda^{\prime}\right) \otimes M(\mu)\right)$ be defined by restriction. Set $\nu^{\prime}=\lambda^{\prime}-\mu$, and define for any $f^{\prime} \in L^{\prime}$ the element $\theta_{f^{\prime}} \in T_{\nu^{\prime}}$ as above. Then for all $f \in L$, we have $\theta_{\psi(f)}(u)=\psi(f)\left(u\left(e_{\lambda^{\prime}} \otimes e_{\mu}\right)\right)=f\left(u\left(e_{\lambda^{\prime}} \otimes e_{\mu}\right)\right)$. Since $a e_{\lambda}=e_{\lambda^{\prime}}$, this gives

$$
\begin{equation*}
(1 \otimes \mathrm{j}(a)) \cdot \theta_{f}=\theta_{\psi(f)} . \tag{*}
\end{equation*}
$$

Similarly let $\psi^{\prime}: L\left(M\left(-\lambda^{\prime}\right) \otimes M(-\mu)\right)^{*} \rightarrow L(M(-\lambda) \otimes M(-\mu))^{*}$ be defined by restriction. Then for each $g^{\prime} \in L\left(M\left(-\lambda^{\prime}\right) \otimes M(-\mu)\right)^{*}$, we have $\theta_{g^{\prime}} \in T_{-\nu^{\prime}}, \theta_{\psi^{\prime}\left(\mathrm{g}^{\prime}\right)} \in T_{-\nu}$, and by (*) and 6.2 we get

$$
(1 \otimes j(\check{a})) \cdot \theta_{g^{\prime}}=s \theta_{\psi^{\prime}\left(g^{\prime}\right)} .
$$

Using ([7], 2.7.7) we have $T_{-\nu} T_{\nu} \subset T_{0}$ and so $L\left(T_{-\nu}\right) L\left(T_{\nu}\right) \subset L\left(T_{0}\right)$. Similarly $L\left(T_{-\nu^{\prime}}\right) L\left(T_{\nu^{\prime}}\right) \subset L\left(T_{0}\right)$. By ([7], 2.7.7), the invariance of $\varphi_{0}$ under $U(\mathfrak{f})_{\text {r }}$ gives (noting $\left.j(\check{a})=j(a)^{v}\right)$ that

$$
\varphi_{0}\left(\left((1 \otimes \mathrm{j}(\check{a})) \cdot \theta_{g^{\prime}}\right) \theta_{f}\right)=\varphi_{0}\left(\theta_{g^{\prime}}\left((1 \otimes j(a)) \cdot \theta_{f}\right)\right) .
$$

Just as in ([7], 9.6.9) using ([7], 2.7.15, 9.6.8) and the reductivity of f ,
one checks that the form $\langle g, f\rangle \mapsto \varphi_{0}\left(\theta_{g} \theta_{f}\right)$ on $L(\lambda, \mu) \times L(-\lambda,-\mu)$ has the properties claimed in 6.1. Furthermore with respect to the above maps we have the

Lemma: The diagram

$$
\begin{gathered}
L\left(\lambda^{\prime}, \mu\right) \times L(-\lambda,-\mu) \xrightarrow{1 \times \psi} L\left(\lambda^{\prime}, \mu\right) \times L\left(-\lambda^{\prime},-\mu\right) \\
\downarrow s \psi^{\prime} \times 1 \\
L(\lambda, \mu) \times L(-\lambda,-\mu) \longrightarrow
\end{gathered}
$$

commutes. That is $s\left\langle\psi^{\prime}\left(g^{\prime}\right), f\right\rangle=\left\langle g^{\prime}, \psi(f)\right\rangle$.
Indeed

$$
\begin{gathered}
s\left\langle\psi^{\prime}\left(g^{\prime}\right), f\right\rangle=\varphi_{0}\left(s \theta_{\psi^{\prime}\left(g^{\prime}\right)} \theta_{f}\right)= \\
\left.\varphi_{0}\left(\left((1 \otimes j(\check{a})) \cdot \theta_{g^{\prime}}\right) \theta_{f}\right)=\varphi_{0}\left(\theta_{g^{\prime}}(1 \otimes j(a)) \cdot \theta_{f}\right)\right)= \\
=\varphi_{0}\left(\theta_{g^{\prime}} \theta_{\psi f)}\right)=\left\langle g^{\prime}, \psi(f)\right\rangle .
\end{gathered}
$$

Remark. A similar result holds for the second variable.
6.4: Take $\lambda, \mu, w_{1}, w_{2}, \alpha$ as in 3.2. Under the hypothesis of 3.2, it follows that $M\left(-w_{2} \lambda\right)$ is a submodule of $M\left(-s_{\alpha} w_{2} \lambda\right)$. Applying the analogue of 6.3 with respect to second variable to 3.6 we obtain

Corollary: The map $L\left(w_{1} \mu, s_{\alpha} w_{2} \lambda\right) \rightarrow L\left(w_{1} \mu, w_{2} \lambda\right)$ defined by restriction is surjective.

Remark. When $\alpha \in B$, this was given in ([4], V, 1.11).
6.5: Both 3.6 and 6.4 admit analogous assertions for the first variable. This gives the commutative diagram of restriction maps

which implies an isomorphism of $L\left(-w_{1} \mu,-w_{2} \lambda\right)$ with $L\left(-s_{\alpha} w_{1} \mu,-s_{\alpha} w_{2} \lambda\right)$. The intertwining operators of ([8], Sect. I, 2) also give an isomorphism between those modules.

## Index of notation

Symbols frequently used are given below in order of appearance.
$1.1 \mathfrak{g}, \mathfrak{h}, R, R^{+}, B, \rho, s_{\alpha}, W, X_{\alpha}, \mathfrak{n}^{+}, \mathfrak{n}^{-}, \mathfrak{b}$.
$1.2 R_{\lambda}, R_{\lambda}^{+}, B_{\lambda}, W_{B^{\prime}}, w_{B^{\prime}}, W_{\lambda}, w_{\lambda}, M(\lambda), \overline{M(\lambda)}, L(\lambda), J(\lambda), e_{\lambda}$.
$1.3 \quad \check{u},{ }^{t} u, U, \mathrm{j}, \mathrm{F}, \mathrm{F}^{\wedge}$.
$1.4 \bigcirc, Z(\mathfrak{g}), \hat{\lambda}, O_{\hat{\lambda}}, p_{\hat{\lambda}}$.
$1.5 L(M, N), L(M \otimes N)^{*}, L(\lambda, \mu)$.
$1.7 \mathscr{H}$.
$1.8 \tau, M^{\tau}, \delta(M)$.
$1.11[M],[M(\lambda): L(\mu)]$.
1.12 $P(R), P(R)^{+}, P(R)^{++}, E(\nu)$.

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