189. On the Bessel Kernel for a Domain

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1. Aronszajn and Smith [2] developed the theory of Bessel potentials from the standpoints of their functional spaces and functional completions [1]. Let $P^{\alpha}(R^n)$ be the functional completion of $C_0^{\infty}(R^n)$ with the norm $\|u\|_{\alpha}^2=\int(1+|\xi|^2)^{\alpha}|\hat{u}(\xi)|^2 d\xi$. They showed that $P^{\alpha}(R^n)$ has the reproducing kernel $G_{2\alpha}(x-y)$ determined by

$$G_{2\alpha}(x)=\frac{1}{2^{(n+2\alpha-2)/2}\pi^{n/2}\Gamma(\alpha)}K_{(n-2\alpha)/2}(|x|)|x|^{(2\alpha-n)/2}$$

where $K_{(n-2\alpha)/2}$ is the modified Bessel function of third kind. The purpose of this paper is to consider the kernel of functional completion $P^{\alpha}(\Omega)$ of $C_0^{\infty}(\Omega)$ with the norm $\|u\|_{\alpha}$. Making use of the methods of general balayage and the theory of α-harmonic functions introduced by M. Itô [6], we define the Green function $G_{2\alpha}^{\Omega}(x,y)$ and α-harmonic functions in the theory of Bessel potentials. Let $E_{2\alpha}(\Omega)$ be the class of all positive measures of finite energy with compact support contained in Ω, $U_{2\alpha}^{\mu}$ be the potential of $\mu\in E_{2\alpha}(\Omega)$ in the functional space $P^{\alpha}(\Omega)$ and $G_{2\alpha}^{\Omega}\mu$ (resp. $\check{G}_{2\alpha}^{\Omega}\mu$) be the potential of μ with respect to the kernel $G_{2\alpha}^{\Omega}(x,y)$(resp. $\check{G}_{2\alpha}^{\Omega}(x,y)=G_{2\alpha}^{\Omega}(y,x)$). We shall prove the following results.

(1) Let Ω be a domain in R^n. Then for every $\mu\in E_{2\alpha}(\Omega)$, there exists an α-harmonic function $H_{2\alpha}^{\mu}(x)$ in Ω such that

$$U_{2\alpha}^{\mu}(x)=G_{2\alpha}^{\Omega}\mu(x)+H_{2\alpha}^{\mu}(x).$$

(2) The following conditions are equivalent:

(a) There exists a bounded domain $\Omega(\neq\emptyset)$ in R^n such that the Green function $G_{2\alpha}^{\Omega}(x,y)$ is the kernel of the functional space $P^{\alpha}(\Omega)$ i.e., $U_{2\alpha}^{\mu}=G_{2\alpha}^{\Omega}\mu$ in $P^{\alpha}(\Omega)$ for every $\mu\in E_{2\alpha}(\Omega)$.

(b) There exist a bounded domain Ω in R^n and a measure $\mu(\neq 0)\in E_{2\alpha}(\Omega)$ such that $G_{2\alpha}^{\Omega}\mu\in P^{\alpha}(\Omega)$ and $G_{2\alpha}^{\Omega}\mu=\check{G}_{2\alpha}^{\Omega}\mu$ in $P^{\alpha}(\Omega)$.

(c) $0<\alpha\leqq 1$.

2. According to Aronszajn and Smith [2], we define the Bessel potentials and summarize the results obtained in [2].

Definition 1. The Bessel potential of order 2α, $\alpha>0$, of a positive measure μ is defined by $G_{2\alpha}\mu(x)=\int G_{2\alpha}(x-y)d\mu(y)$. We denote by $E_{2\alpha}(R^n)$ the class of all positive measures for which the 2α-energy

$$\|\mu\|_{2\alpha}^2=\iint G_{2\alpha}(x-y)d\mu(x)d\mu(y)=\int G_{2\alpha}\mu d\mu$$

is finite.

(I) The following conditions on μ are equivalent:

(a) $\mu\in E_{2\alpha}(R^n)$. (b) $G_\alpha\mu\in L^2(R^n)$. (c) $G_{2\alpha}\mu\in P^\alpha(R^n)$.

Further every function u in $P^\alpha(R^n)$ is μ-integrable and $\int ud\mu=(u, G_{2\alpha}\mu)_\alpha$.

(II) If $0<\beta\leqq\alpha$, then $\|u\|_\beta\leqq\|u\|_\alpha$. Therefore $P^\alpha(R^n)\subset P^\beta(R^n)$, $E_{2\beta}(R^n)\subset E_{2\alpha}(R^n)$ and $\|\mu\|_{2\alpha}\leqq\|\mu\|_{2\beta}$.

Definition 2. The inner and the outer capacities of order 2α with respect to the Bessel kernel $G_{2\alpha}(x-y)$ are defined as usual. Every analytic set is capacitable. We denote by $\mathfrak{A}_{2\alpha}$ the class of all sets of 2α-capacity 0. Then $\mathfrak{A}_{2\alpha}\subset\mathfrak{A}_{2\beta}$ if $0<\beta\leqq\alpha$. A property is said to hold except $\mathfrak{A}_{2\alpha}$ (to be written exc. $\mathfrak{A}_{2\alpha}$) if the set where it fails to hold belongs to $\mathfrak{A}_{2\alpha}$.

Finally we summarize the properties of the kernel $G_{2\alpha}$.

(III) $G_{2\alpha}(x)\in L^1(R^n)$. $\hat{G}_{2\alpha}(\xi)=(2\pi)^{-n/2}(1+|\xi|^2)^{-\alpha}$.

(IV) $G_{\alpha+\beta}(x)=G_\alpha * G_\beta(x)$.

(V) If $0<\alpha\leqq 1$, then $G_{2\alpha}(x-y)$ satisfies the balayage principle. For any positive measure μ and any closed set F, there exists a positive measure μ' supported by F such that $G_{2\alpha}\mu'(x)=G_{2\alpha}\mu(x)$ on F except $\mathfrak{A}_{2\alpha}$ and $G_{2\alpha}\mu'(x)\leqq G_{2\alpha}\mu(x)$ everywhere in R^n (see, for example, M. Kishi [8]).

3. First, we define the potentials in the functional space $P^\alpha(\Omega)$.

Definition 3. We denote by $(u, v)_\alpha$ the inner product in the Hilbert space $P^\alpha(\Omega)$ corresponding to the norm $\|u\|_\alpha$. Let $\mu\in E_{2\alpha}(\Omega)$, then there exists a function $U_{2\alpha}^\mu$ in $P^\alpha(\Omega)$ such that $(\varphi, U_{2\alpha}^\mu)_\alpha=\int\varphi d\mu$ for every $\varphi\in C_0^\infty(\Omega)$. We call $U_{2\alpha}^\mu$ the Bessel potential of μ in $P^\alpha(\Omega)$. In the following sections, we consider the kernel of $U_{2\alpha}^\mu$.

4. If $0<\alpha\leqq 1$, $G_{2\alpha}(x-y)$ satisfies the balayage principle by (V). But we also want to treat with the case that $\alpha>1$. Therefore we use here the methods of balayage introduced by M. Itô [6].

Theorem 1. *Let $\alpha>0$ and p be the integer such that $0<\alpha-p\leqq 1$. Then for any closed set F and for any positive measure μ with compact support, there exists a unique system $\{\mu_i'\}_{i=0}^p$ of positive measures supported by F satisfying*

$$G_{2\alpha}\mu(x)=\sum_{i=0}^p G_{2(\alpha-i)}\mu_i'(x) \qquad \text{on } F \text{ exc. } \mathfrak{A}_{2(\alpha-p)}$$

$$G_{2\alpha}\mu(x)\geqq\sum_{i=0}^p G_{2(\alpha-i)}\mu_i'(x) \qquad \text{everywhere in } R^n,$$

$$G_{2q}\mu(x)=\sum_{i=0}^{q-1} G_{2(q-i)}\mu_i'(x) \qquad \textit{on } F \textit{ exc. } \mathfrak{A}_2,$$

$$G_{2q}\mu(x)\geqq\sum_{i=0}^{q-1} G_{2(q-i)}\mu_i'(x) \qquad \textit{everywhere in } R^n,$$

where q is any integer such that $0<q\leqq p$.

Theorem 1′. *Under the same assumptions in Theorem 1, there exists a unique system $\{\mu_i''\}_{i=0}^p$ of positive measures supported by F satisfying*

$$G_{2(\alpha-p)}\mu(x)=G_{2(\alpha-p)}\mu_0''(x) \qquad \textit{on } F \textit{ exc. } \mathfrak{A}_{2(\alpha-p)},$$

$$G_{2(\alpha-p)}\mu(x)\geqq G_{2(\alpha-p)}\mu_0''(x) \qquad \textit{everywhere in } R^n,$$

$$G_{2(\alpha-p+q)}\mu(x)=G_{2(\alpha-p+q)}\mu_0''(x)+\sum_{i=1}^{q} G_{2(q-i+1)}\mu_i''(x) \quad \textit{on } F \textit{ exc. } \mathfrak{A}_2,$$

$$G_{2(\alpha-p+q)}\mu(x)\geqq G_{2(\alpha-p+q)}\mu_0''(x)+\sum_{i=1}^{q} G_{2(q-i+1)}\mu_i''(x) \quad \textit{everywhere in } R^n,$$

where q is any integer such that $1<q\leqq p$.

The proofs of the above theorems are the same as in [6]. The essential parts are based on the decomposition $G_{2\alpha}=G_{2(\alpha-p)}*G_2*G_2*\cdots*G_2$ of $G_{2\alpha}$.

Definition 4. According to M. Itô [6], we call $\{\mu_i'\}_{i=0}^p$ (resp. $\{\mu_i''\}_{i=0}^p$) the system of balayaged measures of μ on F with respect to the system $(G_2, \cdots, G_{2p}, G_{2\alpha})$(resp. $(G_{2(\alpha-p)}, G_{2(\alpha-p+1)}, \cdots, G_{2\alpha})$). We denote by $\{(\varepsilon_{y,C\Omega}')_i\}_{i=0}^p$ and $\{(\varepsilon_{y,C\Omega}'')_i\}_{i=0}^p$ the systems of balayaged measures on $C\Omega$ of the unit measure ε_y on y. The Green function of order 2α in Ω is defined by $G_{2\alpha}^\Omega(x,y)=G_{2\alpha}(x-y)-\sum_{i=0}^p G_{2(\alpha-i)}(\varepsilon_{y,C\Omega}')_i(x)$. If $0<\alpha\leqq 1$, then $G_{2\alpha}^\Omega(x,y)=G_{2\alpha}^\Omega(y,x)$ and hence $G_{2\alpha}^\Omega(x,y)$ is measurable as a function of y.

Remark. In general $G_{2\alpha}^\Omega(x,y)$ is not symmetric, but we can prove by the same way as in [6], that

$$G_{2\alpha}^\Omega(x,y)=\int\cdots\int G_{2(\alpha-p)}^\Omega(x,z_1)G_2^\Omega(z_1,z_2)\cdots G_2^\Omega(z_p,y)dz_1\cdots dz_p$$

where $G_{2(\alpha-p)}^\Omega$ (resp. G_2^Ω) is the Green function of order $2(\alpha-p)$(resp. 2) for Ω and hence $G_{2\alpha}^\Omega(x,y)$ is measurable as a function of y. For any positive measure μ, put

$$G_{2\alpha}^\Omega\mu(x)=\int G_{2\alpha}^\Omega(x,y)d\mu(y) \text{ and } \check{G}_{2\alpha}^\Omega\mu(x)=\int G_{2\alpha}^\Omega(y,x)d\mu(y).$$

Then we have

$$G_{2\alpha}^\Omega\mu(x)=G_{2\alpha}\mu(x)-\sum_{i=0}^p G_{2(\alpha-i)}\mu_i'(x)$$

$$\check{G}_{2\alpha}^\Omega\mu(x)=G_{2\alpha}\mu(x)-G_{2\alpha}\mu_0''(x)-\sum_{i=1}^p G_{2(\alpha-i+1)}\mu_i''(x).$$

5. First, we treat with the case that $0<\alpha\leqq 1$.

Theorem 2. *Let $0<\alpha\leqq 1$ and Ω be a domain in R^n. Then the Green function $G_{2\alpha}^\Omega(x,y)$ is the kernel of the functional space $P^\alpha(\Omega)$, that is, $U_{2\alpha}^\mu=G_{2\alpha}^\Omega\mu$ in $P^\alpha(\Omega)$ for every measure $\mu\in E_{2\alpha}(\Omega)$.*

Proof. Let ω be a relatively compact domain such that $\overline{\omega} \subset \Omega$ and $e(x)$ be a positive function in $C_0^\infty(R^n)$ satisfying $e(x)=0$ for $|x| \geqq 1$ and $\int e(x)dx=1$. Put $e_\rho(x)=\rho^{-n}e(x/\rho)$ for $0<\rho\leqq 1$. Then

$$G_{2\alpha}^{\omega}\mu * e_\rho(x) \in C_0^\infty(R^n) \cap P^\alpha(R^n),$$
$$G_{2\alpha}^{\omega}\mu * e_\rho(x) \to G_{2\alpha}^{\omega}\mu(x) \text{ in } P^\alpha(R^n) \text{ as } \rho\to 0,$$

because $G_{2\alpha}^{\omega}\mu \in P^\alpha(R^n)$(see [2], p. 423). On the other hand

$$G_{2\alpha}^{\omega}\mu(x)=G_{2\alpha}\mu(x)-G_{2\alpha}\mu'_{C\omega}(x)=0 \text{ on } C\omega \text{ exc. } \mathfrak{A}_{2\alpha}.$$

Therefore, for any sufficiently small ρ, $G_{2\alpha}^{\omega} * e_\rho(x) \in P^\alpha(\Omega)$ and hence $G_{2\alpha}^{\omega}\mu \in P^\alpha(\Omega)$. We shall show that $G_{2\alpha}^{\Omega}\mu \in P^\alpha(\Omega)$. Let $\{\omega_n\}$ be an exhaustion of Ω. Then $G_{2\alpha}^{\omega_n}\mu \in P^\alpha(\Omega)$. The sequence $\{\mu'_{C\omega_n}\}$ of balayaged measures of μ on $C\omega_n$ converges strongly to $\mu'_{C\Omega}$ in $E_{2\alpha}(R^n)$(cf. [3]). Therefore $\{G_{2\alpha}^{\omega_n}\mu\}$ converges strongly to $G_{2\alpha}^{\Omega}\mu$ in $P^\alpha(R^n)$ and hence $G_{2\alpha}^{\Omega}\mu \in P^\alpha(\Omega)$. For any function φ in $C_0^\infty(\Omega)$, we have

$$(\varphi, U_{2\alpha}^{\mu})_\alpha=\int\varphi d\mu=\int\varphi d\mu-\int\varphi d\mu'_{C\Omega}=(\varphi, G_{2\alpha}^{\Omega}\mu)_\alpha.$$

This implies that $G_{2\alpha}^{\Omega}\mu=U_{2\alpha}^{\mu}$ in $P^\alpha(\Omega)$, because $C_0^\infty(\Omega)$ is dense in $P^\alpha(\Omega)$.

6. Since $\hat{G}_{2\alpha}(\xi)=(2\pi)^{-(n/2)}(1+|\xi|^2)^{-\alpha}$, there exists a distribution T_α such that $T_\alpha * G_{2\alpha}=\varepsilon_0$ (i.e., $\hat{T}_\alpha=(2\pi)^{n/2}(1+|\xi|^2)^\alpha$). Similarly as in [6], we define α-harmonic functions as follows.

Definition 5. A function $u(x)$ is said to be α-harmonic in Ω in the theory of Bessel potentials if it satisfies the following conditions:

(a) $u(x)$ is defined in R^n exc. $\mathfrak{A}_{2\alpha}$ and locally integrable in R^n.

(b) $T_\alpha * u$ is defined and $T_\alpha * u=0$ in Ω in the sense of distributions.

Lemma 1. *Let μ be a positive measure with finite total mass. Then the potential $G_{2(\alpha-i)}\mu(x)$ is α-harmonic in Cs_μ for any integer i such that $0\leqq i\leqq p$.*

Proof. The equality $(\varepsilon_0-\Delta)^i G_{2\alpha}=G_{2(\alpha-i)}$ holds for every positive integer i and $(\varepsilon_0-\Delta)^i * \mu=(\varepsilon_0-\Delta)^i * T_\alpha * G_{2\alpha}\mu=T_\alpha * G_{2(\alpha-i)}\mu$. Hence the convolution $T_\alpha * G_{2(\alpha-i)}\mu$ is defined. Since $(\varepsilon_0-\Delta)^i * \mu=0$ in Cs_μ, $T_\alpha * G_{2(\alpha-i)}\mu=0$ in Cs_μ. Therefore $G_{2(\alpha-i)}\mu(x)$ is α-harmonic in Cs_μ.

Remark. By this lemma, the Green function $G_{2\alpha}^{\Omega}(x, y)$ is α-harmonic in $\Omega-\{x\}$.

Theorem 3. *Let Ω be a domain in R^n and $\mu \in E_{2\alpha}(\Omega)$. Then there exists an α-harmonic function $H_{2\alpha}^{\mu}(x)$ in Ω such that*

$$U_{2\alpha}^{\mu}(x)=G_{2\alpha}^{\Omega}\mu(x)+H_{2\alpha}^{\mu}(x).$$

Proof. Put $H_{2\alpha}^{\mu}=U_{2\alpha}^{\mu}-G_{2\alpha}^{\Omega}\mu$. Then we have

$$H_{2\alpha}^{\mu}=(U_{2\alpha}^{\mu}-G_{2\alpha}\mu)+\sum_{i=0}^{p} G_{2(\alpha-i)}\mu'_i.$$

Since S_μ is compact, $\int d\mu'_i<+\infty$ for every i such that $0\leqq i\leqq p$. Therefore, by Lemma 1, $\sum_{i=0}^{p} G_{2(\alpha-i)}\mu'_i$ is α-harmonic in Ω. On the other hand, the convolution $T_\alpha * (U^\mu_{2\alpha} - G_{2\alpha}\mu)$ is defined. By the definition of $U^\mu_{2\alpha}$, there exists a sequence $\{\mu_n\}$ of signed measures supported by $C\Omega$ such that $G_{2\alpha}\mu_n \to (U^\mu_{2\alpha} - G_{2\alpha}\mu)$ strongly in $P^\alpha(R^n)$ and hence in the sense of distributions. Since $G_{2\alpha}\mu_n$ is α-harmonic in Ω, $T_\alpha * G_{2\alpha}\mu_n = 0$ in Ω and hence $T_\alpha * (U^\mu_{2\alpha} - G_{2\alpha}\mu) = 0$ in Ω in the sense of distributions. Therefore $(U^\mu_{2\alpha} - G_{2\alpha}\mu)$ is α-harmonic in Ω. This means that $H^\mu_{2\alpha}$ is α-harmonic in Ω.

Lemma 2. *Suppose* $U^\mu_{2\alpha} = G^\Omega_{2\alpha}\mu$ *for every* $\mu \in E_{2\alpha}(\Omega)$. *Then* $\check{G}^\Omega_{2\alpha}\mu = G^\Omega_{2\alpha}\mu$ *exc.* $\mathfrak{A}_{2\alpha}$ *for every* $\mu \in E_{2\alpha}(\Omega)$.

Proof. Let μ and ν be any measures in $E_{2\alpha}(\Omega)$. By the assumption

$$\iint G^\Omega_{2\alpha}(x, y) d\nu(x) d\mu(y) = (U^\mu_{2\alpha}, U^\nu_{2\alpha})_\alpha = (U^\nu_{2\alpha}, U^\mu_{2\alpha})_\alpha = \iint G^\Omega_{2\alpha}(y, x) d\mu(y) d\nu(x)$$

and hence

$$\int (G^\Omega_{2\alpha}\mu(x) - \check{G}^\Omega_{2\alpha}\mu(x)) d\nu(x) = 0.$$

This means that $G^\Omega_{2\alpha}\mu(x) = \check{G}^\Omega_{2\alpha}\mu(x)$ exc. $\mathfrak{A}_{2\alpha}$, because ν is arbitrary.

Theorem 4. *The following conditions are equivalent*:

(a) *There exists a bounded domain* Ω $(\neq \emptyset)$ *in* R^n *such that the Green function* $G^\Omega_{2\alpha}(x, y)$ *is the kernel of* $P^\alpha(\Omega)$ *i.e.*, $U^\mu_{2\alpha} = G^\Omega_{2\alpha}\mu$ *in* $P^\alpha(\Omega)$ *for every* $\mu \in E_{2\alpha}(\Omega)$.

(b) *There exist a bounded domain* Ω *in* R^n *and a measure* $\mu \neq 0$ *in* $E_{2\alpha}(\Omega)$ *such that* $G^\Omega_{2\alpha}\mu \in P^\alpha(\Omega)$ *and* $G^\Omega_{2\alpha}\mu = \check{G}^\Omega_{2\alpha}\mu$ *in* $P^\alpha(\Omega)$.

(c) $0 < \alpha \leqq 1$.

Remark. If there exists a domain which satisfies the condition in (a) or in (b), then by Theorem 2, every domain satisfies the same condition.

Proof. We have already showed the relations (c)$\Rightarrow$(a)(in Theorem 2) and (a)$\Rightarrow$(b)(in Lemma 2) and therefore we shall prove here the relation (b)$\Rightarrow$(c).

First we suppose that $1 < \alpha < 2$ (i.e., $p = 1$). (b) implies

$$G_{2\alpha}\mu - G_{2\alpha}\mu'_0 - G_{2(\alpha-1)}\mu'_1 = G_{2\alpha}\mu - G_{2\alpha}\mu''_0 - G_2\mu''_1.$$

By Lemma 1, this means that $G_2\mu''_1$ is α-harmonic in Ω and hence

$$T_{\alpha-1} * \mu''_1 = T_{\alpha-1} * T_1 * G_2\mu''_1 = T_\alpha * G_2\mu''_1 = 0 \quad \text{in} \quad \Omega.$$

By the definition of $T_{\alpha-1}$, we have $\hat{T}_{\alpha-1} = (2\pi)^{n/2}(1 + |\xi|^2)^{\alpha-1}$. If $1 < \alpha < 2$, then by Levy-Khintchine's theorem on negative definite functions, there exists a symmetric positive measure σ in $R^n - \{0\}$ such that

$$(1) \qquad (1 + |\xi|^2)^{\alpha-1} = 1 + \int_{|y|>0} (1 - e^{i\xi\cdot y}) d\sigma(y)$$

$$(2) \qquad \int_{|x|>r} d\sigma(y) < +\infty \quad \text{and} \quad \int_{0<|y|<r} |y|^2 d\sigma(y) < +\infty \quad \text{for any} \quad r > 0$$

(see, for example, M. Itô [5]). We can see that $\int_{|y|<\delta} |y|^2 d\sigma(y) > 0$ for any $\delta>0$. Because, if $\int_{|y|<\delta_0}|y|^2\,d\sigma(y)=0$, then by (2) $\int_{|y|>0}d\sigma(y)<+\infty$. Therefore $(1+|\xi|^2)^{\alpha-1}$ should be bounded by (1) and hence a contradiction. Let $\varphi\in C_0^\infty(\Omega)$. Then we have

$$0=T_{\alpha-1}*\mu_1''(\varphi)=\int\varphi d\mu_1''+\int(\mu_1''(\varphi)-\mu_1''*\varphi(y))d\sigma(y)$$

and hence $\int\mu_1''*\varphi(y)d\sigma(y)=0$. On the other hand, the fact $\int_{|y|<\delta}|y|^2\,d\sigma(y)>0$ implies that $\int\mu_1''*\varphi(y)\neq 0$ for a suitable choice of φ. This leads us to a contradiction.

Next we suppose that $\alpha\geqq 2$. (b) means that $G_{2\alpha}\mu-G_{2\alpha}\mu_0''-\sum_{i=1}^{p}G_{2(p-i+1)}\mu_i''$ belongs to $P^\alpha(\Omega)\subset P^2(\Omega)$ and hence $G_2*\left(\sum_{i=1}^{p-1}G_{2(p-i)}\mu_i''+\mu_p''\right)$ belongs to $P^2(R^n)$. Therefore by (I) $\sum_{i=1}^{p-1}G_{2(p-i)}\mu_i''+\mu_p''$ is a function in $L^2(R^n)$. This is a contradiction, because μ_p'' is a measure on the boundary $\partial\Omega$ of Ω.

Therefore (b) implies (c). This completes the proof.

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