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ON THE BEST CONSTANT FOR HARDY'S INEQUALITY IN $\mathbb{R}^n$

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ABSTRACT. Let Ω be a domain in $\mathbb{R}^n$ and $p \in (1, \infty)$. We consider the (generalized) Hardy inequality $\int_\Omega |\nabla u|^p \geq K \int_\Omega |u/\delta|^p$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. The inequality is valid for a large family of domains, including all bounded domains with Lipschitz boundary. We here explore the connection between the *value* of the Hardy constant $\mu_p(\Omega) = \inf_{\overset{\circ}{W}_{1,p}(\Omega)} \left(\int_\Omega |\nabla u|^p \,/\, \int_\Omega |u/\delta|^p\right)$ and the *existence* of a minimizer for this Rayleigh quotient. It is shown that for all smooth n-dimensional domains, $\mu_p(\Omega) \leq c_p$, where $c_p = (1 - \frac{1}{p})^p$ is the one-dimensional Hardy constant. Moreover it is shown that $\mu_p(\Omega) = c_p$ for all those domains *not* possessing a minimizer for the above Rayleigh quotient. Finally, for $p = 2$, it is proved that $\mu_2(\Omega) < c_2 = 1/4$ if and only if the Rayleigh quotient possesses a minimizer. Examples show that strict inequality may occur even for bounded smooth domains, but $\mu_p = c_p$ for convex domains.

INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ with non-empty boundary. Let $L_p(\Omega; \varrho)$ denote the weighted Lebesgue space with weight ϱ and set $\tilde{L}_p(\Omega) = L_p(\Omega; \delta^{-p})$, where

$$\delta(x) = \delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega), \quad \forall x \in \mathbb{R}^n.$$

The norm in $\tilde{L_p}(\Omega)$ will be denoted by $|\!|\cdot|\!|_p^\Omega$, but the superscript will be omitted when there is no danger of confusion.

For $1 < p < \infty$, Hardy's inequality reads

$$|\!|u|\!|_p^\Omega \leq \gamma \|\nabla u\|_{L_p(\Omega)}, \quad \forall u \in \overset{\circ}{W}_{1,p}(\Omega), \tag{0.1}$$

where γ is a constant which may depend on the domain. The inequality was discovered by Hardy [10] in the one-dimensional case and later extended to higher dimensions (see [16] for historical background). It is known that, for $n \geq 2$, the inequality holds if Ω is a bounded Lipschitz domain, [16]. The following variational problem is naturally associated with (0.1):

$$\inf_{u \in \overset{\circ}{W}_{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u/\delta|^p}. \tag{0.2}$$

The infimum (which we denote by $\mu_p(\Omega)$) is positive if and only if (0.1) holds in Ω. Note that the ratio is invariant with respect to dilation. In the *one dimensional*

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case it is known that the infimum is independent of Ω and that

$$\mu_p(\Omega) = c_p := (1 - \frac{1}{p})^p. \tag{0.3}$$

This was established by Hardy [10, 11], who also showed that the constant is not attained, i.e. the variational problem has no minimizer.

In the *n-dimensional case* $\mu_p(\Omega)$ varies with the domain. However, if Ω is convex, $\mu_p(\Omega) = c_p$ (see [15]). In [15] a proof is given for $n = 2$; in [7, p. 115] the result is stated for $p = n = 2$. A simple proof of the result in the general case is provided in appendix A below.

The main theme of this paper is the connection between the existence of minimizers for (0.2) and the value of $\mu_p(\Omega)$. Specifically we shall establish the following results, in which Ω stands for a bounded domain in $\mathbb{R}^n$ with C^2 boundary.

Theorem I. *For every $p \in (1, \infty)$, $\mu_p(\Omega) \le c_p$. If problem* (0.2) *has no minimizer then $\mu_p(\Omega) = c_p$.*

Theorem II. *$\mu_2(\Omega) = c_2$ if and only if problem* (0.2) *has no minimizer.*

The main ingredient in the proof of Theorem I is the following concentration result.

Theorem III. *Suppose that* (0.2) *has no minimizer. If $\{u_k\}$ is a minimizing sequence, bounded in $\overset{\circ}{W}_{1,p}(\Omega)$, then each subsequence of $\{\nabla u_k\}$ which converges in the sense of measures concentrates at the boundary. More precisely,*

$$|\nabla u_k| \to 0 \quad \text{in } L_p^{loc}(\Omega). \tag{0.4}$$

The organization of the paper is as follows. In section 1 we establish a concentration result involving a constant $\mu_p^*(\Omega)$ related to the variational problem (0.2). Theorem III is a consequence of this result. In section 2 we prove Theorem I using the previous concentration result. In section 3 we present the proof of Theorem II, which is based on the spectral theory of operators of the form $\rho^2\Delta$ (on $\overset{\circ}{W}_{1,2}(\Omega)$) where ρ is a smooth function comparable to δ_Ω near the boundary, (see [1]–[3]). Finally, in section 4, we discuss the relation between the constants c_p and $\mu_p(\Omega)$ for various families of domains.

1. A concentration effect

Let

$$\widetilde{W}_{1,p}(\Omega) = \{u \in W_{1,p}^{loc}(\Omega) : |\!|\!| u |\!|\!|_{1,p} < \infty\}, \tag{1.1}$$

where

$$|\!|\!| u |\!|\!|_{1,p} := |\!|\!| u |\!|\!|_p + \|\nabla u\|_{L_p(\Omega)}.$$

If Ω is a domain for which (0.1) is valid, then obviously, $\widetilde{W}_{1,p}(\Omega) \supseteq \overset{\circ}{W}_{1,p}(\Omega)$. Furthermore,

Lemma 1. *Let Ω be a bounded domain of class C^2. Then,*

$$\widetilde{W}_{1,p}(\Omega) = \overset{\circ}{W}_{1,p}(\Omega).$$

The proof of the lemma is given in Appendix B. We observe that, in general, the conclusion of the lemma fails if the domain is unbounded. For instance, suppose that Ω is the complement of the unit sphere and $p > n$. In this case, if $u \in C^\infty(\Omega)$, u vanishes in a neighborhood of $\partial\Omega$ and $u \equiv 1$ in a neighborhood of infinity, then $u \in \widetilde{W}_{1,p}(\Omega) \setminus \overset{\circ}{W}_{1,p}(\Omega)$.

For $u \in \widetilde{W}_{1,p}(\Omega)$ s.t. $\|u\|_p \neq 0$, put

$$\chi_p^\Omega(u) := \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u/\delta|^p}. \tag{1.2}$$

As before, the superscript will be dropped if no ambiguity results. Further, denote

$$\begin{aligned} \mu_p^*(\Omega) \quad := \quad & \inf\{\liminf_{k\to\infty} \chi_p(u_k) \,:\, \{u_k\} \subset \overset{\circ}{W}_{1,p}(\Omega), \\ & u_k \overset{weak}{\to} 0 \text{ in } \overset{\circ}{W}_{1,p}(\Omega),\ \liminf \|u_k\|_p > 0\}. \end{aligned} \tag{1.3}$$

Obviously,

$$\mu_p^*(\Omega) \geq \mu_p(\Omega). \tag{1.4}$$

Remark 1. Suppose that Ω is a domain with compact boundary. If $\{u_k\}$ is a sequence satisfying the conditions of (1.3) then,

$$\begin{aligned} &(i)\ u_k \to 0 \text{ in } L_p^{loc}(\Omega), \\ &(ii)\ \int_{\Omega'} |u_k/\delta|^p \to 0 \text{ if } \Omega' \subset \Omega \text{ and } \operatorname{dist}(\Omega', \partial\Omega) > 0. \end{aligned} \tag{1.5}$$

If in addition Hardy's inequality holds in Ω, then

$$u_k \overset{weak}{\to} 0 \text{ in } \tilde{L}_p(\Omega). \tag{1.6}$$

In fact, (1.5) follows from the assumption that $u_k \overset{weak}{\to} 0$ in $\overset{\circ}{W}_{1,p}(\Omega)$. This implies statement (i) (by Rellich's lemma) and the boundedness of $\{u_k\}$ in $W_{1,p}(\Omega)$. These two facts imply (ii). If Hardy's inequality holds then the boundedness of $\{u_k\}$ in $W_{1,p}(\Omega)$ implies its boundedness in $\tilde{L}_p(\Omega)$. This fact and (1.5)(ii) imply (1.6).

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$ with compact boundary of class C^2. If Hardy's inequality* (0.1) *is valid in Ω, then*

$$\mu_p^*(\Omega) = c_p. \tag{1.7}$$

In addition, if $\{u_k\}$ is a sequence as in (1.3) *and*

$$\lim \chi_p(u_k) = \mu_p^*(\Omega), \tag{1.8}$$

then

$$\nabla u_k \to 0 \ \text{ in } \ L_p^{loc}(\Omega). \tag{1.9}$$

For the proof of the theorem we need a computational lemma, the first part of which is due to [6].

Lemma 3. *(i) Given $h, T, \zeta > 0$, let $V = V_{h,T,\zeta}$ be the function defined by*

$$V(t) = h(t/T)^{\epsilon+1-\frac{1}{p}} \quad \forall t \geq 0,$$

where $\epsilon = \frac{h^p}{p\zeta T^{p-1}}$. Then, for $p > 1$,

$$\int_0^T (V(t)/t)^p \, dt = \zeta, \quad \text{and} \quad \int_0^T |V'|^p \, dt = \zeta(\epsilon + 1 - \frac{1}{p})^p. \tag{1.10}$$

(ii) Let $\widehat{V}$ denote the function in $C[0, 2T]$ determined by the following conditions:

$$\widehat{V} = V \ \textit{in } [0,T], \ \ \widehat{V}(2T) = 0 \textit{ and } \widehat{V} \textit{ is linear in } [T,2T].$$

Then $\widehat{V} \in \overset{\circ}{W}_{1,p}(0,2T)$ *and*

$$\frac{\int_0^{2T} |\widehat{V}'|^p}{\int_0^{2T} (\widehat{V}(t)/t)^p} \to c_p \tag{1.11}$$

if either $h \to 0$ *(while* T, ζ *are fixed) or* $\zeta \to \infty$ *(while* T, h *are fixed).*

Proof. The verification of these statements is straightforward and will be omitted. We note that,

$$\begin{aligned} &\left| \int_0^{2T} (\widehat{V}(t)/t)^p \, dt - \zeta \right| \le h^p T^{1-p}, \\ &\int_0^{2T} |\widehat{V}'|^p \, dt = \zeta(\epsilon + 1 - \frac{1}{p})^p + h^p T^{1-p}. \end{aligned} \tag{1.12}$$

□

Proof of Theorem 2. First we show that

$$\mu_p^*(\Omega) \le c_p. \tag{1.13}$$

(This inequality is valid even if Hardy's inequality does not hold in Ω.) Denote $\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}$. If β is sufficiently small, the function δ_Ω is in $C^1(\Omega_\beta)$ and there exists a set of 'flow coordinates' (δ, σ) such that the transformation of coordinates $x \leftrightarrow (\delta(x), \sigma(x))$ is C^1 in Ω_β and its Jacobian J equals 1 on $\partial\Omega$ (see [17]). Let $\widehat{V} = \widehat{V}_{h,\beta/2,1}$ and set

$$v_h(x) = \begin{cases} \widehat{V}(\delta(x)), & \text{if } x \in \Omega_\beta, \\ 0, & \text{if } x \in \Omega \text{ and } \delta(x) \ge \beta. \end{cases} \tag{1.14}$$

Then, $v_h \in \overset{\circ}{W}_{1,p}(\Omega)$ and (as $|\nabla\delta(x)| = 1$) $|\nabla v_h(x)| = |\widehat{V}'(\delta(x))|$ in Ω_β. For fixed β, $v_h \to 0$ and $\nabla v_h \overset{weak}{\to} 0$ in $L_p(\Omega)$ as $h \to 0$. (The second statement follows from the fact that $\{\nabla v_h\}$ is bounded in $L_p(\Omega)$ and tends to zero in $L_p^{loc}(\Omega)$.) Since $J = 1$ on the boundary, these facts and (1.11) imply that

$$\inf \| v_h \|_p = 1 + o(1) \ \text{ and } \ \liminf_{h \to 0} \chi_p(v_h) = (1 + o(1)) c_p,$$

where $o(1)$ denotes a quantity which tends to zero when $\beta \to 0$. This implies (1.13).

Next we prove (1.9). If this statement is not valid, then there exist a sequence $\{u_k\}$ which satisfies the conditions of (1.3) and (1.8), a compact set $K \subset \Omega$ and a positive number α such that

$$\int_K |\nabla u_k|^p \ge \alpha, \quad \forall k \in \mathbb{N}. \tag{1.15}$$

Since $\| u_k \|_p > 0$ and $\liminf \| u_k \|_p > 0$, it follows that $\inf \| u_k \|_p > 0$. Let β be small enough to satisfy the requirements of the first part of the proof and in addition to guarantee that $K \subset \Omega'_\beta := \Omega \setminus \Omega_\beta$. Our assumptions imply that there exists a positive number c (depending on β) s.t.

$$\int_{\Omega_\beta} |u_k/\delta|^p \ge c \ \text{ and } \ \gamma_k := \int_{\Omega'_\beta} |u_k/\delta|^p \to 0. \tag{1.16}$$

(For the second statement see (1.5).) Thus,

$$\chi_p(u_k) \geq \frac{\int_{\Omega_\beta} |\nabla u_k|^p + \alpha}{\int_{\Omega_\beta} |u_k/\delta|^p + \gamma_k}. \tag{1.17}$$

Let (δ, σ) be the flow coordinates mentioned in the first part of the proof and let J be the Jacobian of the transformation $x \leftrightarrow (\delta(x), \sigma(x))$. Note that for any $f \in C^1(\Omega_\beta)$ we have $|\nabla f| \geq |\frac{\partial f}{\partial \delta}|$. Therefore, using the one dimensional Hardy inequality, we obtain

$$\begin{aligned} \int_{\Omega_\beta} |\nabla u_k|^p &= \int_{\partial\Omega} \int_0^\beta |\nabla u_k|^p J(\delta, \sigma)\, d\delta d\sigma \\ &\geq (1 + o(1))c_p \int_{\partial\Omega} \int_0^\beta |u_k/\delta|^p J(\delta, \sigma)\, d\delta d\sigma \\ &= (1 + o(1))c_p \int_{\Omega_\beta} |u_k/\delta|^p, \end{aligned} \tag{1.18}$$

where $o(1)$ is a quantity which tends to zero when $\beta \to 0$. By (1.16)-(1.18) it follows that

$$\liminf_{k\to\infty} \chi_p(u_k) \geq (1 + o(1))c_p + \alpha/A, \tag{1.19}$$

where $A = \sup \|u_k\|_p^\Omega$. (Since, by assumption, Hardy's inequality holds in Ω, the boundedness of the sequence $\{\nabla u_k\}$ in $L_p(\Omega)$ implies the boundedness of $\{\|u_k\|_p^\Omega\}$.) If β is sufficiently small, the right hand side of (1.19) is larger than c_p. But in view of (1.13) this contradicts (1.8). Therefore (1.9) holds.

Now using (1.9), the previous argument shows that

$$\liminf_{k\to\infty} \chi_p(u_k) \geq (1 + o(1))c_p, \tag{1.20}$$

for any sequence $\{u_k\}$ as in (1.3) which satisfies (1.8). This fact together with (1.13) implies (1.7) and completes the proof. $\square$

The previous theorem dealt with sequences which converge weakly to zero. Next we present a similar result concerning sequences which converge to a fixed non-zero element of $\overset{\circ}{W}_{1,p}(\Omega)$. A related question (in the one dimensional case) was studied in [6], where the authors were interested in an associated 'gap phenomenon' (see also [14]).

Given $u \in \overset{\circ}{W}_{1,p}(\Omega)$, denote

$$\begin{aligned} \nu_p(u) \quad := \quad &\inf\{\liminf \chi_p(u_k) : \{u_k\} \subset \overset{\circ}{W}_{1,p}(\Omega), \\ &\text{s.t. } \ u_k \overset{weak}{\to} u \text{ in } \overset{\circ}{W}_{1,p}(\Omega)\}. \end{aligned} \tag{1.21}$$

If $\{u_k\}$ satisfies the conditions of (1.21) then the statements of Remark 1 apply to $u_k - u$. In particular, if Hardy's inequality holds in Ω, then $u_k \overset{weak}{\to} u$ in $\widetilde{W}_{1,p}(\Omega)$ and consequently $\liminf \|u_k\|_p \geq \|u\|_p$.

Theorem 4. *Let Ω be a domain in $\mathbb{R}^n$ with compact boundary of class C^2 and let u be a non-zero element of $\overset{\circ}{W}_{1,p}(\Omega)$. If Hardy's inequality is valid in Ω, then*

$$\nu_p(u) = \min(c_p, \chi_p(u)). \tag{1.22}$$

If $\{u_k\}$ *is an admissible sequence (in the sense of* (1.21)*) such that*

$$\lim \chi_p(u_k) = \nu_p(u), \tag{1.23}$$

then

$$\nabla(u_k - u) \to 0 \quad in \quad L_p^{loc}(\Omega). \tag{1.24}$$

If, in addition, $\chi_p(u) < c_p$, *then* $u_k \to u$ *in* $\widetilde{W}_{1,p}(\Omega)$.

Proof. Let $\{w_k\}$ be a sequence which satisfies the conditions of (1.3) and

$$\lim \chi_p(w_k) = \mu_p^*(\Omega). \tag{1.25}$$

We may assume that $\|w_k\|_p = 1$. Thus, by (1.25) and (1.7),

$$\chi_p(w_k) = \int_\Omega |\nabla w_k|^p \to c_p. \tag{1.26}$$

In view of (1.25), Theorem 2 implies that $\nabla w_k \to 0$ in $L_p^{loc}(\Omega)$.

Let α be a positive number such that $\alpha \neq \|u\|_p$ and set $U_k := u + \alpha w_k$. Then $\{U_k\}$ satisfies the conditions of (1.21). Let β be a positive number and let Ω_β and Ω'_β be defined as in the proof of Theorem 2. We claim that

$$\lim \int_\Omega (|\nabla U_k|^p - |\alpha \nabla w_k|^p) = \int_\Omega |\nabla u|^p.$$

To verify this fact, observe that

$$\limsup_{k\to\infty} \Big| \int_{\Omega_\beta} (|\nabla U_k|^p - |\alpha \nabla w_k|^p) \Big| = o(1) \quad \text{and} \quad \lim_{k\to\infty} \int_{\Omega'_\beta} |\nabla U_k|^p = \int_{\Omega'_\beta} |\nabla u|^p,$$

where $o(1)$ denotes a quantity (depending only on β) which tends to zero as $\beta \to 0$. Similarly,

$$\lim \int_\Omega (|U_k/\delta|^p - |\alpha w_k/\delta|^p) = \int_\Omega |u/\delta|^p.$$

Hence, using (1.26), we obtain

$$\nu_p(u) \leq \lim \chi_p(U_k) = \frac{\alpha^p c_p + \|u\|_p^p \chi_p(u)}{\alpha^p + \|u\|_p^p}. \tag{1.27}$$

Finally, letting α tend to zero or to infinity, we obtain

$$\nu_p(u) \leq \min(c_p, \chi_p(u)). \tag{1.28}$$

Next, let $\{u_k\}$ be an arbitrary sequence satisfying the conditions of (1.21), and denote $v_k := u_k - u$. Thus, by (0.1), $\{u_k\}$ is bounded in $\widetilde{W}_{1,p}(\Omega)$ and $u_k \to u$ weakly in $\tilde{L}_p(\Omega)$ (see Remark 1). We want to estimate $\liminf \chi_p(u_k)$. Therefore, without loss of generality, we may assume that $\{\chi_p(u_k)\}$ and $\{\|u_k\|_p\}$ converge. Denote

$$\hat{a} := \lim \int_\Omega |u_k/\delta|^p - \int_\Omega |u/\delta|^p.$$

Then $\hat{a} \geq 0$, and equality holds iff $u_k \to u$ in $\tilde{L}_p(\Omega)$. Let β be a positive number as before. Then

$$\chi_p(u_k) = \frac{\int_{\Omega_\beta} |\nabla v_k|^p + \int_{\Omega'_\beta} |\nabla u_k|^p + R_{k,\beta}}{\int_{\Omega_\beta} |v_k/\delta|^p + \int_{\Omega'_\beta} |u_k/\delta|^p + R'_{k,\beta}}, \tag{1.29}$$

where

$$|R_{k,\beta}| = |\int_{\Omega_\beta} |\nabla u_k|^p - \int_{\Omega_\beta} |\nabla v_k|^p| \leq o(1),$$
$$|R'_{k,\beta}| = |\int_{\Omega_\beta} |u_k/\delta|^p - \int_{\Omega_\beta} |v_k/\delta|^p| \leq o(1),$$

and $o(1)$ denotes a quantity (depending only on β) which tends to zero as $\beta \to 0$. Indeed, since $\{u_k\}$ is bounded in $\overset{\circ}{W}_{1,p}(\Omega)$ and $u_k - v_k = u$,

$$|\int_{\Omega_\beta} |\nabla u_k|^p - \int_{\Omega_\beta} |\nabla v_k|^p| \leq c|\, \|\nabla u_k\|_{L_p(\Omega_\beta)} - \|\nabla v_k\|_{L_p(\Omega_\beta)}|$$
$$\leq c\|\nabla u\|_{L_p(\Omega_\beta)} = o(1),$$

where c is a constant depending on the bound of $\{u_k\}$ in $\overset{\circ}{W}_{1,p}(\Omega)$. Denote

$$\bar{a}_\beta = \limsup \int_{\Omega_\beta} |v_k/\delta|^p, \quad \underline{a}_\beta = \liminf \int_{\Omega_\beta} |v_k/\delta|^p.$$

Then

$$\lim_{\beta\to 0} \bar{a}_\beta = \lim_{\beta\to 0} \underline{a}_\beta = \hat{a}. \tag{1.30}$$

Indeed, the limits exist because the quantities $\bar{a}_\beta$, and $\underline{a}_\beta$ vary monotonically with β and the equalities can be verified as follows:

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega |u_k/\delta|^p - \int_{\Omega'_\beta} |u/\delta|^p &= \lim_{k\to\infty} \int_{\Omega_\beta} |u_k/\delta|^p \\ &= \lim_{k\to\infty} (\int_{\Omega_\beta} |v_k/\delta|^p + R'_{k,\beta}), \end{aligned} \tag{1.31}$$

with $R'_{k,\beta}$ as before. Consequently, $\bar{a}_\beta - \underline{a}_\beta = o(1)$, and (1.30) follows from (1.31) by taking the limit as $\beta \to 0$.

If β is sufficiently small, the argument employed in (1.18) yields

$$\int_{\Omega_\beta} |\nabla v_k|^p \geq (1 + o(1))c_p \int_{\Omega_\beta} |v_k/\delta|^p. \tag{1.32}$$

From this inequality together with (1.29) and(1.30) we obtain,

$$\begin{aligned} \lim \chi_p(u_k) &\geq \limsup \frac{(1+o(1))c_p \int_{\Omega_\beta} |v_k/\delta|^p + \int_{\Omega'_\beta} |\nabla u_k|^p + R_{k,\beta}}{\int_{\Omega_\beta} |v_k/\delta|^p + \int_{\Omega'_\beta} |u_k/\delta|^p + R'_{k,\beta}} \\ &\geq \frac{(1+o(1))c_p \underline{a}_\beta + \int_{\Omega'_\beta} |\nabla u|^p + o(1)}{\bar{a}_\beta + \int_{\Omega'_\beta} |u/\delta|^p + o(1)}. \end{aligned} \tag{1.33}$$

Hence, taking the limit as $\beta \to 0$, we obtain

$$\lim \chi_p(u_k) \geq \frac{\hat{a} c_p + \int_\Omega |\nabla u|^p}{\hat{a} + \int_\Omega |u/\delta|^p} \geq \min(c_p, \chi_p(u)). \tag{1.34}$$

Inequalities (1.28) and (1.34) imply (1.22).

Finally, suppose that $\{u_k\}$ satisfies (1.23). If (1.24) does not hold, there exist a compact set $K \subset \Omega$, a positive number α and a subsequence of $\{u_k\}$ (which we also denote by $\{u_k\}$) such that

$$\int_K |\nabla u_k|^p \geq \alpha + \int_K |\nabla u|^p, \quad \forall k \in \mathbb{N}. \tag{1.35}$$

Using this fact and repeating the previous argument, we obtain

$$\nu_p(u) = \lim \chi_p(u_k) \geq \frac{\hat{a}c_p + \int_\Omega |\nabla u|^p + \alpha}{\hat{a} + \int_\Omega |u/\delta|^p} > \min(c_p, \chi_p(u)), \tag{1.36}$$

which contradicts (1.22). If in addition, $\chi_p(u) < c_p$, this argument shows that (assuming (1.23)) $\hat{a} = 0$. As mentioned before, this implies that $u_k \to u$ in $\tilde{L}_p(\Omega)$. Since $\chi_p(u_k) \to \chi_p(u)$, this in turn implies that $\|\nabla u_k\|_{L_p(\Omega)} \to \|\nabla u\|_{L_p(\Omega)}$. Since $\nabla u_k \to \nabla u$ weakly in $L_p(\Omega)$, these facts imply that $u_k \to u$ in $\widetilde{W}_{1,p}(\Omega)$. $\square$

Remark 2. The proof of the theorem also demonstrates the following fact. If u is a non-zero element of $\overset{\circ}{W}_{1,p}(\Omega)$ such that $\chi_p(u) \leq c_p$, then there exists an admissible sequence $\{u_k\}$ (in the sense of (1.21)) such that (1.23) holds. If $\chi_p(u) > c_p$, then there is no sequence for which the value $\nu_p(u) = c_p$ is attained.

2. Estimates of μ_p

Theorem 2 and (1.4) imply that

$$\mu_p(\Omega) \leq c_p, \tag{2.1}$$

for domains of class C^2. Actually this inequality remains valid under much weaker assumptions on $\partial\Omega$, as was shown by Davies [8] in the case $p = 2$. The following is a similar result for $p > 1$. Our proof is different from Davies' proof, although the latter could also be extended to the more general case.

Theorem 5. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that at some point $P \in \partial\Omega$ there exists a tangent hyperplane Π. More precisely, assume that there exists a neighborhood U of P such that*

$$|\operatorname{dist}(x, \Pi) - \delta_\Omega(x)| \leq o(1)(\operatorname{dist}(x, P)), \quad \forall x \in U \cap \Omega, \tag{2.2}$$

where $o(1)$ is a quantity which tends to zero as $x \to P$, and $U \cap \Omega$ contains a segment $\overline{PQ}$ perpendicular to Π. Then (2.1)*) is valid.*

Proof. Put $x' = (x_1, \cdots, x_{n-1})$ so that a generic point in $\mathbb{R}^n$ can be written in the form $x = (x', x_n)$. Without loss of generality, we assume that $P = 0, \Pi = \{x_n = 0\}$ and that Ω contains a segment $\{(0, x_n) : 0 < x_n < b\}$. Then condition (2.2) implies that for every $A > 0$ there exists $\alpha > 0$ such that

$$K_{A,\alpha} = \{x : 0 < x_n < \alpha, |x'| < Ax_n\} \subset \Omega. \tag{2.3}$$

Let $H := \{x_n > 0\}$ and let $\epsilon \in (0, 1)$. Since $\mu_p(H) = c_p$, there exists $\phi \in C_0^\infty(H)$ such that $|\chi_p^H(\phi) - c_p| < \epsilon$. There exists $A > 0$ such that

$$\operatorname{supp} \phi \subset K_A = \{x : x_n > 0, |x'| < Ax_n\}.$$

The ratio χ_p^H and K_A are invariant with respect to transformations of the form $x \to ax$ with $a > 0$. Therefore, in view of (2.2) and (2.3) we may assume that

$$\operatorname{supp} \phi \subset U \cap \Omega \text{ and } \delta_\Omega(x) < (1 + \epsilon)x_n, \; \forall x \in \operatorname{supp} \phi.$$

These facts imply that

$$\chi_p^\Omega(\phi) \le (1+\epsilon)\chi_p^H(\phi) \le (1+\epsilon)(c_p+\epsilon).$$

Since $\epsilon \in (0,1)$ is arbitrary, (2.1) follows. $\square$

Remark. In connection with this result it is natural to ask whether the estimate (2.1) remains valid for arbitrary domains, with no assumption of regularity. In the case that Ω is the punctured space $\mathbb{R}^n \setminus \{0\}$, it is known that $\mu_p(\Omega) = |1-n/p|^p$, which may be larger than c_p (see Example 1). However the question remains open in the case of irregular bounded domains or exterior domains (i.e. complements of bounded domains). Davies [8] conjectured that in this case (2.1) is valid, at least for $p=2$.

The next result describes a connection between the value of $\mu_p(\Omega)$ and the existence of minimizers for problem (0.2).

Theorem 6. *Let Ω be a domain in $\mathbb{R}^n$ with compact boundary of class C^2. Suppose that Hardy's inequality* (0.1) *is valid in Ω. If problem* (0.2) *has no minimizer, then*

$$\mu_p(\Omega) = c_p. \tag{2.4}$$

Proof. Let $\{u_k\}$ be a minimizing sequence for (0.2), normalized so that $\|u_k\|_p = 1$ for all k. Then the sequence is bounded in $\overset{\circ}{W}_{1,p}(\Omega)$, and consequently there is a subsequence (which we continue to denote by $\{u_k\}$) which converges weakly to some element U in this space. If $U=0$, then by Theorem 2,

$$\mu_p = \liminf \chi_p(u_k) \ge \mu_p^*(\Omega) = c_p,$$

and therefore, by (1.4), $\mu_p = c_p$. On the other hand, if $U \ne 0$ then, by Theorem 4,

$$\mu_p(\Omega) = \lim \chi_p(u_k) \ge \nu_p(U) = \min(c_p, \chi_p(U)) \ge \mu_p(\Omega).$$

Thus, either (2.4) holds or $\mu_p(\Omega) = \chi_p(U)$. Since, by assumption, there is no minimizer, it follows that (2.4) is valid. $\square$

3. Proof of Theorem II

Throughout this section, Ω denotes a bounded domain of class C^2 in $\mathbb{R}^n$, $n \ge 2$, $p > 2$ and $\delta = \delta_\Omega$. We start with a brief discussion of some spectral properties of the operator $P = -\delta^2(x)\Delta$ in Ω. The proof of Theorem II will be based on these properties and the asymptotic behavior of positive solutions of minimal growth at $\partial\Omega$.

First, we turn Ω into a Riemannian manifold M equipped with the metric $ds^2 = \delta^{-2}(x)\sum_{i=1}^n dx_i^2$. In accordance with previous notation we put $\tilde{L}_2(M) = \tilde{L}_2(\Omega)$ and $\tilde{H}^1(M) = \widetilde{W}_{1,2}(\Omega)$ with the norm $\|\cdot\|_{1,2}$ (see (1.1)). The closure of $C_0^1(\Omega)$ under this norm will be denoted by $\tilde{H}_0^1(M)$.

Let $\tilde{P}$ be the Friedrichs extension of the operator P considered as a symmetric operator in $\tilde{L}_2(M)$ with domain $C_0^1(\Omega)$ (see [2]). Since M is a complete Riemannian manifold, the operator $\tilde{P}$ is the unique selfadjoint realization of P in $\tilde{L}_2(M)$. Actually, $\tilde{P}$ coincides with the Dirichlet realization of P with domain of definition given by

$$D(\tilde{P}) = \{u \mid u \in \tilde{L}_2(M) \cap H_{loc}^1(M), Pu \in \tilde{L}_2(M)\}.$$

We denote by $\sigma(\tilde{P})$, $\sigma_{ess}(\tilde{P})$, $\sigma_{point}(\tilde{P})$, the spectrum, essential spectrum and point spectrum of $\tilde{P}$, respectively. Finally, the convex cone of all positive solutions of the equation $Pu = 0$ in Ω will be denoted by $\mathcal{C}_P(\Omega)$.

It is well known that

$$\lambda_0 := \inf \sigma(\tilde{P}) = \mu_2(\Omega) = \inf_{u \in \tilde{H}_0^1(M)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u/\delta|^2 dx}$$

and

$$\begin{aligned} \lambda_0 &= \sup\{\lambda \in \mathbb{R} \,:\, \mathcal{C}_{P-\lambda}(\Omega) \neq \emptyset\} \\ &= \sup\{\lambda \in \mathbb{R} \,:\, \exists u \in H^1_{loc}(\Omega), u > 0, (P - \lambda)u \geq 0 \text{ in } \Omega\}, \end{aligned} \tag{3.1}$$

and the supremum λ_0 is achieved. Further,

$$\lambda_\infty := \inf \sigma_{ess}(\tilde{P}) = \sup_{K \subset\subset \Omega} \lambda_0(M \setminus K), \tag{3.2}$$

where $\lambda_0(M \setminus K)$ is defined in the same way as λ_0 with M replaced by $M \setminus K$ (but $\delta = \delta_\Omega$), and

$$\begin{aligned} \lambda_\infty &= \sup\{\lambda \in \mathbb{R} \,:\, \exists K \subset\subset \Omega \text{ s.t. } \mathcal{C}_{P-\lambda}(\Omega \setminus K) \neq \emptyset\} \\ &= \sup\{\lambda \in \mathbb{R} \,:\, \exists K \subset\subset \Omega \text{ and } u \in H^1_{loc}(\Omega \setminus K), \text{ s.t.} \\ &\qquad u > 0 \text{ and } (P - \lambda)u \geq 0 \text{ in } \Omega \setminus K\} \end{aligned} \tag{3.3}$$

(see [1]). Clearly, $\lambda_\infty \geq \lambda_0$.

If problem (0.2) is solvable then it possesses a positive minimizer. Since every minimizer is a solution of the equation $(P - \lambda_0)u = 0$ in Ω, it follows that problem (0.2) possesses a minimizer if and only if $\lambda_0 = \mu_2(\Omega)$ and $\mathcal{C}_{P-\lambda_0}(\Omega) \cap \tilde{L}_2(\Omega) \neq \emptyset$.

Let $\epsilon > 0$. Since Ω is a smooth bounded domain, there exists $\beta = \beta(\epsilon) > 0$ such that the function $\delta^{1/2}$ is a positive supersolution of the equation $(P + \epsilon - 1/4)u = 0$ in $\Omega_\beta = \{x \in \Omega \,:\, \delta(x) < \beta\}$. Therefore, (3.3) implies that $\lambda_\infty \geq 1/4$. On the other hand, the proof of Theorem 5 and (3.2) imply that $\lambda_\infty \leq 1/4$. Therefore,

$$\lambda_\infty = c_2 = 1/4.$$

Note that if Ω is unbounded or nonsmooth it may happen that $0 \leq \lambda_\infty < 1/4$ (see Section 4).

Now, a general theorem of Agmon [3] implies that if $\partial\Omega$ is real-analytic then $\sigma_{ess}(\tilde{P}) = [1/4, \infty)$, and every eigenvalue is an isolated point of the spectrum and has finite multiplicity. Therefore, (0.2) possesses a minimizer if and only if $\mu_2 = \lambda_0 < \lambda_\infty = 1/4$, and Theorem II follows. We present here a direct and elementary proof of Theorem II which does not assume the analyticity of the boundary. We were later informed (Agmon [4]) that the results quoted above can also be established without the assumption that the boundary is analytic.

The proof of the theorem is based on several lemmas, of which the first is due to Agmon while the second is a modification of Agmon's proof of the analogous theorem for Schrödinger operators in $\mathbb{R}^n$ (see [2, Theorem 2.7]).

Lemma 7 (Agmon)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2. Assume that $0 < \lambda \leq 1/4$ and let $s = s(\lambda)$ be the positive number such that $s(1 - s) = \lambda$ and $1/2 \leq s < 1$. Consider the functions*

$$v_s(x) = \delta^s(x) - \delta(x)/2, \quad u_s(x) = \delta^s(x) + \delta(x)/2.$$

Then for every $\epsilon > 0$ there exists $\beta > 0$ such that:

(a) for each $\epsilon \le \lambda \le 1/4$ the function v_s is a positive supersolution and u_s is a positive subsolution of the equation $(P - \lambda)u = 0$ in Ω_β;

(b) for $0 < \lambda < 1/4$, $v_s, u_s \in \tilde{L}^2(\Omega_\beta)$.

Proof. Let the normal curvature of $\partial\Omega$ be bounded in absolute value by K and let $\beta_1 = 1/K$. Since Ω is of class C^2, it follows that $\delta \in C^2(\Omega_{\beta_1})$ (see [9, Lemma 14.15]). Let $\alpha > 0$; then in Ω_{β_1} we have

$$\Delta\delta^\alpha(x) = \alpha\delta(x)^{\alpha-1}\Delta\delta(x) + \alpha(\alpha-1)\delta(x)^{\alpha-2}|\nabla\delta(x)|^2.$$

Moreover, $|\Delta\delta|$ is bounded in a neighborhood of the boundary and $|\nabla\delta(x)| \to 1$ as $x \to \partial\Omega$. These facts together with some elementary calculations imply the lemma. □

Lemma 8. *Let Ω be a domain in $\mathbb{R}^n$ with compact Lipschitz boundary. Let $\lambda \le 1/4$ and let w be a positive continuous supersolution of the equation $(P - \lambda)u = 0$ in Ω_β for some $\beta > 0$. Let v be a continuous subsolution of the same equation in Ω_β such that*

$$\liminf_{k\to\infty} \int_{D_k} |\frac{v(x)}{\delta}|^2 dx = 0, \tag{3.4}$$

where $D_k = \{x \in \Omega : 2^{-(k+1)} < \delta(x) < 2^{-k}\}$. Then there exists a positive constant C, which depends only on the values of w and v on $\{x : \delta(x) = \beta/2\}$, such that

$$v(x) \le Cw(x) \tag{3.5}$$

on $\Omega_{\beta/2}$.

Proof. Let C be a positive constant such that

$$v(x) < Cw(x) \tag{3.6}$$

on $\{x \in \Omega \mid \delta(x) = \beta/2\}$. Define a function u_0 in $\Omega_{\beta/2}$ by

$$u_0(x) = (v(x) - Cw(x))_+. \tag{3.7}$$

We claim that $u_0 = 0$ in $\Omega_{\beta/2}$. Note that u_0 is a subsolution of the equation $(P - \lambda)u = 0$ in $\Omega_{\beta/2}$. Moreover, it follows from (3.6) and (3.7) that there exists $\epsilon > 0$ such that $u_0 = 0$ in $\Omega_{\beta/2} \setminus \Omega_{\beta/2-\epsilon}$. Extend u_0 by zero in $\Omega \setminus \Omega_{\beta/2}$. It follows that u_0 is a non-negative subsolution in Ω, and we have

$$\int_{\Omega_{\beta/2}} \left(\nabla u_0 \cdot \nabla(\zeta^2 u_0) - \lambda\frac{(u_0\zeta)^2}{\delta^2}\right) dx \le 0 \tag{3.8}$$

for any real function $\zeta \in C_0^\infty(\Omega)$. Using the identity

$$\nabla u_0 \cdot \nabla(\zeta^2 u_0) = |\nabla(\zeta u_0)|^2 - u_0^2|\nabla\zeta|^2, \tag{3.9}$$

we infer from (3.8) that

$$\int_{\Omega_{\beta/2}} \left(|\nabla(\zeta u_0)|^2 - \lambda\frac{(u_0\zeta)^2}{\delta^2}\right) \le \int_{\Omega_{\beta/2}} u_0^2|\nabla\zeta|^2 dx. \tag{3.10}$$

On the other hand, since w is a positive supersolution,

$$\int_{\Omega_{\beta/2}} \nabla w \cdot \nabla\left(\frac{\zeta^2 u_0^2}{w}\right) \ge \int_{\Omega_{\beta/2}} \frac{\lambda}{\delta^2}\zeta^2 u_0^2. \tag{3.11}$$

Therefore, noting that by (3.9)

$$\nabla w \cdot \nabla(\frac{\zeta^2 u_0^2}{w}) = |\nabla(\zeta u_0)|^2 - w^2 |\nabla(\frac{\zeta u_0}{w})|^2,$$

we obtain

$$\int_{\Omega_{\beta/2}} w^2 |\nabla(\frac{\zeta u_0}{w})|^2 dx \le \int_{\Omega_{\beta/2}} \left(|\nabla(\zeta u_0)|^2 - \lambda \frac{(u_0\zeta)^2}{\delta^2} \right) dx. \tag{3.12}$$

Combining (3.10) and (3.12), we find that

$$\int_{\Omega_{\beta/2}} w^2 |\nabla(\frac{\zeta u_0}{w})|^2 \le \int_{\Omega_{\beta/2}} u_0^2 |\nabla \zeta|^2 dx. \tag{3.13}$$

Note that, by (3.4),

$$\liminf_{k\to\infty} \ 2^{2k} \int_{D_k} |u_0(x)|^2 dx = 0. \tag{3.14}$$

Pick $\chi_k \in C_0^\infty(\Omega)$ such that $0 \le \chi_k \le 1$ and

$$\begin{array}{ll} \chi_k(x) = 1 & \text{if } \ \delta(x) \ge 2^{-k}, \\ \chi_k(x) = 0 & \text{if } \ \delta(x) \le 2^{-k-1}, \\ |\nabla \chi_k| \le \gamma\, 2^k. & \end{array}$$

Applying (3.13) with $\zeta = \chi_k$, using (3.14) and Fatou's lemma, we find that

$$\begin{aligned} \int_{\Omega_{\beta/2}} w^2 |\nabla(\frac{u_0}{w})|^2 &\le \liminf_{k\to\infty} \int_{\Omega_{\beta/2}} u_0^2 |\nabla \chi_k|^2 dx \\ &\le \liminf_{k\to\infty} \ \gamma\, 2^{2k} \int_{D_k} u_0^2 dx = 0. \end{aligned} \tag{3.15}$$

Therefore, $u_0 = cw$ in $\Omega_{\beta/2}$ for some constant c. Since w is positive and u_0 vanishes in an open set of $\Omega_{\beta/2}$, it follows that $c = 0$ and $u_0 = 0$ in $\Omega_{\beta/2}$. □

Remark 3. Suppose that Ω is a domain whose boundary contains a compact connected component Γ of class C^2. Let $\beta > 0$ be sufficiently small and assume that u is a positive continuous supersolution and v is a continuous subsolution in $\Gamma_\beta = \{x \in \Omega \mid \operatorname{dist}(x,\Gamma) < \beta\}$, and v satisfies (3.4) with $D_k^\Gamma := D_k \cap \Gamma_\beta$ replacing D_k. The proof of Lemma 8 demonstrates that there exists $C > 0$ such that $v \le Cu$ in $\Gamma_{\beta/2}$.

Let $K \subset \Omega$ be a compact set. The set $\Omega \setminus K$ is called a *neighborhood of infinity in* Ω. Let v be a positive solution of the equation $(P-\lambda)u = 0$ in some neighborhood of infinity in Ω. Then v is said to be a *positive solution of minimal growth in a neighborhood of infinity in* Ω if for each positive supersolution w of the equation $(P-\lambda)u = 0$ in some neighborhood of infinity in Ω there exist a positive constant C and a neighborhood of infinity $\Omega_1 \subset \Omega$ such that $v(x) \le Cw(x)$ in Ω_1.

Lemma 9. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2 and let $\epsilon, \beta > 0$. Let $\epsilon < \lambda \le 1/4$. Suppose that v is a positive solution of the equation $(P-\lambda)u = 0$ in Ω_β. which has minimal growth at infinity in Ω. Then there are positive constants C and β_1 such that*

$$C^{-1}\delta^s(x) \le v(x) \le C\delta^s(x) \quad \text{in} \ \ \Omega_{\beta_1}, \tag{3.16}$$

with s as in Lemma 7. The constant C depends only on ϵ, β_1 and the maximum of v on $\partial\Omega_{\beta_1}$.

Proof. Let $\epsilon > 0$ be fixed. By Lemma 7 there exists $\beta_0 > 0$ such that for each $\epsilon < \lambda \le 1/4$ the functions $v_s(x) = \delta^s(x) - \frac{1}{2}\delta(x)$ and $u_s(x) = \delta^s(x) + \frac{1}{2}\delta(x)$ are positive supersolution and, respectively, subsolution of the equation $(P-\lambda)u = 0$ in Ω_{β_0}. By the definition of positive solution of minimal growth at infinity it follows that there exist positive constants C and β_1 such that

$$v(x) \le C v_s(x) \le C\delta^s(x) \text{ in } \Omega_{\beta_1}. \tag{3.17}$$

If $\lambda < 1/4$ then $s > 1/2$, and consequently $u_s \in \tilde{L}^2(\Omega_{\beta_0})$. Now, if $\epsilon < \lambda < 1/4$, Lemma 8 implies that there exist β_1 and $C > 0$ such that

$$C^{-1}\delta^s(x) \le v(x) \text{ in } \Omega_{\beta_1}. \tag{3.18}$$

Next suppose that $\lambda = 1/4$. A simple calculation shows that there exists $\beta_0 > 0$ such that, for every $0 < \eta < 1/4$, the function $u_{1/2+\eta}$ is a subsolution of the equation $(P - 1/4)u = 0$ in Ω_{β_0} and $u_{1/2+\eta} \in \tilde{L}^2(\Omega_{\beta_0})$. Therefore, by Lemma 8 there exist positive constants c and β_1 such that

$$\tfrac{1}{2}c\delta^{1/2+\eta}(x) \le cu_{1/2+\eta}(x) \le v(x) \text{ in } \Omega_{\beta_1}$$

for every $0 < \eta < 1/4$. By letting $\eta \to 0$ we obtain

$$\tfrac{1}{2}c\delta^{1/2}(x) \le v(x) \text{ in } \Omega_{\beta_1}. \quad \square \tag{3.19}$$

Remark. Note that the latter inequality holds true for *any* positive supersolution of $(P-1/4)u = 0$ in some neighborhood of infinity in Ω. In particular if $u \in \mathcal{C}_{P-1/4}(\Omega)$ then $u \notin \tilde{L}_2(M)$.

Theorem 10. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2 and let λ_0 be as in* (3.1).

- (i) *If $0 < \lambda_0 < 1/4$ then λ_0 is a simple eigenvalue of $\tilde{P}$. Consequently problem* (0.2) *possesses a minimizer which is unique up to a multiplicative constant and the minimizer satisfies estimate* (3.16). *Furthermore, $\mathcal{C}_{P-\lambda_0}(\Omega)$ is a one dimensional cone.*
- (ii) *If $\lambda_0 = 1/4$ then problem* (0.2) *has no minimizer, and if $u \in \mathcal{C}_{P-1/4}(\Omega)$ then there exists a constant $c > 0$ such that $u(x) \ge c\delta^{1/2}(x)$. Moreover, λ_0 is not an eigenvalue of $\tilde{P}$.*

Proof. First we establish the following claim. Let $\lambda \in (0, 1/4]$ and suppose that $\mathcal{C}_{P-\lambda}(\Omega)$ is not empty. If $u \in \mathcal{C}_{P-\lambda}(\Omega)$ and $w \in \tilde{L}_2(M)$ is a solution of $(P-\lambda)w = 0$, then w is a constant multiple of u. Indeed, if w is a non-trivial solution, then Lemma 8 implies that there exists $\epsilon > 0$ such that $\epsilon w(x) \le u(x)$ in Ω. Define

$$\epsilon_0 = \sup\{\epsilon \;:\; u(x) - \epsilon w(x) \ge 0 \text{ in } \Omega\}.$$

If $u - \epsilon_0 w \in \mathcal{C}_{P-1/4}(\Omega)$, then by Lemma 8 there exists $\epsilon_1 > 0$ such that $u - \epsilon_0 w > \epsilon_1 w$, which contradicts the maximality of ϵ_0. Therefore, $u = \epsilon_0 w$.

Now in part (i) we assume that $\lambda_0 < \lambda_\infty = 1/4$. Since λ_0 is achieved, $\mathcal{C}_{P-\lambda_0}(\Omega) \neq \emptyset$. In addition, if $\lambda_0 < \lambda_\infty$, it is known that $u \in \mathcal{C}_{P-\lambda_0}(\Omega)$ if and only if u is a positive solution in Ω which has minimal growth at infinity (see, for example, [1, Theorem 5.5]). Therefore, by Lemma 9, every u in $\mathcal{C}_{P-\lambda_0}(\Omega)$ satisfies (3.16), so that λ_0 is an eigenvalue, and in view of the previous claim, it is a simple eigenvalue. This implies statement (i).

Finally we observe that if $u \in \mathcal{C}_{P-1/4}(\Omega)$ then, by the remark following Lemma 9, there exists $C > 0$ such that $\delta^{1/2}(x) \le Cu(x)$ in Ω, and therefore $u \notin \tilde{L}_2(M)$.

Thus, u is not a minimizer of (0.2). Furthermore, if λ_0 is an eigenvalue then problem (0.2) possesses a minimizer, and consequently there exists a positive eigenfunction corresponding to λ_0; but this contradicts the previous conclusion. This proves statement (ii). □

Remark 4. For the second statement of the theorem, the assumptions on the domain can be relaxed. In fact, if Ω is an arbitrary domain (not necessarily bounded) such that $\partial\Omega$ contains a compact connected component Γ of class C^2, then statement (ii) remains valid. Indeed, if $\lambda_0 = 1/4$ and $u \in \mathcal{C}_{P-1/4}(\Omega)$, then by Remark 3 there exist constants $\beta, c > 0$ such that $u \geq c\delta^{1/2}$ in Γ_β. Thus $u \notin \tilde{L}_2(M)$.

4. Some Examples

In this section we discuss the value of the best Hardy constant for various specific domains. Recall that, by Theorem 5, if Ω is a bounded Lipschitz domain then $0 < \mu_p \leq c_p$. On the other hand, if $\Omega \neq \mathbb{R}^n$ is convex then $\mu_p = c_p$ (see Appendix A).

Example 1. The punctured space: Let $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$. It is easy to see that

$$\mu_p(\mathbb{R}^n_*) = \inf\{\chi_p^{\mathbb{R}^n_*}(u) \,:\, u \in \overset{\circ}{W}_{1,p}(\mathbb{R}^n_*) \text{ and } u \text{ radially symmetric}\}.$$

In fact this is valid in every radially symmetric domain Ω. Therefore, by [11, 13] (see also [16]), $\mu_p(\mathbb{R}^n_*) = c^*_{p,n} := |\frac{n-p}{p}|^p$.

Example 2. Exterior domain: Let Ω be an exterior domain such that Ω has at least one boundary point which admits a tangent hyperplane. It follows from Theorem 5 that $\mu_p(\Omega) \leq c_p$. Without loss of generality, we may assume that $0 \notin \bar{\Omega}$. Consider the sequence of domains $\Omega_k = \frac{1}{k}\Omega$, $k \geq 1$. Then, $\mu_p(\Omega_k) = \mu_p(\Omega)$. On the other hand, by Lemma 12, $\limsup_{k\to\infty} \mu_p(\Omega_k) \leq \mu_p(\mathbb{R}^n_*) = c^*_{p,n}$. Thus, $\mu_p(\Omega) \leq c^*_{p,n}$, and consequently

$$\mu_p(\Omega) \leq \min(c_p, c^*_{p,n}). \tag{4.1}$$

In particular, if $p = n$, then $\mu_p(\Omega) = 0$. Note that $c^*_{p,n} \leq c_p$ if and only if $p \geq (n+1)/2$.

Example 3. Punctured domain: Let Ω be a Lipschitz domain such that $0 \in \Omega$ and consider the punctured domain $\Omega^* = \Omega \setminus \{0\}$.

Note that, in view of scale invariance, $\mu_p(\mathbb{R}^n_*)$ can be approximated by test functions with supports in arbitrarily small neighborhoods of the origin. Consequently,

$$\mu_p(\Omega^*) \leq \mu_p(\mathbb{R}^n_*) = c^*_{p,n}. \tag{4.2}$$

Thus, for $p = n$, $\mu_p(\Omega^*) = 0$. If $p < n$ then $\overset{\circ}{W}_{1,p}(\Omega) = \overset{\circ}{W}_{1,p}(\Omega^*)$. Since $\delta_\Omega(x) \geq \delta_{\Omega^*}(x)$ it follows that

$$\mu_p(\Omega^*) \leq \mu_p(\Omega). \tag{4.3}$$

Moreover, if we assume also that $\mu_p(\Omega) < c_p$, then there exists a strictly positive minimizer u of the variational problem for the domain Ω. Clearly, $\chi_p^{\Omega^*}(u) < \chi_p^{\Omega}(u)$. Hence, $\mu_p(\Omega^*) < \mu_p(\Omega)$.

Example 4. ANNULAR DOMAIN: Let Ω_1, Ω_2 be two bounded Lipschitz domains in $\mathbb{R}^n$, such that $\Omega_1 \subset\subset \Omega_2$, and set $\Omega_0 = \mathbb{R}^n \setminus \Omega_1$. Consider the domain $\Omega = \Omega_0 \cap \Omega_2$. Set $\mu_{p,i} = \mu_p(\Omega_i)$ and $\alpha = \frac{\mu_{p,2}}{\mu_{p,0}+\mu_{p,2}}$. If $u \in \overset{\circ}{W}_{1,p}(\Omega)$, then

$$(\alpha + (1-\alpha))\|\nabla u\|^p_{L_p(\Omega)} \geq \alpha\mu_{p,0}(|\!|u|\!|_p^{\Omega_0})^p + (1-\alpha)\mu_{p,2}(|\!|u|\!|_p^{\Omega_2})^p$$
$$= \alpha\mu_{p,0}\left\{(|\!|u|\!|_p^{\Omega_0})^p + (|\!|u|\!|_p^{\Omega_2})^p\right\} \geq \alpha\mu_{p,0}(|\!|u|\!|_p^{\Omega})^p.$$

Hence,

$$\frac{\mu_{p,0}\mu_{p,2}}{\mu_{p,0}+\mu_{p,2}} \leq \mu_p(\Omega). \tag{4.4}$$

Assume that $0 \in \Omega_1$ and put $\Omega^k = \Omega_0 \cap (k\Omega_2)$. Then $\{\Omega^k\}_{k=1}^\infty$ is a normal approximating sequence for Ω_0 (see Definition 1) and consequently, by Lemma 12 and (4.1),

$$\limsup_{k\to\infty} \mu_p(\Omega^k) \leq \mu_p(\Omega_0) \leq \min(c_p, c^*_{p,n}). \tag{4.5}$$

In particular, if $p = n$, $\lim \mu_p(\Omega^k) = 0$. In this connection we recall that, in the case $p = n = 2$, Ancona [5, p.278] proved that if Ω is a *simply connected* domain then $\mu_2(\Omega) \geq 1/16$. Our result shows that for annular domains μ_2 may be arbitrarily small.

On the other hand, if $\hat{\Omega}^k := (\frac{1}{k}\Omega_0) \cap \Omega_2$, then $\{\hat{\Omega}^k\}_{k=1}^\infty$ is a normal approximating sequence for the punctured domain Ω_2^*. Hence, if $p \leq n$,

$$\limsup_{k\to\infty} \mu_p(\hat{\Omega}^k) \leq \mu_p(\Omega_2^*) \leq \min(\mu_p(\Omega), c^*_{p,n}). \tag{4.6}$$

In particular, if $p = n$, $\lim \mu_p(\hat{\Omega}^k) = 0$. Furthermore, by (4.3) and the previous convexity argument,

$$\frac{c^*_{p,n}\mu_p(\Omega_2)}{c^*_{p,n} + \mu_p(\Omega_2)} \leq \mu_p(\Omega_2^*) \leq \mu_p(\Omega_2). \tag{4.7}$$

Example 5. RADIALLY SYMMETRIC DOMAINS: In the case of radially symmetric domains, more precise information concerning μ_2 can be obtained through the construction of explicit solutions.

We start with the exterior of a ball $\Omega = \mathbb{R}^n \setminus B(0,R)$. We know that if $n = 2$, $\mu_2(\Omega) = 0$. Suppose now that $n \geq 3$. Consider the radial equation

$$y'' + \frac{(n-1)y'}{r} + \frac{y}{4(r-R)^2} + \frac{(n-1)(n-3)y}{4r^2} = 0,$$

where $r > R$. It has a positive solution of the form $y(r) = \sqrt{(r-R)r^{1-n}}$. Therefore, $y(r)$ is a positive supersolution of the equation $(P - \frac{1}{4})u = 0$ in Ω, and therefore, $\mu_2 \geq 1/4$. Hence, by Example 2, $\mu_2(\Omega) = 1/4$ for $n \geq 3$.

Note that if $n = 3$ then $y(r)$ is a positive solution of the equation $(P - 1/4)u = 0$ in Ω which has minimal growth in a neighborhood of infinity in Ω. Therefore, in this case $y(r)$ is the *unique* positive solution of $(P - 1/4)u = 0$ in Ω and can be thought of as a *virtual minimizer* of (0.2).

Next we consider an annulus, $\Omega = A(r,R) = B(0,R) \setminus B(0,r), 0 < r < R$. If $n = 2$ then $\lim_{R\to\infty} \mu_2(A(r,R)) = 0$. Now assume that $n \geq 3$. Then it is easily checked that

$$v(x) = |x|^{(n-1)/2} \min\{(|x| - r)^{1/2}, (R - |x|)^{1/2}\}$$

is a positive supersolution of the equation $(P-\frac{1}{4})u=0$ in Ω. Therefore, $\mu_2(A(r,R)) \geq 1/4$ and hence (in view of Theorem 5) $\mu_2(A(r,R))=1/4$.

Finally we consider the punctured ball $B(0,R)^*$. By Example 3, if $n=2$ then $\mu_2(B(0,R))^*=0$ and consequently $\lim_{r\to 0}\mu_2(A(r,R))=0$. Suppose now that $n\geq 3$, in which case, $\mu_2(A(r,R))=1/4$ for every annulus. By Lemma 12 and (4.3), $1/4=\lim_{r\to 0}\mu_2(A(r,R))\leq \mu_2(B(0,R)^*)\leq 1/4$. Thus $\mu_2(B(0,R)^*)=1/4$.

Example 6. Almost conical domains: Consider for simplicity a conical domain in $\mathbb{R}^3$ (the same argument applies also for $n>3$). So, let $\gamma>0$ and let $x\in\mathbb{R}^3$ be given in polar coordinates by (r,θ,ϕ). Consider the domain

$$D_\gamma := \{x\in\mathbb{R}^3 \mid 0<r<1, \gamma<\theta\leq\pi, 0\leq\phi\leq 2\pi\}.$$

Davies [8] has proved that $\lim_{\gamma\to 0}\mu_2(\Omega_\gamma)=0$. Hence, taking a smooth normal approximating sequence of domains for D_γ, we see that for every $\epsilon>0$ there exists a smooth bounded domain A_ϵ such that $0<\mu_2(A_\epsilon)\leq\epsilon$.

Further, using a smooth approximation of a sector in $\mathbb{R}^2$ and employing an estimate of Davies [8] concerning such sectors, we find that for every $1/4.86\leq\beta\leq 1/4$ there exists a smooth bounded domain $E_\beta\subset\mathbb{R}^2$ such that $\mu_2(E_\beta)\leq\beta$.

Appendix A.

In this appendix we prove the following result.

Theorem 11. *Let $\Omega\neq\mathbb{R}^n$ be a convex domain. Then*

$$\mu_p(\Omega)=c_p. \tag{A.1}$$

The proof will be based on three lemmas. We start with a definition.

Definition 1. *Let Ω be a domain in $\mathbb{R}^n$. A sequence of domains $\{\Omega_k\}$ is a* normal approximating sequence *for Ω if it satisfies the following conditions:*

$$\delta_{\Omega_k}(x)\to\delta_\Omega(x), \quad \forall x\in\Omega, \tag{A.2}$$

and for every compact subset K of Ω there exists an integer j such that

$$K\subset\bigcap_{k=j}^{\infty}\Omega_k.$$

Note that every increasing sequence of subdomains whose limit is Ω is a normal approximating sequence in the sense of this definition.

Lemma 12. *Let Ω be a domain in $\mathbb{R}^n$ and let $\{\Omega_k\}$ be a normal approximating sequence of domains. Then*

$$\limsup\mu_p(\Omega_k)\leq\mu_p(\Omega). \tag{A.3}$$

If in addition to the above, there exists a point $P\in\partial\Omega$ such that $\partial\Omega$ has a tangent hyperplane at P, then

$$\limsup\mu_p(\Omega_k)=c_p\Longrightarrow\mu_p(\Omega)=c_p. \tag{A.4}$$

Proof. If $u\in C_0^\infty(\Omega)$ then for sufficiently large k, $u\in C_0^\infty(\Omega_k)$. Condition (A.2) implies that $\chi_p^{\Omega_k}(u)\to\chi_p^{\Omega}(u)$. This fact and the definition of $\mu_p(\Omega)$ imply (A.3). In view of Theorem 5, (A.4) follows from (A.3). □

Let $x' = (x_1, \cdots, x_{n-1})$ denote a generic point in $\mathbb{R}^{n-1}$, so that $x = (x', x_n)$ is a point in $\mathbb{R}^n$. Let D be a domain in $\mathbb{R}^{n-1}$ and let Ω be a domain in $\mathbb{R}^n$ of the form $\{(x', x_n) : x' \in D, 0 < x_n < A(x')\}$. When this is the case we shall say that Ω *covers* D.

Lemma 13. *Let Ω be a bounded domain of the form described above. If $u \in C^1(\overline{\Omega})$ and u vanishes on $x_n = 0$, then*

$$\int_\Omega |\nabla u|^p \geq c_p \int_\Omega |u/x_n|^p. \tag{A.5}$$

Proof. Let $(0, A)$ be a bounded interval. If $f \in W_{1,p}(0, A)$ and $f(0) = 0$, then

$$\int_0^A |f'| \geq c_p \int_0^A |f(t)/t|^p \, dt. \tag{A.6}$$

Using this fact we obtain,

$$\begin{aligned}
\int_\Omega |\nabla u|^p &\geq \int_D \int_0^{A(x')} |\partial u/\partial x_n|^p \, dx_n dx' \\
&\geq c_p \int_D \int_0^{A(x')} |u/x_n|^p \, dx_n dx' \\
&= c_p \int_\Omega |u/x_n|^p.
\end{aligned}$$

□

Let S be a bounded polytope and let $\Gamma_1, \ldots, \Gamma_q$ denote the (open) faces of S. Further, denote

$$\begin{gathered}
\Gamma_j' := \partial S \setminus \Gamma_j, \\
S_j := \{x \in S : \operatorname{dist}(x, \Gamma_j) < \operatorname{dist}(x, \Gamma_j')\}, \\
S^* := \textstyle\bigcup_{j=1}^q S_j.
\end{gathered} \tag{A.7}$$

Lemma 14. *If S is a bounded convex polytope, then* (A.1) *holds.*

Proof. Let $u \in C_0^\infty(S)$. By Lemma 13,

$$\int_{S_j} |\nabla u|^p \geq c_p \int_{S_j} |u(x)/\delta_S(x)|^p \, dx. \tag{A.8}$$

Since S is convex, $S \setminus S^*$ is a set of measure zero. Therefore, in view of Theorem 5, (A.8) implies (A.1). □

Proof of Theorem 11. We may assume that $n \geq 2$. Since Ω is convex, it follows that for almost all points $x \in \partial\Omega$ there exists a tangent hyperplane at x (see, for example, [12, Theorem 2.1.22]). Thus, by Theorem 5, $\mu_p(\Omega) \leq c_p$.

On the other hand, the convexity of Ω implies that there exists a normal approximating sequence of domains $\{\Omega_k\}$ such that, for each k, Ω_k is a bounded convex polytope. Therefore Lemma 12 and Lemma 14 imply that $\mu_p(\Omega) \geq c_p$. Hence we obtain (A.1). □

Appendix B.

In this appendix we give a proof of Lemma 1 and mention some related questions.

Proof of Lemma 1. Let $f \in W_{1,p}(0,1)$ and suppose that $\int_0^1 |f(t)/t|^p < \infty$. Clearly this implies that $f(0) = 0$. Therefore the statement of the lemma is valid in the one-dimensional case. Next, using the notation introduced in the previous appendix, let D be a domain in $\mathbb{R}^{n-1}$ and let Ω be a bounded domain in $\mathbb{R}^n$ which covers D. Thus

$$\Omega = \{(x', x_n) : x' \in D, 0 < x_n < A(x')\},$$

where $A(\cdot)$ is a bounded function in D. Suppose that $u \in W_{1,p}(\Omega)$ and that

$$\int_\Omega |u(x)/x_n|^p \, dx < \infty.$$

Then one can take for u an appropriate element of its equivalence class, so that, for almost every $x' \in D$,

$$u(x', \cdot) \in W_{1,p}(0, A(x')) \quad \text{and} \quad \int_0^{A(x')} |u(x', x_n)/x_n|^p \, dx_n < \infty.$$

By the previous remarks, $\lim_{x_n \to 0} u(x', x_n) = 0$ for almost every $x_n \in D$. This implies that the trace of u on D is zero.

Now let Ω be a bounded domain of class C^2 and let $u \in \widetilde{W}_{1,p}(\Omega)$. Using the fact established above, together with a standard partitioning technique plus local 'flattening' of the boundary, we conclude that the trace of u on the boundary of Ω is zero. Hence $u \in \overset{\circ}{W}_{1,p}(\Omega)$. □

Remarks. The inclusion $\widetilde{W}_{1,p}(\Omega) \supset \overset{\circ}{W}_{1,p}(\Omega)$ is valid in any domain (not necessarily bounded) in which Hardy's inequality holds. However, in general, equality does not hold in unbounded domains. For instance, if $n \geq 3$, $p = 2$ and Ω is the exterior of the unit ball, then Hardy's inequality is valid. However, in this case, it is easy to construct functions which behave like a negative power of $|x|$ at infinity and belong to $\widetilde{W}_{1,p}(\Omega)$ but not to $\overset{\circ}{W}_{1,p}(\Omega)$.

Conversely, if Ω is bounded and $\widetilde{W}_{1,p}(\Omega) \supset \overset{\circ}{W}_{1,p}(\Omega)$, then Hardy's inequality holds in Ω. This is a consequence of the closed graph theorem and the Poincaré inequality.

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