limit point (with respect to the topology induced by X^{ω^2+2}) of $\{f_{(\sigma_j,1)}\}_{j=1}^{\infty}$. Then

$$\|x_\sigma\|,\ \|f_\sigma\| \leqslant 2, \quad f_\sigma(x_\tau) = \begin{cases} 1 & \text{if } \tau > \sigma, \\ 0 & \text{if } \tau < \sigma. \end{cases} \tag{4.5}$$

In particular, $\|x_{\sigma_1} - x_{\sigma_2}\| \geqslant 1/2$ for every $\sigma_1 \neq \sigma_2$, and this concludes the proof of the theorem.

COROLLARY 1. *For every separable non-reflexive Banach space X there is an ordinal α ($\alpha \leqslant \omega^2$) so that X^α is separable but $X^{\alpha+2}$ is non-separable.*

Proof. Let β be the first even ordinal so that X^β is non-separable. Then $\beta \leqslant \omega^2+2$ and β cannot be a limit ordinal. Hence $\beta = \alpha+2$ and this α has the desired property.

COROLLARY 2. *For every non-reflexive Banach space X the quotient space $X^{\omega^2+2}/X^{\omega^2}$ is non-separable.*

Proof. Use Corollary 1, the fact that if $Y \subset X$ then Y^{**}/Y is isomorphic to a subspace of X^{**}/X and that every non-reflexive space has a separable non-reflexive subspace.

It was observed in [1] that if J is the classical example of James for a quasireflexive space then J^{ω^2} is separable. This shows that the ordinals appearing in Theorem 4 and its corollaries are the best possible (i.e. cannot be replaced in general by smaller ordinals).

Added in proof: J. Farahat recently extended the result of Section 3 by proving that, for every integer k and every $p<2$, there is a space with k-structure and type p. Hence, for every k, there is a space with k-structure which does not have $k+1$-structure.

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On the best constants in the Khinchin inequality*

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Abstract. Let (r_j) denote the sequence of Rademacher functions. It is shown that

$$\int_0^1 \Big|\sum_{j=1}^{\infty} c_j r_j(t)\Big| dt \geqslant \frac{1}{\sqrt{2}} \Big(\sum_{j=1}^{\infty} |c_j|^2\Big)^{1/2}$$

for every square summable sequence of scalars (c_j). The constant $1/\sqrt{2}$ is the best (the largest) possible.

1. Introduction. Let r_n denote the nth Rademacher function, i.e.

$$r_n(t) = \operatorname{sign}\sin 2^n\pi t \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad (n = 1, 2, \ldots).$$

The classical Khinchin inequality states that, for every $p \in [1, \infty)$, there exist positive constants a_p and b_p such that, for every finite sequence of scalars (c_j),

$$a_p\Big(\sum_j |c_j|^2\Big)^{1/2} \leqslant \Big(\int_0^1 \Big|\sum_j c_j r_j(t)\Big|^p dt\Big)^{1/p} \leqslant b_p\Big(\sum_j |c_j|^2\Big)^{1/2}. \tag{0}$$

Let us denote by A_p and B_p, respectively, the largest a_p and the smallest b_p satisfying (0). B. Tomaszewski has observed that the values of A_p and B_p are independent of the choice of the scalar field, i.e. they are the same for real sequences as well as for complex sequences (cf. also Remark 3 in Section 3).

Therefore in the sequel we shall consider inequality (0) for real sequences only.

Obviously, $A_p = 1$ for $p \geqslant 2$ and $B_p = 1$ for $1 \leqslant p \leqslant 2$. Stečkin [6] has shown that

$$B_{2m} = \big((2m-1)!!\big)^{1/2m} \quad \text{for} \quad m = 1, 2, 3, \ldots$$

* This is a part of the author's masters thesis written under the supervision of Professor A. Pełczyński at the Warsaw University.

In the paper we show that $A_1 = 1/\sqrt{2}$. A part of our argument is a modification of the method used in [1] where it is shown that $A_1^{-1} < 1.5$. Precisely, our main result is

THEOREM 1. *We have*

$$\int_0^1 \Big|\sum_{j=1}^{\infty} c_j r_j(t)\Big|\,dt \geqslant \frac{1}{\sqrt{2}}\Big(\sum_{j=1}^{\infty} c_j^2\Big)^{1/2} \tag{1}$$

for every real $c_1, c_2, \ldots$ *with* $\sum_{j=1}^{\infty} c_j^2 < \infty$.

Moreover, the equality holds iff there exist indices i *and* k *with* $1 \leqslant i < k < \infty$ *such that* $|c_i| = |c_k|$ *and* $c_s = 0$ *for* $i \neq s \neq k$.

Let us recall that the condition $\sum_{j=1}^{\infty} c_j^2 < \infty$ implies that the series $\sum_{j=1}^{\infty} c_j r_j(t)$ converges almost everywhere (cf. e.g. [2]).

Theorem 1 implies in particular that, for the real Banach spaces l^1 and l^2, we have $\pi_1(I_{1,2}) = \sqrt{2}$, where $\pi_1(I_{1,2})$ denotes the absolutely summing norm of the natural injection $I_{1,2}\colon l^1 \to l^2$. Indeed, using (1) the same argument as in [5], 2.4.2, shows that $\pi_1(I_{1,2}) \leqslant \sqrt{2}$ while a direct computation shows that if $\mathbf{x}_1 = (1, 1, 0, 0, \ldots)$ and $\mathbf{x}_2 = (1, -1, 0, 0, \ldots)$ then

$$\|I\mathbf{x}_1\|_2 + \|I\mathbf{x}_2\|_2 = \sqrt{2}\max(\|\mathbf{x}_1+\mathbf{x}_2\|_1, \|\mathbf{x}_1-\mathbf{x}_2\|_1);$$

hence $\pi_1(I_{1,2}) \geqslant 2$.

2. Proof of the main result. We shall employ the following notation l^2 — the real space of real square summable sequences $\mathbf{c} = (c_j)_{j=1}^{\infty}$ with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|_2$ defined by

$$(\mathbf{c}, \mathbf{d}) = \sum_{j=1}^{\infty} c_j d_j; \qquad \|\mathbf{c}\|_2 = \Big(\sum_{j=1}^{\infty} c_j^2\Big)^{1/2} \quad \text{for} \quad \mathbf{c}, \mathbf{d} \in l^2.$$

$l_n^2 = \{\mathbf{c} \in l^2\colon\ c_j = 0 \text{ for } j > n\}$,
$D^n = \{\boldsymbol{\varepsilon} \in l_n^2\colon\ |\varepsilon_j| = 1 \text{ for } j = 1, 2, \ldots, n\}$,
$T(n) = \{\mathbf{c} \in l_n^2\colon\ c_1 + c_2 = \sqrt{2} \text{ and } c_1 \geqslant c_2 \geqslant \ldots \geqslant c_n \geqslant 0\}$ $(n = 1, 2, \ldots)$,
T = closure $\bigcup_{n=1}^{\infty} T(n)$, where the closure is taken in l^2,
$D_+^n(\mathbf{c}) = \{\boldsymbol{\varepsilon} \in D^n\colon\ (\boldsymbol{\varepsilon}, \mathbf{c}) > 0\}$
$D_0^n(\mathbf{c}) = \{\boldsymbol{\varepsilon} \in D^n\colon\ (\boldsymbol{\varepsilon}, \mathbf{c}) = 0\}$. for $\mathbf{c} \in l^2$ and for $n = 1, 2, \ldots$

We shall be dealing with the positive function f defined on $l^2 \setminus \{0\}$ by

$$f(\mathbf{c}) = \|\mathbf{c}\|_2^{-1}\int_0^1 \Big|\sum_{j=1}^{\infty} c_j r_j(t)\Big|\,dt.$$

By $f_{|n}$ we denote the restriction of f to $l_n^2 \setminus \{0\}$ for $n = 1, 2, \ldots$ Clearly, f is homogeneous, moreover, for every permutation $p(\cdot)$ of the indices, if $\mathbf{d} \in l^2$ is such that $|c_j| = |d_{p(j)}|$ for all j then $f(\mathbf{c}) = f(\mathbf{d})$; this follows for instance from formula (3) below.

LEMMA 1. *Let* $n = 1, 2, \ldots$, *let* $\mathbf{c} \in l_n^2$ *with* $\|\mathbf{c}\|_2 = 1$. *Then*

1° *For every* $\mathbf{h} \in l_n^2$ *with* $(\mathbf{h}, \mathbf{c}) = 0$ *and* $\|\mathbf{h}\|_2 = 1$ *and for every real* t

$$\begin{aligned} f_{|n}(\mathbf{c}+t\mathbf{h}) &\\ \geqslant (1+t^2)^{-1/2}\Big(f_{|n}(\mathbf{c}) &+ 2^{-n+1}t\sum_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{c})}(\boldsymbol{\varepsilon}, \mathbf{h}) + 2^{-n}|t|\sum_{\boldsymbol{\varepsilon}\in D_0^n(\mathbf{c})}|(\boldsymbol{\varepsilon}, \mathbf{h})|\Big). \end{aligned} \tag{2}$$

Moreover, there exists $\delta = \delta(\mathbf{c}) > 0$ *such that for* $|t| < \delta$ *the inequality becomes the equality.*

2° *If* $f_{|n}$ *has at* $\mathbf{c}$ *a local minimum, then* $D_0^n(\mathbf{c})$ *contains* $n-1$ *linearly independent vectors.*

Proof. Let $\mathbf{d} \in l_n^2 \setminus \{0\}$. Then

$$f_{|n}(\mathbf{d}) = \|\mathbf{d}\|_2^{-1}2^{-n}\sum_{\boldsymbol{\varepsilon}\in D^n}|(\boldsymbol{\varepsilon}, \mathbf{d})| = \|\mathbf{d}\|_2^{-1}\cdot 2^{-n+1}\Big(\sum_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{d})}(\boldsymbol{\varepsilon}, \mathbf{d})\Big). \tag{3}$$

Hence

$$\begin{aligned} f_{|n}(\mathbf{c}+t\mathbf{h}) &= (1+t^2)^{-1/2}2^{-n}\sum_{\boldsymbol{\varepsilon}\in D^n}|(\boldsymbol{\varepsilon}, \mathbf{c}) + t(\boldsymbol{\varepsilon}, \mathbf{h})| \\ &\geqslant (1+t^2)^{-1/2}2^{-n}\Big(2\sum_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{c})}(\boldsymbol{\varepsilon}, \mathbf{c}) + 2t\sum_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{c})}(\boldsymbol{\varepsilon}, \mathbf{h}) + \\ &\qquad + |t|\sum_{\boldsymbol{\varepsilon}\in D_0^n(\mathbf{c})}|(\boldsymbol{\varepsilon}, \mathbf{h})|\Big). \end{aligned} \tag{4}$$

Since $\|\mathbf{c}\|_2 = 1$, it follows from (3) that

$$2\sum_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{c})}(\boldsymbol{\varepsilon}, \mathbf{c}) = 2^n f_{|n}(\mathbf{c}).$$

Moreover, if $|t| \leqslant n^{-1/2}\min_{\boldsymbol{\varepsilon}\in D_+^n(\mathbf{c})}(\boldsymbol{\varepsilon}, \mathbf{c})$ then the inequality in (4) may be replaced by the equality. Therefore (4) implies (2). This completes the proof of 1°.

To prove 2° assume to the contrary that there exists a $\mathbf{c}$ in l_n^2 with $\|\mathbf{c}\|_2 = 1$ such that $f_{|n}$ has at $\mathbf{c}$ a local minimum and the dimension of the linear manifold spanned by $D_0^n(\mathbf{c})$ is less than $n-1$. Then there exists an $\mathbf{h} \in l_n^2$ with $\|\mathbf{h}\|_2 = 1$ such that $(\mathbf{h}, \mathbf{c}) = 0$ and $(\mathbf{h}, \boldsymbol{\varepsilon}) = 0$ for every $\boldsymbol{\varepsilon} \in D_0^n(\mathbf{c})$. Let $g(t) = f_{|n}(\mathbf{c}+t\mathbf{h})$. Then, by 1°,

$$g(t) = \frac{\beta + at}{\sqrt{1+t^2}} \quad \text{for} \quad |t| \leqslant \delta(c)$$

where $a = \sum_{\varepsilon \in D^n_+(c)} (\varepsilon, \mathbf{h})$ and $\beta = f_{|n}(\mathbf{c}) \neq 0$. Therefore g does not have a local minimum at the point $t = 0$, thus the function $f_{|n}$ does not have a local minimum at $\mathbf{c}$, a contradiction.

COROLLARY. *Let* $\mathbf{e}_1 = (1, 1, 0, 0, \ldots)$, $\mathbf{e}_2 = (1, 1, 1, 1, 0, 0, \ldots)$, $\mathbf{e}_3 = (2, 1, 1, 1, 1, 0, 0, \ldots)$, $\mathbf{e}_4 = (3, 3, 2, 2, 1, 1, 0, \ldots)$, $\mathbf{e}_5 = (1, 1, 1, 1, 1, 1, 0, \ldots)$, $\mathbf{e}_6 = (3, 1, 1, 1, 1, 1, 0, \ldots)$, $\mathbf{e}_7 = (3, 2, 2, 1, 1, 1, 0, \ldots)$, $\mathbf{e}_8 = (2, 2, 1, 1, 1, 1, 0, \ldots)$.

Then

$$f_{|6}(\mathbf{e}_1) = \frac{1}{\sqrt{2}}, \quad f_{|6}(\mathbf{e}_i) \geq \frac{3}{4} \quad \text{for} \quad 2 \leq i \leq 8.$$

Moreover, if $f_{|6}$ has a local minimum at a point $\mathbf{c} \in l_6^2$, then there exists an index i with $1 \leq i \leq 8$ such that $\mathbf{e}_i$ is proportional to the sequence whose coordinates are some permutation of absolute values of the coordinates of $\mathbf{c}$; in particular, $f(\mathbf{c}) = f(\mathbf{e}_i)$.

The corollary is proved by examining all the points in $T(6)$ which are orthogonal to some five linearly independent vectors in D^6. There exist at most $\binom{64}{5}$ points with the above property.

Let us put $\mathbf{e} = \|\mathbf{e}_1\|_2^{-1}\mathbf{e}_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, \ldots)$. Our next lemma provides an information on the behaviour of the function f in a neighbourhood of the point $\mathbf{e}$.

LEMMA 2. *Suppose that for some $n = 2, 3, 4, \ldots$ and for every $\mathbf{h}' \in l^2_{n-1} \setminus \{0\}$ we have $f_{|n-1}(\mathbf{h}') \geq 1/\sqrt{2}$. Then for every $\mathbf{h} \in l_n^2$ with $\|\mathbf{h}\|_2 = 1$ and $(\mathbf{e}, \mathbf{h}) = 0$*

(i) *if $0 < t < 4/3$, then $f(\mathbf{e}+t\mathbf{h}) > 1/\sqrt{2}$,*

(ii) *if $1/7 \leq t \leq 1$, then $f(\mathbf{e}+t\mathbf{h}) \geq 3/4$.*

Proof. Since $D^n_+(\mathbf{e}) = \{\varepsilon = (\varepsilon_j) \in D^n\colon \varepsilon_1 = \varepsilon_2\}$, we have

$$\sum_{\varepsilon \in D^n_+(\mathbf{e})} \varepsilon = 2^{n-2}\mathbf{e}.$$

Thus $\left(\sum_{\varepsilon \in D^n_+(\mathbf{e})} (\varepsilon, \mathbf{h})\right) = 0$ whenever $(\mathbf{e}, \mathbf{h}) = 0$.

Similarly, $D^n_0(\mathbf{e}) = \{\varepsilon \in D^n\colon \varepsilon_1 = -\varepsilon_2\}$. Therefore, by (3),

$$\sum_{\varepsilon \in D^n_0(\mathbf{e})} |(\varepsilon, \mathbf{h})| = \sum_{\varepsilon' \in D^{n-1}} |(\varepsilon', \mathbf{h}')| = 2^{n-1} f(\mathbf{h}')\|\mathbf{h}'\|_2$$

where $\mathbf{h}' = (2h_1, h_3, h_4, \ldots) \in l^2_{n-1} \setminus \{0\}$, because if $\mathbf{h} \in l_n^2$ and $(\mathbf{e}, \mathbf{h}) = 0$ then $h_1 = -h_2$ and for $\varepsilon \in D^n_0(\mathbf{e})$ $(\varepsilon, \mathbf{h}) = 2h_1\varepsilon_1 + h_3\varepsilon_3 + h_4\varepsilon_4 + \ldots = (\varepsilon', \mathbf{h}')$ where $\varepsilon' = (\varepsilon_1, \varepsilon_3, \varepsilon_4, \ldots) \in D^{n-1}$. Now using (2) for $\mathbf{c} = \mathbf{e}$ and the assumption that $f(\mathbf{h}') \geq 1/\sqrt{2}$ for $\mathbf{h}' \in l^2_{n-1} \setminus \{0\}$ we get

$$f(\mathbf{e}+t\mathbf{h}) \geq \frac{f(\mathbf{e}) + 2^{-1}|t| f(\mathbf{h}')\|\mathbf{h}'\|_2}{\sqrt{1+t^2}} \geq \frac{\frac{1}{\sqrt{2}}\left(1 + \frac{|t|}{2}\right)}{\sqrt{1+t^2}}. \tag{5}$$

Comparing the right side of (5) with $1/\sqrt{2}$ and $3/4$ we obtain (i) and (ii), respectively.

Remark. Let Z_n be the set of all points in l_n^2 whose absolute values of coordinates are some permutation of coordinates of $\mathbf{e}$. Then Lemma 2 remains true after replacing $\mathbf{e}$ by some $\mathbf{e}' \in Z_n$.

Before stating the next lemma we shall introduce some notation. For $m = 1, 2, \ldots$ and for fixed $\mathbf{c} \in T(2m)$ we put

$$x = 2\|\mathbf{c}\|_2^{-2} c_{2m}c_{2m-1}, \quad y = 2c_{2m}c_{2m-2}\|\mathbf{c}\|_2^{-2},$$

$$z = 2\|\mathbf{c}\|_2^{-2} c_{2m-1}c_{2m-2}, \quad v = 2c_{2m-2}c_{2m-3}\|\mathbf{c}\|_2^{-2},$$

$$q_m(\mathbf{c}) = \tfrac{1}{4}\left(\sqrt{1+x+v} + \sqrt{1+x-v} + \sqrt{1-x+(z-y)} + \sqrt{1-x-(z-y)}\right),$$

$$Q_m = \inf_{\mathbf{c} \in T(2m)} q_{2m}(\mathbf{c}).$$

LEMMA 3. *We have*

$$\frac{3}{4}\prod_{m=4}^{\infty} Q_m = K > \frac{1}{\sqrt{2}}.$$

The tedious numerical proof of Lemma 3 is given at the end of this paper.

Proof of Theorem 1. Let us put $K_n = \frac{3}{4}$ for $1 \leq n \leq 6$, $K_{2m-1} = K_{2m} = \frac{3}{4}\prod_{j=4}^{m} Q_j$ for $m \geq 4$.

Observe first that the sequence (K_n) is non-increasing because the function $\sqrt{t}$ is concave and therefore $q_m(\mathbf{c}) \leq 1$ for every $\mathbf{c} \in T(2m)$ and for every $m = 4, 5, \ldots$ Hence, by Lemma 3,

$$K_n \geq K > \frac{1}{\sqrt{2}} \quad \text{for every } n = 1, 2, 3, \ldots \tag{*}$$

Next observe that in order to prove inequality (1) it is enough to show that for $n = 1, 2, \ldots$

$(\mathrm{iii})_n$ $f(\mathbf{c}) \geq 1/\sqrt{2}$ for $\mathbf{c} \in l_n^2 \setminus \{0\}$.

For this purpose we shall formulate for $n = 1, 2, \ldots$

$(\mathrm{iv})_n$ if $\mathbf{c} \in T(n)$ and $\|\mathbf{c}-\mathbf{e}\|_2 \geq 1$, then $f(\mathbf{c}) \geq K_n$ and prove $(\mathrm{iii})_n$ and $(\mathrm{iv})_n$ by induction.

To achieve this we observe that for $n \leqslant 6$ $(\text{iii})_n$ follows immediately from Corollary.

To prove $(\text{iv})_n$ for $n \leqslant 6$ let us fix such an n and assume that, for some $\mathbf{c} \in T(n)$, $f(\mathbf{c}) < 3/4$. Then, by Corollary, $(\text{iii})_{n-1}$ and Remark, there exists some $\mathbf{e}' \in Z_n$ such that

$$\tan\alpha(\mathbf{e}', \mathbf{c}) < 1/7$$

where $\alpha(\mathbf{x}, \mathbf{y})$ denotes the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Now, taking into account the formula

$$\alpha(\mathbf{e}, \mathbf{c}) \leqslant \alpha(\mathbf{e}', \mathbf{c}) \quad \text{for every } \mathbf{e}' \in Z_n$$

which is a direct consequence of the assumption $\mathbf{c} \in T(n)$, we obtain

$$\tan\alpha(\mathbf{e}, \mathbf{c}) < 1/7 \text{ iff } \|\mathbf{e}-\mathbf{c}\|_2 < 1/7 \text{ as that } \mathbf{c} \in T(n).$$

Thus $(\text{iv})_n$ is proved.

Next observe that the implication

$$(\text{iii})_n \text{ and } (\text{iv})_{n+1} \Rightarrow (\text{iii})_{n+1}$$

follows immediately from $(*)$, Lemma 2 and the formula

$$\inf_{\mathbf{c} \in T(n)} f(\mathbf{c}) = \inf_{\mathbf{c} \in l_n^2 \setminus \{0\}} f(\mathbf{c}).$$

Thus to complete the inductive proof of $(\text{iii})_n$ and $(\text{iv})_n$ it is enough to establish the implications

I. $(\text{iv})_{2m-2}$ and $(\text{iii})_{2m-1} \Rightarrow (\text{iv})_{2m}$;

II. $(\text{iv})_{2m-2}$ and $(\text{iii})_{2m-2} \Rightarrow (\text{iv})_{2m-1}$ $(m = 4, 5, \ldots)$.

Proof of I. Let us suppose to the contrary that, for some $m \geqslant 4$, $(\text{iv})_{2m-2}$ and $(\text{iii})_{2m-1}$ holds but there exists a $\mathbf{c} = (c_1, c_2, \ldots, c_{2m}, 0, 0, \ldots) \in T(2m)$ with $\|\mathbf{c}-\mathbf{e}\|_2 \geqslant 1$ such that $f(\mathbf{c}) < K_{2m}$. Let us define $\mathbf{c}_j \in l_{2m-2}^2 \setminus \{0\}$ for $j = 1, 2, 3, 4$ by

$$\mathbf{c}_1 = (c_1, c_2, \ldots, c_{2m-4}, c_{2m-3}+c_{2m-2}, c_{2m-1}+c_{2m}, 0, 0, \ldots),$$

$$\mathbf{c}_2 = (c_1, c_2, \ldots, c_{2m-4}, c_{2m-3}-c_{2m-2}, c_{2m-1}+c_{2m}, 0, 0, \ldots),$$

$$\mathbf{c}_3 = (c_1, c_2, \ldots, c_{2m-4}, c_{2m-3}, c_{2m-2}+c_{2m-1}-c_{2m}, 0, 0, \ldots),$$

$$\mathbf{c}_4 = (c_1, c_2, \ldots, c_{2m-4}, c_{2m-3}, c_{2m-2}-c_{2m-1}+c_{2m}, 0, 0, \ldots).$$

Then, in view of (3),

$$f(\mathbf{c}) = \tfrac{1}{4}\sum_{j=1}^{4} f(\mathbf{c}_j)\|\mathbf{c}_j\|_2\|\mathbf{c}\|_2^{-1}.$$

Now, remembering that $q_m(\mathbf{c}) \geqslant Q_m$ and using the identity

$$\tfrac{1}{4}\sum_{j=1}^{4}\|\mathbf{c}_j\|_2 = q_m(\mathbf{c})\|\mathbf{c}\|_2$$

which can be verifying by a direct checking, we get

$$Q_m \cdot \min_{1 \leqslant j \leqslant 4} f(\mathbf{c}_j) \leqslant f(\mathbf{c}) < K_{2m} = Q_m \cdot K_{2m-2}.$$

Thus

$$\min_{1 \leqslant j \leqslant 4} f(\mathbf{c}_j) < K_{2m-2}.$$

Let, for instance, $f(\mathbf{c}_2) < K_{2m-2}$. Denote by $\mathbf{c}^*$ the vector in $T(2m-2)$ which is obtained from $\mathbf{c}_2$ by rearrangement of the coordinates of $\mathbf{c}_2$ in the decreasing order and multiplying by an appropriate constant λ (note that the coordinates of $\mathbf{c}_2$ are non-negative). Clearly, $f(\mathbf{c}^*) = f(\mathbf{c}_2) < K_{2m-2}$. Now, by $(\text{iii})_{2m-1}$, we may apply Lemma 2(ii) which combined with $(\text{iv})_{2m-2}$ gives $\|\mathbf{c}^*-\mathbf{e}\|_2 < 1/7$. Hence

$$c_1^* \geqslant c_2^* > \frac{1}{\sqrt{2}} - \frac{1}{7},$$

$$\frac{1}{7} > c_3^* \geqslant c_4^* \geqslant \ldots \geqslant c_{2m-2}^*.$$

Observe that neither c_1^* nor c_2^* is equal to $\lambda(c_{2m-1}+c_{2m})$. Otherwise we would have contradictory inequality

$$\frac{1}{\sqrt{2}} - \frac{1}{7} < \lambda(c_{2m-1}+c_{2m}) \leqslant 2\lambda c_3 < \frac{2}{7}$$

because one of the numbers $c_3^*, c_4^*, \ldots, c_{2m-2}^*$ would be equal to c_3. Hence $c_1^* = c_1$ and $c_2^* = c_2$ and therefore

$$\|\mathbf{c}_2-\mathbf{e}\|_2 = \|\mathbf{c}^*-\mathbf{e}\|_2 < \frac{1}{7}.$$

Combining this inequality with the assumption $\|\mathbf{c}-\mathbf{e}\|_2 \geqslant 1$ we get

$$1 - \frac{1}{49} < \|\mathbf{c}-\mathbf{e}\|_2^2 - \|\mathbf{c}_2-\mathbf{e}\|_2^2 = 2c_{2m-3}c_{2m-2} - 2c_{2m-1}c_{2m}$$

$$\leqslant 2c_{2m-3}c_{2m-2} \leqslant 2c_{2m-4}^2 < \frac{2}{49}$$

because $c_{2m-4} = c_i^*$ for some $i \geqslant 3$, a contradiction.

Similarly we show that each of the assumptions $f(\mathbf{c}_j) < K_{2m-2}$ $(j = 1, 3, 4)$ leads to a contradiction; this completes the proof of implication I.

The proof of implication II is exactly the same as the proof of I because $T(2m-1) \subset T(2m)$; the only difference is that the application of Lemma 2 is based upon $(\mathrm{iii})_{2m-2}$ instead of $(\mathrm{iii})_{2m-1}$. This completes the proof of (1).

To prove the second part of Theorem 1 note that from the validity of $(\mathrm{iv})_n$ for all n one obtains by a standard limit procedure

(iv) if $\mathbf{c} \in T$ and $\|\mathbf{c}-\mathbf{e}\|_2 \geqslant 1$, then $f(\mathbf{c}) \geqslant K$.

Using again a limit procedure and applying inequality (1) we conclude that (5) is valid for every $\mathbf{h} \in l^2$ with $(\mathbf{h}, \mathbf{e}) = 0$ and $\|\mathbf{h}\|_2 = 1$. Thus for every $\mathbf{h}$ with the above properties the assertions (i) and (ii) of Lemma 2 hold. Combining (iv) with (i) we infer that if $\mathbf{c} \in T$ and $f(\mathbf{c}) = 1/\sqrt{2}$ then $\mathbf{c} = \mathbf{e}$. This clearly implies that if $f(\mathbf{c}) = 1/\sqrt{2}$ for some $\mathbf{c} \in l^2 \setminus \{0\}$ then $\mathbf{c}$ is of the form described in the second part of Theorem 1.

3. Remarks.

Remark 1. Theorem 1 admits the following generalization:

THEOREM 1a. *There exists a* $p_0 > 1$ *such that*

$$A_p = 2^{1/2-1/p} \quad \textit{for} \quad 1 \leqslant p \leqslant p_0,$$

i.e. for every real sequence (c_j)

$$\Big(\int_0^1 \Big|\sum c_j r_j(t)\Big|^p dt\Big)^{1/p} \geqslant 2^{1/2-1/p}\Big(\sum c_j^2\Big)^{1/2}.$$

Proof. We shall show that the assertion of Theorem 1a holds for p satisfying the conditions

(j) $2^{1/2-1/p} \leqslant K$,

(jj) $p \leqslant 7^{2-p}$.

Similarly as in the proof of Theorem 1 it is enough to consider $\mathbf{c} \in T(n)$ for $n = 1, 2, \ldots$ Let us set for $1 \leqslant p < \infty$

$$f_p(\mathbf{c}) = \frac{\Big(\int_0^1 \big|\sum c_j r_j(t)\big|^p dt\Big)^{1/p}}{\big(\sum c_j^2\big)^{1/2}} \qquad (\mathbf{c} \in l^2 \setminus \{0\}).$$

Observe first that for every $p \geqslant 1$ satisfying (j) we have

(jjj) if $\mathbf{c} \in T(n)$ for some n and $\|\mathbf{c}-\mathbf{e}\|_2 \geqslant 1/7$, then $f_p(\mathbf{c}) \geqslant 2^{1/2-1/p}$.

This follows from (iv), the implications (i) and (ii) of Lemma 2 and from the fact that, for every fixed $\mathbf{c} \in l^2 \setminus \{0\}$, $f_p(\mathbf{c})$ is a non-decreasing function of p.

Next observe that for $1 \leqslant p \leqslant 2$ the following analogue of Lemma 2 holds:

If $f_p(\mathbf{h}') \geqslant 2^{1/2-1/p}$ for all $\mathbf{h}' \in l^2_{n-1} \setminus \{0\}$, then for every $\mathbf{h} \in l^2_n$ with $(\mathbf{h}, \mathbf{e}) = 0$ and $\|\mathbf{h}\|_2 = 1$ we have

$$f_p(\mathbf{e}+t\mathbf{h}) \geqslant \left(\frac{2^{p/2-1} + 2^{p/2-1}\dfrac{|t|^p}{2}}{(1+t^2)^{p/2}}\right)^{1/p} \quad \text{for every real } t.$$

The proof of this fact is similar to the proof of (5) in Lemma 2. Hence

$$f_p(\mathbf{e}+t\mathbf{h}) \geqslant 2^{1/2-1/p} \quad \text{for} \quad |t| \leqslant p^{1/(p-2)}.$$

Thus, by (jjj), if p satisfies (jj) then the assumption $f_p(\mathbf{h}') \geqslant 2^{1/2-1/p}$ for every $\mathbf{h}' \in l^2_{n-1} \setminus \{0\}$ implies that $f_p(\mathbf{c}) \geqslant 2^{1/2-1/p}$ for every $\mathbf{c} \in T(n)$ and therefore also for every $\mathbf{c} \in l^2_n \setminus \{0\}$. Now the desired inequality follows by induction. Obviously, $A_p \leqslant 2^{1/2-1/p} = f_p(\mathbf{e})$.

Remark 2. For $p < 2$ but sufficiently close to 2, $A_p < 2^{1/2-1/p}$. This follows from a result of Stečkin [6] who has shown that

$$A_p \leqslant \sqrt{2}\left(\frac{\Gamma\left(\dfrac{p+1}{2}\right)}{\sqrt{\pi}}\right)^{1/p} \quad \text{for every } 1 \leqslant p \leqslant 2.$$

Hence $A_p < 2^{1/2-1/p}$ whenever $\Gamma\left(\dfrac{p+1}{2}\right) < \dfrac{\sqrt{\pi}}{2}$ which holds in some interval $2-\delta < p < 2$.

Remark 3. Let $g_1, g_2, \ldots, g_n$ be arbitrary real valued functions in $L^1 = L^1([0,1])$. We shall repeat an argument of Orlicz [4]. Using the Fubini theorem, (1), and the Schwartz inequality we get

$$\begin{aligned}\int_0^1 \Big\|\sum_{j=1}^n r_j(t) g_j\Big\|_{L^1} dt &= \int_0^1\int_0^1 \Big|\sum_{j=1}^n r_j(t) g_j(s)\Big|\, dt\, ds \\ &\geqslant \frac{1}{\sqrt{2}}\int_0^1 \Big(\sum_{j=1}^n g_j(s)^2\Big)^{1/2} ds \geqslant \frac{1}{\sqrt{2}}\int_0^1 \sum_{j=1}^n a_j |g_j(s)|\, ds \\ &= \frac{1}{\sqrt{2}}\Big(\sum_{j=1}^n \Big(\int_0^1 |g_j(s)|\, ds\Big)^2\Big)^{1/2} = \frac{1}{\sqrt{2}}\Big(\sum_{j=1}^n \|g_j\|_{L^1}^2\Big)^{1/2}\end{aligned}$$

where the reals $a_1, a_2, \ldots, a_n$ are chosen so that

$$\sum_{j=1}^n a_j \|g_j\|_{L^1} = \Big(\sum_{j=1}^n \|g_j\|_{L^1}^2\Big)^{1/2} \quad \text{with} \quad \sum_{j=1}^n a_j^2 = 1.$$

Thus we get

THEOREM 1b. *If E is a real Banach space which is isometrically isomorphic to a subspace of L^1, then*

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t)\mathbf{x}_j \Big\|_E dt \geqslant \frac{1}{\sqrt{2}} \Big(\sum_{j=1}^n \|\mathbf{x}_j\|_E^2 \Big)^{1/2} \tag{1b}$$

for arbitrary $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in E $(n = 1, 2, 3, \ldots)$.

Observe that every Euclidean space and, by a result of Lindenstrauss [3], Corollary 2, every two-dimensional Banach space is isometric to a subspace of L^1. Therefore for this spaces we have inequality (1b). In particular, we have (1) for complex sequences (c_j), because the complex plane can be regarded as the two-dimensional Euclidean vector space. An inspection of Orlicz's argument yields that the second part of Theorem 1 is also true for complex valued sequences.

4. Proof of Lemma 3. Observe first that for $m \geqslant 4$ we have

$$0 \leqslant x \leqslant y \leqslant z \leqslant v, \tag{6}$$

$$\max(|x+v|, |x-v|, |-x+z-y|, |-x-z+y|) < 1, \tag{7}$$

$$x^2+(z-y)^2 \leqslant x^2+v^2 \leqslant 2m^{-2}, \tag{8}$$

$$2x^2+(z-y)^2+v^2 \leqslant x^2+z^2+v^2 \leqslant 3m^{-2}. \tag{9}$$

Inequalites (6)–(9) either follow immediately from the definition of x, y, z, v or are obtained by the standard argument involving Lagrange multipliers. Next we show that for every $\mathbf{c} \in T(2m)$

$$q_m(\mathbf{c}) \geqslant 1 - \frac{3}{16} m^{-2} - \frac{85}{168} m^{-4}. \tag{10}$$

To this end we expand q_m into the power series with respect to x, y, z, v. For $|t| < 1$ we have

$$\sqrt{1+t} = 1 + \frac{1}{2}t - \frac{t^2}{8} + \frac{t^3}{16} - \frac{5}{128}t^4 + \sum_{k=5}^{\infty} (-1)^{k-1} a_k t^k \tag{11}$$

for some a_k with $0 < a_k < 5/128$ $(k \geqslant 5)$. Replacing in (11) t by $x+v$, $x-v$, $-x+z-y$, $-x-z+y$, respectively (it is admissible by (7)) and adding all four expansions together and dividing by 4 we obtain

$$q_m(\mathbf{c}) = \sum_{k=0}^{\infty} B_k^{(m)}$$

where

$$B_0^{(m)} = 1, \quad B_1^{(m)} = 0,$$

$$B_{2n}^{(m)} = -\frac{1}{2} a_{2n} \sum_{j=0}^{n} \binom{2n}{2j} x^{2j} \big(v^{2(n-j)} + (z-y)^{2(n-j)}\big) \quad \text{for} \quad n = 1, 2, \ldots,$$

$$B_{2n+1}^{(m)} = \frac{1}{2} a_{2n+1} \sum_{j=1}^{n} \binom{n}{j} x^{2n-2j+1} \big(v^{2j} - (z-y)^{2j}\big) \quad \text{for} \quad n = 1, 2, \ldots$$

Clearly, $B_{2n}^{(m)} \leqslant 0$ and, by (6), $B_{2n+1}^{(m)} \geqslant 0$ for $n = 1, 2, \ldots$ Hence to prove (10) it is enough to show

$$-B_2^{(m)} \leqslant \frac{3}{16} m^{-2}, \tag{12}$$

$$-\sum_{n=2}^{\infty} B_{2n}^{(m)} \leqslant \frac{85}{168} m^{-4}. \tag{13}$$

Clearly, (12) follows from (9). To prove (13) observe first that, for $n \geqslant 2$,

$$\begin{aligned}
-B_{2n}^{(m)} &\leqslant 5\cdot 2^{-8} \sum_{j=0}^{n} \binom{2n}{2j} x^{2j} \big(v^{2(n-j)} + (z-y)^{2(n-j)}\big) \\
&\leqslant 5\cdot 2^{-8}\cdot 2^n \sum_{j=0}^{n} \binom{n}{j} (x^2)^j \big[(v^2)^{n-j} + ((z-y)^2)^{n-j}\big] \\
&= 5\cdot 2^{-8}\cdot 2^n \big[(x^2+v^2)^n + \big(x^2+(z-y)^2\big)^n\big].
\end{aligned}$$

Hence, by (8) and (9),

$$-B_{2n}^{(m)} \leqslant 5\cdot 2^{-8} [(4m^{-2})^n + (2m^{-2})^n].$$

Thus, for, $m \geqslant 4$,

$$-\sum_{n=2}^{\infty} B_{2n}^{(m)} \leqslant 5\cdot 2^{-8} \Big(\frac{16}{m^2(m^2-4)} + \frac{4}{m^2(m^2-2)} \Big) < \frac{85}{168} m^{-4}.$$

This completes the proof of (10).

Finally, we shall show

$$\sum_{m=4}^{\infty} \Big(\frac{3}{16} m^{-2} + \frac{85}{168} m^{-4} \Big) < 1 - \frac{2\sqrt{2}}{3} \tag{14}$$

which obviously implies

$$\prod_{m=4}^{\infty} \Big(1 - \frac{3}{16} m^{-2} - \frac{85}{168} m^{-4} \Big) > \frac{4}{3} \frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{3}. \tag{15}$$

We have

$$\frac{3}{16}\sum_{m=4}^{\infty} m^{-2} = \frac{3}{16}\left(\frac{\pi^2}{6}-1-\frac{1}{4}-\frac{1}{9}\right) < 0.0532,$$

$$\frac{85}{168}\sum_{m=4}^{\infty} m^{-4} = \frac{85}{168}\left(\frac{\pi^4}{90}-1-\frac{1}{16}-\frac{1}{81}\right) < 0.0038,$$

$$\frac{2\sqrt{2}}{3} < 0.9429.$$

Clearly, the last three estimations imply (14).

The assertion of Lemma 3 is an obvious consequence of (10) and (15).

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