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ON THE BETTI NUMBERS OF THE REAL PART OF A THREE-DIMENSIONAL TORUS EMBEDDING

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Let X be the three-dimensional, complete, nonsingular, complex torus embedding corresponding to a fan $S \subseteq \mathbb{R}^3$ and let V be the real part of X (for definitions see [1] or [3]). The aim of this note is to give a simple combinatorial formula for calculating the Betti numbers of V.

1. Let us recall some basic definitions concerning torus embeddings (for details see [1]–[3]). For a fixed lattice M of rank n and for the lattice N dual to M let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. There are pairings

$$\langle\, ,\, \rangle : N \times M \to \mathbb{Z} \quad \text{and} \quad \langle\, ,\, \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{Z}\,.$$

1.1. Definition. A *convex rational polyhedral cone* in $M_{\mathbb{R}}$ ($N_{\mathbb{R}}$) is a set

$$\sigma = \Big\{x \in M_{\mathbb{R}} : x = \sum_{i=1}^{k} r_i \alpha_i\Big\} \quad \text{where} \quad r_i \in \mathbb{R},\ r_i \geq 0\,,$$

and $\alpha_1, \ldots, \alpha_k$ are some primitive vectors in M (N). The dimension of σ is by definition the dimension of the linear space spanned by $\alpha_1, \ldots, \alpha_k$ in $M_{\mathbb{R}}$ ($N_{\mathbb{R}}$).

In this article we consider only convex, rational, polyhedral cones in $N_{\mathbb{R}}$ which do not include any line. If $\sigma \subset N_{\mathbb{R}}$ is a cone then the set $\widehat{\sigma} = \{y \in M_{\mathbb{R}} : \langle x, y\rangle \geq 0\}$ is also a cone. We call it the *dual cone*. The *face* of the cone σ is the set

$$\tau = \{x \in \sigma : \langle x, m\rangle \geq 0 \text{ for some } m \in \widehat{\sigma}\}\,.$$

1.2. Definition. A *fan* S in $N_{\mathbb{R}}$ is a set of cones in $N_{\mathbb{R}}$ which satisfies the following conditions:

(a) If $\sigma \in S$ and τ is the face of σ then $\tau \in S$.

(b) If $\sigma_1, \sigma_2 \in S$ then $\sigma_1 \cap \sigma_2 \in S$.

If the union of all cones $\sigma \in S$ is the whole space $N_{\mathbb{R}}$ the fan S is said to be *complete*. If every cone $\sigma \subset N_{\mathbb{R}}$ is spanned by a subset of a base of N the fan S is said to be *nonsingular* (see 1.1).

1.3. Let k be an algebraically closed field, S a fan in $N_{\mathbb{R}}$ and $\sigma \in S$ a cone. Since M is a group and $\widehat{\sigma} \subseteq M_{\mathbb{R}}$ is a semigroup we have an embedding $k[\widehat{\sigma} \cap M] \to k[M]$ of a semigroup algebra into a group algebra and this embedding gives us a morphism of affine varieties

$$\operatorname{Spec} k[M] \to \operatorname{Spec} k[\widehat{\sigma} \cap M] .$$

It follows from 1.2 (see [2] or [3]) that one can glue the varieties $\operatorname{Spec} k[\widehat{\sigma} \cap M]$, $\sigma \in S$, to obtain a new algebraic variety X_S containing $(k^*)^n$ as a dense subset. Moreover, X_S is nonsingular and complete if and only if S is nonsingular and complete.

1.4. Definition. Let $k = \mathbb{C}$, and let S be a nonsingular complete fan.

(a) The *real part* V of X_S is the closure of $(\mathbb{R}^*)^n \subset (\mathbb{C}^*)^n$ in the variety X_S.

(b) The *real nonnegative part* V_+ of X_S is the closure of $(\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n$ in X_S.

It is known that V is a real nonsingular compact manifold and $V_+ \subset V$ is a real variety with corners (see [2], [3]).

1.5. Theorem ([1], 4.4.3). *Let $\alpha_1, \ldots, \alpha_k$ be the primitive vectors spanning* 1*-cones of a fan S. The variety V is nonorientable iff there exists a subset $\{\alpha_{i1}, \ldots, \alpha_{is}\}$ of $\{\alpha_1, \ldots, \alpha_k\}$ such that s is odd and $\alpha_{i1} + \ldots + \alpha_{is} \equiv 0 \bmod 2$.*

2. Consider the case of a nonsingular complete fan S of dimension 3. For primitive vectors $\alpha_1, \ldots, \alpha_n$ spanning one-dimensional cones $\sigma_1, \ldots, \sigma_n$ which belong to S we put

$$I = \{(i,j) : \alpha_i \text{ and } \alpha_j \text{ span a 2-cone } \sigma_{ij} \text{ in } S\} ,$$
$$J = \{(i,j,k) : \alpha_i, \alpha_j \text{ and } \alpha_k \text{ span a 3-cone } \sigma_{ijk} \text{ in } S\} .$$

For a given $v = (v_1, v_2, v_3) \in \mathbb{Z}^3$ we define

$$(-1)^v = ((-1)^{v_1}, (-1)^{v_2}, (-1)^{v_3}) \in \mathbb{Z}^3 \subseteq (\mathbb{R}^*)^3 .$$

2.1. Let D be a sphere in $M_{\mathbb{R}}$ with center at zero. The intersection of D with the fan S defines some triangulation of D and some graph G on D called the *triangulation graph*. The vertices v_i of G correspond to the 1-cones σ_i spanned by α_i in $M_{\mathbb{R}}$, the edges of G correspond to the 2-cones σ_{ij} spanned by α_i and α_j in $M_{\mathbb{R}}$. Two vertices v_i and v_j are connected by an edge in G if and only if the cone σ_{ij} spanned by α_i and α_j is in S. We define a new graph H on the sphere in the following way:

(a) With every vertex v_i of G we associate three vertices of H: v_{i1}, v_{i2}, v_{i3}. Each of them corresponds to a four-element subgroup of $\mathbb{Z}_2^3$ containing $(-1)^{\alpha_i}$. Note that there are exactly three four-element subgroups of $\mathbb{Z}_2^3$

containing a given nonzero element of this group; $(-1)^{\alpha_i}$ is clearly nonzero since α_i is primitive.

(b) For $i \neq j$, v_{ik} and v_{jl} are connected by an edge in H if and only if v_i and v_j are connected in G and the subgroups of $\mathbb{Z}_2^3$ corresponding to v_{ik} and v_{jl} are the same.

Let $d = \pi_0(H)$ denote the number of connected components of the graph H and let $b_i = \dim H_i(V, \mathbb{Q})$.

2.2. Theorem. *Let V be the real part of a three-dimensional, complete, nonsingular, complex torus embedding.*

(a) *If V is an orientable manifold then $b_0 = b_3 = 1$, $b_1 = b_2 = d - 6$.*

(b) *If V is a nonorientable manifold then $b_0 = 1$, $b_1 = d - 6$, $b_2 = d - 7$, $b_3 = 0$.*

2.3. Proof. We use the cellular decomposition of V described in [1]. Let V_+ be the real nonnegative part of X. The cellular decomposition of V_+ is dual to the decomposition of the fan S into open faces. Let D be the 3-cell of V_+ corresponding to the 0-cone in S and let $S_1, \ldots, S_n$ be the 2-cells of V_+ corresponding to the 1-cones $\sigma_1, \ldots, \sigma_n$ spanned by $\alpha_1, \ldots, \alpha_n$, respectively; moreover, K_{ij}, $(i,j) \in I$, denotes the 1-cell of V_+ corresponding to the 2-cone σ_{ij} spanned by α_i, α_j, and W_{ijk}, $(i,j,k) \in J$, denotes the 0-cell of V_+ corresponding to the 3-cone σ_{ijk} spanned by α_i, α_j, α_k. We know that $\mathbb{Z}_2^3 \subseteq (\mathbb{R}^*)^3$ acts on V. Then V_+ is a fundamental domain for this action and the orbits of k-cells of V_+ give a cellular decomposition of V (see [1]).

Moreover, it follows from [1], 2.4, that the isotropy groups of S_i, K_{ij}, W_{ijk} are $\langle(-1)^{\alpha_i}\rangle$, $\langle(-1)^{\alpha_i}, (-1)^{\alpha_j}\rangle$ and $\langle(-1)^{\alpha_i}, (-1)^{\alpha_j}, (-1)^{\alpha_k}\rangle$, respectively ($\langle g\rangle$ denotes the subgroup generated by g). We define an orientation in cells of V as in [1], 4.4. It follows that $\partial D = S_1 + \ldots + S_n$ and $\partial S_i = \sum \operatorname{sgn}(i-j) K_{ij}$ where we sum over j such that $(i,j) \in I$. The action of $\mathbb{Z}_2^3$ commutes with the boundary operator, that is, $\partial(\alpha(K)) = \alpha(\partial(K))$ for every $\alpha \in \mathbb{Z}_2^3$ and any cell K of V. For any $\alpha \in \mathbb{Z}_2^3$ we set

$$L^\alpha = D - \alpha(D), \qquad L_i^\alpha = S_i - \alpha(S_i), \qquad L_{ij}^\alpha = K_{ij} - \alpha(K_{ij})\,.$$

We have the following chain complex for V:

$$(1) \qquad 0 \longrightarrow \langle\langle D\rangle\rangle \oplus \bigoplus \langle\langle L^\alpha\rangle\rangle \xrightarrow{\partial_3} \bigoplus \langle\langle S_i\rangle\rangle \oplus \bigoplus \langle\langle L_i^\alpha\rangle\rangle \xrightarrow{\partial_2} \bigoplus \langle\langle K_{ij}\rangle\rangle \oplus \bigoplus \langle\langle L_{ij}^\alpha\rangle\rangle \xrightarrow{\partial_1} \bigoplus \langle\langle W_{ijk}\rangle\rangle \longrightarrow 0$$

where the sums are taken over $\alpha \in \mathbb{Z}_2^3$, $i = 1, \ldots, n$, with $(i,j) \in I$ and $(i,j,k) \in J$. The above complex is a direct sum of the chain complex for V_+ and the complex

$$(2) \qquad 0 \longrightarrow \langle\langle L^\alpha\rangle\rangle \xrightarrow{\partial_3} \langle\langle L_i^\alpha\rangle\rangle \xrightarrow{\partial_2} \langle\langle L_{ij}^\alpha\rangle\rangle \xrightarrow{\partial_1} 0\,.$$

Since V_+ is contractible we can calculate $H_i(V,\mathbb{Q})$ for $i>0$ from the complex (2). It follows from (1) that the Euler characteristic of V is zero. Moreover, $\dim\langle\langle L^\alpha\rangle\rangle = 7$, and $\dim\ker\partial_3$ is 0 if V is orientable, and 1 if V is nonorientable. Therefore to prove the theorem it suffices to show that $\dim\ker\partial_2 = d = \pi_0(H)$. It is easy to see that

$$\begin{aligned} L_i^\alpha = L_i^\beta &\quad \text{if and only if} \quad (-1)^{\alpha-\beta} \in \langle(-1)^{\alpha_i}\rangle\,,\\ L_{ij}^\alpha = L_{ij}^\beta &\quad \text{if and only if} \quad (-1)^{\alpha-\beta} \in \langle(-1)^{\alpha_i},(-1)^{\alpha_j}\rangle\,,\\ L_{ij}^\alpha = 0 &\quad \text{if and only if} \quad (-1)^{\alpha} \in \langle(-1)^{\alpha_i},(-1)^{\alpha_j}\rangle \end{aligned}$$

and

$$\partial L_i^\alpha = \sum \operatorname{sgn}(i-j) L_{ij}^\alpha$$

where we sum over j such that $(i,j)\in I$. It follows that for a given i we have three different nonzero chains $L_i^{\beta_{i1}}$, $L_i^{\beta_{i2}}$, $L_i^{\beta_{i3}}$ while for a given $(i,j)\in I$ we have only one chain $L_{ij}^\alpha \neq 0$ (in this case we will write L_{ij} instead of L_{ij}^α). Set

$$z = \sum_{i=1}^n (a_{i1}L_i^{\beta_{i1}} + a_{i2}L_i^{\beta_{i2}} + a_{i3}L_i^{\beta_{i3}})\,, \qquad \partial z = \sum_{(i,j)\in I} b_{ij}L_{ij}\,.$$

We calculate that

$$b_{ij} = \operatorname{sgn}(j-i)(a_{ik}+a_{il}) + \operatorname{sgn}(i-j)(a_{jp}+a_{jr})$$

where

$$(-1)^{\beta_{ik}}, (-1)^{\beta_{il}} \notin \langle(-1)^{\alpha_i},(-1)^{\alpha_j}\rangle$$

and

$$(-1)^{\beta_{jp}}, (-1)^{\beta_{jr}} \notin \langle(-1)^{\alpha_i},(-1)^{\alpha_j}\rangle\,.$$

Clearly $\partial z = 0$ if and only if $b_{ij}=0$ for all $(i,j)\in I$. Set

$$p_{im} = a_{ik}+a_{il}, \qquad p_{js} = a_{jp}+a_{jr} \qquad \text{for } \{k,l,m\} = \{p,r,s\} = \{1,2,3\}\,.$$

We obtain a system of linear equations

$$\forall (i,j)\in I \qquad p_{im} = p_{js} \quad \text{iff} \quad \langle(-1)^{\alpha_i},(-1)^{\beta_{im}}\rangle = \langle(-1)^{\alpha_j},(-1)^{\beta_{js}}\rangle\,.$$

There is a one-to-one correspondence between the set $\{p_{im} : i=1,\ldots,n, m=1,2,3\}$ and the set of vertices of the graph H. Namely, p_{im} corresponds to v_{is} ($s=s(m)$) if and only if the group $\langle(-1)^{\alpha_i},(-1)^{\beta_{im}}\rangle$ is associated with the vertex v_{is}. This correspondence has the following property: the equation $p_{ik}=p_{jl}$ appears in the system (3) if and only if the vertices $v_{is(k)}$ and $v_{js(l)}$ corresponding to p_{ik} and p_{jl} are connected by an edge in H. Thus we have a bijection between some basis of solutions of (3) and the set of connected components of H. Therefore $\dim\ker\partial_2 = \pi_0(H)$, which concludes the proof.

2.4. Remark. In the case $\dim S = 2$ our method is in fact the same as that used in [1], Theorem 4.5.1.

3. The fundamental group of V. In this section we use additive notation for the group $\mathbb{Z}_2^3$ and identify the vectors α_i with their images $(-1)^{\alpha_i}$ in $\mathbb{Z}_2^3$.

3.1. Let P be a graph with eight vertices $v_e, v_{g1}, v_{g2}, \ldots, v_{g7}$ labeled by eight elements of $\mathbb{Z}_2^3$. For a pair $(i, \overline{\alpha})$, $i \in \{1, \ldots, n\}$, $\alpha \in \mathbb{Z}_2^3$, $\overline{\alpha} \in \mathbb{Z}_2^3/\langle\alpha_i\rangle$, the edge $e_i^{\overline{\alpha}}$ links v_α with $v_{\alpha_i+\alpha}$. The group $\mathbb{Z}_2^3$ acts on the set of vertices and on the set of edges of P:

$$\alpha(v_\beta) = v_{\alpha+\beta}\,, \qquad \alpha(e_i^{\overline{\beta}}) = e_i^{\overline{\alpha+\beta}}\,.$$

For $(i,j) \in I$ let R_{ij} be the graph which is the orbit of a pair of edges $e_i^{\alpha_i}$ and $e_i^{\alpha_j}$, and let $\Phi_{ij} : R_{ij} \to P$ be the inclusion.

3.2. Proposition. (a) *The fundamental group of V is isomorphic to the fundamental group of the graph P modulo the relations given by the images $\Phi(R_{ij})$.*

(b) $\pi_1(V)$ *is generated by $4n$ elements $g_1, g_2, \ldots, g_{4n}$ and there are two types of relations between g_j in $\pi_1(V)$:*

- $r_i = g_i$ *for* $i = 1, \ldots, 7$,
- $s_i = g_j^{\varepsilon j} g_k^{\varepsilon k} g_l^{\varepsilon l} g_m^{\varepsilon m}$ *for* $i = 1, \ldots, 2 \cdot \#I$,

where j, k, l, m depend on i and εj, εk, εl, εm are ± 1.

3.3. Proof. Let T be a tubular neighbourhood of the 1-skeleton of V. The decomposition $\overline{T} \cup \overline{V - T}$ is the Heegard splitting of V. Using this fact we can calculate $\pi_1(V)$ (see [4]). First we observe that the graph P is homotopy equivalent to $\overline{V - T}$ (vertices of P correspond to 3-cells of V and edges of P correspond to 2-cells of V, see [1], proof of 4.3.1). It is not difficult to see that the graphs R_{ij} are "meridians" in $\overline{V - T}$ which can be contracted in $\overline{T}$. This proves (a).

The graph P has $4n$ edges. A maximal tree in P has seven edges. Contraction of these elements gives relations in $\pi_1(P)$ and consequently in $\pi_1(V)$. So we have seven relations of type r_i.

For $(i,j) \in I$ the graph R_{ij} is the orbit of the pair of edges $e_i^{\alpha_i}$ and $e_i^{\alpha_j}$ and consists of eight edges. These edges form two loops and each loop is glued from four edges. In this way we obtain relations of type s_i. By properly labeling the edges of P we obtain a presentation of $\pi_1(V)$ in the form described in (b).

3.4. Remark. Let $\dim X_S = 2$. The fundamental group of V is generated by the one-dimensional orbits of $(\mathbb{R}^*)^3$, call them $E_1, \ldots, E_n$, modulo the relations

$$\prod_{i \in I_1} E_i\,, \quad \prod_{i \in I_2} E_i\,, \quad \prod_{i \in I_3} E_i$$

where $I_1 = \{i : \alpha_i \neq (1,0)\}$, $I_2 = \{i : \alpha_i \neq (0,1)\}$, $I_3 = \{i : \alpha_i \neq (1,1)\}$ and in each product the index set is a monotonic sequence.

3.5. Remark. In the case $\dim V = 3$ let $V_1, \ldots, V_n$ be the two-dimensional orbits of the action of $(\mathbb{R}^*)^3$. Each V_i is the real part of a 2-dimensional torus embedding and the fan S_i corresponding to V_i can be easily obtained from S. Using 3.4 we can describe $\pi_1(V_i)$ as the group generated by the 1-dimensional orbits E_{ij} of the action of $(\mathbb{R}^*)^3$ on V. (For $(i,j) \in I$, E_{ij} is a one-dimensional orbit of the action of some $(\mathbb{R}^*)^2$ on V_i). It is not difficult to see that the fundamental group of V is the free product of $\pi_1(V_1), \ldots, \pi_1(V_n)$ modulo the relations $E_{ij} = E_{ji}^{-1}$.

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