119. *On the Bi-ideals in Associative Rings*

By S. LAJOS*) and Ferenc A. SZÁSZ**)

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By a ring we shall mean an arbitrary associative ring. For the terminology not defined here we refer to N. Jacobson [3] and N. H. McCoy [8]. We announce some properties of bi-ideals in rings which are analogous to some properties of bi-ideals in semigroups.

For the subsets X and Y of a ring A by the product XY we mean the subring of A which is generated by the set of all products xy, where $x \in X$, $y \in Y$. By a bi-ideal B of A we mean a subring B of A satisfying the condition

(1) $$BAB \subseteq B.$$

Obviously every one-sided ideal of A is a bi-ideal and the intersection of a left and a right ideal of A is also a bi-ideal. It may be remarked that the notion of the bi-ideal in semigroups is a special case of the (m, n)-ideal introduced by S. Lajos [4]. The notion of bi-ideal for associative rings was earlier mentioned by S. Lajos [5]. He noted that the set of all bi-ideals of a regular ring is a multiplicative semigroup. The concept of the bi-ideal was introduced by R. A. Good and D. R. Hughes [1]. An interesting particular case of the bi-ideal is the notion of quasi-ideal due to O. Steinfeld [9] which is defined as follows. A submodule Q of a ring A is called a quasi-ideal of A if the following condition holds:

(2) $$AQ \cap QA \subseteq Q.$$

It is known that the product of any two quasi-ideals is a bi-ideal (see S. Lajos [5]). It may be remarked that in case of regular rings the notions of bi-ideal and quasi-ideal coincide.

In the following we formulate some general properties of bi-ideals in rings and characterize two important classes of rings in terms of bi-ideals.

Proposition 1. *The intersection of an arbitrary set of bi-ideals B_i $(i \in I)$ of a ring A is again a bi-ideal of A.*

Proposition 2. *The intersection of a bi-ideal B of a ring A and a subring S of A is a bi-ideal of the ring S.*

Proposition 3. *For an arbitrary subset T of a ring A and for a bi-ideal B of A the products BT and TB are bi-ideals of A.*

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*) K. Marx University of Economics, Budapest, Hungary.
**) Math. Institute of the Hungarian Academy of Sciences, Budapest, Hungary.

In analogy with the case of semigroups (cf. S. Lajos [7]) we obtain the following result.

Proposition 4. *Let B be an arbitrary bi-ideal of the ring A, and C be a bi-ideal of the ring B such that $C^2=C$. Then C is a bi-ideal of A.*

Proposition 5. *An arbitrary associative ring A contains no non-trivial bi-ideal if and only if A is either a zero ring of prime order or else a division ring.*

Proposition 6. *Let T be a non-empty subset of the ring A. Then the bi-ideal $T_{(1,1)}$ of A generated by T is of the form*

$$T_{(1,1)}=IT+IT^2+TAT, \tag{3}$$

where I denotes the ring of rational integers.

Proposition 7. *For any associative ring A denote $\bar{A}$ the set of all subrings of A, and A_1 the set of all bi-ideals of A. Then $\bar{A}$ and A_1 are semigroups under the multiplication of subsets (defined in the introduction) and A_1 is a two-sided ideal of $\bar{A}$.*

Remark. The multiplicative semigroup of all non-empty subsets of an arbitrary semigroup was formerly investigated by S. Lajos [7]. He proved, among others, that the set of all the bi-ideals of a semigroup S is a two-sided ideal of the multiplicative semigroup of all non-empty subsets of S.

The following result is in complete analogy with a semigroup-theoretical theorem of S. Lajos [4].

Theorem 1. *For an arbitrary non-empty subset B of an associative ring A the following conditions are pairwise equivalent:*

(i) *B is a bi-ideal of A.*

(ii) *B is a left ideal of a right ideal of A.*

(iii) *B is a right ideal of a left ideal of A.*

Theorem 2. *For an associative ring A the following conditions are equivalent:*

(I) *A is regular.*

(II) *$L\cap R=RL$ for every left ideal L and every right ideal R of A.*

(III) *For any elements a, b of A.*

$$(a)_r\cap(b)_l=(a)_r(b)_l.$$

(IV) *For any element a of A*

$$(a)_r\cap(a)_l=(a)_r(a)_l.$$

(V) *$(a)_{(1,1)}=(a)_r(a)_l$ for any element a of A.*

(VI) *$(a)_{(1,1)}=aAa$ for any element a of A.*

(VII) *$QAQ=Q$ for every quasi-ideal Q of A.*

(VIII) *$BAB=B$ for each bi-ideal B of A.*

The equivalence of the above conditions (I)–(VI) in case of semigroups was proved by S. Lajos [6] and K. Iséki [2].

Theorem 3. *The following conditions for an associative ring A are equivalent*:

(I) *A is a regular two-sided ring.*[1)]

(II) *A is a subcommutative regular ring.*[2)]

(III) *A is strongly regular.*

(IV) *$B^2=B$ for any bi-ideal B of A.*

(V) *$Q^2=Q$ for any quasi-ideal Q of A.*

(VI) *$L\cap R=LR$ for every left ideal L and every right ideal R of A.*

(VII) *$L\cap L'=LL'$ for any two left ideals L, L' of A.*

(VIII) *A is regular and it is a subdirect sum of division rings.*

(IX) *A is a regular ring with no non-zero nilpotent elements.*

(X) *The multiplicative semigroup of A is a semilattice of groups.*

References

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1) An associative ring A is said to be two-sided (or duo) if every one-sided (left or right) ideal of A is two-sided.

2) An associative ring A is called subcommutative if $aA=Aa$ for any element a of A.