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ON THE BIAS OF THE LEAST SQUARES ESTIMATOR FOR THE FIRST ORDER AUTOREGRESSIVE PROCESS

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Abstract. The paper provides an exact formula for the bias of the parameter estimator of the first order autoregressive process and derives the asymptotic bias.

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1. Introduction

Let $\{X_t; t \geq 1\}$ be the first order autoregressive process

$$X_t = \alpha X_{t-1} + e_t, \quad t \geq 1, \quad X_0 = 0, \tag{1.1}$$

where α is a real parameter and $\{e_t; t \geq 1\}$ is a Gaussian white noise with mean zero and variance σ^2. The least squares estimator (LSE), which is also the maximum likelihood estimator, α_T of α, based on the observations $X_1, \ldots, X_T$ is given by

$$\alpha_T = \left(\sum_{t=2}^{T} X_{t-1}^2 \right)^{-1} \left(\sum_{t=2}^{T} X_{t-1} X_t \right), \quad T \geq 2. \tag{1.2}$$

Consistency and asymptotic distribution of α_T have been thoroughly studied. For all α, α_T is strongly consistent (Rubin (1950) and Anderson (1971)) and for a suitable normalizing sequence $\{S_\alpha(T); T > 1\}$ the random variable $S_\alpha(T)(\alpha_T - \alpha)$ has a limiting distribution $\mathscr{L}$. In the stable case, i.e., $|\alpha| < 1$, with $S_\alpha(T) = [T/(1 - \alpha^2)]^{1/2}$, $\mathscr{L}$ is standard normal (Anderson (1959, 1971)). In the unstable case, i.e., $|\alpha| = 1$, with $S_\alpha(T) = 2T\alpha$, $\mathscr{L}$ is the distribution of $(W_1^2 - 1)\Big/\left(\int_0^1 W_s^2 ds\right)$ where $\{W_s; 0 \leq s \leq 1\}$ is a standard Brownian motion (Rao (1966, 1978)). Finally, in the explosive case, i.e., $|\alpha| > 1$, with $S_\alpha(T) = |\alpha|^T(\alpha^2 - 1)^{-1}$, $\mathscr{L}$ is Cauchy (White (1958, 1959) and Rao (1961, 1966)).

The present paper considers the bias of α_T. Series expansions of this up to term T^{-3} have been provided in the stable case by several authors (Bartlett (1946), Hurwicz (1950), Kendall (1954), Mariott and Pope (1954) and White (1961)). As far as we know, similar results have not been obtained for the unstable or explosive process. Here these cases are investigated. Section 2 gives an exact formula for the bias analogous to that given by Liptser and Shiryayev ((1978), Theorem 17.3) for continuous time processes. Section 3 derives the asymptotic bias when T goes to infinity.

2. The bias of the LSE

Let $\boldsymbol{X}_T=(X_1,\ldots,X_T)'$ be an observation from the process (1.1). We shall suppose that $\sigma^2=1$ since the distribution of α_T, as is seen from (1.2), does not depend on σ^2. The distribution of $\boldsymbol{X}_T$ is thus specified by α only. For any measurable real function γ on $\mathbb{R}^T$, the expectation of $\gamma(\boldsymbol{X}_T)$, when it exists, will be denoted by $E_\alpha(\gamma)$. Then from (1.1) and (1.2) the bias, when it is well defined, is given by

$$b_T(\alpha)=E(\alpha_T)-\alpha=E_\alpha(\gamma_1^{-1}\gamma_2)\,, \tag{2.1}$$

where γ_1 and γ_2 are the functions defined for $\boldsymbol{x}=(x_1,\ldots,x_T)'\in\mathbb{R}^T$ by

$$\gamma_1(\boldsymbol{x})=\sum_{t=2}^{T}x_{t-1}^2,\qquad \gamma_2(\boldsymbol{x})=\sum_{t=2}^{T}x_{t-1}(x_t-\alpha x_{t-1})\,. \tag{2.2}$$

THEOREM 2.1. *Let $\psi_T(\alpha,\theta)$ be defined for $T\geq 1$, $\alpha\in\mathbb{R}$ and $\theta>0$ by*

$$\psi_T(\alpha,\theta)=[\det(I_T+2\theta\Gamma_T^\alpha)]^{-1/2}\,, \tag{2.3}$$

where I_T is the $T\times T$ identity matrix and Γ_T^α is the covariance matrix of $\boldsymbol{X}_T$. Then for $T\geq 4$ and all α, the bias is well defined and

$$b_T(\alpha)=\frac{\partial}{\partial\alpha}\int_0^\infty\psi_{T-1}(\alpha,\theta)\,d\theta\,. \tag{2.4}$$

PROOF. The proof parallels that of Theorem 17.3 in Liptser and Shiryayev (1978). We first show that if the function γ satisfies

$$E_\alpha(|\gamma|)<\infty,\qquad \alpha\in\mathbb{R}\,, \tag{2.5}$$

$$\sup_{\alpha_1\leq\alpha\leq\alpha_2}E_\alpha(|\gamma|\,|\gamma_2|^p)<\infty,\qquad p=1,2,\qquad -\infty<\alpha_1<\alpha_2<\infty\,, \tag{2.6}$$

then the function $\alpha\to E_\alpha(\gamma)$ is differentiable and

$$\frac{\partial}{\partial\alpha} E_\alpha(\gamma) = E_\alpha(\gamma\gamma_2) . \tag{2.7}$$

Indeed, the likelihood function $g_T(\alpha, \boldsymbol{x})$ corresponding to $\boldsymbol{X}_T$ is differentiable with respect to α, with derivative $\gamma_2(\boldsymbol{x})g_T(\alpha, \boldsymbol{x})$. Hence, for $-\infty < \alpha_1 < \alpha_2 < \infty$,

$$\begin{aligned} E_{\alpha_2}(\gamma) - E_{\alpha_1}(\gamma) &= \int_{\mathbf{R}^T} \gamma(\boldsymbol{x})[g_T(\alpha_2, \boldsymbol{x}) - g(\alpha_1, \boldsymbol{x})]dx_1 \cdots dx_T \\ &= \int_{\mathbf{R}^T} \gamma(\boldsymbol{x}) \left[\int_{\alpha_1}^{\alpha_2} \gamma_2(\boldsymbol{x}) g_T(\alpha, \boldsymbol{x}) d\alpha \right] dx_1 \cdots dx_T . \end{aligned}$$

Therefore, using (2.6) with $p = 1$, and the Fubini Theorem, one gets

$$E_{\alpha_2}(\gamma) - E_{\alpha_1}(\gamma) = \int_{\alpha_1}^{\alpha_2} E_\alpha(\gamma\gamma_2) d\alpha ,$$

and hence, the function $\alpha \to E_\alpha(\gamma)$ is absolutely continuous. But using (2.6) with $p = 2$, one can apply what just has been proved, with γ replaced by $\gamma\gamma_2$, to obtain that the function $\alpha \to E_\alpha(\gamma\gamma_2)$ is continuous, giving the desired result.

We now show that (2.5) and (2.6) hold for $\gamma = \gamma_1^{-1}$. For this, observe that

$$E_\alpha(\gamma_1^{-1}) = E_\alpha \left[\int_0^\infty \exp(-\theta\gamma_1) d\theta \right] = \int_0^\infty E\left[\exp\left(-\theta \sum_{t=2}^T X_{t-1}^2 \right) \right] d\theta ,$$

and the expectation in the last right-hand side is the moment generating function of $\sum_{t=2}^T X_{t-1}^2$. Thus

$$E_\alpha(\gamma_1^{-1}) = \int_0^\infty \psi_{T-1}(\alpha, \theta) d\theta , \tag{2.8}$$

where ψ_T is defined in (2.3). A representation of the right-hand side of (2.8) as an integral of a simple function will be derived in (2.13) below, and from this, it can be seen that

$$\sup_{\alpha_1 \le \alpha \le \alpha_2} E_\alpha(\gamma_1^{-1}) < \infty \quad \text{for all} \quad \alpha_1 < \alpha_2 . \tag{2.9}$$

Thus (2.5) holds for $\gamma = \gamma_1^{-1}$. On the other hand, from $\gamma_2(\boldsymbol{x}) = x_1x_2 + \cdots + x_{T-1}x_T - \alpha\gamma_1(\boldsymbol{x})$ and Schwarz inequality

$$|\gamma_2(\boldsymbol{x})| \le \gamma_1^{1/2}(\boldsymbol{x})[\gamma_1(\boldsymbol{x}) + x_T^2]^{1/2} + |\alpha|\gamma_1(\boldsymbol{x}) . \tag{2.10}$$

Therefore

$$E_\alpha|(\gamma_1^{-1}\gamma_2)| \le E[1 + \gamma_1^{-1}(X_T)X_T^2]^{1/2} + |\alpha|$$
$$\le \{1 + E[\gamma_1^{-1}(X_T)X_T^2]\}^{1/2} + |\alpha| \, .$$

The expectation in the last right-hand side equals

$$E[\gamma_1^{-1}(X_T)E(X_T^2|X_1,\dots,X_{T-1})] = \alpha^2 E[\gamma_1^{-1}(X_T)X_{T-1}^2] + E_\alpha(\gamma_1^{-1})$$
$$\le \alpha^2 + E_\alpha(\gamma_1^{-1}) \, .$$

Thus by (2.9), (2.6) holds for $\gamma = \gamma_1^{-1}$ with $p = 1$. Finally, from (2.10)

$$\gamma_1^{-1}\gamma_2^2(x) \le 2[(1+\alpha^2)\gamma_1(x) + x_T^2] \le 2(1+\alpha^2)\sum_{t=1}^{T} x_t^2 \, ,$$

and $E\left(\sum_{t=1}^{T} X_t^2\right)$ is a continuous function of α since $X_t = e_t + \alpha e_{t-1} + \cdots + \alpha^{t-1}e_1$. This implies that (2.6) holds for $\gamma = \gamma_1^{-1}$ with $p = 2$. The result of Theorem 2.1 now follows from (2.1), (2.7) and (2.8).

COROLLARY 2.1. *For $T \ge 4$ and all α, the bias $b_T(\alpha)$ is given by*

$$b_T(\alpha) = \alpha\left[\frac{\partial}{\partial a}\int_{\max(a,1)}^{\infty} h_T(a,\lambda)\lambda^{-(T+4)/2}d\lambda\right]_{a=\alpha^2}, \tag{2.11}$$

where

$$h_T(a,\lambda) = (\lambda^2 - a)^{3/2}[(\lambda - a) + (\lambda - 1)a^T\lambda^{-2T+1}]^{-1/2} \, . \tag{2.12}$$

PROOF. Let B be the $T \times T$ matrix with 1 on the main diagonal, $-\alpha$ on the diagonal below it and 0 elsewhere. Then from (1.1), the random vector BX_T has a unit covariance matrix. Therefore, $\Gamma_T^\alpha = B^{-1}(B^{-1})'$ and it follows that

$$\det(I_T + 2\theta\Gamma_T^\alpha) = \det\begin{pmatrix} 1+2\theta & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+2\theta+\alpha^2 & -\alpha & \cdots & 0 \\ 0 & -\alpha & 1+2\theta+\alpha^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\theta+\alpha^2 \end{pmatrix}.$$

Denote the above left-hand side by $D_T(\alpha,\theta)$ with $D_0(\alpha,\theta) = 1$ by convention. It can be seen that the following recurrence holds:

$$D_{T+1}(\alpha,\theta)=(1+2\theta+\alpha^2)D_T(\alpha,\theta)-\alpha^2 D_{T-1}(\alpha,\theta),\qquad T\geq 1\,.$$

The solution of this recurrence equation is $D_T(\alpha,\theta)=c_+\lambda_+^T+c_-\lambda_-^T$ where λ_+, λ_- are the roots of the trinomial $\lambda^2-(1+2\theta+\alpha^2)\lambda+\alpha^2$, and c_+, c_- are the solutions of $c_++c_-=1$ and $c_+\lambda_++c_-\lambda_-=1+2\theta$. Thus

$$\int_0^\infty \psi_T(\alpha,\theta)\,d\theta=\int_0^\infty (c_+\lambda_+^T+c_-\lambda_-^T)^{-1/2}d\theta\,.$$

Taking $\lambda=\lambda_+$, the greatest root, then $\lambda\geq\max(\alpha^2,1)$, $\lambda_-=\alpha^2/\lambda$, $c_+=\lambda(\lambda-\alpha^2)/(\lambda^2-\alpha^2)$, $c_-=\alpha^2(\lambda-1)/(\lambda^2-\alpha^2)$ and $\theta=(\lambda-1)(\lambda-\alpha^2)/(2\lambda)$. The change of variable from θ to λ leads to

$$\int_0^\infty \psi_T(\alpha,\theta)\,d\theta=\frac{1}{2}\int_{\max(\alpha^2,1)}^\infty h_{T+1}(\alpha^2,\lambda)\lambda^{-(T+5)/2}d\lambda\,. \tag{2.13}$$

The result then follows from Theorem 2.1.

3. The asymptotic bias of the LSE

The following result provides the first order expansion of $b_T(\alpha)$ as T goes to infinity. It should be possible to obtain higher order expansions in a similar way.

THEOREM 3.1. *As T goes to infinity,*
(i) *if $|\alpha|<1$, $Tb_T(\alpha)$ converges to* -2α,
(ii) *if $|\alpha|=1$, $Tb_T(\alpha)$ converges to* $-2^{1/2}I\alpha$ *where*

$$I=\int_0^\infty (1+e^{-4x})^{-3/2}(1+3e^{-4x}+4xe^{-4x})e^{-x}dx\,,$$

(iii) *if $|\alpha|>1$, $T^{-1/2}|\alpha|^T b_T(\alpha)$ converges to* $-2^{-1/2}\pi^{1/2}\alpha^{-1}(\alpha^2-1)^{3/2}$.

PROOF. We shall prove only (ii) and (iii) since the proof of (i) is similar (see also Pantula and Fuller (1985)). We first note that $h_T(a,\lambda)$ is differentiable with respect to a, with derivative

$$-\frac{3}{2}(\lambda^2-a)^{1/2}[(\lambda-a)+(\lambda-1)a^T\lambda^{-2T+1}]^{-1/2}$$

$$-\frac{1}{2}(\lambda^2-a)^{3/2}[(\lambda-a)+(\lambda-1)a^T\lambda^{-2T+1}]^{-3/2} \tag{3.1}$$

$$\cdot [- 1 + T(\lambda - 1)a^{T-1}\lambda^{-2T+1}] .$$

By an argument similar to that in the proof of Theorem 2.1, one may exchange the order of differentiation and integration in (2.11), yielding, for $|\alpha| \le 1$,

$$b_T(\alpha) = \alpha \int_1^\infty \left[\frac{\partial}{\partial a} h_T(a,\lambda) \right]_{a=\alpha^2} \lambda^{-(T+4)/2} d\lambda .$$

Consider the case $|\alpha| = 1$. Making the change of variable $\lambda = e^{2x/(T+2)}$ in the last integral, one gets

$$\text{(3.2)} \qquad \frac{T+2}{2\alpha} b_T(\alpha) = \int_0^\infty \left[\frac{\partial}{\partial a} h_T(a, e^{2x/(T+2)}) \right]_{a=1} e^{-x} dx .$$

For fixed x and T tending to infinity, $\lambda \to 1$, $\lambda^{-2T+1} \to e^{-4x}$ and $T(\lambda - 1) \downarrow 2x$. Hence, from (3.1)

$$\left[\frac{\partial}{\partial a} h_T(a, e^{2x/(T+2)}) \right]_{a=1} \to - 2^{-1/2}(1 + e^{-4x})^{-3/2}(1 + 3e^{-4x} + 4xe^{-4x}) .$$

Moreover, since $T(\lambda - 1)\lambda^{-2T+1} \le 1$ for $T \ge 4$, $\lambda \ge 1$, the above left-hand side may be shown to be bounded in absolute value by $Ke^{x/2}$, K being a constant. Hence, applying the Lebesgue dominated convergence theorem to the right-hand side of (3.2) yields the result (ii).

Consider now the case $|\alpha| > 1$. Making the change of variable $\lambda = a\mu$ in (2.11) and, as before, exchanging the order of differentiation and integration, one gets

$$b_T(\alpha) = \alpha \int_1^\infty \left[\frac{\partial}{\partial a} h_T(a, a\mu) a^{-(T+2)/2} \right]_{a=\alpha^2} \mu^{-(T+4)/2} d\mu .$$

Making further change of variable $\mu = e^{2x/(T+2)}$, the above right-hand side equals $b_{1T}(\alpha) + b_{2T}(\alpha)$ where

$$\text{(3.3)} \qquad b_{1T}(\alpha) = - \alpha |\alpha|^{-(T+4)} \int_0^\infty h_T(\alpha^2, \alpha^2 e^{2x/(T+2)}) e^{-x} dx ,$$

$$\text{(3.4)} \qquad b_{2T}(\alpha) = \frac{2}{T+2} \alpha |\alpha|^{-(T+2)} \int_0^\infty \left[\frac{\partial}{\partial a} h_T(a, ae^{2x/(T+2)}) \right]_{a=\alpha^2} e^{-x} dx .$$

Put $k_T(a,\mu) = a(\mu - 1) + (a\mu - 1)a^{-T+1}\mu^{-2T+1}$, then

$$h_T(a, a\mu) = (a^2\mu^2 - a)^{3/2} k_T^{-1/2}(a,\mu) ,$$

$$\frac{\partial}{\partial a} h_T(a, a\mu) = \frac{3}{2}(2a\mu^2 - 1)(a^2\mu^2 - a)^{1/2} k_T^{-1/2}(a,\mu)$$

$$- \frac{1}{2}(a^2\mu^2 - a)\{\mu - 1 + [\mu a^{-T+1} + (a\mu - 1)(1 - T)a^{-T}]$$

$$\cdot \mu^{-2T+1}\} k_T^{-3/2}(a,\mu) .$$

For fixed x and T tending to infinity, $Tk_T(a, e^{2x/(T+2)}) \to 2ax$, and hence

$$T^{-1/2} h_T(a, ae^{2x/(T+2)}) \to a(a-1)(2x)^{-1/2} , \tag{3.5}$$

$$T^{-3/2} \frac{\partial}{\partial a} h_T(a, ae^{2x/(T+2)}) \to 0 . \tag{3.6}$$

Moreover, using the fact that $Tk_T(a, e^{2x/(T+2)}) \geq 2x$ and $k_T(a,\mu) \geq \max[\mu - 1, (a-1)a^{T-1}\mu^{-2T+1}]$, the left-hand sides of (3.5) and (3.6) can be seen to be bounded in absolute values for all $T \geq 5$, $x \geq 0$ by $Me^{6x/7}x^{-1/2}$, M being a constant. Therefore, applying the Lebesgue dominated convergence to (3.3), (3.4) yields the result (iii).

Notes.

(1) In every case the asymptotic bias has the opposite sign as α.

(2) For $|\alpha| < 1$, the normalizing factor is of the order $T^{1/2}S_\alpha(T)$; this is reasonable since the limiting distribution of $S_\alpha(T)(\alpha_T - \alpha)$ has zero mean. Our result is consistent with Kendall (1954), White (1961), Yamamoto and Kunitomo (1984) and Pantula and Fuller (1985).

(3) For $|\alpha| > 1$, the normalizing factor is of the order $T^{-1/2}S_\alpha(T)$ which is coherent with the fact that the limiting distribution of $S_\alpha(T)(\alpha_T - \alpha)$ has undefined mean since it is Cauchy.

(4) For $|\alpha| = 1$, the normalizing factor is of the order $S_\alpha(T)$ which is also coherent with the fact that the limiting distribution of $S_\alpha(T)(\alpha_T - \alpha)$ has non zero mean. In fact, the constant $-2^{-1/2}I$ appearing in (ii) is just the mean of the random variable $Z = (W_1^2 - 1)\big/\left(2\int_0^1 W_s^2 ds\right)$ where $\{W_s; s \geq 0\}$ is a standard Brownian motion. This can be seen as follows. First, it may be shown that

$$-2^{-1/2}I = -\int_0^\infty 2x(\cosh 2x)^{-1/2}dx + 1 = -\frac{1}{2}\int_0^\infty u(\cosh u)^{-1/2}du + 1 . \tag{3.7}$$

On the other hand, from Liptser and Shiryayev ((1978), Chapter 17.3.1)

$$EZ = \int_0^\infty \left[\frac{\partial}{\partial \theta} \exp(-\theta/2)p(\theta, a)\right]_{\theta=0} da , \tag{3.8}$$

where

$$p(\theta, a) = [2\gamma(\theta, a)]^{1/2}\{[\gamma(\theta, a) + \theta]e^{-\gamma(\theta, a)} + [\gamma(\theta, a) + \theta]e^{-\gamma(\theta, a)}\}^{-1/2} ,$$

with $\gamma(a, \theta) = (\theta^2 + 2a)^{1/2}$. Making the change of variable $u = (2a)^{1/2}$ in the integral in the right-hand side of (3.8), it can be seen that it equals the last right-hand side of (3.7), giving the result.

(5) The constant $2^{1/2}I$ can be computed from (3.7) by the series

$$2^{3/2} \sum_{n=0}^{\infty} (-1)^n (2n)!/[2^n n!/(4n+1)^2] - 1 \simeq 1.78143 .$$

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REFERENCES

Anderson, T. W. (1959). On the asymptotic distribution of estimates of parameters of stochastic difference equations, *Ann. Math. Statist.*, **30**, 676–687.

Anderson, T. W. (1971). *The Statistical Analysis of Time Series*, Wiley, New York.

Bartlett, M. S. (1946). On the theoretical specification and sampling properties of auto-correlated time series, *J. Roy. Statist. Soc. Suppl.*, **8**, 27.

Hurwicz, L. (1950). Least squares bias in time series, *Statistical Inference in Dynamic Economic Models*, (ed. T. C. Koopmans), Wiley, New York.

Kendall, M. G. (1954). Note on the bias in the estimation of autocorrelation, *Biometrika*, **41**, 403–404.

Liptser, R. S. and Shiryayev, A. N. (1978). *Statistics of Random Processes* I and II, Springer, New York-Heidelberg-Berlin.

Mariott, F. H. C. and Pope, J. A. (1954). Bias in the estimation of autocorrelation, *Biometrika*, **41**, 390–396.

Pantula, S. G. and Fuller, W. A. (1985). Mean estimation bias in least squares autoregressive processes, *J. Econometrics*, **27**, 99–121.

Rao, M. M. (1961). Consistency and limit distribution of estimators of parameters in explosive stochastic difference equations, *Ann. Math. Statist.*, **32**, 195–218.

Rao, M. M. (1966). Inference in stochastic processes II, *Z. Wahrsch. Verw. Gebiete*, **5**, 317–335.

Rao, M. M. (1978). Asymptotic distribution of an estimator of the boundary parameter of an unstable process, *Ann. Statist.*, **6**, 185–190.

Rubin, H. (1950). Consistency of maximum likelihood estimates in the explosive case, *Statistical Inference in Dynamic Economic Models*, (ed. T. C. Koopmans), Wiley, New York.

White, J. S. (1958). The limiting distribution of the serial correlation coefficient in the explosive case, *Ann. Math. Statist.*, **29**, 1188–1197.

White, J. S. (1959). The limiting distribution of the serial correlation coefficient in the explosive case, II, *Ann. Math. Statist.*, **30**, 831–834.

White, J. S. (1961). Asymptotic expansion for the mean and variance of the serial correlation coefficient, *Biometrika*, **48**, 85–94.

Yamamoto, T. and Kunitomo, N. (1984). Asymptotic bias of the least squares estimator for multivariate autoregressive models, *Ann. Inst. Statist. Math.*, **36**, 419–430.