28. *Classification of Algebraic Varieties*

By Shigeru IITAKA
Department of Mathematics, Faculty of Science,
University of Tokyo, Hongo, Tokyo 113

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§ 0. **Introduction.** The classification of projective algebraic surfaces was obtained by G. Castelnuovo and F. Enriques at the beginning of this century. Recently, a number of mathematicians have started to develop the classification theory of complete algebraic varieties up to birational equivalence.

Our purpose here is to introduce the classification theory of arbitrary (possibly incomplete) algebraic varieties up to a suitable kind of birational equivalence. When dealing with incomplete algebraic varieties, birational equivalence seems too loose but isomorphism in the category of schemes is too tight. For instance G_a and G_m are birationally equivalent to $\boldsymbol{P}^1$. However, when one uses canonical divisors and regular forms, it is indispensable to introduce a certain notion of birational equivalence. With this in mind, proper birational map will be introduced in § 2 and proper birational equivalence between V_1 and V_2 will be taken to mean the existence of a proper birational map between V_1 and V_2. Moreover, logarithmic plurigenera and logarithmic irregularity will be introduced, which are invariants of proper birational equivalence classes. In § 3, we shall raise a couple of conjectures concerning the classification of algebraic varieties, in which *logarithmic Kodaira dimension* will play the most important role. Note that our classification contains that of affine rings. For instance, Theorem 7 is a numerical characterization of $\boldsymbol{C}[x, y, x^{-1}, y^{-1}]$, which is the counterpart of the Enriques criterion on abelian surfaces.

§ 1. **Notation.** We shall work in the category of schemes over $k=\boldsymbol{C}$. Let V be an algebraic variety of dimension n. Then by Nagata we have a completion $\bar{V}$ of V, and by Hironaka we have a non-singular model $\mu: \bar{V}^* \to \bar{V}$, in other words, $\bar{V}^*$ is a complete non-singular algebraic variety and μ a proper birational morphism. Furthermore, we may assume that $D^* = \bar{V}^* - \mu^{-1}(V)$ is a divisor of normal crossing type (i.e., D is a sum of non-singular subvarieties D_j and ΣD_j has only normal crossings). Then $l(m(K(\bar{V}^*)+D^*))$ and $\dim \Gamma(\bar{V}^*, \Omega^1 \log D^*)$ are independent of the choice of $\bar{V}^*$. Hence we write

$$\bar{P}_m(V) = l(m(K(\bar{V}^*)+D^*)),$$
$$\bar{q}(V) = \dim \Gamma(\bar{V}^*, \Omega^1 \log D^*), \text{ and}$$

$$\bar{\kappa}(V)=\kappa(K(\bar{V}^*)+D^*, \bar{V}^*).$$

These are called logarithmic m-genera, logarithmic irregularity and *logarithmic Kodaira dimension*, respectively (see [1]).

Proposition 1. *Let $f: V \to W$ be a proper birational morphism. Then*

$$\bar{P}_m(V)=\bar{P}_m(W),\ \bar{q}(V)=\bar{q}(W) \quad \text{and} \quad \bar{\kappa}(V)=\bar{\kappa}(W).$$

A proper birational map $\varphi: V_1 \to V_2$ is defined to be $f \cdot g^{-1}$ where $f: W \to V_2$ and $g: W \to V_1$ are proper birational morphisms. When such a φ exists, V_1 is proper birationally equivalent to V_2.

§ 2. Fundamental theorems. Let V be an n-dimensional algebraic variety with $\bar{\kappa}=\bar{\kappa}(V) \geqq 0$. Then take $m \gg 0$ such that $\bar{P}_m(V)>0$. Denote by Φ_m the rational map associated with $|m(K(\bar{V}^*)+D^*)|$. Then $k(\Phi_m(\bar{V}))$ is algebraically closed in $k(V)=k(\bar{V})$. Choosing a suitable non-singular model $\tilde{\mu}: \bar{V}^\# \to \bar{V}^*$, we may assume that $f=\Phi_m \cdot \tilde{\mu}$ is a morphism. Writing $V^\#=\tilde{\mu}^{-1}(V^*)$ we obtain a proper birational morphism $\tilde{\mu}|V^\#$.

Theorem 1. *For a general point $w \in \bar{W}=\Phi_m(\bar{V})$, the fiber $V^\#_w$ is connected and $\bar{\kappa}(V^\#_w)=0$.*

Theorem 2. *Let $f: V_1 \to V_2$ be an étale proper morphism. Then*

$$\bar{\kappa}(V_1)=\bar{\kappa}(V_2).$$

Theorem 3. *Let $f: V_1 \to V_2$ be a dominant morphism with connected fibers. Take a general point $v \in V_2$. Then*

$$\bar{\kappa}(V_1) \leqq \bar{\kappa}(f^{-1}(v))+\dim(V_2).$$

Theorem 4 (Y. Kawamata [7]). *Under the same condition as in Theorem* 3,*we further assume that* $\dim f^{-1}(v)=1$. *Then*

$$\bar{\kappa}(f^{-1}(v))+\bar{\kappa}(V_2) \leqq \bar{\kappa}(V_1).$$

§ 3. Conjectures. Let V be an n-dimensional non-singular algebraic variety.

Conjecture $\bar{A}_n$. *$\bar{P}_m(V)$ and $\bar{\kappa}(V)$ are deformation invariants.*

Deformation of an incomplete algebraic variety is understood as logarithmic deformation defined by Kawamata [8]. When V is complete, we write Conjecture A_n instead of $\bar{A}_n$.

Conjecture $\bar{B}_n$. *If $\bar{\kappa}(V)=0$, then $\bar{q}(V) \leqq n$.*

Conjecture $\bar{C}_n$. *Notations being as in Theorem* 3 *with $V=V_1$, we have $\bar{\kappa}(f^{-1}(v))+\bar{\kappa}(V_2) \leqq \bar{\kappa}(V_1)$.*

Conjecture $\bar{D}_n$. *If $\bar{\kappa}(V)=0$ and the quasi-Albanese map $\alpha_V: V \to \tilde{\mathcal{A}}_V$ is dominant, then α_V is birational.*

Note that $\bar{C}_n$ implies $\bar{B}_n$ by Ueno's Theorem [2]. Assuming $\bar{C}_n$, we see that $\bar{D}_n$ is equivalent to

Conjecture $\bar{D}^*_n$: *If $\bar{\kappa}(V)=0$ and $\bar{q}(V)=n$, then α_V is birational.*

C_2 and D_2 were proved by the Italian school. A_2 was proved by the author. C_n and $\bar{C}_n$ for the case $\dim f^{-1}(v)=1$ were established by E. Viehweg and Y. Kawamata [8], respectively. C_3 for the case $\kappa(V)$

$\geqq 0$ was proved by K. Ueno [9] and T. Fujita. Ueno showed D_3 ([9]). The author proved $\bar{D}_n$ for the case $q(V)=0$ (Theorem 12 [3]).

§ 4. **Further results.** Theorem 5. *Let V be an algebraic variety of hyperbolic type, namely, $\bar{\kappa}(V)=\dim V$. Then the group generated by proper birational maps is a finite group* (see [1], [4]).

Theorem 6. *Let V be an algebraic variety with $\bar{\kappa}(V)\geqq 0$. Then* $\dim \mathrm{Aut}\,(V)\leqq n=\dim V$. *If* $\dim \mathrm{Aut}\,(V)=n$, *then V is a quasi-abelian variety* (see [1]).

Theorem 7. *Let S be an affine normal surface. Then $S=G_m^2$ if and only if $\bar{\kappa}(S)=0$ and $\bar{q}(S)=2$* (see [5]).

Theorem 8. *Let A be an integrally closed affine ring of dimension n. Suppose that $\bar{\kappa}$* Spec $A=n$. *Then any dominant endomorphism of A turns out to be an isomorphism* (see [3]).

Theorem 9. *Let A be an integrally closed affine ring that is algebraic over $k[T_1,\cdots,T_n,T_1^{-1},\cdots,T_n^{-1}]$. Then* $\bar{\kappa}$ Spec $(A)\geqq 0$. *If* $\bar{\kappa}$ Spec $A=0$, *then* $A=k[X_1,\cdots,X_n,X_1,\cdots,X_n^{-1}]$ (see [3]).

Theorem 10. *Let $F\in k[x,y]$ and assume that* Aut $k[x,y,1/F]$ *is not a finite group. For any $m>0$, the function field defined by $z^m=F(x,y)$ is ruled* (see [3]).

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