doi: 10.2478/umcsmath-2014-0002

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXVIII, NO. 1, 2014 SECTIO A 11–20

E. BALLICO

On the birational gonalities of smooth curves

Abstract. Let C be a smooth curve of genus g. For each positive integer r the birational r-gonality $s_r(C)$ of C is the minimal integer t such that there is $L \in \mathrm{Pic}^t(C)$ with $h^0(C, L) = r + 1$. Fix an integer $r \geq 3$. In this paper we prove the existence of an integer g_r such that for every integer $g \geq g_r$ there is a smooth curve C of genus g with $s_{r+1}(C)/(r + 1) > s_r(C)/r$, i.e. in the sequence of all birational gonalities of C at least one of the slope inequalities fails.

1. Introduction. Let C be a smooth curve of genus g. For each positive integer r the birational r-gonality $s_r(C)$ of C is the minimal integer t such that there is $L \in \mathrm{Pic}^t(C)$ with $h^0(C, L) = r + 1$ ([1], §2). In this paper we prove the following result.

Theorem 1. *Fix an integer $r \geq 3$. Then there exists an integer g_r such that for every integer $g \geq g_r$ there is a smooth curve C of genus g with $s_{r+1}(C)/(r + 1) > s_r(C)/r$.*

Theorem 1 means that for the curve C at least one slope inequality fails. For any integer $r \geq 1$ the r-gonality of C is the minimal degree of a line bundle L on C with $h^0(C, L) \geq r + 1$. Obviously $s_r(C) \geq d_r(C)$ if $r \geq 2$. Equality holds if $d_r(C) < r \cdot d_1(C)$ and C has no non-trivial morphism onto a smooth curve of positive genus. In [6] H. Lange and G. Martens studied

2000 *Mathematics Subject Classification.* 14H45; 14H50; 32L10.
Key words and phrases. Birational gonality sequence, smooth curve, nodal curve, Hirzebruch surface.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

the slope inequality for the usual gonality sequence of smooth curves (it may fail for some C, but not for a general C).

We work over an algebraically closed base field with characteristic zero.

2. Working inside a Hirzebruch surface. Fix $e \in \mathbb{N}$. Let $F_e \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ denote the Hirzebruch surface ([4], Chapter V, §2). We call $\pi : F_e \to \mathbb{P}^1$ a ruling of F_e. We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^2$ and take as a basis of $\mathrm{Pic}(F_e)$ a fiber f of π and a section h of π with $h^2 = -e$ (π and h are unique if $e > 0$). For any finite set $S \subset F_e$ let $2S$ denote the first infinitesimal neighborhood of S in F_e, i.e. the closed subscheme of F_e with $(\mathcal{I}_S)^2$ has its ideal sheaf. We have $(2S)_{red} = S$ and $\deg(2S) = 3 \cdot \sharp(S)$. Fix an integer $a \geq 0$. The line bundle $\mathcal{O}_{F_e}(ah + bf)$ is spanned (resp. very ample) if and only if $b \geq ea$ (resp. $b > ea$ and $a > 0$) ([4], V.2.18). We have $h^1(F_e, \mathcal{O}_{F_e}(ah + bf)) = 0$ if and only if $b \geq -1$. If $b \geq ae$, then

$$h^0(F_e, \mathcal{O}_{F_e}(ah + bf)) = (a + 1)(2b - ea + 2)/2$$

([5], Proposition 2.3). Assume $a > 0$ and $b \geq ae$; if $e = 0$, then assume $b > 0$. Fix any $Y \in |\mathcal{O}_{F_e}(ah + eaf)|$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-e - 2)f)$, the adjunction formula gives

$$\omega_Y \cong \mathcal{O}_Y((a - 2)h + (ea - e - 2)f).$$

Hence $p_a(Y) = 1 + a(ea - e - 2)/2$. We have

$$h^0(F_e, \mathcal{O}_{F_e}(ah + eaf)) = (ea + 2)(a + 1)/2.$$

To prove Theorem 1 for the integer r we will use as C the normalization of a nodal curve $Y \in |\mathcal{O}_{F_e}(ah + eaf)|$, where $e := r - 1$.

Notation 1. For all integers $a \geq 1$ and $e \geq 1$ set $g_{a,e} := 1 + a(ae - 2 - e)/2$.

Notice that if $a \geq 2$, then $g_{a,e} - g_{a-1,e} = ae - e - 1$.

Lemma 1. *Assume $e \geq 2$. Fix integers a, x. If $x = 0$, assume $a \geq 1$. If $x > 0$, assume $a \geq 5$ and $3x \leq (ea - 2e + 1)(a - 1)/2$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$. Then*

$$h^1(F_e, \mathcal{I}_{2S}(ah + eaf)) = 0,\ h^0(F_e, \mathcal{I}_{2S}(ah + eaf)) = (ea + 2)(a + 1)/2 - 3x,$$

a general $Y \in |\mathcal{I}_{2S}(ah + eaf)|$ is integral, nodal and with $\mathrm{Sing}(Y) = S$.

Proof. First assume $x = 0$. Since $\mathcal{O}_{F_e}(ah + aef)$ is spanned, Bertini's theorem gives that a general $Y \in |\mathcal{O}_{F_e}(ah + aef)|$ is smooth. Since

$$h^0(F_e, \mathcal{O}_{F_e}(h+ef)) + h^0(F_e, \mathcal{O}_{F_e}((c-1)h+(c-1)rf)) < h^0(F_e, \mathcal{O}_{F_e}(ch+cf))$$

for every integer $c \in \{1, \dots, a - 1\}$ and $|\mathcal{O}_{F_e}(uh + vf)|$ has h in the base locus if $u > 0$ and $v < eu$, Y is also irreducible.

Now assume $x > 0$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$. Since

$$3x \leq h^0(F_e, \mathcal{O}_{F_e}((a - 2)h + e(a - 2)f)),$$

$e \geq 2$ and $a-2 \geq 3$, a theorem of A. Laface gives

$$h^1(F_e, \mathcal{I}_{2S}((a-2)h + e(a-2)f)) = 0$$

([5], Proposition 5.2 and case $m = 2$ of Theorem 7.2). Hence

$$h^1(F_e, \mathcal{I}_{2S}((a-i)h + e(a-i)f)) = 0$$

for $i = 0, 1$. Hence

$$h^0(F_e, \mathcal{I}_{2S}(ah + eaf)) = (ea+2)(a+1)/2 - 3x.$$

Fix $P \in F_e \setminus S$ and a general $A \in |\mathcal{O}_{F_e}(h+e)f)|$ containing P. The curve A is smooth if $P \notin h$, while $A = h \cup F$ with $F \in |\mathcal{O}_{F_e}(f)|$ if $P \in h$. In all cases we see that $\mathcal{O}_A(ah + eaf)$ is spanned at P (in the case $P \in h$ use the following facts: $\mathcal{O}_h(ah + eah) \cong \mathcal{O}_h$, $F \cong \mathbb{P}^1$, and $\mathcal{O}_{\mathbb{P}^1}(a)$ is spanned). Since $h^1(F_e, \mathcal{O}_{F_e}((a-1)h + e(a-1)f)) = 0$, $P \in A$ and $\mathcal{O}_A(ah+eaf)$ is spanned at P, the exact sequence

$$\begin{aligned} 0 \to \mathcal{I}_{2S}((a-1)h + e(a-1)f) &\to \mathcal{I}_{2S}((a-1)h + e(a-1)f) \\ &\to \mathcal{O}_A(ah + eaf) \to 0 \end{aligned} \tag{1}$$

gives that $\mathcal{I}_{2S}(ah + eaf)$ is spanned at P. Since this is true for all $P \notin S$, Bertini's theorem gives $\mathrm{Sing}(Y) = S$. In particular Y has no multiple component. Fix $P \in S$. Since S is general, we have $P \notin h$. Since $|\mathcal{O}_{F_e}(h + ef)|$ induces a morphism with injective differential at P, $|\mathcal{O}_{F_e}(2h + 2af)|$ spans the jets at P of $\mathcal{O}_{F_e}$ up to order 2. Hence we may find $Y' \in |\mathcal{O}_{F_e}(2h + 2ef)|$ with an ordinary node at P. Since

$$h^1(F_e, \mathcal{I}_{2S}((a-2)h + e(a-2)f)) = 0,$$

we have

$$h^1(F_e, \mathcal{I}_{\{P\} \cup 2(S\setminus\{P\})}((a-2)h + e(a-2)f)) = 0.$$

Hence

$$\begin{aligned} h^0(F_e, &\mathcal{I}_{\{P\} \cup 2(S\setminus\{P\})}((a-2)h + e(a-2)f)) \\ &= h^0(F_e, \mathcal{I}_{2(S\setminus\{P\})}((a-2)h + e(a-2)f)) - 1. \end{aligned}$$

Hence there is $Y'' \in |\mathcal{I}_{2(S\setminus\{P\})}((a-2)h + e(a-2)f)|$ such that $P \notin Y''$. Hence $Y'' \cup Y'$ has an ordinary node at P. Since $Y'' \cup Y' \in |\mathcal{I}_{2S}(ah+eaf)|$, S is finite and Y is general, Y is nodal. Recall that $\mathrm{Sing}(Y) = S$ and that S is general. Since S is general, no pair of points of S is on the same fiber of the ruling of F_e. Hence no fiber of F_e may be an irreducible component of Y. Since $\mathcal{O}_{F_e}(ch + ecf) \cdot \mathcal{O}_{F_e}((a-c)h + e(a-c)f) = ec(a-c)$, we immediately see that Y is irreducible. □

Lemma 2. *Assume $e \geq 2$. Fix integers a, x. If $x = 0$, assume $a \geq 1$. If $x > 0$, assume $a \geq 5$ and $3x \leq (ea - 2e + 1)(a-1)/2$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$ and a general $Y \in |\mathcal{I}_{2S}(ah + eaf)|$. Let $u : C \to Y$ denote the normalization map. The line bundle $u^*(\mathcal{O}_Y(f))$ is spanned* *and* $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. *Let* $\rho : C \to \mathbb{P}^1$ *be the morphism induced by* $|u^*(\mathcal{O}_Y(f))|$. *Then* ρ *is not composed with an involution, i.e. there are no* (C', ρ', ρ'') *with* C' *a smooth curve,* $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, $\deg(\rho') \geq 2$ *and* $\deg(\rho'') \geq 2$.

Proof. Obviously $u^*(\mathcal{O}_Y(f))$ is spanned. Since $ae+1-e-2 \geq e(a-2)-1$, Serre's duality gives

$$h^1(F_e, \mathcal{O}_{F_e}(-ah-(ae+1)f)) = h^1(F_e, \mathcal{O}_{F_e}((a-2)h+(ae+1-e-2)f)) = 0.$$

Hence $h^0(Y, \mathcal{O}_Y(f)) = 2$. Since $h^i(F_e, \mathcal{O}_{F_e}) = 0$, $i = 1, 2$, $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-e-2)f))$, Y is nodal and $S = \mathrm{Sing}(Y)$, we have

$$H^0(Y, \omega_Y) \cong H^0(F_e, \mathcal{O}_{F_e}((a-2)h + (ae-e-2)f))$$

and $H^0(C, \omega_C)$ is induced (after deleting the base points) from

$$H^0(F_e, \mathcal{I}_S((a-2)h + (ae-2-e)f)).$$

Hence $h^0(C, u^*(\mathcal{O}_Y(f))) = 2 = h^0(Y, \mathcal{O}_Y(f))$ if and only if

$$h^1(C, u^*(\mathcal{O}_Y(f))) = x + h^1(Y, \mathcal{O}_Y(f)),$$

i.e. if and only if $h^1(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f)) = 0$. The last equality is true, because S is general and $x \leq (a-1)(ea-2-2e)/2 = h^0(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f))$.

For any $P \in F_e$ let F_P be the fiber of the ruling of F_e containing P. We fix $P \in F_e \setminus h$ such that $F_P \cap S = \emptyset$. Let $Z \subset F_P$ be the degree two effective divisor with P as its support. Take any $S_1 \subset F_P \setminus \{P, h \cap F_P\}$ such that $\sharp(S_1) = a-2$ and set $Z' := Z \cup S_1$. Taking the inclusion $F_P \hookrightarrow F_e$, we may also see Z' as a degree a zero-dimensional subscheme of F_e.

Claim. $h^1(F_e, \mathcal{I}_{2S \cup Z'}(ah + aef)) = 0$.

Proof of the Claim. Set $T := h \cup F_P \in |\mathcal{O}_{F_e}(h+f)|$. Since $S \cap h = \emptyset$ and $S \cap F_P = \emptyset$, we have $S \cap T = \emptyset$. Hence $(2S \cup Z') \cap T = Z'$. We proved during the proof of Lemma 1 that $h^1(F_e, \mathcal{I}_{2S}((a-1)h + (a-1)ef))) = 0$. Hence $h^1(F_e, \mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f)) = 0$. Notice that

$$\mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f) \cong \mathcal{I}_{2S}(ah + aef)(-T).$$

Since $h^1(F_e, \mathcal{I}_{2S}(ah + aef)) = 0$ (Lemma 1), the Claim is true if

$$h^1(T, \mathcal{I}_{Z',T}(ah + aef)) = 0.$$

The nodal curve T has two irreducible components, h and F_P, and both components are isomorphic to $\mathbb{P}^1$. Since $Z' \cap h = \emptyset$, we have $Z' \cap h \cap F_P = \emptyset$ and hence the $\mathcal{O}_T$-sheaf $\mathcal{I}_{Z'}(ah + aef)$ is a line bundle. Since $Z' \cap h = \emptyset$ and $\mathcal{O}_h(ah + aef) \cong \mathcal{O}_h$, we have $\mathcal{I}_{Z',T}(ah + aef)|h \cong \mathcal{O}_h$. Since $\deg(Z') = a$, we have $\mathcal{I}_{Z',T}(ah + aef) \cap F_P \cong \mathcal{O}_{F_P}$. Hence a Mayer–Vietoris exact sequence gives $h^1(T, \mathcal{I}_{Z',T}(ah + aef)) = 0$, concluding the proof of the Claim.

The Claim is equivalent to

$$h^0(F_e, \mathcal{I}_{2S \cup Z'}(ah + aef)) = h^0(F_e, \mathcal{I}_{2S}(ah + aef)) - a.$$

Set $\Gamma := \bigcup_{Q\in S} F_Q$. We take all $Y \in |\mathcal{I}_{2S}(ah+eaf)|$ containing some Z'. The set of all $P \in F_e$ has dimension 2. For fixed P the set of all $S_1 \subset F_P \setminus F_P \cap (\{P\} \cup h)$ with $\sharp(S_1) = a-2$ has dimension $a-2$. Each Y may contain only finitely many schemes Z', because each non-constant morphism $C \to \mathbb{P}^1$ has only finitely many ramification points. Varying first $P \in F_e \setminus (h \cup \Gamma)$ and then all $S_1 \subset F_P \setminus (h \cap F_P \cup \{P\})$ with $\sharp(S_1) = a-2$, we get that a general $Y \in |\mathcal{I}_{2S}(ah+aef)|$ contains some Z' for some $P \in F_e \setminus (h \cup \Gamma)$. Let $u : C \to \mathbb{P}^1$ be the normalization of any such Y, say containing $Z' = Z \cup S_1$ with $Z \subset F_P$. We saw that $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $\rho : C \to \mathbb{P}^1$ be the morphism associated to $|u^*(\mathcal{O}_Y(f))|$. Notice that ρ is induced by the ruling $\rho_1 : F_e \to \mathbb{P}^1$. Set $Q := \rho_1(P)$. By the construction $\rho^{-1}(Q) \cong Z \cup S_1$, i.e. the fiber of ρ over Q contains a point with multiplicity two and $a-2$ points with multiplicity one. Hence there are no (C', ρ', ρ'') with C' a smooth curve, $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, $\deg(\rho') \geq 2$ and $\deg(\rho'') \geq 2$. □

Lemma 3. *Fix S, Y, C, u as in Lemma 1 and take any spanned line bundle L of degree > 0. Fix a general $A \in |L|$ and set $B := u(A)$. Then $S \cap B = \emptyset$ and $h^1(F_e, \mathcal{I}_{S\cup B}((a-2)h + (ae-e-2)f)) > 0$.*

Proof. Since $\deg(L) > 0$, $A \neq \emptyset$. Since L is spanned, $h^0(C, L(-Q)) = h^0(C, L) - 1$ for each $Q \in C$ and in particular for each $Q \in A$. Riemann–Roch gives $h^1(C, \mathcal{O}_C(A \setminus \{Q\}) = h^1(C, \mathcal{O}_C(A)$ for every $Q \in A$. Since $H^0(C, \omega_C) \cong H^0(F_e, \mathcal{I}_S((a-2)h + (ae-e-2)f))$, we get

$$h^0(F_e, \mathcal{I}_{S\cup(B\setminus\{P\}}((a-2)h + (ae-e-2)f)) \\ = h^0(F_e, \mathcal{I}_{S\cup B}((a-2)h + (ae-e-2)f)$$

for every $P \in B$. Hence $h^1(F_e, \mathcal{I}_{S\cup B}((a-2)h + (ae-e-2)f)) > 0$. □

Lemma 4. *Take e, a, x, S, Y, C as in Lemma 2. Then $d_1(C) = a$.*

Proof. The line bundle $u^*(\mathcal{O}_Y(f))$ gives $d_1(C) \leq a$. Assume $z := d_1(C) < a$ and take $L \in \mathrm{Pic}^z(C)$ evincing $d_1(C)$, i.e. evincing the gonality of C. Fix a general $A \in |L|$ and set $B := u(A)$. Lemma 3 gives

$$h^1(F_0, \mathcal{I}_{S\cup B}((a-2)h + (ae-2-e)f)) > 0.$$

Since L is spanned and A is general, we have $S \cap B = B \cap h = \emptyset$. Lemma 2 gives $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $v : C \to \mathbb{P}^1$ be the morphism induced by $|L|$ and $v' : C \to \mathbb{P}^1$ the morphism induced by $|u^*(\mathcal{O}_Y(f))|$. Since v' is not composed with an involution (Lemma 3), the induced map $(v, v') : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. Hence for general B we have $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$. Since $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) > z$, there is $A_1 \in |\mathcal{O}_{F_e}(h+ef)|$ containing B. Since $B \cap h = \emptyset$ and $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$, A_1 is irreducible. Hence $E \cong \mathbb{P}^1$. Since S is general and $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = e+2$, we have $\sharp(S \cap A_1) \le e+1$. Hence

$$\sharp(A_1 \cap (S \cup B)) \le z+e+1 \le a+e.$$

Since $\deg(\mathcal{O}_{A_1}((a-2)h+(ae-e-2)f)) = ae-e-2 \ge a+e-1$, we have

$$h^1(A_1, \mathcal{I}_{A_1 \cap (S \cup B), A_1}((a-2)h+(ae-e-2)f)) = 0.$$

Hence the case $i=1$ of (1) gives

$$h^1(F_e, \mathcal{I}_{S \setminus S \cap A_1}((a-3)h+((a-1)e-e-2)f)) > 0.$$

Since $S \setminus S \setminus S \cap A_1$ is general and

$$x \le e(a-2)(ea-3e+2)/2 \le h^0(F_e, \mathcal{O}_{F_e}((a-3)h+((a-1)e-e-2)f)),$$

we have

$$h^1(F_e, \mathcal{I}_{S \setminus S \cap A_1}((a-3)h+((a-1)e-e-2)f)) = 0,$$

a contradiction. □

Lemma 5. *Fix integers $e \ge 2$ and $a \ge 2$. Fix any integral $Y \in |\mathcal{O}_{F_e}(ah+eaf)|$ and call $u : C \to Y$ the normalization map. Then $s_{e+1+2j}(C) \le ae+je$ for every integer $j \ge 0$.*

Proof. We have $h^0(F_e, \mathcal{O}_{F_e}(h+(e+j)ef)) = e+2+2j$, for every integer $j \ge 0$. Since $a \ge 2$, we have $h^0(F_e, \mathcal{I}_Y(h+yf)) = 0$ for any y. We have $\mathcal{O}_{F_e}(h+(e+j)f) \cdot \mathcal{O}_{F_e}(ah+eaf) = a(e+j)$. Since for any $j \ge 0$ the linear system $|\mathcal{O}_{F_e}((h+(e+j)f)|$ embeds $F_e \setminus h$, the spanned line bundle $u^*(\mathcal{O}_Y((h+(e+j)f))$ gives $s_{e+1+2j}(C) \le ae+je$. □

Lemma 6. *Fix an integer $e \ge 2$. There is an integer $A_e \ge 5$ with the following property. Fix integers a, x such that $a \ge A_e$ and $0 \le x \le ae-e-2$. Moreover, every base point free linear system on C with degree $\le ae$ and birationally very ample is induced (after deleting the base points) from a linear subspace of $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$.*

Proof. Fix an integer $z \le ae$ such that there is a spanned $L \in \mathrm{Pic}^z(C)$ such that the morphism $v : C \to \mathbb{P}^k$, $k := h^0(C,L)-1$, induced by $|L|$ is birational onto its image. Fix a general $A \in |L|$ and set $B := u(A)$. Since L is spanned and A is general, we have $S \cap B = \emptyset$ and $B \cap h = \emptyset$. Lemma 3

$$h^1(F_0, \mathcal{I}_{S \cup B}((a-2)h+(ae-2-e)f)) > 0.$$

(a) Since the monodromy group G of the general hyperplane section of $v(C)$ is the full symmetric group S_z, B is in uniform position in F_e and in particular for all integers c, t such that $0 \le c \le a$ and $t \ge ec$ and any $B' \subset B$, either $h^0(F_e, \mathcal{I}_{B'}(ch+tf)) = \max\{0, (c+1)(t+1) - \sharp(B')\}$ or $h^0(F_e, \mathcal{I}_B(ch+tf)) > 0$. In particular, $\sharp(D \cap B) \le 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$.

(b) In this step we assume $h^0(F_e, \mathcal{I}_B(h+ef)) > 0$. Let t be the minimal non-negative integer such that $h^0(F_e, \mathcal{I}_B(h+tf)) > 0$. By assumption we have $t \le e$. Varying A in $|L|$, we get that $|L|$ is obtained (after deleting the base locus) from a linear subspace of $|\mathcal{O}_{F_e}(h+tf)|$. Since $|\mathcal{O}_{F_e}(h+tf)|$ sends $F_e \setminus h$ onto $\mathbb{P}^1$ if $t < e$, while v is birational onto its image, we get $t = e$. Since $h^0(F_e, \mathcal{I}_B(h+(e-1)f)) = 0$, step (a) gives $\sharp(D \cap B) \le e-1$ for every $\Gamma \in |\mathcal{I}_B(h+(e-1)f))|$. Since $\sharp(D \cap B) \le 1$ for every $D \in |\mathcal{O}_{F_e}(1)|$ and $z > e$, T is irreducible. Hence $T \cong \mathbb{P}^1$. Since $\sharp(B) \le Y \cdot T = ae$, we have $z \le ae$ and if inequality holds, then $|L|$ is induced without deleting any base point from $|\mathcal{O}_{F_e}(h+ef)|$. Hence $k \le e+1$ and v is induced (after deleting the base points) from a linear subspace of $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$. We get that if L evinces $s_{e+1}(C)$ and the assumption of this step holds, then $s_{e+1}(C) = ae$ and $L \cong u^*(\mathcal{O}_Y(h+ef))$.

(c) From now on we assume $h^0(F_e, \mathcal{I}_B(h+ef)) = 0$. To conclude the proof of the lemma it is sufficient to find a contradiction for $a \gg 0$ and any $x \le ae-e-2$. Set $c := \lfloor z/(e+1) \rfloor$. Set $S_0 := S$ and $B_0 := B$. Fix $A_1 \in |\mathcal{O}_{F_e}(h+ef)|$ such that $a_1 := \sharp(A_1 \cap B_0)$ is maximal. Set $S_1 := S_0 \setminus S_0 \cap A_1$ and $B_1 := B_0 \setminus B_0 \cap A_1$. For each integer $i \ge 2$ define recursively the curve $A_i \in |\mathcal{O}_{F_e}(h+ef)|$, the integer a_i, and the sets S_i, B_i in the following way. Fix $A_i \in |\mathcal{O}_{F_e}(h+ef)|$ such that $a_i := \sharp(A_i \cap B_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap A_i$ and $B_i := B_{i-1} \setminus B_{i-1} \cap A_i$. Since $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = e+2$ and $h^0(F_e, \mathcal{I}_B(h+ef)) = 0$, step (a) gives $a_i \le e+1$ for all i. Since $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = e+2$ and a_i is maximal, either $a_i = e+1$ or $B_i = \emptyset$. Hence $a_i = e+1$ for $i \le c$, $a_{c+1} = z - c(a+1) \le e+1$ and $a_i = 0$ for all $i \ge c+2$. Assume $a \ge 4e$. Hence $(e+1)^2(a-3) \ge e(e+2)a$. Since $z \le ea$, we get $c \le a-4$. For each integer $i = 1, \dots, c+1$ we have an exact sequence

$$\begin{aligned} &0 \to \mathcal{I}_{S_i \cup B_i}((a-2-i)f + (e(a-i)-e-2)f) \\ (2) \qquad &\to \mathcal{I}_{S_{i-1} \cup B_{i-1}}((a-1-i)h + (e(a-i+1)-e-2)f) \\ &\to \mathcal{I}_{A_i \cap (S_{i-1} \cup B_{i-1}, A_i}((a-1-i)h + (e(a-i+1)-e-2)f) \to 0. \end{aligned}$$

Fix $i \in \{1, \dots, c\}$. By step (a) we have $\sharp(D \cap B) \le 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$. Hence A_i is irreducible. Hence $A_i \cong \mathbb{P}^1$. Since $\sharp(D \cap B) \le 1$ for every $B \in |\mathcal{O}_{F_e}(f)|$ and $B \cap h = \emptyset$, even if $a_{c+1} \le a$ we may take an irreducible $A_{c+1} \in |\mathcal{O}_{F_e}(f)|$ containing B_{c+1}. Assume for a moment $c+1 \le a-5$. Since $e \ge 2$, we have $e(a-c+1)-e-2 \ge 2e+1$. Set $x_i := \sharp(S_{i-1} \cap A_i)$. Since S is general, we have $x_i \le e+1$. Hence $x_i + a_i \le 2e+2$. Since $A_i \cong \mathbb{P}^1$ and

$$\begin{aligned} \deg(\mathcal{O}_{A_i}((a-1-i)h + (e(a-i+1)-e-2)f)) &= e(a-i+1)-e-2) \\ \ge e(a-c+1)-e-2 \ge 2e+1, \end{aligned}$$

we have

$$h^1(A_i, \mathcal{I}_{A_i \cap (S_{i-1} \cup B_{i-1}, A_i}((a-1-i)h + (e(a-i+1)-e-2)f)) = 0.$$

Hence applying (2) first for $i = 1$, then for $i = 2$, and so on up to $i = c+1$, we get

$$h^1(F_e, \mathcal{I}_{S_{c+1}}((a-3-c)f + (e(a-c-1)-e-2)f)) > 0.$$

Since $2e \geq e+1$, we have

$$h^1(F_e, \mathcal{O}_{F_e}((a-3-c)f + (e(a-c-1)-e-2)f)) = 0.$$

Since S is general and $S_c \subseteq S$, to have $h^1(F_e, \mathcal{I}_{S_{c+1}}((a-3-c)f + (e(a-c-1)-e-2)f)) = 0$ (and hence a contradiction), it is sufficient to have

$$\sharp(S_c) \leq h^0(F_e, \mathcal{O}_{F_e}((a-3-c)f + (e(a-c-1)-e-2)f)).$$

Since $\sharp(S_c) \leq x$, it is sufficient to have $x \leq (a-3-c)(e(a-3-c)+2e-2)/2$. Since $x \leq ae-e-2$, it is sufficient to have $(a-c-3)^2e/2 \geq ae$. Thus it is sufficient to have $c \leq a-3-\sqrt{2a}$. Since $c \leq ea/(e+1)$, it is sufficient to have $a-(e+1)\sqrt{2a}-3e-3 \geq 0$. Hence we may take $A_e = 32(e+1)^2$. Notice that we also checked the assumption $a-c-1 \leq a-5$. □

Lemma 7. *Take $e \geq 2$, A_e, $a \geq A_e$, $0 \leq x \leq ea-e-2$, S, Y and C as in Lemma 5.*

(a) *We have $s_e(C) = ea-1-\min\{1,x\}$.*

(b) *If $x > 0$, then each $L \in Pic(C)$ evincing $s_e(C)$ is induced by $|\mathcal{I}_{\{P\}}(h+ef)|$ (after deleting the degree 2 base locus $u^{-1}(P)$) for some $P \in S$. For an arbitrary x any spanned and birationally very ample line bundle M of degree $ea-1$ is induced by $|\mathcal{I}_{\{P\}}(h+ef)|$ (after deleting the degree 1 base locus $u^{-1}(P)$) for some $P \in Y \setminus (S \cup h)$.*

Proof. The linear systems described in part (b) shows that $s_e(C) \leq ea-1-\min\{1,x\}$. By Lemma 7 any such birationally very ample and spanned complete linear system $|L|$ is induced (after deleting the base locus) from a codimension 1 linear subspace V of $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$. Call $\mathcal{B} \subset F_e$ the base scheme of V as a linear system on F_e and $\mathbb{B}$ the base locus of $u^*(V)$ on C. Since $h^0(C, u^*(\mathcal{O}_Y(h+ef))) \geq e+2$, we have $\mathbb{B} \neq \emptyset$. Obviously $\mathbb{B}_{red} = u^{-1}(\mathcal{B} \cap Y)$. Hence $\mathcal{B} \cap Y \neq \emptyset$. Since $\mathcal{O}_h(h+ef) \cong \mathcal{O}_h$,

$$h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = 2 + h^0(F_e, \mathcal{O}_{F_e}(ef))$$

and V has codimension 1 in $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$, we have $h \cap \mathcal{B} = \emptyset$. Since $|\mathcal{O}_{F_e}(h+ef)|$ induces an embedding of $F_e \setminus h$, the scheme $\mathcal{B}$ must be a single point, P, with its reduced structure. Since $\mathcal{B} \cap Y \neq \emptyset$, we have $P \in Y$. We have $\deg(L) = ae-1$ if $P \notin S$ and $\deg(L) = ae-2$ if $P \in S$. □

3. Proof of Theorem 1. We fix the integer $r \geq 3$ for which we want to prove Theorem 1 and set $e := r-1$. Hence $e \geq 2$. Fix A_e as in Lemma 6 and any integer $g \geq eA_e^2/2 - eA_e + e + 2$. Let a be the minimal integer such that $g \leq g_{a,e}$. Since $g_{a,e} - g_{a-1,e} = ae-e-1$, we have $a \geq A_e$ and there is a unique integer x such that $0 \leq x \leq ae-e-2$ and $g = g_{a,e} - x$. Take C as in Lemmas 6 and 7. Lemma 6 gives $s_{e+1}(C)) = ae$. Hence it is sufficient to prove that $s_{e+2}(C) > (e+2)ea/(e+1)$. Assume $z := s_{e+2}(C) \leq (e+2)ea/(e+1)$ and fix $L \in \mathrm{Pic}^z(C)$ evincing $s_{e+2}(C)$. The line bundle L is spanned, $h^0(C,L) = e+3$ and $|L|$ induces a morphism

$v : C \to \mathbb{P}^{e+2}$ birationally onto its image and with $v(C)$ a degree z non-degenerate curve with arithmetic genus $\geq g$. Set $m_1 := \lfloor (z-1)/(e+2) \rfloor$, $\epsilon_1 = z - 1 - m_1(e+2)$, $\mu_1 := 0$ if $\epsilon_1 \neq e+1$ and $\mu_1 := 1$ if $\epsilon_1 = e+1$. Set $\pi_1(z, e+2) = (e+2)m_1(m_1-1)/2 + m_1(\epsilon_1+1) + \mu_1$. Notice that

$$\pi_1(z,e+2) \leq z(z+2)/2(e+2) \leq ea(e+2)(ea(e+2)+2e+2)/(2(e+2)(e+1)^2).$$

Notice that $e^2(e+2)^2/(2(e+2)(e+1)^2) < e/2$. Since $g > g_{a-1,e} = 1 + (a-1)(ae-2-2e)/2$, we have $g > \pi(z, e+2)$ if $a \gg 0$, say if $a \geq A'_e$. Hence [3], Theorem 3.15, gives that $v(C)$ is contained in a degree $e+1$ surface $T \subset \mathbb{P}^{e+2}$. By the classification of all minimal degree surfaces ([2]), either T is a cone over a rational normal curve or $T \cong F_m$ embedded by the complete linear system $|\mathcal{O}_{F_{e+1}}(h + (e+1+m)f)|$ for some integer $m \equiv e+1 \pmod 2$ with $0 \leq m \leq e-1$. In the latter case we set $E := v(C)$. In the former case T is the image of F_{e+1} by the complete linear system $|\mathcal{O}_{F_{e+1}}(h + (e+1)f)|$; in this case set $m := e+1$ and call E the strict transform of $v(C)$ in F_{e+1}. In both cases E is a curve contained in F_m with C as its normalization. Call $u' : C \to E$ the normalization map. Hence there are integers c, y such that $E \in |\mathcal{O}_{F_m}(ch + yf)|$ with $y \geq mc$ and $c > 0$. Lemma 4 gives $c \geq a$; if $m = 0$ it also gives $y \geq a$.

(a) Here we assume $m \leq e-1$. Let $T' \subset \mathbb{P}^e$ be the image of F_m by the complete linear system $|\mathcal{O}_{F_m}(h + (e+m)f)|$. Since either $T' \cong F_m$ (case $m \neq e-1$) or T' is the blowing down of h (case $m = e-1$), the image of E in T' gives $s_e(C) \leq \mathcal{O}_{F_m}(h+(e+m)f) \cdot \mathcal{O}_{F_m}(ch+yf) = z - c$. Since $c \geq a$, Lemma 7 gives $z \geq c + ae - 2 \geq a(e+1) - 2$, contradicting the assumption $z \leq ea(e+2)/(e+1)$ (with $a > 2(e+1)^2$).

(b) Now assume $m = e+1$. Since $y \geq mc = (e+1)c$ and $c \geq a$ (Lemma 6), this case is impossible.

The proof of Theorem 1 is complete.

References

[1] Coppens, M., Martens, G., *Linear series on 4-gonal curves*, Math. Nachr. **213**, no. 1 (2000), 35–55.

[2] Eisenbud, D., Harris, J., *On varieties of minimal degree (a centennial account)*, *Algebraic Geometry*, Bowdoin, 1985 (Brunswick, Maine, 1985), 3–13, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[3] Harris, J., Eisenbud, D., *Curves in projective space*, Séminaire de Mathématiques Supérieures, 85, Presses de l'Université de Montréal, Montréal, Que., 1982.

[4] Hatshorne, R., *Algebraic Geometry*, Springer-Verlag, Berlin, 1977.

[5] Laface, A., *On linear systems of curves on rational scrolls*, Geom. Dedicata **90**, no. 1 (2002), 127–144; generalized version in arXiv:math/0205271v2.

[6] Lange, H., Martens, G., *On the gonality sequence of an algebraic curve*, Manuscripta Math. **137** (2012), 457–473.

E. Ballico
Dept. of Mathematics
University of Trento
38123 Povo (TN)
Italy
e-mail: ballico@science.unitn.it

Received May 17, 2012