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A WELL-POSEDNESS RESULT FOR A STOCHASTIC CAHN-HILLIARD EQUATION

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ABSTRACT. This paper is about the study of the well-posedness of a stochastic Cahn-Hilliard equation driven by white noise induced by a Q-Brownian motion. The proof of the existence of a unique global solution relies on the Galerkin method together with a monotonicity method.

1. INTRODUCTION

The Cahn-Hilliard equation was presented by J. W. Cahn and J. E. Hilliard [3] in 1958, describing spinodal decomposition of binary mixtures that apears, for example, in cooling process of alloys, glasses or polymer mixtures. The unknown is the concentration u which satisfies the equation

$$\frac{\partial u}{\partial t} + \Delta^2 u + \Delta f(u) = 0. \tag{1.1}$$

The function f is the derivative of the homogeneous free energy F. In its original form, F contains a logarithmic term with singularities, which makes the study of this equation delicate. In some circumstances, F can be approximated by a polynomial of even degree with a strictly positive dominant coefficient. Equation

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(1.1) has been studied by a number of researchers and the reader is refered to [2, 7, 9–11, 18, 19, 23].

In 1996 G. DaPrato and A. Debussche [6] considered a stochastic version of the Cahn-Hilliard equation

$$du + (\Delta^2 u - \Delta f(u))dt = dW, \tag{1.2}$$

defined in a domain $G = \Pi_{i=1}^{n}]0, L_i[\subset \mathbb{R}^n$ with a polynomial free energy. This equation was associated with a Neumann boundary condition or a periodic boundary condition and supplemented with an initial condition $u(0, x) = u_0(x)$.

The existence results for Problem (1.2) was given by the Galerkin method in two cases: the first one was when W is a cylindrical white noise process, then for any initial data u_0, $\mathcal{F}_0$ measurable with values in $H^1(G)$, $u \in C([0, T]; H^1(G))$ a.s. The second result of existence was given when the Weiner process has a covariance matrix Q satifying

$$Tr(A^{\delta-1}Q) < +\infty, \tag{1.3}$$

for some $\delta > 0$, where A is the realization of the Laplace operator on $L^2(G)$ with Neumann boundary condition.

In this work we study the former problem

$$(P) \begin{cases} v_t + \Delta^2 v - \Delta f(v) = W_t, & x \in D, t > 0 \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial \Delta v}{\partial \nu} = 0, & x \in \partial D, t \geq 0 \\ v(x, 0) = v_0(x), & x \in D \end{cases},$$

where D is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂D. The function f is such that $f(s) = s^3 - s$ and the function $W = W(x, t)$ is a Q-Brownian motion. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(D)$, $\{e_l\}_{l\geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q and $\{\lambda_l\}_{l\geq 1}$ be the corresponding eigenvalues, so that

$$Qe_l = \lambda_l e_l,$$

for all $l \geq 1$. Since Q is of trace-class, it follows that

(1.4) $$TrQ = \sum_{l=1}^{\infty} \langle Qe_l, e_l \rangle_{L^2(D)} = \sum_{l=1}^{\infty} \lambda_l \leq \Lambda_0,$$

for some positive constant Λ_0. We suppose furthermore that $e_l \in H^1(D) \cap L^\infty(D)$, for $l = 1, 2, \dots$ and that there exist positive constants Λ_1 and Λ_2 such that

(1.5) $$\sum_{l=1}^{\infty} \lambda_l \|e_l\|^2_{L^\infty(D)} \leq \Lambda_1$$

and

(1.6) $$\sum_{l=1}^{\infty} \lambda_l \|\nabla e_l\|^2_{L^2(D)} \leq \Lambda_2.$$

Let (Ω, F, P) be a probability space equipped with a filtration (F_t) and $\{B_l(t)\}_{l\geq 1}$ be a sequence of independent (F_t)-Brownian motions on (Ω, F, P). The Q-Wiener process W is defined by

(1.7) $$W(x,t) = \sum_{l=1}^{\infty} B_l(t) Q^{1/2} e_l(x) = \sum_{l=1}^{\infty} \sqrt{\lambda_l} B_l(t) e_l(x),$$

in $L^2(D)$. We recall that a Brownian motion $B(t)$ is called an (F_t) Brownian motion if it is F_t-adapted and the increment $B(t) - B(s)$ is independent of F_s for every $0 \leq s < t$. where ν is the outer normal vector to ∂D.

The initial function v_0 is such that $v_0 \in L^2(D)$.

We denote by $m(v) = \dfrac{1}{|D|} \int_D v(x)dx$ and define the following spaces $H = \{v \in L^2(\Omega \times D), m(v) = 0\}$, $V = \{v \in H^2(D), \dfrac{\partial v}{\partial \nu} = 0$ on $\partial D\}$ and $Z = V \cap L^4(D)$, where $\|.\|$ is the norm corresponding to the space H. We also define $\langle .,. \rangle_{Z^*,Z}$ as the duality product between Z and its dual space $Z^* = V^* + L^{\frac{4}{3}}(D)$. The proof of the existence and uniqueness of the solution of Problem (P) is based on a Galerkin method together with a monotonicity argument simillar to that used in [16].

The rest of this paper is organized as follows: In section 2 a stochastic auxiliary problem is introduced with a change of function to obtain an equation without the noise term, which simplifies later the use of the Galerkin method. Then the existence and uniqueness of the solution of the auxiliary problem is proved. In section 3, uniform bounds are obtained for the approximate solution in $L^\infty(0,T;H) \cap L^2(\Omega \times (0,T); V) \cap L^4(\Omega \times D \times (0,T))$, to deduce that the approximate weak solution converges weakly along a subsequence to a limit. After that the limit of the reaction term should be identified by the monotonicity method.

Finally the uniqueness of the weak solution which in turn implies the convergence of the whole sequence is proved.

2. A PRELIMINARY CHANGE OF FUNCTIONS

Define the linear unbounded operator on H by

$$Av = -\Delta v, \quad v \in D(A) = V,$$

and consider the linear equation

$$(P_1) \begin{cases} \dfrac{\partial W_A}{\partial t} + A^2 W_A = W_t, & x \in D, t > 0 \\ \dfrac{\partial W_A}{\partial \nu} = \dfrac{\partial A W_A}{\partial \nu} = 0, & x \in \partial D, t \geq 0 \\ W_A(x,0) = 0, & x \in D \end{cases} .$$

Definition 2.1. *We say that W_A is a strong solution of Problem (P_1) if*

$(i)\, W_A \in L^\infty(0,T; L^2(\Omega \times D)) \cap L^2(\Omega \times (0,T); H^2(D))$;
$(ii)\, W_A \in L^2(\Omega; C([0,T]; L^2(D)))$;
$(iii)\, W_A$ *satisfies a.s. for all* $t \in (0,T)$ *the problem*

$$\begin{cases} W_A(t) = -\displaystyle\int_0^t A^2 W_A(s)ds + W(t), \textit{in } L^2(D), \\ \dfrac{\partial W_A}{\partial \nu} = \dfrac{\partial A W_A}{\partial \nu} = 0, \textit{ in suitable sense of trace on } \partial D. \end{cases} \tag{2.1}$$

The unique solution W_A of Problem (P_1) exists strongly, more precisely

$$W_A \in L^\infty(0,T; L^4(\Omega \times D)). \tag{2.2}$$

A result of existence and uniqueness of the solution W_A of Problem (P_1) will be proved in this section. To that puropose some a priori estimates for a Galerkin approximation in $L^\infty(0,T; L^2(\Omega\times D)) \cap L^2(\Omega\times(0,T); H^1(D)) \cap L^2(\Omega\times(0,T); H^2(D))$ as in [13](p.2363 (2.13)) will be used.

Theorem 2.1. *There exists a unique solution to Problem (P_1).*

Proof. We use a Galerkin approximation. Let $m \in \mathbb{N}$ and define P_m as the orthogonal projection on $H_m = span\{\omega_1, \dots, \omega_m\}$, where $P_m a := \sum_{j=1}^m (\int_D a\omega_j)\omega_j$ for

$a \in L^2(D)$. The Galerkin approximation of W_A is given by $W_A^m = P_m W_A$, such that

$$W_A^m(t) = -\int_0^t P_m[A^2 W_A^m(s)]ds + \sum_{l=1}^m P_m(\sqrt{\lambda_l} e_l)\beta_l(t) \ a.s. \tag{2.3}$$

Note that (cf. [4] p.193)

$$\|P_m a\|_{H^1(D)} \leq \|a\|_{H^1(D)} \tag{2.4}$$

and that (cf. [13] Remark 2.3)

$$P_m a \longrightarrow a, \text{in } H^1(D) \text{ as } m \longrightarrow \infty. \tag{2.5}$$

This implies in particular that

$$P_m a \longrightarrow a, \text{ in } L^2(D) \text{ as } m \longrightarrow \infty. \tag{2.6}$$

In addition

$$P_m A u_m = A u_m. \tag{2.7}$$

The next Lemma gives a priori estimates for a Galerkin approximation.

Lemma 2.1. *There exists a positive constant K such that*

$$\sup_{t\in(0,T)} \mathbb{E} \int_D (W_A^m)^2 dx \leq K, \tag{2.8}$$

$$\mathbb{E} \int_0^T \int_D (W_A^m)^2 dxdt \leq K, \tag{2.9}$$

$$\mathbb{E} \int_0^T \int_D |AW_A^m|^2 dxdt \leq K, \tag{2.10}$$

$$\mathbb{E} \int_0^T \|P_m(A^2 W_A^m)\|^2_{L^2(D)} \leq K. \tag{2.11}$$

Proof. Recall that Itô's formula is established, as in [20] p 16-17, which is based on [14] (p 153, Theorem 3.6). It is applicable to systems of stochastic ordinary differential equations (S.O.D.E), presented in the following Lemma, which is equivalent to Lemma 1.5.3 of [8].

Lemma 2.2. *For a smooth vector function h and an adapted process $(g(t), t \geq 0)$ with $\int_0^T |g(t)|dt < \infty$ a.s., set*

$$X(t) := \int_0^t g(s)ds + \int_0^t hdW(s), \quad 0 \leq t \leq T, \text{ for all } T > 0,$$

where h is a vector of components h_l, $l = 1, .., m$ and dW is a vector of components $d\beta_l$, $l = 1, .., m$ with β_l a one-dimensional Brownian motion. Then, for F twice continuously differentiable in X and continuously differentiable in t, one has

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) &+ \int_0^t F_t(s, X(s))ds + \int_0^t F_x(X(s))g(s)ds \\ &+ \int_0^t F_x(X(s))h(s)dW(s) + \frac{1}{2}\sum_{l=1}^m \int_0^t F_{xx}(X(s))h_l^2 ds. \end{aligned} \tag{2.12}$$

Next we apply Lemma (2.2) to equation (2.3) with $hdW = \sum_{l=1}^m P_m\sqrt{\lambda_l}e_l d\beta_l(s)$ and $h_l = P_m\sqrt{\lambda_l}e_l$. Supposing that F does not depend on time and setting

$$\begin{aligned} X(t) =& W_A^m(t), \\ F(X(t)) =& (X(t))^2, \\ F_x(X(t)) =& 2X(t), \\ F_{xx}(X(t)) =& 2, \\ g(s) =& -P_m(A^2 W_A^m(s)), \end{aligned}$$

where in this case F does not depend on t. We integrate on D and we obtain almost surely, for all $t \in [0, T]$

$$\begin{aligned} \int_D W_A^m(x,t)^2 dx = -2\int_0^t \int_D & W_A^m P_m[A^2 W_A^m(s)]dxds \\ &+ 2\sum_{l=1}^m \int_0^t \int_D W_A^m P_m(\sqrt{\lambda_l}e_l)dxd\beta_l(s) \\ &+ \int_0^t \sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|_{L^2(D)}^2 dxds. \end{aligned} \tag{2.13}$$

Substituting (2.7) into (2.13) and taking the expectation with the relation $2\mathbb{E}[\sum_{l=1}^m \int_0^t \int_D W_A^m \sqrt{\lambda_l}e_l dxd\beta_l(s)] = 0$ ([15] Theorem (2.3.4) p 11), we obtain

$$\begin{aligned}(2.14)\qquad &\mathbb{E}\|W_A^m(t)\|^2_{L^2(D)} + 2\mathbb{E}\int_0^t\int_D (AW_A^m(s))^2dxds\\ &= \mathbb{E}\int_0^t\sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}dxds.\end{aligned}$$

Using (2.4), (1.4) and (1.6) yields

$$(2.15)\quad \sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} \le \sum_{l=1}^m(\|\sqrt{\lambda_l}e_l\|^2_{L^2(D)}+\|\nabla(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)}) \le (\Lambda_0+\Lambda_2).$$

Taking the supremum of equation (2.14) and substituting (2.15) into (2.14) we obtain

$$\sup_{t\in(0,T)}\mathbb{E}\|W_A^m(t)\|^2_{L^2(D)} \le T(\Lambda_0+\Lambda_2) \le K.$$

This completes the proof of (2.8).

From equation (2.3), we have for $x \in D$

$$(2.16)\qquad AW_A^m(t) = -\int_0^t A\{P_m(A^2W_A^m)\}ds + \sum_{l=1}^m\int_0^t A\{P_m[\sqrt{\lambda_l}e_l]\}d\beta_l(s).$$

Fix $x \in D$ and apply again Itô's formula in Lemma (2.2) to the integral equation (2.16), for $hdW = -\sum_{l=1}^m A\{P_m\sqrt{\lambda_l}e_l\}d\beta_l(s)$ and $h_l = -A\{P_m\sqrt{\lambda_l}e_l\}$

$$\begin{aligned}X(t) &= -AW_A^m(x,t),\\ F(X(t)) &= (X(t))^2,\\ F_x(X(t)) &= 2X(t),\\ F_{xx}(X(t)) &= 2,\\ g(s) &= A\{P_m(A^2W_A^m(x,s))\}.\end{aligned}$$

After integrating over D, we obtain almost surely, for all $t \in [0,T]$

$$\begin{aligned}\int_D (AW_A^m(x,t))^2dx = &-2\int_0^t\int_D A^2W_A^m(x,s)P_mA^2W_A^m(x,s)dxds\\ &-2\sum_{l=1}^m\int_0^t\int_D \nabla(AW_A^m(x,s))\nabla\{P_m(\sqrt{\lambda_l}e_l)\}dxd\beta_l(s)\\ &+\sum_{l=1}^m\int_0^t\int_D |\nabla P_m(\nabla(\sqrt{\lambda_l}e_l))|^2dxds.\end{aligned}$$

In view of (2.7) and (2.4) we have that

$$
\begin{aligned}
\int_D (AW_A^m(t))^2 dx \leq & -2\int_0^t \|P_m A^2 W_A^m(s)\|^2_{L^2(D)} ds \\
& - 2\sum_{l=1}^m \int_0^t \int_D \nabla(AW_A^m(x,s))\nabla\{P_m(\sqrt{\lambda_l}e_l)\} dx d\beta_l(s) \\
& + \sum_{l=1}^m \int_0^t \int_D |\nabla(\nabla(\sqrt{\lambda_l}e_l))|^2 dx ds.
\end{aligned} \tag{2.17}
$$

Thus, taking the expectation of (2.17) and using (1.6) knowing that

$$
\mathbb{E}[\sum_{l=1}^m \int_0^t \int_D \nabla(AW_A^m(x,s))\nabla\{P_m(\sqrt{\lambda_l}e_l)\} dx d\beta_l(s)] = 0,
$$

we obtain

$$
\begin{aligned}
& \mathbb{E}\int_D (AW_A^m(t))^2 dx + \mathbb{E}\int_0^t \|P_m A^2 W_A^m(s)\|^2_{L^2(D)} ds \\
& \leq \Lambda_2 \sum_{l=1}^m \mathbb{E}\int_0^t \|\nabla(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} ds.
\end{aligned} \tag{2.18}
$$

Adding (2.14) to (2.18), using again (2.4) and (1.6) yields

$$
\begin{aligned}
& \mathbb{E}\int_D (AW_A^m(t))^2 dx + \mathbb{E}\|W_A^m(t)\|^2_{L^2(D)} \\
& + 2\mathbb{E}\int_0^t \int_D (AW_A^m(s))^2 dx ds + 2\mathbb{E}\int_0^t \|P_m A^2 W_A^m(s)\|^2_{L^2(D)} ds \\
& \leq \mathbb{E}\int_0^t \sum_{l=1}^m \|P_m(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} dx ds + \Lambda_2 \sum_{l=1}^m \mathbb{E}\int_0^t \|\nabla(\sqrt{\lambda_l}e_l)\|^2_{L^2(D)} ds \\
& \leq c_0 \mathbb{E}\int_0^t (\sum_{l=1}^m \lambda_l \|e_l\|^2_{L^2(D)} + \sum_{l=1}^m \lambda_l \|\nabla e_l\|^2_{L^2(D)}) ds \\
& \leq c_0 T(\Lambda_0 + \Lambda_2),
\end{aligned}
$$

where $c_0 = \max(1;\Lambda_2)$. Which completes the proof of (2.9)-(2.11) and ends the proof of Lemma (2.1).

□

Hence there exists a subsequence which we denote again by W_A^m and a function $W_A \in L^2(\Omega \times (0,T); H^2(D)) \bigcap L^\infty(0,T; L^2(\Omega \times D))$ such that as $m \longrightarrow \infty$

$$W_A^m \rightharpoonup W_A, \text{ weakly in } L^2(\Omega \times (0,T); H^2(D)), \tag{2.19}$$

$$W_A^m \rightharpoonup W_A, \text{ weakly star in } L^\infty(0,T; L^2(\Omega \times D)), \tag{2.20}$$

$$P_m A^2 W_A^m \rightharpoonup A^2 W_A, \text{ weakly in } L^2(\Omega \times (0,T); L^2(D)). \tag{2.21}$$

In addition, we have the following result.

Lemma 2.3. *[8]*

$$\sum_{l=1}^{m} P_m(\sqrt{\lambda_l} e_l)\beta_l(t) \longrightarrow \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t), \textit{ in } L^\infty((0,T); L^2(\Omega; L^2(D))), \tag{2.22}$$

as $m \longrightarrow \infty$.

Let y be an arbitrary bounded random variable and let ψ be an arbitrary bounded function on $(0,T)$. We multiply equation (2.3) by the product $y\psi$, integrate on D between 0 and T and take the expectation to obtain

$$\mathbb{E}\int_0^T \int_D y\psi(t) W_A^m w_j dx dt = \mathbb{E}\int_0^T y\psi(t)\{-\int_0^t \langle P_m(A^2 W_A^m), w_j\rangle ds\} dt$$
$$+ \mathbb{E}\int_0^T y\psi(t)\{\int_D \sum_{l=1}^{m} P_m(\sqrt{\lambda_l} e_l)\beta_l(t) w_j dx\} dt.$$

Passing to the limit when $m \longrightarrow \infty$, using (2.19)-(2.22) (where the linear combinitions of w_j are dense in $H^2(D)$), yields

$$\mathbb{E}\int_0^T \int_D y\psi(t) W_A \tilde{w} dx dt = -\mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle A^2 W_A, \tilde{w}\rangle ds\} dt$$
$$+ \mathbb{E}\int_0^T y\psi(t)\{\int_D \sum_{l=1}^{\infty} (\sqrt{\lambda_l} e_l)\beta_l(t)\tilde{w} dx\} dt,$$

for all $\tilde{w} \in H^2(D)$. Therefore, we deduce that

$$W_A(t) = -\int_0^t A^2 W_A(s) ds + \sum_{l=1}^{\infty} \sqrt{\lambda_l} e_l \beta_l(t), \text{ on } \Omega \times (0,T) \times D. \tag{2.23}$$

One finally concludes that W_A satisfies Definition (2.1).

We prove bellow that W_A is in $L^\infty(0,T; L^4(\Omega \times D))$ (this result is based on [1] and [8]).

Theorem 2.2. *Let W_A be a solution of Problem (P_1) then $W_A \in L^\infty(0,T;L^4(\Omega\times D))$.*

Proof. We define the function $\Phi_k : \mathbb{R} \to \mathbb{R}$ for all positive constant k, by

$$\Phi_k(y) = \begin{cases} |y|^4, & \text{if } 0 \le |y| < k \\ 6k^2y^2 - 8k^3|y| + 3k^4, & \text{if } k \le |y| \end{cases},$$

where Φ_k is a convex C^2 function and Φ_k' is a Lipschitz-continuous function with $\Phi_k'(0) = 0$. The function Φ_k satisfies

$$0 \le \Phi_k'(y) \le c(k)y \text{ and } 0 \le \Phi_k(y) = \int_0^y \Phi_k'(\zeta)d\zeta \le \frac{c(k)}{2}y^2 \text{for all } y \in \mathbb{R}^+.$$

Then, we deduce from the Definition (2.1) (i) that

$$\mathbb{E}\int_D \Phi_k(W_A(x,t))dx \le \frac{c(k)}{2}\mathbb{E}\int_D W_A^2(x,t)dx \le \bar{c}(k), \quad \text{for a.e. } t \in [0,T].$$

Lemma 2.4.

(i) *One has $0 \le \Phi_k''(y) \le c(k)$ for all $y \in \mathbb{R}$ where c_k is a positive constant depending on k.*

(ii) *For any $y \in \mathbb{R}$, one has $0 \le \Phi_k''(y) \le 12(1 + \Phi_k(y))$.*

Proof. (i) We have

$$\Phi_k''(y) = \begin{cases} 12|y|^2, & \text{if } 0 \le |y| < k \\ 12k^2, & \text{if } k \le |y| \end{cases}.$$

Thus $\Phi_k''(y) \le 12k^2 =: c_k$.

(ii) If $|y| < k$, $\Phi_k''(y) = 12|y|^2$. But if $1 \le |y| < k$, $|y|^2 \le |y|^4$ which gives the result and if $0 \le |y| < 1$, $|y|^4 \le |y|^2 < 1$ then $|y|^2 \le 1 + |y|^4$. If $|y| \ge k$, $\Phi_k''(y) - 12(1 + \Phi_k(y)) \le 0$. ☐

Lemma 2.5. *Let h be an $L^2(D)$-valued progressively measurable Bochner integrable process. Consider the following well defined process*

$$X(t) = \int_0^t h(s)ds + W(t), \quad t \in [0,T].$$

Assume that a function $F : [0,T] \times L^2(D) \longrightarrow \mathbb{R}$ and its partial derivatives F_t, F_x, F_{xx} are uniformly continuous on bounded subsets of $[0,T] \times L^2(D)$ and that $F(X(0),0) = 0$. Then, a.s. for all $t \in [0,T]$

$$F(t,X(t)) = \int_0^t F_t(s,X(s))ds + \int_0^t \langle F_x(s,X(s)), h(s)\rangle_{L^2(D)}ds$$
$$+ \int_0^t \langle F_x(s,X(s)), dW(s)\rangle_{L^2(D)}ds + \frac{1}{2}\int_0^t Tr[F_{xx}(s,X(s))Q]ds,$$

where

$$Tr[F_{xx}(X(s))Q] = \sum_{l=1}^{\infty}\langle F_{xx}(s,X(s))Qe_l, e_l\rangle_{L^2(D)}$$

and

$$\langle u,v\rangle_{L^2(D)} = \int_D u(x)v(x)dx,$$

where $TrA = \sum_{l=1}^{\infty}\langle Ae_l, e_l\rangle_{L^2(D)}$ is a bounded linear operator on $L^2(D)$.

Applying Lemma 2.5 to equation (2.1), with the supposition that F does not depend on time and setting

$$X(t) = W_A(t),$$
$$F(X(t)) = \int_D \Phi_k(X(t))dx,$$
$$F_x(X(t)) = \Phi_k'(X(t)),$$
$$h = -A^2W_A,$$
$$F_{xx}(X(t)) = \Phi_k''(X(t)).$$

Taking the expectation, applying Green's formula and considering the fact that $\mathbb{E}\int_0^t\int_D \Phi_k'(W_A)dW(s) = 0$ (see [15] Theorem (2.3.4) p 11), gives

$$\mathbb{E}\int_D \Phi_k(W_A(t))dx \leq \frac{1}{2}\Lambda_1\mathbb{E}\int_0^t\int_D \Phi_k''(W_A(t))dxds.$$

Using Lemma (2.4) (ii) and the Gronwall's Lemma we obtain

$$\mathbb{E}\int_D \Phi_k(W_A(t))dx \leq 6\Lambda_1 t|D|\exp(6\Lambda_1 t).$$

Since $\Phi_k(W_A(x,t))$ converges to $|W_A(x,t)|^4$ for a.e. x and t when k goes to infinity, it follows from Fatou's Lemma for all $t > 0$ that

$$\mathbb{E}\int_D |W_A(x,t)|^4 dx = \lim_{k\to\infty}\inf \mathbb{E}\int_D \Phi_k(W_A(x,t))dx \le 6\Lambda_1 t|D|\exp(6\Lambda_1 t).$$

Therefore, $W_A \in L^\infty((0,T);L^4(\Omega\times D))$. □

To prove the uniqueness of the solution W_A of Problem (P_1), suppose the existence of two solutions of Problem (P_1), $W_A^1 = W_A^1(\omega,x,t)$ and $W_A^2 = W_A^2(\omega,x,t)$ satisfying

$$W_A^i(\omega,.,.) \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^2(D)), \text{for } i=1,2.$$

So in $L^2(D\times(0,T))$

$$W_A^1 - W_A^2 = -\int_0^t \{A^2 W_A^1(s) - A^2 W_A^2(s)\}ds. \tag{2.24}$$

Taking the duality product of this equation with $W_A^1 - W_A^2 \in L^2((0,T);H^2(D))$ gives

$$\|W_A^1 - W_A^2\|^2_{L^2(D)} \le -\int_0^t \|A(W_A^1 - W_A^2)\|^2_{L^2(D)}, \tag{2.25}$$

which implies that

$$W_A^1 = W_A^2 \quad \text{a.e. in } D\times(0,T).$$

This completes the proof of Theorem (2.1). □

We perform the change of functions $u(t) := v(t) - W_A(t)$, then v is a solution of (P) if and only if u satisfies

$$(P_2)\begin{cases} \dfrac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u+W_A) = 0, & x\in D, t>0 \\ \dfrac{\partial(u+W_A)}{\partial\nu} = \dfrac{\partial\Delta(u+W_A)}{\partial\nu} = 0, & x\in\partial D, t\ge 0. \\ u(x,0) = v_0(x), & x\in D \end{cases}$$

This model is mass conserved, namely

$$\frac{1}{|D|}\int_D u(x,t)dx = \frac{1}{|D|}\int_D v_0(x)dx, \text{ a.s. for a.e. } t\in\mathbb{R}^+.$$

The function f satisfies the following hypotheses

(H_1) $-C_3 s^4 - C_4 \le -f(s)s \le -C_1 s^4 + C_2$,

(H_2) $|f(s)| \leq C_5|s|^3 + C_6,$
(H_3) $f'(s) \geq C_1' s^2 - C_2',$
(H_4) $|f'(s)| \leq C_3' s^2 + C_4',$
(H_5) $f'(s) > C_5',$

for $C_i > 0$, $\forall i = 1, \cdots, 6$ and $C_j' > 0$, $\forall j = 1, \cdots, 5$.

Definition 2.2. *We say that u is a solution of Problem (P_2) if*

(i) $u \in L^\infty(0,T;H) \cap L^2(\Omega \times (0,T);V) \cap L^4(\Omega \times (0,T) \times D)$;

(ii) u *satisfies almost surely the problem, for all $t \in (0,T)$:*

$$(2.26)\qquad \begin{cases} u(t) = v_0 - \displaystyle\int_0^t \Delta^2 u ds + \int_0^t \Delta f(u + W_A) ds, \\ \qquad \textit{in the sense of distributions,} \\ \dfrac{\partial(u + W_A)}{\partial \nu} = \dfrac{\partial \Delta(u + W_A)}{\partial \nu} = 0, \\ \qquad \textit{in a suitable sense of trace on } \partial D. \end{cases}$$

3. Existence and Uniqueness of a solution of Problem (P_2)

3.1. Existence of a solution to Problem (P_2).

The following theorem contains the main result of this paper.

Theorem 3.1. *There exists a unique solution of Problem (P_2).*

Proof. We apply the Galerkin method to prove the existence of a solution to Problem (P_2).

Denote by $0 < \gamma_1 < \gamma_2 \leq \ldots \leq \gamma_{\tilde{k}} \leq \ldots$ the eignvalues of the operator $-\Delta$ with homogenous boundary conditions and by $w_{\tilde{k}}$, $\tilde{k} = 0, \ldots$ the corresponding unit eigenfunctions in $L^2(D)$. Note that they are smooth functions.

We look for an approximate solution of the form

$$u_m(x,t) = \sum_{i=1}^m d_{im}(t) w_i,$$

such that the function u_m satisfies the equations

$$(3.1)\qquad \int_D \frac{\partial u_m(x,t)}{\partial t} w_i dx + \int_D \Delta^2 u_m w_i dx - \int_D \Delta f(u_m + W_A) w_i dx = 0,$$

for all $i = 1, \ldots, m$. Remark that $u_m(x,0) = \sum_{i=1}^{m} \langle v_0, w_i \rangle w_i$ converges strongly to v_0 in $L^2(D)$ as $m \to \infty$.

Problem (3.1) is an initial value problem for a system of m ordinary differential equations. So that, it has a unique solution on some interval $(0, T_m), T_m > 0$. We multiply (3.1) by $d_{im} = d_{im}(t)$ and sum on $i = 1, \ldots, m$ to obtain

$$\int_D \frac{\partial u_m(x,t)}{\partial t} u_m dx + \int_D \Delta^2 u_m u_m dx = \int_D \Delta f(u_m + W_A) u_m dx, \tag{3.2}$$

an integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\int_D u_m^2 dx + \int_D (\Delta u_m)^2 dx = \int_D f(u_m + W_A)\Delta u_m dx. \tag{3.3}$$

Using the Green's formula and by the Neumann boundary condition we have

$$\begin{aligned}\int_D f(u_m + W_A)\Delta u_m dx = &- \int_D f'(u_m + W_A)|\nabla u_m|^2 \\ &- \int_D f'(u_m + W_A)\nabla u_m \nabla W_A dx \\ = &\; I_1 + I_2.\end{aligned}$$

For I_1 an application of (H_3) and the Poincare's inequality: $\{\exists C' = C_\Omega > 0 : \int_D u_m^2 dx \leq C' \int_D |\nabla u_m|^2\}$ yields

$$I_1 \leq -\frac{C_1'}{C'}\int_D u_m^4 dx + \frac{C_1'}{C'}\int_D u_m^2 W_A^2 dx + \int_D |(\frac{2C_1'}{C'}W_A)u_m + \frac{C_2'}{C'}|u_m^2 dx$$

and by Cauchy Shwartz's inequality and Young's inequality it will be

$$\begin{aligned}I_1 \leq &-\frac{C_1'}{C'}\int_D u_m^4 dx + (\frac{3}{2} + \varepsilon)\int_D u_m^4 dx + \frac{1}{2}[(\frac{C_1'}{C'})^2 \\ &+ 2(\frac{C_1'}{C'})^4]\int_D W_A^4 dx + [C_\varepsilon + \frac{1}{2}(\frac{C_2'}{C'})^4]|D|.\end{aligned}$$

Moreover by (H_5), Green's formula and Young's inequality I_2 satifies

$$I_2 \leq \varepsilon \int_D (\Delta u_m)^2 dx + \frac{(C_5')^4}{2}\int_D W_A^4 dx + C_\varepsilon |D|.$$

Thus

$$\begin{aligned} I_1 + I_2 &= \int_D f(u_m + W_A)\Delta u_m \\ &\le -(\frac{C_1'}{C'} - \frac{3}{2} - \varepsilon)\int_D u_m^4 dx + \varepsilon \int_D (\Delta u_m)^2 dx + [\frac{(C_5')^4}{2} + \frac{1}{2}(\frac{C_1'}{C'})^2 \\ &\quad + \frac{1}{2}(\frac{C_1'}{C'})^4]\int_D W_A^4 dx + [\frac{1}{2}(\frac{C_2'}{C'})^4 + 2C_\varepsilon]|D|, \end{aligned}$$

where $\varepsilon > 0$ is given such that $(\frac{C_1'}{C'} - \frac{3}{2} - \varepsilon) > 0$. Set $\tilde{C}_\varepsilon = \frac{C_1'}{C'} - \frac{3}{2} - \varepsilon$, $\quad \tilde{C}_1 = \frac{(C_5')^4}{2} + \frac{1}{2}(\frac{C_1'}{C'})^2 + 2(\frac{C_1'}{C'})^4$, $\quad \tilde{C}_2 = \frac{1}{2}(\frac{C_2'}{C'})^4 + 2C_\varepsilon$, to conclude that

$$\begin{aligned} \int_D f(u_m + W_A)\Delta u_m &\le -\tilde{C}_\varepsilon \int_D u_m^4 dx + \varepsilon \int_D (\Delta u_m)^2 dx \\ &\quad + \tilde{C}_1 \int_D W_A^4 dx + \tilde{C}_2 |D|, \end{aligned} \tag{3.4}$$

then (3.3) will be

$$\frac{1}{2}\frac{d}{dt}\int_D u_m^2 dx + \bar{C}_\varepsilon \int_D (\Delta u_m)^2 dx + \tilde{C}_\varepsilon \int_D u_m^4 dx \le \tilde{C}_1 \int_D W_A^4 dx + \tilde{C}_2 |D|, \tag{3.5}$$

with $\varepsilon > 0$ is such that $\bar{C}_\varepsilon = 1 - \varepsilon, (\varepsilon < 1 \ \text{ and } \ \varepsilon < \frac{C_1'}{C'} - \frac{3}{2})$.

***(i)* A priori estimates**

In what follows, we derive a priori estimates for the function u_m.

Lemma 3.1. *There exists a positive constant ϱ such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left(\int_D (u_m)^2 dx\right) \le \varrho, \tag{3.6}$$

$$\mathbb{E}\left(\int_0^T \int_D |\Delta u_m|^2 dx dt\right) \le \varrho, \tag{3.7}$$

$$\mathbb{E}\left(\int_0^T \int_D (u_m)^4 dx dt\right) \le \varrho, \tag{3.8}$$

$$\mathbb{E}\left(\int_0^T \int_D (\Delta f(u_m + W_A))^{\frac{4}{3}} dx dt\right) \le \varrho. \tag{3.9}$$

Proof. Integrating in time (3.5) and taking the expectation, we deduce thanks to (2.2) that for all $t \in [0,T]$

$$\frac{1}{2}\mathbb{E}\left(\int_D u_m^2(t)dx\right) + \bar{C}_\varepsilon \mathbb{E}\left(\int_0^t \int_D (\Delta u_m)^2 dxds\right) + \tilde{C}_\varepsilon \mathbb{E}\left(\int_0^t \int_D u_m^4 dxds\right) \leq \frac{1}{2}\|u_0\|^2_{L^2(D)}dx + \tilde{C}_2|D|T + \tilde{c_1}T \leq K,$$

thus

$$\mathbb{E}\left(\int_D u_m^2(t)dx\right) \leq 2K, \quad \forall t \in [0,T],$$
$$\mathbb{E}\left(\int_0^T \int_D (\Delta u_m)^2 dxdt\right) \leq \frac{K}{\bar{C}_\varepsilon},$$
$$\mathbb{E}\left(\int_0^T \int_D u_m^4 dxds\right) \leq \frac{K}{\tilde{C}_\varepsilon}.$$

Therefore u_m is bounded independently of m in $L^\infty((0,T);H) \cap L^2(\Omega \times (0,T);V) \cap L^4(\Omega \times D \times (0,T))$.

By hypothesis (H_2) we deduce that

$$\mathbb{E}\left(\|f(u_m+W_A)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(D\times(0,T))}\right) \leq 2^{\frac{1}{3}}\mathbb{E}\left(\int_0^T \int_D \tilde{C_5}\left[(|u_m|+|W_A|)^3\right]^{\frac{4}{3}} dxdt\right) + \tilde{C_6}|D|T,$$

where $\tilde{C_5} = C_5^{4/3}$ and $\tilde{C_6} = 2^{\frac{1}{3}} \times \tilde{C_6}^{4/3}$. By Minkowski's inequality for $c_5 = 2^{\frac{1}{3}} \times \tilde{C_5}$

$$\mathbb{E}\left(\|f(u_m+W_A)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(D\times(0,T))}\right) \leq c_5\mathbb{E}\left(\int_0^T \int_D |u_m|^4 dxdt\right) + c_5\mathbb{E}\left(\int_0^T \int_D |W_A|^4 dxdt\right) + \tilde{C_6}|D|T.$$

In view of (3.8) and (2.2) we have

$$\mathbb{E}\left(\|f(u_m+W_A)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(D\times(0,T))}\right) \leq K_1.$$

Then

$$\mathbb{E}\left(\|\Delta f(u_m+W_A)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(D\times(0,T))}\right) \leq K_1.$$

□

Hence there exists a subsequence which we denote again by $\{u_m\}$ and a function $u \in L^2(\Omega \times (0,T);V) \cap L^4(\Omega \times D \times (0,T)) \cap L^\infty(0,T;H)$ such that as $m \longrightarrow \infty$

(3.10) $u_m \rightharpoonup u$ weakly in $L^2(\Omega \times (0,T);V)$ and $L^4(\Omega \times (0,T) \times D)$,

(3.11) $u_m \rightharpoonup u$ weakly star in $L^\infty(0,T;H)$,

(3.12) $\Delta f(u_m + W_A) \rightharpoonup \Delta\chi$ in $L^{\frac{4}{3}}(\Omega \times (0,T) \times D)$.

Integrating in time equation (3.1) yields

$$\int_D u_m(x,t) w_j = \int_D u_m(0) w_j dx - \int_0^t \langle \Delta^2 u_m, w_j \rangle + \int_0^t \int_D \Delta f(u_m + W_A) w_j dxds, \tag{3.13}$$

for all $j = 1, \ldots, m$. Let $y = y(\omega)$ be an arbitrary bounded random variable and let ψ be an arbitrary bounded function on $(0,T)$. We multiply equation (3.13) by the product $y\psi$, integrate between 0 and T and take the expectation to deduce that

$$\begin{aligned} &\mathbb{E}\int_0^T \int_D y\psi(t) u_m(t) w_j dxdt \\ &= \mathbb{E}\int_0^T \int_D y\psi(t) u_m(0) w_j dxdt \\ &- \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \langle \Delta^2 u_m, w_j\rangle ds\} dt \\ &+ \mathbb{E}\int_0^T y\psi(t)\{\int_0^t \int_D \Delta f(u_m + W_A) w_j dxds\} dt. \end{aligned} \tag{3.14}$$

for all $j = 1, \ldots, m$. Next we pass to the limit in (3.14) (we only give the proof of convergence for the last term using the a priori estimates and Hölder's inequality) to get

$$\begin{aligned} &|\psi(t)\mathbb{E}\int_0^t \int_D \Delta f(u_m + W_A) y w_j dxds| \\ &\leq \|y\|_{L^\infty(\Omega)} |\psi(t)| \left(\mathbb{E}\int_0^t \int_D |\Delta f(u_m + W_A)|^{\frac{4}{3}} dxds\right)^{\frac{3}{4}} \left(\mathbb{E}\int_0^t \int_D |w_j|^4 dxds\right)^{\frac{1}{4}} \\ &\leq \bar{C}\|y\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(0,T)}. \end{aligned}$$

This shows that $|\psi(t)\mathbb{E}\int_0^t\int_D \Delta f(u_m + W_A)yw_jdxds|$ is uniformly bounded by a function belonging to $L^1(0,T)$. In addition using (3.12) we have that

$$\psi(t)\mathbb{E}\int_0^t\int_D \Delta f(u_m + W_A)yw_jdxds \longrightarrow \psi(t)\mathbb{E}\int_0^t\int_D \Delta\chi yw_jdxds$$

for a.e. $t \in (0,T)$. Applying Lebesgue's dominated convergence gives

$$\begin{aligned}
&\lim_{m\longrightarrow+\infty}\int_0^T \psi(t)dt\mathbb{E}\int_0^t\int_D \Delta f(u_m + W_A)yw_jdxds\\
&=\int_0^T \lim_{m\longrightarrow+\infty}\psi(t)\mathbb{E}\int_0^t\int_D \Delta f(u_m + W_A)yw_jdxdsdt\\
&=\mathbb{E}\int_0^T y\psi(t)dt\{\int_0^t\int_D \Delta\chi w_jdxds\}.
\end{aligned}$$

Performing a similar proof for each term in (3.14), passing to the limit by using Lebesgue's dominated convergence Theorem, yields

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_D y\psi(t)u(t)w_jdxdt\\
&=\mathbb{E}\int_0^T\int_D y\psi(t)v_0w_jdxdt\\
&-\mathbb{E}\int_0^T y\psi(t)\{\int_0^t\langle\Delta^2u, w_j\rangle ds\}dt\\
&+\mathbb{E}\int_0^T y\psi(t)\{\int_0^t\int_D \Delta\chi w_jdxds\}dt, \quad \text{for all } \ j = 1,\dots,m.
\end{aligned} \tag{3.15}$$

We remark that the linear combinations of w_j are dense in $V \cap L^4(D)$, so that

$$\begin{aligned}
\mathbb{E}\int_0^T\int_D y\psi(t)u(t)\tilde{w}dxdt &= \mathbb{E}\int_0^T\int_D y\psi(t)v_0\tilde{w}dxdt\\
-\mathbb{E}\int_0^T y\psi(t)\{\int_0^t\langle\Delta^2u, \tilde{w}\rangle ds\}dt &+ \mathbb{E}\int_0^T y\psi(t)\{\int_0^t\int_D \Delta\chi\tilde{w}dxds\}dt,
\end{aligned}$$

for all $\tilde{w} \in V \cap L^4(D), y \in L^\infty(\Omega)$ and $\psi \in L^\infty(0,T)$. This implies that for a.e. $(w,t) \in \Omega \times (0,T)$

$$\langle u(t), \tilde{w}\rangle = \langle v_0, \tilde{w}\rangle - \int_0^t\langle\Delta^2u, \tilde{w}\rangle ds + \int_0^t\langle\Delta\chi, \tilde{w}\rangle ds \tag{3.16}$$

for all $\tilde{w} \in V \cap L^4(D)$.

Lemma 3.2. *[22] The function u is such that $u \in C([0,T]; L^2(D))$ a.s.*

It remains to prove that $\langle \Delta\chi, \tilde{w}\rangle = \langle \Delta f(u + W_A(t)), \tilde{w}\rangle$, for all $\tilde{w} \in V \cap L^4(D)$; which is by the monotonicity method.

***(ii)* The monotonicity argument**

Let ω be such that $\omega \in L^2(\Omega \times (0,T); V) \cap L^4(\Omega \times (0,T) \times D)$ and let c be a positive constant which will be fixed later. We define

$$\begin{aligned} O_m = \mathbb{E}[\int_0^T &\exp(-cs)\{2\langle -\Delta^2 u_m - (-\Delta^2\omega), u_m - \omega\rangle_{Z^\star,Z} \\ &+ 2\langle \Delta f(u_m + W_A) - \Delta f(\omega + W_A), u_m - \omega\rangle_{Z^\star,Z} \\ &- c\|u_m - \omega\|^2\}ds] \\ =& J_1 + J_2 + J_3, \end{aligned}$$

where

$$J_1 = \mathbb{E}\int_0^T \exp(-cs)\{2\langle -\Delta^2 u_m - (-\Delta^2\omega), u_m - \omega\rangle_{Z^\star,Z}\}ds,$$
$$J_2 = \mathbb{E}\int_0^T \exp(-cs)\{2\langle \Delta f(u_m + W_A) - \Delta f(\omega + W_A), u_m - \omega\rangle_{Z^\star,Z}\}ds,$$
$$J_3 = \mathbb{E}\int_0^T \exp(-cs)\{-c\|u_m - \omega\|^2\}ds.$$

Lemma 3.3.

$$O_m \leq 0.$$

Proof. First we estimate J_1

$$\begin{aligned} J_1 = \mathbb{E}\int_0^t \exp(-cs)\{-2\langle \Delta u_m - \Delta\omega, \Delta u_m \\ - \Delta\omega\rangle_{Z^\star,Z}\} = -2\mathbb{E}\int_0^T \exp(-cs)\|\Delta(u_m - \omega)\|^2 \leq 0. \end{aligned}$$

To estimate J_2, we use the Green's formula and (H_5), where

$$\begin{aligned} J_2 = -2\mathbb{E}\int_0^T &\exp(-cs)\langle f'(u_m + W_A)\nabla(u_m + W_A) \\ &- f'(\omega + W_A)\nabla(\omega + W_A), \nabla u_m - \nabla\omega\rangle_{Z^\star,Z} ds \\ &\leq -2C_5'\mathbb{E}\int_0^T \exp(-cs)||\nabla(u_m - \omega)||^2 ds \leq 0. \end{aligned}$$

Finally,

$$J_3 = -c\mathbb{E}\int_0^T \exp(-cs)\|u_m - \omega\|^2 ds \le 0.$$

□

We integrate in time equation (3.1) to obtain

$$\int_D u_m(x,T)w_j dx = \int_D u_m(0)w_j dx - \int_0^T \langle \Delta^2 u_m, w_j\rangle + \int_0^T\int_D \Delta f(u_m + W_A)w_j dxdt, \tag{3.17}$$

for all $j = 1, \dots, m$. Next we recall a chain rule formula, which can be viewed as a simplified **Itô's formula.**

Proposition 3.1. *Let X be a real valued function such that*

$$X(t) = X(0) + \int_0^t h(s)ds, \quad 0 \le s \le t$$

and suppose that h is measurable in time such that $h \in L^1(0,T)$. Suppose that the function $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ and its partial derivatives $\dfrac{\partial F}{\partial t}$ and $\dfrac{\partial F}{\partial X}$ are continuous on $[0,T]\times\mathbb{R}$. Then for all $t \in [0,T]$

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial t}(s, X(s))ds + \int_0^t \frac{\partial F}{\partial X}(s, X(s))h(s)ds. \tag{3.18}$$

Applying (3.18) to the m equations in (3.17) with

$$X_j = \int_D u_m w_j,\ j = 1, \dots, m,\ F(s,q) = \exp(-cs)q^2$$

and

$$h(s) = \langle -\Delta^2 u_m + f(u_m + W_A), w_j\rangle_{Z^\star, Z},$$

gives

$$\begin{aligned}&\exp(-cT)(\int_D u_m(x,T)w_j)^2\\ &= (\int_D u_m(0)w_j)^2 - c\int_0^T \exp(-cs)(\int_D u_m w_j)^2 ds -\end{aligned} \tag{3.19}$$

$$- 2\int_0^T \exp(-cs)\{\int_D u_m w_j\}\langle \Delta^2 u_m, w_j\rangle +$$
$$+ 2\int_0^T \exp(-cs)\{\int_D u_m w_j\}\langle f(u_m + W_A), w_j\rangle,$$

for all $j = 1, \ldots, m$.

In what follows, we will use the identity

Lemma 3.4. *Let* $F \in Z^\star$ *and* $B_m = \sum_{j=1}^m \langle B_m, w_j\rangle w_j$. *Then*

$$\sum_{j=1}^m \langle F, w_j\rangle\langle B_m, w_j\rangle = \langle F, B_m\rangle. \tag{3.20}$$

Summing each term of (3.19) on $j = 1, \ldots, m$ and applying (3.20) yields

$$\exp(-cT)\sum_{j=1}^m \langle u_m(x,T), w_j\rangle\langle u_m(x,T), w_j\rangle = \exp(-cT)||u_m(x,T)||^2.$$

Doing the same to obtain $||u_m(0)||^2$ and $-c\int_0^T \exp(-cs)||u_m(s)||^2 ds$. Also

$$-2\int_0^T \exp(-cs)\sum_{j=1}^m \langle u_m, w_j\rangle\langle \Delta^2 u_m, w_j\rangle = -2\int_0^T \exp(-cs)\langle \Delta^2 u_m, u_m\rangle$$

and

$$2\int_0^T \exp(-cs)\sum_{j=1}^m \langle u_m, w_j\rangle\langle \Delta f(u_m + W_A), w_j\rangle$$
$$= 2\int_0^T \exp(-cs)\langle u_m, \Delta f(u_m + W_A)\rangle.$$

Then

$$\begin{aligned}\mathbb{E}(\exp(-cT)\|u_m(T)\|^2) = E(\|u_m(0)\|^2) - c\mathbb{E}(\int_0^T \exp(-cs)\|u_m(s)\|^2 ds)\\ - 2\mathbb{E}(\int_0^T \exp(-cs)\langle \Delta^2 u_m, u_m\rangle_{Z^\star,Z} ds)\\ + 2\mathbb{E}(\int_0^T \exp(-cs)\langle \Delta f(u_m + W_A), u_m\rangle_{Z^\star,Z}).\end{aligned} \tag{3.21}$$

Set O_m in the form $O_m = O_m^1 + O_m^2$ where

$$O_m x‘x^1 = \mathbb{E}\Big[\int_0^T \exp(-cs)\{2\langle -\Delta u_m, u_m\rangle_{Z^*,Z} \tag{3.22}$$

$$+ 2\langle \Delta f(u_m + W_A), u_m\rangle_{Z^*,Z} - c\|u_m\|^2\}ds\Big]. \tag{3.23}$$

It follows from (3.23) and (3.21) that

$$\limsup_{m\to\infty} O_m^1 = \mathbb{E}[\exp(-cT)\|u(T)\|^2] - \mathbb{E}[\|u(0)\|^2] + \delta \exp(-cT), \tag{3.24}$$

where

$$\delta = \lim_{m\to\infty} \sup \mathbb{E}[\|u_m(T)\|^2] - \mathbb{E}[\|u(T)\|^2] \geq 0.$$

On the other hand, equation (3.16) implies that

$$u(t) = v_0 - \int_0^t \Delta^2 u ds + \int_0^t \Delta\chi ds,\ \forall t \in [0,T] \tag{3.25}$$

a.s. in $Z^\star = V^\star + L^{\frac{4}{3}}(D)$.

Next we recall a second variant of the chain rule formula, which can be viewed as a simplifide Itô's formula as in [21](p 75 Theorem (4.2.5)). Consider the Gefland triple

$$Z \subset H \subset Z^\star,$$

where $Z = V \cap L^4(D)$ and $Z^\star$ are defined in the introduction.

Proposition 3.2. *Let* $X \in L^2(0,T;V) \cap L^4(0,T;L^4(D))$ *and* $Y \in L^2(0,T;V^*) + L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(D))$ *be such that*

$$X(t) := X_0 + \int_0^t Y(s)ds, \quad t \in [0,T].$$

Suppose that the function $F : [0,T]\times Z \to \mathbb{R}$ *and its partial derivatives* $\dfrac{\partial F}{\partial t}$ *and* $\dfrac{\partial F}{\partial X}$ *are continuous on* $[0,T]\times Z$*. Then for all* $t \in [0,T]$

$$F(t,X(t)) = F(0,X(0)) + \int_0^t \frac{\partial F}{\partial t}(s,X(s))ds + \int_0^t \langle Y(s), \frac{\partial F}{\partial X}(s,X(s))\rangle_{Z^\star,Z} ds. \tag{3.26}$$

Applying Proposition 3.2 to equation (3.25) and seting

$$X(t) = u(t),\ F(s,q) = \exp(-cs)\|q\|^2 \text{ and } Y(s) = -\Delta^2 u + \Delta\chi,$$

in (3.26) to deduce that

$$\begin{aligned}\mathbb{E}[\exp(-cT)\|u(T)\|^2] = E[\|v_0\|^2] - c\mathbb{E}[\int_0^T \exp(-cs)\|u(s)\|^2 ds] \\ - 2\mathbb{E}[\int_0^T \exp(-cs)\langle \Delta^2 u, u\rangle_{Z^\star,Z} ds] \\ + 2\mathbb{E}[\int_0^T \exp(-cs)\langle \Delta\chi, u\rangle_{Z^\star,Z} ds],\end{aligned}$$

which we combine with (3.24) to deduce that

(3.27)

$$\begin{aligned}\lim_{m\to\infty}\sup O_m^1 = -2\mathbb{E}[\int_0^T \exp(-cs)\langle \Delta^2 u, u\rangle_{Z^*,Z} ds] + 2\mathbb{E}[\int_0^T \exp(-cs)\langle \Delta\chi, u\rangle_{Z^*,Z} ds] \\ - c\mathbb{E}[\int_0^T \exp(-cs)\|u(s)\|^2 ds] + \delta\exp(-cT).\end{aligned}$$

It remains to compute the limit of O_m^2. For this reason, we first simplify O_m

$$\begin{aligned}O_m = &-2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 u_m, u_m-\omega\rangle_{Z^\star,Z} + 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 \omega, u_m-\omega\rangle_{Z^\star,Z} \\ &+ 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(u_m+W_A), u_m-\omega\rangle_{Z^\star,Z} \\ &- 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(\omega+W_A), u_m-\omega\rangle_{Z^\star,Z} - c\mathbb{E}\int_0^T \exp(-cs)\|u_m-\omega\|^2 \\ = &-2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 u_m, u_m\rangle_{Z^\star,Z} + 2E\int_0^T \exp(-cs)\langle \Delta^2 u_m, \omega\rangle_{Z^\star,Z} \\ &+ 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 \omega, u_m\rangle_{Z^\star,Z} - 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 \omega, \omega\rangle_{Z^\star,Z} \\ &+ 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(u_m+W_A), u_m\rangle_{Z^\star,Z} \\ &- 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(u_m+W_A), \omega\rangle_{Z^\star,Z} \\ &- 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(\omega+W_A), u_m\rangle_{Z^\star,Z}\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(\omega + W_A), \omega\rangle_{Z^\star,Z} \\
&- c\mathbb{E}\int_0^T \exp(-cs)\|u_m\|^2 ds - cE\int_0^T \exp(-cs)\|\omega\|^2 ds \\
&+ 2c\mathbb{E}\int_0^T \exp(-cs)\langle u_m, \omega\rangle_{Z^\star,Z} ds.
\end{aligned}
$$

Here

$$
\begin{aligned}
-O_m^1 = {}& 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta^2 u_m, u_m\rangle_{Z^\star,Z} - 2\mathbb{E}\int_0^T \exp(-cs)\langle \Delta f(u_m + W_A), u_m\rangle_{Z^\star,Z} \\
&+ c\mathbb{E}\int_0^T \exp(-cs)\|u_m\|^2 ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
O_m^2 = O_m - O_m^1 = \mathbb{E}\int_0^T \exp(-cs)\{&2\langle \Delta^2 u_m, \omega\rangle_{Z^\star,Z} + 2\langle \Delta^2 \omega, u_m\rangle_{Z^\star,Z} \\
&- 2\langle \Delta^2 \omega, \omega\rangle_{Z^\star,Z} - 2\langle \Delta f(u_m + W_A), \omega\rangle_{Z^\star,Z} \\
&- 2\langle \Delta f(\omega + W_A), u_m\rangle_{Z^\star,Z} + 2\langle \Delta f(\omega + W_A), \omega\rangle_{Z^\star,Z} \\
&- c\|\omega\|^2 ds + 2c\langle u_m, \omega\rangle_{Z^\star,Z}\} ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
O_m^2 = {}& \mathbb{E}\int_0^T \exp(-cs)\{2\langle \Delta^2 \omega, u_m\rangle_{Z^\star,Z} + 2\langle \Delta^2 u_m - \Delta^2 \omega, \omega\rangle_{Z^\star,Z} \\
&- 2\langle \Delta f(\omega + W_A), u_m\rangle_{Z^\star,Z} - 2\langle \Delta f(u_m + W_A) - \Delta f(\omega + W_A), \omega\rangle_{Z^\star,Z} \\
&- c\|\omega\|^2 ds + 2c\langle u_m, \omega\rangle_{Z^\star,Z}\} ds.
\end{aligned}
$$

In view of (3.10) and (3.12), we deduce that

$$
\begin{aligned}
\lim_{m\to\infty} O_m^2 = \mathbb{E}\int_0^T \exp(-cs)\{&2\langle \Delta^2 \omega, u\rangle_{Z^\star,Z} + 2\langle \Delta^2 u - \Delta^2 \omega, \omega\rangle_{Z^\star,Z} \\
&- 2\langle \Delta f(\omega + W_A), u\rangle_{Z^\star,Z} - 2\langle \Delta \chi - \Delta f(\omega + W_A), \omega\rangle_{Z^\star,Z} \\
&- c\|\omega\|^2 ds + 2c\langle u, \omega\rangle_{Z^\star,Z}\} ds.
\end{aligned} \tag{3.28}
$$

Combining (3.27) and (3.28), (where $O_m \leq 0$), yields

$$\mathbb{E}\int_0^T \exp(-cs)\{-2\langle \Delta^2 u - \Delta^2\omega, u-\omega\rangle_{Z^\star,Z} + 2\langle \Delta\chi - \Delta f(\omega + W_A), u-\omega\rangle_{Z^\star,Z} - c\|u-\omega\|^2\} + \delta\exp(-cT) \leq 0.$$

Let $v \in L^2(\Omega\times(0,T);V)\cap L^4(\Omega\times D\times(0,T))$ be arbitrary and set

$$\omega = u - \lambda v, \text{ with } \lambda \in \mathbb{R}_+.$$

We obtain

$$\mathbb{E}\int_0^T \exp(-cs)\big\{-2\langle\lambda\Delta^2 v, \lambda v\rangle_{Z^\star,Z} + 2\langle\Delta\chi - \Delta f(u-\lambda v + W_A), \lambda v\rangle_{Z^\star,Z} - c\|\lambda v\|^2\big\} \leq 0.$$

Dividing by λ and letting $\lambda \to 0$, we find that

$$\mathbb{E}\int_0^T \exp(-cs)\langle\Delta\chi - \Delta f(u+W_A), v\rangle_{Z^*,Z}dt \leq 0.$$

Thus, for all $v \in L^2(\Omega\times(0,T);V)\cap L^4(\Omega\times D\times(0,T))$

$$\Delta\chi = \Delta f(u+W_A) + \theta(\omega,t), \tag{3.29}$$

a.s. a.e. in $D\times(0,T)$. Taking the duality product of (3.29) with $\tilde{\omega} \in V\cap L^4(D)$, to obtain

$$\begin{aligned}\langle\Delta\chi,\tilde{\omega}\rangle_{Z^*,Z} &= \langle\Delta f(u+W_A) + \theta(\omega,t),\tilde{\omega}\rangle_{Z^*,Z}\\ &= \langle\Delta f(u+W_A),\tilde{\omega}\rangle_{Z^*,Z}.\end{aligned} \tag{3.30}$$

Substituting (3.30) in (3.16) we deduce that for a.e. $(\omega,t)\in\Omega\times(0,T)$,

$$\langle u(t),\tilde{\omega}\rangle = \langle v_0,\tilde{\omega}\rangle - \int_0^t\langle\Delta^2 u,\tilde{\omega}\rangle_{Z^\star,Z}ds + \int_0^t\langle\Delta f(u+W_A),\tilde{\omega}\rangle_{Z^\star,Z}ds, \tag{3.31}$$

for all $\tilde{\omega}\in V\cap L^4(D)$.

This completes the identification of the limit reaction term by the monotonicity method.

Next, we prove that u satisfies equation (2.26) in Definition (2.2). Set $\mathcal{V} = H^2(D)\cap L^4(D)$. Equation (3.31) implies that a.s. in $\mathcal{V}^* = (H^2(D))' + L^{\frac{4}{3}}(D)$

$$u(t) = v_0 - \int_0^t \Delta^2 u ds + \int_0^t \Delta f(u + W_A) ds, \tag{3.32}$$

for all $t \in [0, T]$.

3.2. **Uniqueness of the solution to Problem** (P_2)**.**

Let ω be given such that there exists two pathwise solutions of Problem (P_2), $u_i = u_i(\omega, x, t)$ for $i = 1, 2$, satisfying

$$u_i(\omega, ., .) \in L^\infty(0, T; L^2(D)) \cap L^2((0, T); H^2(D)) \cap L^4(D \times (0, T)),$$

$$\Delta f(u_i + W_A) \in L^{\frac{4}{3}}((0, T) \times D),$$

where $u_1(., 0) = u_2(., 0) = v_0$. Then, for $i = 1, 2$

$$u_i(x, t) = u_i(x, 0) - \int_0^t \Delta^2 u_i + \int_0^t \Delta f(u_i + W_A).$$

Thus,

$$\begin{aligned} u_1(t) - u_2(t) = &- \int_0^t [\Delta^2 u_1 - \Delta^2 u_2] ds \\ &+ \int_0^t [\Delta f(u_1 + W_A) - \Delta f(u_2 + W_A)] ds \end{aligned} \tag{3.33}$$

in $L^2((0, T); V^*) + L^{\frac{4}{3}}((0, T) \times D)$.

Next we recall a simplified **Itô's formula** as in [17] (Theorem (4.2.5)).

Theorem 3.2. *Let* $X \in L^2(0, T; V) \cap L^4(D \times (0, T))$ *and* $Y \in L^2(0, T; V^*) + L^{\frac{4}{3}}(D \times (0, T))$, $\varsigma \in L^2(D \times (0, T))$, *both progressively measurable, such that*

$$X(t) := X(0) + \int_0^t Y(s) ds + \int_0^t \varsigma(s) dW(s), \quad t \in [0, T].$$

Then the following Itô's formula holds for the square of its H*-norm P-a.s.*

$$\begin{aligned} ||X(t)||_H^2 := ||X_0||_H^2 + &\int_0^t (2\langle Y(s), X(s)\rangle_{Z^*, Z} + ||\varsigma(s)||^2_{L^2(D \times (0,T))}) ds \\ &+ 2 \int_0^t \langle X(s), \varsigma(s) dW(s)\rangle_H. \end{aligned} \tag{3.34}$$

Applying (3.34) to equation (3.33) with

$$\begin{aligned}X(t) =&u_1(t) - u_2(t),\\ Y(s) =&-[\Delta^2 u_1 - \Delta^2 u_2] + [\Delta f(u_1 + W_A) - \Delta f(u_2 + W_A)],\\ \varsigma(s) =&0,\end{aligned}$$

yields for all $t \in (0, T)$

$$\begin{aligned}\|u_1 - u_2\|^2_{L^2(D)} = &-2\int_0^t \int_D (\Delta(u_1 - u_2))^2 dxds - 2\int_0^t \langle f'(u_1 + W_A)\nabla(u_1 + W_A)\\ &- f'(u_2 + W_A)\nabla(u_2 + W_A), \nabla u_1 - \nabla u_2\rangle ds\\ \le &-2\int_0^t \|\Delta(u_1 - u_2)\|^2_{L^2(D)} ds - 2C_5' \int_0^t \|\nabla(u_1 - u_2)\|^2_{L^2(D)} ds. \qquad (3.35)\end{aligned}$$

By the Poincare's inequality

$$-2C_5' \int_0^t \|\nabla(u_1 - u_2)\|^2_{L^2(D)} \le -\frac{2C_5'}{C'} \int_0^t \|u_1 - u_2\|^2_{L^2(D)}, \qquad (3.36)$$

with $\alpha = |\frac{-2C_5'}{C'}|$, relation (3.35) will be

$$\int_D (u_1 - u_2)^2(x, t)dx \le \alpha \int_0^t \int_D (u_1 - u_2)^2(x, t)dxds \quad \text{for all } t \in (0, T),$$

which in turn implies by Gronwall's Lemma that $u_1 = u_2 \quad a.e.$ in $D \times (0, T)$. □

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