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ON THE BLOWUP AND LIFESPAN OF SMOOTH SOLUTIONS TO A CLASS OF 2-D NONLINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA

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Abstract. We are concerned with a class of two-dimensional nonlinear wave equations $\partial_t^2 u - \mathrm{div}(c^2(u)\nabla u) = 0$ or $\partial_t^2 u - c(u)\mathrm{div}(c(u)\nabla u) = 0$ with small initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$, where $c(u)$ is a smooth function, $c(0) \neq 0$, $x \in \mathbb{R}^2$, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$ depend only on $r = \sqrt{x_1^2 + x_2^2}$, and $\varepsilon > 0$ is sufficiently small. Such equations arise in a pressure-gradient model of fluid dynamics, as well as in a liquid crystal model or other variational wave equations. When $c'(0) \neq 0$ or $c'(0) = 0$, $c''(0) \neq 0$, we establish blowup and determine the lifespan of smooth solutions.

1. Introduction and main results. In this paper, we shall focus on two-dimensional nonlinear wave equations of the form

$$\begin{cases} \partial_t^2 u - \mathrm{div}(c^2(u)\nabla u) = 0, \\ u(0,x) = \varepsilon u_0(x), \quad \partial_t u(0,x) = \varepsilon u_1(x), \end{cases} \tag{1.1}$$

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where $c(u)$ is a smooth function with $c(0) \neq 0$, $x \in \mathbb{R}^2$, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$ depend only on $r = \sqrt{x_1^2 + x_2^2}$, and $\varepsilon > 0$ is sufficiently small. We will assume $c(u) = 1+u+O(u^2)$ or $c(u) = 1 + u^2 + O(u^3)$ without loss of generality.

Equation (1.1) has an interesting physical background. In [1], [28], a pressure-gradient model for the positive pressure function P derived from the 2-D compressible full Euler system takes the form

$$\partial_t \left(\frac{\partial_t P}{P} \right) - \Delta P = 0.$$

When initial data $(P(0,x), \partial_t P(0,x)) = (1 + \varepsilon P_0(x), \varepsilon P_1(x))$ is given and one sets $u(t,x) = \ln P(t,x)$, then one obtains $\partial_t^2 u - \operatorname{div}(e^u \nabla u) = 0$ with initial data $u(0,x) = \ln(1+\varepsilon P_0(x))$ and $\partial_t u(0,x) = \varepsilon P_1(x)(1+\varepsilon P_0(x))^{-1}$. This is the case of $c(u) = \exp(\frac{u}{2})$ in (1.1). By [7], [13], [23], [27], a 2-D liquid crystal equation, or variational wave equation, takes the form $\partial_t^2 u - c(u)\operatorname{div}(c(u)\nabla u) = 0$. Especially for the nematic liquid crystal equation, one has $c(u) = \alpha \cos^2 u + \beta \sin^2 u$ with positive constants α and β satisfying $\alpha \neq \beta$. In this case, $c(u) = \alpha + (\beta - \alpha)\sin^2 u = \alpha + (\beta - \alpha)u^2 + O(u^3)$, which essentially corresponds to $c(u) = 1 + u^2 + O(u^3)$ in (1.1).

There has been extensive and remarkable work concerning the global existence or blowup and lifespan of smooth solutions to n-dimensional $(n \geq 2)$ nonlinear wave equations of the form

$$\begin{cases} \displaystyle\sum_{i,j=0}^{n} g_{ij}(u, \nabla u)\partial_{ij}^2 u = f(u, \nabla u, \nabla^2 u), \\ u(0,x) = \varepsilon u_0(x), \quad \partial_{x_0} u(0,x) = \varepsilon u_1(x), \end{cases} \tag{1.2}$$

where $x_0 = t$, $x = (x_1, \dots, x_n)$, $\nabla = \nabla_{x_0,x}$, g_{ij} and f are smooth functions of their arguments that satisfy $g_{ij}(u, \nabla u) = c_{ij} + O(|u| + |\nabla u|)$ and $f(u, \nabla u, \nabla^2 u) = O(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2)$, respectively, the c_{ij} are constants, and the linear operator $\sum_{i,j=0}^{n} c_{ij}\partial_{ij}^2$ is strictly hyperbolic with respect to time t. For the functions g_{ij} and f independent of u, for $n \geq 4$, it has been shown that (1.2) admits a global smooth solution (see [9], [10], [17]). For $n = 2, 3$, the authors of [3], [5], [12], [15] obtained the global existence if null conditions hold. Otherwise, if these null conditions do not hold, then smooth solutions blow up in finite time and their lifespan can be explicitly determined in terms of the initial data (see [2], [3], [8], [11], [14], [25], and the references therein). If the functions g_{ij} and f depend on u and the derivatives of u, then problem (1.2) is much harder and there are some partial results on the global existence or blowup and lifespan of smooth solutions when $2 \leq n \leq 4$ (for $n \geq 5$, (1.2) admits a global solution; see [18], [22]). For $2 \leq n \leq 4$, lower bounds on the lifespan under some suitable restrictions were obtained in [18], [19], [22], and the references therein. We especially point out that if the equation in (1.2) has the form $\partial_t^2 u - (1 + u)\Delta u = 0$, then, for $n = 3$, the authors of [4] and [20], [21] established the global existence of smooth solution. These solutions, however, often exhibit a behavior at infinity much different from that of solutions to a linear wave equation.

In this paper, we will concentrate on the nonlinear wave equation (1.1), show the finite-time blowup of smooth solution, and give an explicit expression for the lifespan T_ε as $\varepsilon \to 0$. The main result reads:

THEOREM 1.1. Let $u_0(x)$, $u_1(x) \in C_0^\infty(\mathbb{R}^2)$ depend only on $r = \sqrt{x_1^2 + x_2^2}$. If $u_0(x) \not\equiv 0$ or $u_1(x) \not\equiv 0$, then problem (1.1) possesses a C^∞-solution for $0 \le t < T_\varepsilon$, where T_ε stands for the lifespan of smooth solution $u(t, x)$.

(*i*) For $c(u) = 1 + u + O(u^2)$,

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0 \equiv -\frac{1}{2 \min\limits_{\sigma} F_0'(\sigma)}. \tag{1.3}$$

(*ii*) For $c(u) = 1 + u^2 + O(u^3)$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon = \nu_0 \equiv -\frac{1}{2 \min\limits_{\sigma} \{F_0(\sigma) F_0'(\sigma)\}}. \tag{1.4}$$

Here $F_0(\sigma)$ is the Friedlander radiation field for the 2-D linear wave equation $\Box w = 0$ with initial data $(w(0, x), \partial_t w(0, x)) = (u_0(r), u_1(r))$.

Recall that $F_0(\sigma) = \dfrac{1}{2\pi\sqrt{2}} \displaystyle\int_\sigma^{+\infty} \frac{R(s; u_1) - R_s'(s; u_0)}{\sqrt{s - \sigma}}\, ds$, where

$$R(s; u_i) = \int \delta(s - \langle \omega, x \rangle) u_i(x)\, dx = \int_{-\infty}^{\infty} u_i(\sqrt{s^2 + y^2})\, dy$$

is the Radon transform of $u_i(r)$ $(i = 0, 1)$.

Henceforth, we shall also assume that u_0, u_1 are supported in the disk $B(0, M)$, where $M > 0$.

REMARK 1.1. It follows from Theorem 1.1 that smooth solutions to (1.1) blow up in finite time provided that $u_0(x) \not\equiv 0$ or $u_1(x) \not\equiv 0$ because $F_0 \not\equiv 0$, $F_0(M) = 0$, and $\lim\limits_{\sigma \to -\infty} F_0(\sigma) = 0$. Further properties of the function $F_0(\sigma)$ can be found in [10, Theorem 6.2.2].

REMARK 1.2. For $c(u) = 1 + O(u^3)$, (1.1) admits a global smooth solution (see [18]).

REMARK 1.3. If $c(u) = 1 + c_1 u + O(u^2)$ or $c(u) = 1 + c_2 u^2 + O(u^3)$ in (1.1), with $c_1 \neq 0$ and $c_2 \neq 0$, then one also has finite-time blowup of smooth solutions, and one can establish an explicit expressions for T_ε as in Theorem 1.1.

REMARK 1.4. For the Cauchy problem for the 1-D liquid crystal equation $\partial_t^2 u - c(u)\partial_x(c(u)\partial_x u) = 0$ with $(u(0, x), \partial_t u(0, x)) = (u_0(x), u_1(x))$, it has been shown in [7, Theorem 1] that if there exist positive constants $c_0 < c_1$ and an $m_0 \in \mathbb{R}$ such that $c_0 \le c(u) \le c_1$ for all $u \in \mathbb{R}$ and $c'(m_0) \neq 0$, then the C^1-solution $u(t, x)$ with special initial data $u_0(x) = m_0 + \varepsilon\phi(\dfrac{x}{\varepsilon})$ and $u_1(x) = -\operatorname{sgn}(c'(m_0)) c(u_0(x)) u_0'(x)$ blows up in finite time; here $\phi \in C_0^1(0, 1)$ and $\varepsilon > 0$ is sufficiently small. However, for small and in general smooth initial data $(\varepsilon u_0(x), \varepsilon u_1(x))$ with compact support, which certainly do not satisfy the assumptions on the initial data made in [7, Theorem 1], there are no blowup results and precise estimates of the lifespan for the 1-D or 2-D liquid crystal equations available in the literature so far. In fact, from the proof of Theorem 1.1 and Remark 2.1(ii), it follows that smooth solutions to the 2-D liquid crystal equation

$\partial_t^2 u - c(u)\text{div}(c(u)\nabla u) = 0$ develop singularities in finite time for arbitrarily given small and smooth, spherically symmetric initial data with compact support.

REMARK 1.5. From the proof of Theorem 1.1, we infer that $\lim_{t\to T_\varepsilon-} \|\nabla u(t,\cdot)\|_{L^\infty} = \infty$. Note that this is different from the geometric blowup $\lim_{t\to T_\varepsilon-} \|\nabla^2 u(t,\cdot)\|_{L^\infty} = \infty$ that occurs for solutions $u \in C^2([0,T_\varepsilon]\times\mathbb{R}^2)$ to the nonlinear wave equation $\sum_{i,j=0}^{2} g_{ij}(\nabla u)\partial_{ij}^2 u = 0$ with small initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$ (see [2], [3]). In this latter case, a local shock is formed at blowup time for the unsteady potential flow equation (see [24]).

Let us comment on the proof of Theorem 1.1. To show (1.3) or (1.4), first we study the lower bound on the lifespan T_ε for problem (1.1). As in [10, Chapter 6] or [4], [6], by constructing a suitable approximate solution u_a to (1.1) and then considering the difference of the exact solution u and u_a, applying the Klainerman-Sobolev inequality, and further establishing some delicate energy estimate, we obtain the desired lower bound on the lifespan T_ε. On the other hand, the solution u to (1.1) is spherically symmetric for $t < T_\varepsilon$ due to the spherical symmetry of the initial data (u_0, u_1). Based on this, we can change (1.1) to a 2×2 equation system in the coordinates (t,r), with u still appearing in the coefficients. Thanks to the good properties of the difference of the real solution u and the approximate solution u_a before blowup time T_ε, we can treat the solution u and its derivatives simultaneously to obtain a precise estimate on the upper bound of T_ε. We point out that the methods in this paper are partly motivated by [9] and [14], where equations of the form $\partial_t^2 u - c^2(\partial_t u)\Delta u = 0$ with $c'(0) \neq 0$ were studied, but only the estimates of the first-order derivatives of u were required.

The paper is organized as follows. In Section 2, we construct an approximate solution u_a to (1.1) in the two cases $c(u) = 1 + u + O(u^2)$ and $c(u) = 1 + u^2 + O(u^3)$, and we establish some related estimates. In Section 3, we obtain the lower bound on the lifespan T_ε by continuous induction, studying the nonlinear equation satisfied by $u - u_a$. In Section 4, we change the second-order equation in (1.1) to a 2×2 first-order partial differential system and further establish some delicate estimates on u and ∇u. From this, together with the blowup lemma of Hörmander [10, Lemma 1.3.2], we obtain the upper bound on T_ε and thus complete the proof of Theorem 1.1. Some useful auxiliary lemmas and conclusions are given in the appendix.

In what follows, we will make use of the following notation:

- Z stands for one of the Klainerman vector fields in the symmetric case,

$$\partial_r,\ \partial_t,\ S = t\partial_t + r\partial_r,\ H = r\partial_t + t\partial_r.$$

- ∂ represents ∂_r or ∂_t.
- The norm $\|f\|_{L^2}$ stands for $\|f(t,\cdot)\|_{L^2(\mathbb{R}^2)}$.

2. Construction of an approximate solution to (1.1). In this section, we construct a suitable approximate solution u_a to (1.1). Then the lower bound on T_ε is obtained, in Section 3, by a continuous induction argument through estimating the difference of the solution u and u_a. As $c(u)$ assumes the two different forms of $c(u) =$

$1 + u + O(u^2)$ and $c(u) = 1 + u^2 + O(u^3)$, respectively, and since the constructions will be slightly different in these two cases, we divide this section into two parts.

2.1. *Construction of u_a when $c(u) = 1 + u + O(u^2)$.* As in [10, Chapter 6], we introduce the slow time variable $\tau = \varepsilon\sqrt{1+t}$ and assume that the solution to (1.1) can be approximated by

$$\varepsilon r^{-\frac{1}{2}} V(\tau, \sigma), \quad r > 0,$$

where $\sigma = r - t$.

Let $V(\tau, \sigma)$ solve the equation

$$\begin{cases} \partial^2_{\tau\sigma} V + 2V\partial^2_\sigma V + 2(\partial_\sigma V)^2 = 0, & (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ V(0, \sigma) = F_0(\sigma), \\ \text{supp} V \subseteq \{(\tau, \sigma) : \sigma \le M\}; \end{cases} \tag{2.1}$$

$F_0(\sigma)$ was introduced in Theorem 1.1.

Recall that $\tau_0 \equiv -\dfrac{1}{2\min\limits_\sigma F_0'(\sigma)} > 0$. With regard to problem (2.1), one then has

Lemma 2.1. Problem (2.1) admits a C^∞-solution $V(\tau, \sigma)$ for $0 \le \tau < \tau_0$, and $V(\tau, \sigma)$ blows up as $\tau \to \tau_0-$.

Proof. Set $W = \partial_\sigma V$. Then it follows from (2.1) that

$$\begin{cases} \partial_\tau W + 2V\partial_\sigma W + 2W^2 = 0, & (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ W(0, \sigma) = F_0'(\sigma). \end{cases} \tag{2.2}$$

The characteristic curve $\sigma = \sigma(\tau, s)$ of (2.1) emanating from $(0, s)$ is defined by

$$\begin{cases} \dfrac{d\sigma}{d\tau}(\tau, s) = 2V(\tau, \sigma(\tau, s)), \\ \sigma(0, s) = s. \end{cases} \tag{2.3}$$

Along characteristic curves, one has

$$\begin{cases} \dfrac{dW}{d\tau}(\tau, \sigma(\tau, s)) + 2W^2(\tau, \sigma(\tau, s)) = 0, \\ W(0, \sigma(0, s)) = F_0'(s), \end{cases}$$

which yields, for $\tau < \tau_0$,

$$W(\tau, \sigma(\tau, s)) = \frac{F_0'(s)}{1 + 2F_0'(s)\tau}. \tag{2.4}$$

Note that the equation in (2.1) is equivalent to $\partial_\sigma(\partial_\sigma V + 2V\partial_\sigma V) = 0$. Together with the boundary condition for V in (2.1), this yields

$$\begin{cases} \dfrac{dV}{d\tau}(\tau, \sigma(\tau, s)) = 0, \\ V(0, \sigma(0, s)) = F_0(s). \end{cases}$$

This means

$$V(\tau, \sigma(\tau, s)) = F_0(s). \tag{2.5}$$

From (2.5) and (2.3), one concludes

$$\sigma(\tau, s) = s + 2F_0(s)\tau, \tag{2.6}$$

which implies that $\partial_s \sigma(\tau, s) = 1 + 2F_0'(s)\tau > 0$ for $\tau < \tau_0$. Thus it follows from the implicit function theorem that $s = s(\tau, \sigma)$ is a smooth function of the variables τ and σ. Consequently, $V(\tau, \sigma) = F_0(s(\tau, \sigma))$ is a smooth solution to (2.1) for $\tau < \tau_0$, and as $\tau \to \tau_0-$, the derivative $V_\sigma(\tau, \sigma)$ blows up due to (2.4). Lemma 2.1 is proved. □

From [10, Chapter 6], one has that $F_0(\sigma) \in C^\infty(\mathbb{R})$ is supported in $(-\infty, M]$ and obeys the estimates

$$|F_0^{(k)}(\sigma)| \le C_k(1 + |\sigma|)^{-\frac{1}{2}-k}, \quad k \in \mathbb{N}_0. \tag{2.7}$$

From (2.7), we now derive a decay estimate for $V(\tau, \sigma)$ in (2.1) for $\tau < \tau_0$ and $\sigma \to -\infty$.

Lemma 2.2. For any positive constant $b < \tau_0$ and $0 \le \tau \le b$, the smooth solution V to (2.1) satisfies the estimates

$$|Z^\alpha \partial_\tau^l \partial_\sigma^m V(\tau, \sigma)| \le C_{\alpha b}^{lm}(1 + |\sigma|)^{-\frac{1}{2}-l-m}, \quad \alpha, l, m \in \mathbb{N}_0, \tag{2.8}$$

where $C_{\alpha b}^{lm}$ are positive constants depending on b, α, l, and m.

Proof. When $\tau \le b$, it follows from (2.6)–(2.7) that $\frac{|s|}{2} \le |\sigma| \le 2|s|$ for large $|s|$. Together with (2.4)–(2.5), this yields

$$|V(\tau, \sigma)| \le C_b(1 + |\sigma|)^{-\frac{1}{2}}, \quad |\partial_\sigma V(\tau, \sigma)| \le C_b(1 + |\sigma|)^{-\frac{3}{2}}. \tag{2.9}$$

By (2.6) and (2.4), one has

$$\partial_\sigma s(\tau, \sigma) = \frac{1}{1 + 2F_0'(s)\tau}$$

and

$$\partial_\sigma^2 V(\tau, \sigma(\tau, s)) = \frac{F_0''(s)}{(1 + 2F_0'(s)\tau)^2} - \frac{2F_0'(s)F_0''(s)}{(1 + 2F_0'(s)\tau)^3},$$

which yields

$$|\partial_\sigma^2 V(\tau, \sigma)| \le C_b(1 + |\sigma|)^{-\frac{5}{2}}. \tag{2.10}$$

On the other hand, it follows from (2.1) and (2.10) that

$$|\partial_{\tau\sigma}^2 V(\tau, \sigma)| \le C_b(1 + |\sigma|)^{-\frac{7}{2}}$$

and further

$$|\partial_\tau V(\tau, \sigma)| \le C_b(1 + |\sigma|)^{-\frac{5}{2}}. \tag{2.11}$$

Based on (2.9)–(2.11), by an inductive argument, one arrives at

$$|\partial_\tau^l \partial_\sigma^m V(\tau, \sigma)| \le C_b^{lm}(1 + |\sigma|)^{-\frac{1}{2}-l-m}, \quad l, m \in \mathbb{N}_0.$$

Due to $S = \sigma\partial_\sigma + \frac{\varepsilon t}{2\sqrt{1+t}}\partial_\tau$ and $H = -\sigma\partial_\sigma + \frac{\varepsilon r}{2\sqrt{1+t}}\partial_\tau$ by Lemma A.1(ii), one analogously obtains

$$|Z^\alpha \partial_\tau^l \partial_\sigma^m V(\tau, \sigma)| \le C_{\alpha b}^{lm}(1 + |\sigma|)^{-\frac{1}{2}-l-m}, \quad \alpha, l, m \in \mathbb{N}_0,$$

and this completes the proof of Lemma 2.2. □

Next we construct an approximate solution u_a^I to (1.1) for $0 \le \tau = \varepsilon\sqrt{1+t} < \tau_0$.

Let w_0 be the solution of the linear wave equation

$$\begin{cases} \partial_t^2 w_0 - \triangle w_0 = 0, \\ w_0(0,x) = u_0(x), \\ \partial_t w_0(0,x) = u_1(x). \end{cases}$$

It follows from [10, Theorem 6.2.1] that, for any constants $l > 0$ and $0 < m < 1$,

$$|Z^\alpha(w_0(t,x) - r^{-\frac{1}{2}}F_0(\sigma))| \le C_{\alpha l}(1+t)^{-\frac{3}{2}}(1+|\sigma|)^{\frac{1}{2}}, \quad r \ge lt, \tag{2.12}$$

$$|\partial^k w_0(t,x)| \le C_{km}(1+t)^{-1-|k|}, \quad r \le mt. \tag{2.13}$$

Choose a C^∞-function $\chi(s)$ such that $\chi(s) = 1$ for $s \le 1$ and $\chi(s) = 0$ for $s \ge 2$. For $0 \le \tau = \varepsilon\sqrt{1+t} < \tau_0$, we take the approximate solution u_a^I to (1.1) to be

$$u_a^I(t,x) = \varepsilon\Big(\chi(\varepsilon t)w_0(t,x) + r^{-\frac{1}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)V(\sigma,\tau)\Big). \tag{2.14}$$

By Lemma 2.2 and [10, Theorem 6.2.1], one has that, for a fixed positive constant $b < \tau_0$,

$$|Z^\alpha u_a^I(t,x)| \le C_{\alpha b}\varepsilon(1+t)^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{1}{2}}, \quad \tau \le b. \tag{2.15}$$

Set $J_a^I = \partial_t^2 u_a^I - c^2(u_a^I)\triangle u_a^I - 2c(u_a^I)c'(u_a^I)|\nabla u_a^I|^2$.

LEMMA 2.3. One has

$$\int_0^{\frac{b^2}{\varepsilon^2}-1} \|Z^\alpha J_a^I\|_{L^2}\, dt \le C_{\alpha b}\varepsilon^{\frac{3}{2}}.$$

Proof. We divide the proof into three parts.

(i) $0 \le t \le \dfrac{1}{\varepsilon}$**.** In this case, $\chi(\varepsilon t) = 1$ and $u_a^I = \varepsilon w_0$. This yields

$$J_a^I = \varepsilon(1 - c^2(\varepsilon w_0))\Delta w_0 - 2\varepsilon^2 c(\varepsilon w_0)c'(\varepsilon w_0)|\nabla w_0|^2.$$

It follows from (2.15) and a direct computation that

$$\|Z^\alpha J_a^I\|_{L^2} \le C\varepsilon^2(1+t)^{-\frac{1}{2}}, \quad 0 \le t \le \frac{1}{\varepsilon}. \tag{2.16}$$

(ii) $\dfrac{1}{\varepsilon} \le t \le \dfrac{2}{\varepsilon}$**.** Now we rewrite u_a^I as

$$u_a^I = \varepsilon w_0(t,x) + \varepsilon(1-\chi(\varepsilon t))\big(r^{-\frac{1}{2}}\chi(-3\varepsilon\sigma)V(\tau,\sigma) - w_0(t,x)\big).$$

Then

$$J_a^I = J_1 + J_2 + J_3 + J_4, \tag{2.17}$$

where

$$\begin{aligned} J_1 &= (1 - c^2(u_a^I))\Delta u_a^I - 2c(u_a^I)c'(u_a^I)|\nabla u_a^I|^2, \\ J_2 &= \varepsilon(\partial_t^2 - \Delta)\big\{(1-\chi(\varepsilon t))r^{-\frac{1}{2}}\chi(-3\varepsilon\sigma)\big(V(\tau,\sigma) - F_0(\sigma)\big)\big\}, \\ J_3 &= \varepsilon(\partial_t^2 - \Delta)\big\{\chi(-3\varepsilon\sigma)\big(r^{-\frac{1}{2}}F_0(\sigma) - w_0(t,x)\big)\big\}, \\ J_4 &= \varepsilon(\partial_t^2 - \Delta)\big\{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\sigma) - 1)w_0(t,x)\big\}. \end{aligned}$$

We treat each J_i $(1 \le i \le 4)$ in (2.17) separately.

From (2.15) one obtains

$$\|Z^\alpha J_1^I\|_{L^2} \le C_{\alpha b}\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{2.18}$$

Since

$$\begin{aligned}
J_2 &= \varepsilon r^{-\frac{1}{2}}(\partial_t-\partial_r)(\partial_t+\partial_r)\Big\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(V(\tau,\sigma)-F_0(\sigma)\big)\Big\}\\
&\quad -\frac{\varepsilon}{4}r^{-\frac{5}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)\big(V(\tau,\sigma)-F_0(\sigma)\big)\\
&= O(\varepsilon^3)r^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{1}{2}} + O(\varepsilon^2)r^{-\frac{1}{2}}\int_0^\tau \partial^2_{\tau\sigma}V(s,\sigma)\,ds\\
&\quad + O(\varepsilon^2)(1+t)^{-\frac{1}{2}}r^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{3}{2}} + O(\varepsilon)(1+t)^{-\frac{5}{2}}
\end{aligned}$$

and

$$\left|Z^\alpha\big(r^{-\frac{1}{2}}\int_0^\tau \partial^2_{s\sigma}V(s,\sigma)ds\big)\right| \le C_{\alpha b}\varepsilon(1+t)^{\frac{1}{2}}r^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{3}{2}},$$

one has

$$\|Z^\alpha J_2\|_{L^2} \le C_{\alpha b}\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{2.19}$$

Note that $-\dfrac{2}{3\varepsilon} \le \sigma \le M$ holds in the support of J_3, which implies $r \ge \dfrac{1}{3}t$. Together with (2.12), this yields

$$\begin{aligned}
J_3 =&O(\varepsilon^3)(1+|\sigma|)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} + O(\varepsilon^2)\partial\big(r^{-\frac{1}{2}}F_0(\sigma)-w_0\big)\\
&+O(\varepsilon)(1+t)^{-\frac{5}{2}}(1+|\sigma|)^{-\frac{1}{2}}.
\end{aligned}$$

On the other hand, it follows from property (i) of Lemma A.1 that

$$\begin{aligned}
&|Z^\alpha\partial\big(r^{-\frac{1}{2}}F_0(\sigma)-w_0(t,x)\big)|\\
\le& C_\alpha|\partial Z^\alpha\big(r^{-\frac{1}{2}}F_0(\sigma)-w_0(t,x)\big)|\\
&\le C_\alpha(1+|\sigma|)^{-1}|ZZ^\alpha\big(r^{-\frac{1}{2}}F_0(\sigma)-w_0(t,x)\big)|\\
&\le C_\alpha(1+t)^{-\frac{3}{2}}(1+|\sigma|)^{-\frac{1}{2}}.
\end{aligned}$$

One then obtains

$$\|Z^\alpha J_3\|_{L^2} \le C_\alpha\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{2.20}$$

Analogously, together with (2.13), one arrives at

$$\|Z^\alpha J_4\|_{L^2} \le C_{\alpha b}\varepsilon^2(1+t)^{-2}. \tag{2.21}$$

Collecting (2.18)–(2.21) yields

$$\|Z^\alpha J_a^I\|_{L^2} \le C_{\alpha b}\varepsilon^2(1+t)^{-\frac{1}{2}}, \quad \frac{1}{\varepsilon}\le t\le\frac{2}{\varepsilon}. \tag{2.22}$$

(iii) $\frac{2}{\varepsilon} \le t \le \frac{b^2}{\varepsilon^2} - 1$. Together with (2.1), by a direct computation one has

$$\begin{aligned} J_a^I = &- \varepsilon^2 r^{-\frac{1}{2}} \partial_{\tau\sigma}^2 \hat{V} \bigg(\frac{1}{\sqrt{1+t}} - r^{-\frac{1}{2}} \bigg) \\ &- \varepsilon^2 r^{-1} \Big(\partial_{\tau\sigma}^2 \hat{V} + 2\hat{V} \partial_\sigma^2 \hat{V} + 2(\partial_\sigma \hat{V})^2 \Big) \\ &+ O(\varepsilon^3) r^{-\frac{1}{2}} (1+t)^{-1} \partial \hat{V} + O(\varepsilon^2)(1+t)^{-\frac{3}{2}} r^{-\frac{1}{2}} \partial \hat{V}, \end{aligned} \tag{2.23}$$

where $\hat{V}(\tau, \sigma) = \chi(-3\varepsilon\sigma) V(\tau, \sigma)$.

It follows from (2.1) that

$$\begin{aligned} &\varepsilon^2 r^{-1} \Big(\partial_{\tau\sigma}^2 \hat{V} + 2\hat{V} \partial_\sigma^2 \hat{V} + 2(\partial_\sigma \hat{V})^2 \Big) \\ =&O(\varepsilon^3) r^{-1} (1+|\sigma|)^{-\frac{3}{2}} - \varepsilon^2 r^{-1} \chi(-3\varepsilon\sigma)(1-\chi(-3\varepsilon\sigma)) \partial_{\tau\sigma}^2 V \\ =&O(\varepsilon^3) r^{-1} (1+|\sigma|)^{-\frac{3}{2}}; \end{aligned} \tag{2.24}$$

here we have used the fact that $\chi(-3\varepsilon\sigma)(1-\chi(-3\varepsilon\sigma))$ is supported in the interval $[-\frac{2}{3\varepsilon}, -\frac{1}{3\varepsilon}]$.

Substituting (2.24) into (2.23) yields

$$\|Z^\alpha J_a^I\|_{L^2} \le C_{\alpha b} \Big(\varepsilon^3 (1+t)^{-\frac{1}{2}} + \varepsilon^2 (1+t)^{-\frac{3}{2}} \Big). \tag{2.25}$$

Consequently, combining (2.16), (2.22), and (2.25), one obtains

$$\int_0^{\frac{b^2}{\varepsilon^2}-1} \|Z^\alpha J_a^I\|_{L^2}\, dt \le C_{\alpha b} \varepsilon^{\frac{3}{2}},$$

which finishes the proof of Lemma 2.3. □

2.2. *Construction of u_a when $c(u) = 1 + u^2 + O(u^3)$.* When $c(u) = 1 + u^2 + O(u^3)$, set the slow time variable to $\tau = \varepsilon^2 \ln(1+t)$ as in [4], and assume that the solution to (1.1) can be approximated by

$$\varepsilon r^{-\frac{1}{2}} G(\tau, \sigma), \quad r > 0,$$

where $\sigma = r - t$ and where $G(\tau, \sigma)$ solves the equation

$$\begin{cases} \partial_{\tau\sigma}^2 G + G^2 \partial_\sigma^2 G + 2G(\partial_\sigma G)^2 = 0, \quad (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ G(0, \sigma) = F_0(\sigma), \\ \mathrm{supp} G \subseteq \{(\tau, \sigma) \colon \sigma \le M\}. \end{cases} \tag{2.26}$$

Recall that $\nu_0 \equiv -\frac{1}{2\min_\sigma \{F_0(\sigma) F_0'(\sigma)\}} > 0$. With regard to problem (2.26), one has

Lemma 2.4. Problem (2.26) admits a C^∞-solution $G(\tau, \sigma)$ for $0 \le \tau < \nu_0$, and $G(\tau, \sigma)$ blows up as $\tau \to \nu_0-$.

Proof. Set $Q = \partial_\sigma G$. Then it follows from (2.26) that

$$\begin{cases} \partial_\tau Q + G^2\partial_\sigma Q + 2GQ^2 = 0, \quad (\tau,\sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ Q(0,\sigma) = F_0'(\sigma), \\ \mathrm{supp} Q \subseteq \{(\tau,\sigma) \colon \sigma \le M\}. \end{cases} \tag{2.27}$$

The characteristic curve $\sigma = \sigma(\tau, s)$ of (2.26) emanating from $(0, s)$ is defined by

$$\begin{cases} \dfrac{d\sigma}{d\tau}(\tau,s) = G^2(\tau,\sigma(\tau,s)), \\ \sigma(0,s) = s. \end{cases} \tag{2.28}$$

Along characteristic curves, one has

$$\begin{cases} \dfrac{dQ}{d\tau}(\tau,\sigma(\tau,s)) + 2(GQ^2)(\tau,\sigma(\tau,s)) = 0, \\ Q(0,\sigma(0,s)) = F_0'(s) \end{cases}$$

and

$$\begin{cases} \dfrac{dG}{d\tau}(\tau,\sigma(\tau,s)) = 0, \\ G(0,\sigma(0,s)) = F_0(s). \end{cases}$$

Then, for $\tau < \nu_0$,

$$\begin{cases} G(\tau,\sigma(\tau,s)) = F_0(s), \\ Q(\tau,\sigma(\tau,s)) = \dfrac{F_0'(s)}{1 + 2F_0(s)F_0'(s)\tau}. \end{cases}$$

Together with (2.28), this yields

$$\sigma(\tau,s) = s + F_0^2(s)\tau, \tag{2.29}$$

which means that, for $\tau < \nu_0$,

$$\partial_s\sigma(\tau,s) = 1 + 2F_0(s)F_0'(s)\tau > 0.$$

Therefore, it follows from the implicit function theorem that $s = s(\tau,\sigma)$ can be taken as a smooth function of τ and σ. Thus $G(\tau,\sigma) = F_0(s(\tau,\sigma))$ is a smooth solution of (2.26) when $\tau < \nu_0$. One then completes the proof of Lemma 2.4 as the one of Lemma 2.1. □

Parallel to Lemma 2.2, one has

LEMMA 2.5. *For any positive constant $b < \nu_0$ and $0 \le \tau \le b$, the smooth solution G of (2.26) satisfies the estimates*

$$|Z^\alpha \partial_\tau^l \partial_\sigma^m G(\tau,\sigma)| \le C_{\alpha b}^{lm}(1+|\sigma|)^{-\frac{1}{2}-l-m}, \quad \alpha, l, m \in \mathbb{N}_0. \tag{2.30}$$

Proof. Since the proof is similar to that of Lemma 2.2, it is omitted. □

Next we construct an approximate solution u_a^{II} to (1.1) for $0 \le \tau = \varepsilon^2\ln(1+t) < \nu_0$. As in the first part, we take the approximate solution u_a^{II} of (1.1) to be

$$u_a^{II}(t,x) = \varepsilon\Big(\chi(\varepsilon t)w_0(t,x) + r^{-\frac{1}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\sigma)G(\tau,\sigma)\Big). \tag{2.31}$$

By Lemma 2.5 and [10, Theorem 6.2.1], one has

$$|Z^\alpha u_a^{II}(t,x)| \le C_{\alpha b}\varepsilon(1+t)^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{1}{2}}, \quad \tau \le b < \nu_0. \tag{2.32}$$

Set $J_a^{II} = \partial_t^2 u_a^{II} - c^2(u_a^{II})\triangle u_a^{II} - 2c(u_a^{II})c'(u_a^{II})|\nabla u_a^{II}|^2$. Then one has

LEMMA 2.6.

$$\int_0^{e^{\frac{b}{\varepsilon^2}-1}} \|Z^\alpha J_a^{II}\|_{L^2}\, dt \le C_{\alpha b}\varepsilon^{\frac{3}{2}}|\ln\varepsilon|.$$

Proof. We divide this proof procedure into two parts.

(i) $0 \le t \le \dfrac{2}{\varepsilon}$**.** As in (i) and (ii) of the proof of Lemma 2.3, one has

$$\|Z^\alpha J_a^{II}\|_{L^2} \le C_{\alpha b}\varepsilon^2|\ln\varepsilon|(1+t)^{-\frac{1}{2}}. \tag{2.33}$$

Note that the factor $\ln\varepsilon$ in (2.33) appears due to

$$\begin{aligned} & G(\tau,\sigma) - F_0(\sigma) \\ =& \tau\int_0^\tau \partial_s G(s,\sigma)\, ds \\ =& \varepsilon^2\ln(1+t)\int_0^\tau \partial_s G(s,\sigma)\, ds \\ =& O(\varepsilon^2|\ln\varepsilon|), \quad \frac{1}{\varepsilon} \le t \le \frac{2}{\varepsilon}. \end{aligned}$$

(ii) $\dfrac{2}{\varepsilon} \le t \le e^{\frac{b}{\varepsilon^2}} - 1$. It follows from a direct computation that

$$\begin{aligned} J_a^{II} =& -2\varepsilon^3 r^{-\frac{1}{2}}\partial^2_{\tau\sigma}\hat{G}\Big(\frac{1}{1+t} - \frac{1}{r}\Big) \\ & - 2\varepsilon^3 r^{-\frac{3}{2}}\Big(\partial^2_{\tau\sigma}\hat{G} + \hat{G}^2\partial^2_\sigma\hat{G} + 2\hat{G}(\partial_\sigma\hat{G})^2\Big) + O(\varepsilon^3)(1+t)^{-\frac{5}{2}}, \end{aligned} \tag{2.34}$$

where $\hat{G}(\tau,\sigma) = \chi(-3\varepsilon\sigma)G(\tau,\sigma)$.

Note that

$$\begin{aligned} & \varepsilon^3 r^{-\frac{3}{2}}\Big(\partial^2_{\tau\sigma}\hat{G} + \hat{G}^2\partial^2_\sigma\hat{G} + 2\hat{G}(\partial_\sigma\hat{G})^2\Big) \\ =& O(\varepsilon^4)r^{-\frac{3}{2}}(1+|\sigma|)^{-\frac{3}{2}} + \varepsilon^3 r^{-\frac{3}{2}}\chi(-3\varepsilon\sigma)(1-\chi^2(-3\varepsilon\sigma))\partial^2_{\tau\sigma}G \\ =& O(\varepsilon^4)(1+t)^{-1}r^{-\frac{1}{2}}(1+|\sigma|)^{-\frac{3}{2}}. \end{aligned}$$

One then obtains

$$\|Z^\alpha J_a^{II}\|_{L^2} \le C_{\alpha b}\Big(\varepsilon^4(1+t)^{-1} + \varepsilon^3(1+t)^{-\frac{3}{2}}\Big). \tag{2.35}$$

Combining (2.35) and (2.33) yields

$$\int_0^{e^{\frac{b}{\varepsilon^2}-1}} \|Z^\alpha J_a^{II}\|_{L^2}\, dt \le C_{\alpha b}\varepsilon^{\frac{3}{2}}|\ln\varepsilon|.$$

This completes the proof of Lemma 2.6. □

REMARK 2.1. Consider the 2-D variational wave equation $\partial_t^2 u - c(u)\mathrm{div}(c(u)\nabla u) = 0$ with initial data $(u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$.

(i) Let $c(u) = 1 + u + O(u^2)$. As in the first part, u can then be approximated by $\varepsilon r^{-\frac{1}{2}} V(\tau,\sigma)$, where $\tau = \varepsilon\sqrt{1+t}$, $\sigma = r - t$, and $V(\tau,\sigma)$ solves the equation

$$\begin{cases} \partial^2_{\tau\sigma} V + 2V\partial^2_\sigma V + (\partial_\sigma V)^2 = 0, \quad (\tau,\sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ V(0,\sigma) = F_0(\sigma), \\ \operatorname{supp} V \subseteq \{(\tau,\sigma) \colon \sigma \le M\}. \end{cases} \tag{2.36}$$

Applying the method of characteristics, one easily proves that (2.36) admits a smooth solution only for $0 \le \tau < \tau_0 = -\dfrac{1}{\min F_0'(\sigma)}$.

(ii) Let $c(u) = 1 + u^2 + O(u^3)$. As in Section 2.2, u can then be approximated by $\varepsilon r^{-\frac{1}{2}} G(\tau,\sigma)$, where $\tau = \varepsilon^2 \ln(1+t)$, $\sigma = r - t$, and $G(\tau,\sigma)$ solves the equation

$$\begin{cases} \partial^2_{\tau\sigma} G + G^2\partial^2_\sigma V + G(\partial_\sigma G)^2 = 0, \quad (\tau,\sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\ G(0,\sigma) = F_0(\sigma), \\ \operatorname{supp} G \subseteq \{(\tau,\sigma) \colon \sigma \le M\}. \end{cases} \tag{2.37}$$

As shown in [26, Theorem 1.2], for some special class of initial data $G(0,\sigma)$, C^1-solutions G to (2.37) blow up in finite time. The Friedlander radiation field $F_0(\sigma)$, however, does not meet the assumptions of [26] due to $F_0^{(k)}(M) = 0$ for any $k \in \mathbb{N}_0$, which is different from $F_0'(M) \ne 0$ assumed in [26]. Nonetheless, one can still obtain the finite-time blowup of smooth solution to (2.37) (see Lemma A.5).

3. The lower bound on the lifespan T_ε. In this section, based on the preparations in Section 2, we establish the lower bound on the lifespan T_ε by utilizing continuous induction and the energy method.

First we deal with the case that $c(u) = 1 + u + O(u^2)$ in (1.1).

LEMMA 3.1. Let $c(u) = 1 + u + O(u^2)$. Then, for sufficiently small ε and $0 \le \tau = \varepsilon\sqrt{1+t} \le b < \tau_0$, (1.1) admits a C^∞-solution u which satisfies the estimate

$$|Z^\kappa \partial(u - u_a^I)| \le C_b \varepsilon^{\frac{3}{2}} (1+t)^{-\frac{1}{2}} (1 + |t-r|)^{-\frac{1}{2}} \tag{3.1}$$

for $|\kappa| \le 2$; here u_a^I was introduced in (2.14).

Proof. Set $v = u - u_a^I$. Then

$$\begin{cases} \partial_t^2 v - c^2(u)\Delta v = F, \\ v(0,x) = \partial_t v(0,x) = 0, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F = &-J_a^I + (c^2(u) - c^2(u_a^I))\Delta u_a^I + 2c(u)c'(u)|\nabla v|^2 + 4c(u)c'(u)\nabla v \cdot \nabla u_a^I \\ &+ 2\big(c(u)c'(u) - c(u_a^I)c'(u_a^I)\big)|\nabla u_a^I|^2. \end{aligned} \tag{3.3}$$

We will use continuous induction to prove (3.1). To this end, we assume that, for some $T \le \dfrac{b^2}{\varepsilon^2} - 1$,

$$|Z^\kappa \partial v| \le \varepsilon (1+t)^{-\frac{1}{2}} (1 + |t-r|)^{-\frac{1}{2}}, \quad |\kappa| \le 2, \ \ t \le T, \tag{3.4}$$

holds and then prove that

$$|Z^\kappa \partial v| \le \frac{1}{2}\varepsilon(1+t)^{-\frac12}(1+|t-r|)^{-\frac12}, \quad |\kappa| \le 2,\ \ t \le T. \tag{3.5}$$

Note that from (3.4) one has

$$|Z^\kappa v| \le C\varepsilon(1+t)^{-\frac12}(1+|t-r|)^{\frac12}, \quad |\kappa| \le 2,\ \ t \le T. \tag{3.6}$$

Applying Z^α to both sides of (3.2) yields, for $|\alpha| \le 4$,

$$\begin{aligned}(\partial_t^2 - c^2(u)\Delta)Z^\alpha v =& G\\ \equiv& \sum_{|\beta|\le|\alpha|} C_{\alpha\beta}Z^\beta F + \big[Z^\alpha, (c^2(u)-1)\Delta\big]v\\ &+ \sum_{|\beta|<|\alpha|} C'_{\alpha\beta}Z^\beta\big((c^2(u)-1)\Delta v\big);\end{aligned} \tag{3.7}$$

here the commutator relation $[Z^\alpha, \partial_t^2 - \triangle] = \sum_{|\beta|<|\alpha|} C''_{\alpha\beta}Z^\beta(\partial_t^2 - \triangle)$ with suitable constants $C_{\alpha\beta}, C'_{\alpha\beta}, C''_{\alpha\beta}$ has been made use of.

Next we derive an estimate of $\|\partial Z^\alpha v\|_{L^2}$ from (3.7). Define the energy

$$E(t) = \frac12 \sum_{|\alpha|\le 4}\int_{\mathbb{R}^2}(|\partial_t Z^\alpha v|^2 + c^2(u)|\nabla Z^\alpha v|^2)\,dx.$$

Multiplying both sides of (3.7) by $\partial_t Z^\alpha v$ ($|\alpha| \le 4$), integrating by parts in $\mathbb{R}^2$, and noting that $|\partial u| = |\partial u_a^I + \partial v| \le C_b\varepsilon(1+t)^{-\frac12}$ from the construction of u_a^I and assumption (3.4), one arrives at

$$E'(t) \le \frac{C_b\varepsilon}{\sqrt{1+t}}E(t) + \sum_{|\alpha|\le 4}\int_{\mathbb{R}^2}|G|\cdot|\partial_t Z^\alpha v|\,dx. \tag{3.8}$$

Moreover, due to the inductive hypothesis (3.4) and (2.15), one has

$$|Z^\kappa u| \le C_b\varepsilon(1+t)^{-\frac12}(1+|\sigma|)^{\frac12} \le C_b\varepsilon, \quad |\kappa| \le 2,\ \ t \le T. \tag{3.9}$$

We now treat each term in the sum $\sum_{|\alpha|\le 4}\int_{\mathbb{R}^2}|G|\cdot|\partial_t Z^\alpha v|\,dx$ separately.

(A) Treatment of $\sum_{|\beta|<|\alpha|}\int_{\mathbb{R}^2}|Z^\beta\big((c^2(u)-1)\Delta v\big)|\cdot|\partial_t Z^\alpha v|\,dx$. It follows from (3.9) that, for $|\beta| < |\alpha|$,

$$\begin{aligned}&\int_{\mathbb{R}^2}|Z^\beta((c^2(u)-1)\Delta v)|\cdot|\partial_t Z^\alpha v|\,dx\\ &\quad\le C_b\sum_{|\beta_1|+|\beta_2|=|\beta|}\int_{\mathbb{R}^2}|Z^{\beta_1}u|\cdot|Z^{\beta_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\quad\le C_b\sum_{|\beta_1|+|\beta_2|=|\beta|}\int_{\mathbb{R}^2}|Z^{\beta_1}v|\cdot|Z^{\beta_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\qquad+ C_b\sum_{|\beta_1|+|\beta_2|=|\beta|}\int_{\mathbb{R}^2}|Z^{\beta_1}u_a^I|\cdot|Z^{\beta_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx.\end{aligned} \tag{3.10}$$

The troublesome term in (3.10) is $Z^{\beta_1}v$ since a term of the form $Z^{\beta_1}v$ might not be contained in the energy $E(t)$. However, thanks to property (i) of Lemma A.1, one has

$$|Z^{\beta_2}\Delta v| \le \frac{2}{1+|t-r|}\sum_{|\beta_2'|=|\beta_2|+1}|Z^{\beta_2'}\partial v|.$$

Due to $|\beta|<|\alpha|\le 4$, by (3.4) and Lemma A.2, the first term in the right-hand side of (3.10) can then be estimated as

$$\begin{aligned}\int_{\mathbb{R}^2}&|Z^{\beta_1}v|\cdot|Z^{\beta_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\le C_b\sum_{|\beta_2'|=|\beta_2|+1}\int_{\mathbb{R}^2}\left|\frac{1}{1+|t-r|}Z^{\beta_1}v\right|\cdot|Z^{\beta_2'}\partial v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\le \frac{C_b\varepsilon}{\sqrt{1+t}}E(t).\end{aligned}\tag{3.11}$$

Analogously,

$$\int_{\mathbb{R}^2}|Z^{\beta_1}u_a^I|\cdot|Z^{\beta_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\le\frac{C_b\varepsilon}{\sqrt{1+t}}E(t).$$

Therefore, one obtains

$$\sum_{|\beta|<|\alpha|}\int_{\mathbb{R}^2}|Z^\beta\big((c^2(u)-1)\Delta v\big)|\cdot|\partial_t Z^\alpha v|\,dx\le\frac{C_b\varepsilon}{\sqrt{1+t}}E(t).\tag{3.12}$$

(B) Treatment of $\int_{\mathbb{R}^2}|\big[Z^\alpha,\big(c^2(u)-1\big)\Delta\big]v|\cdot|\partial_t Z^\alpha v|\,dx$**.** For

$$\begin{aligned}\int_{\mathbb{R}^2}&|\big[Z^\alpha,\big(c^2(u)-1\big)\Delta\big]v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\le C_b\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\|\alpha_1|\ge1}}\int_{\mathbb{R}^2}|Z^{\alpha_1}u|\cdot|Z^{\alpha_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\le C_b\Big(\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\|\alpha_1|\ge1}}\int_{\mathbb{R}^2}|Z^{\alpha_1}u_a^I|\cdot|Z^{\alpha_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\\ &\qquad+\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\|\alpha_1|\ge1}}\int_{\mathbb{R}^2}|Z^{\alpha_1}v|\cdot|Z^{\alpha_2}\Delta v|\cdot|\partial_t Z^\alpha v|\,dx\Big),\end{aligned}$$

by the same argument as in (3.11), one has

$$\int_{\mathbb{R}^2}|\big[Z^\alpha,\big(c^2(u)-1\big)\Delta\big]v|\cdot|\partial_t Z^\alpha v|\,dx\le\frac{C_b\varepsilon}{\sqrt{1+t}}E(t).\tag{3.13}$$

Next we treat each term $\int_{\mathbb{R}^2}|Z^\beta F|\cdot|\partial_t Z^\alpha v|\,dx$, $|\beta|\le|\alpha|$, that is included in $\sum_{|\alpha|\le4}\int_{\mathbb{R}^2}|G|\cdot|\partial_t Z^\alpha v|\,dx$.

(C) Treatment of $\int_{\mathbb{R}^2} |Z^\beta J_a^I| \cdot |\partial_t Z^\alpha v|\, dx$**.** In this case, one has

$$\int_{\mathbb{R}^2} |Z^\beta J_a^I| \cdot |\partial_t Z^\alpha v|\, dx \le \|Z^\beta J_a\|_{L^2} \cdot \sqrt{E(t)}. \tag{3.14}$$

(D) Treatment of $\int_{\mathbb{R}^2} |Z^\beta \big((c^2(u) - c^2(u_a^I))\Delta u_a^I\big)| \cdot |\partial_t Z^\alpha v|\, dx$**.** Due to (3.9) and Lemmas A.1 and A.2, a direct computation yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} |Z^\beta \big((c^2(u) - c^2(u_a^I))\Delta u_a^I\big)| \cdot |\partial_t Z^\alpha v|\, dx \\
\le& C_b \sum_{|\beta_1|+|\beta_2|=|\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} v| \cdot |Z^{\beta_2} \Delta u_a^I| \cdot |\partial_t Z^\alpha v|\, dx \\
\le& C_b \sum_{\substack{|\beta_1|+|\beta_2|=|\beta| \\ |\beta_2'|=|\beta_2|+1}} \int_{\mathbb{R}^2} \frac{1}{1+|t-r|} |Z^{\beta_1} v| \cdot |Z^{\beta_2'} \partial u_a^I| \cdot |\partial_t Z^\alpha v|\, dx \\
\le& \frac{C_b \varepsilon}{\sqrt{1+t}} E(t).
\end{aligned} \tag{3.15}$$

(E) Treatment of $\int_{\mathbb{R}^2} |Z^\beta \big(c(u)c'(u)|\nabla v|^2\big)| \cdot |\partial_t Z^\alpha v|\, dx$**.** Similarly to (D), one has

$$\int_{\mathbb{R}^2} |Z^\beta \big(c(u)c'(u)|\nabla v|^2\big)| \cdot |\partial_t Z^\alpha v| dx \le \frac{C_b \varepsilon}{\sqrt{1+t}} E(t). \tag{3.16}$$

(F) Treatment of $\int_{\mathbb{R}^2} |Z^\beta \big(c(u)c'(u)\nabla v \cdot \nabla u_a^I\big)| \cdot |\partial_t Z^\alpha v|\, dx$**.** It follows from a direct computation that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |Z^\beta \big(c(u)c'(u)\nabla v \cdot \nabla u_a^I\big)| \cdot |\partial_t Z^\alpha v|\, dx \\
\le& C_b \sum_{|\beta_1|+|\beta_2|\le|\beta|} |Z^{\beta_1} \partial v| \cdot |Z^{\beta_2} \partial u_a^I| \cdot |\partial_t Z^\alpha v|\, dx \\
\le& \frac{C_b \varepsilon}{\sqrt{1+t}} E(t).
\end{aligned} \tag{3.17}$$

(G) Treatment of $\int_{\mathbb{R}^2} |Z^\beta \big(\big(c(u)c'(u) - c(u_a^I)c'(u_a^I)\big)|\nabla u_a^I|^2\big)| \cdot |\partial_t Z^\alpha v|\, dx$**.** This case is also similar to (D). In particular, one has

$$\int_{\mathbb{R}^2} |Z^\beta \big(\big(c(u)c'(u) - c(u_a^I)c'(u_a^I)\big)|\nabla u_a^I|^2\big)| \cdot |\partial_t Z^\alpha v|\, dx \le \frac{C_b \varepsilon}{\sqrt{1+t}} E(t). \tag{3.18}$$

Substituting (3.12)–(3.18) into (3.8) yields

$$E'(t) \le \frac{C_b \varepsilon}{\sqrt{1+t}} E(t) + \sum_{|\beta|\le 4} \|Z^\beta J_a^I\|_{L^2} \sqrt{E(t)}.$$

Thus, by Lemmas 2.3 and A.3, one obtains

$$\|\partial Z^\alpha v\|_{L^2} \le C_b \varepsilon^{\frac{3}{2}}, \quad |\alpha| \le 4,$$

and further

$$\|Z^\alpha \partial v\|_{L^2} \le C_b \varepsilon^{\frac{3}{2}}, \quad |\alpha| \le 4. \tag{3.19}$$

By (3.19) and the Klainerman-Sobolev inequality (see [10], [16]), one has

$$|Z^\kappa \partial v| \le C_b \varepsilon^{\frac{3}{2}} (1+t)^{-\frac{1}{2}} (1+|t-r|)^{-\frac{1}{2}}, \quad |\kappa| \le 2, \ t \le T, \tag{3.20}$$

which means that, for small ε,

$$|Z^\kappa \partial v| \le \frac{1}{2} \varepsilon (1+t)^{-\frac{1}{2}} (1+|t-r|)^{-\frac{1}{2}}, \quad |\kappa| \le 2, \ t \le T.$$

This completes the proofs of (3.6) and (3.1). □

When $c(u) = 1 + u^2 + O(u^3)$ in (1.1), then similarly to Lemma 3.1, one has

LEMMA 3.2. Let $c(u) = 1 + u^2 + O(u^3)$. Then, for sufficiently small ε and $0 \le \tau = \varepsilon^2 \ln(1+t) \le b < \nu_0$, (1.1) admits a C^∞-solution u which satisfies the estimate

$$|Z^\kappa \partial (u - u_a^{II})| \le C_b \varepsilon^{\frac{3}{2}} |\ln \varepsilon| (1+t)^{-\frac{1}{2}} (1+|t-r|)^{-\frac{1}{2}}, \tag{3.21}$$

for all $|\kappa| \le 2$.

Proof. As in the proof of Lemma 3.1, we define the energy

$$E(t) = \frac{1}{2} \sum_{|\alpha| \le 4} \int_{\mathbb{R}^2} (|\partial_t Z^\alpha v|^2 + c^2(u) |\nabla Z^\alpha v|^2) \, dx$$

and obtain

$$E'(t) \le \frac{C_b \varepsilon^2}{1+t} E(t) + \sum_{|\beta| \le 4} C_b \|Z^\beta J_a^{II}\|_{L^2} \sqrt{E(t)}.$$

Due to Lemmas 2.6 and A.3, one then obtains (3.21) as in the proof of Lemma 3.1. □

REMARK 3.1. Lemma 3.1 implies that $\varliminf_{\varepsilon \to 0} \varepsilon \sqrt{1 + T_\varepsilon} \ge \tau_0$ holds for the lifespan T_ε of solutions to (1.1) in the case $c(u) = 1 + u + O(u^2)$. Hence,

$$\varliminf_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \ge \tau_0. \tag{3.22}$$

Similarly, Lemma 3.2 implies for the lifespan T_ε of (1.1) in the case $c(u) = 1 + u^2 + O(u^3)$ that

$$\varliminf_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon \ge \nu_0. \tag{3.23}$$

4. The upper bound on the lifespan T_ε.

In this section, we establish the upper bound on T_ε. Some of our ideas are inspired by [9], [14], and [6]. Since, in contrast to [9], [14], $c(u)$ in (1.1) contains the solution u, and not the derivatives of u, our derivation has to be more careful. Thanks to estimates of $Z^\alpha(u - u_a^I)$ and $Z^\alpha(u - u_a^{II})$ with $|\alpha| \le 2$ in Lemmas 3.1 and 3.2, respectively, one observes that $|Z^\alpha(u - u_a^I)| \le C_b \varepsilon^{\frac{3}{2}} (1+t)^{-\frac{1}{2}}$ for $t \le \frac{b^2}{\varepsilon^2} - 1$ and $|Z^\alpha(u - u_a^{II})| \le C_b \varepsilon^{\frac{3}{2}} |\ln \varepsilon| (1+t)^{-\frac{1}{2}}$ for $t \le e^{\frac{b}{\varepsilon^2}} - 1$, respectively, near the light cone. This will play a crucial role in the analysis later on.

Set $U = r^{\frac{1}{2}}u$. Because of the spherical symmetry of u, (1.1) can be written as

$$\begin{cases} \partial_t^2 U - c^2(u)\partial_r^2 U = \frac{1}{4}r^{-\frac{3}{2}}c^2(u)u + 2r^{-\frac{1}{2}}c(u)c'(u)\Big(\partial_r U - \frac{1}{2}r^{-\frac{1}{2}}u\Big)^2, \\ U(0,r) = \varepsilon r^{\frac{1}{2}}u_0, \\ \partial_t U(0,r) = \varepsilon r^{\frac{1}{2}}u_1. \end{cases} \tag{4.1}$$

Define the operators L_1 and L_2 by

$$L_1 = \partial_t + c(u)\partial_r, \qquad L_2 = \partial_t - c(u)\partial_r.$$

We also set

$$w_1 = L_2 U = (\partial_t - c(u)\partial_r)U, \qquad w_2 = L_1 U = (\partial_t + c(u)\partial_r)U,$$

which means $\partial_t U = \dfrac{w_2 + w_1}{2}$ and $\partial_r U = \dfrac{w_2 - w_1}{2c(u)}$.

For

$$L_1L_2 = \partial_t^2 - c^2(u)\partial_r^2 - (L_1 c(u))\partial_r, \qquad L_2L_1 = \partial_t^2 - c^2(u)\partial_r^2 + (L_2 c(u))\partial_r,$$

one has

$$\begin{aligned} L_1 w_1 = {} & \frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)w_1^2 + \frac{c'(u)}{4r^{\frac{1}{2}}c(u)}\Big(\frac{3}{r^{\frac{1}{2}}}c(u)u - 2w_2\Big)w_1 \\ & + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 - \frac{3}{4r}c'(u)uw_2, \end{aligned} \tag{4.2}$$

$$\begin{aligned} L_2 w_2 = {} & \frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)w_2^2 - \frac{c'(u)}{4r^{\frac{1}{2}}c(u)}\Big(\frac{3}{r^{\frac{1}{2}}}c(u)u + 2w_1\Big)w_2 \\ & + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 + \frac{3}{4r}c'(u)uw_1. \end{aligned} \tag{4.3}$$

Because $\partial_r c(u) = c'(u)\partial_r u = \dfrac{c'(u)}{2r^{\frac{1}{2}}c(u)}(w_2 - w_1) - \dfrac{1}{2r}c'(u)u$, one also has

$$L_1 w_1 + w_1\partial_r c(u) = \frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 - \frac{3}{4r}c'(u)uw_2,$$

$$L_2 w_2 - w_2\partial_r c(u) = -\frac{1}{4r}c'(u)uw_2 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 + \frac{3}{4r}c'(u)uw_1$$

and

$$\begin{aligned} & d(|w_1|(dr - cdt)) \\ & = \mathrm{sgn} w_1(L_1 w_1 + w_1\partial_r c)\, dt \wedge dr \\ & = \mathrm{sgn} w_1\Big[\frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 - \frac{3}{4r}c'(u)uw_2\Big]\, dt \wedge dr, \end{aligned} \tag{4.4}$$

$$\begin{aligned} & d(|w_2|(dr + cdt)) \\ & = \mathrm{sgn} w_2(L_2 w_2 - w_2\partial_r c)\, dt \wedge dr \\ & = \mathrm{sgn} w_2\Big[-\frac{1}{4r}c'(u)uw_2 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 + \frac{3}{4r}c'(u)uw_1\Big]\, dt \wedge dr. \end{aligned} \tag{4.5}$$

We first deal with the case $c(u) = 1 + u + O(u^2)$.

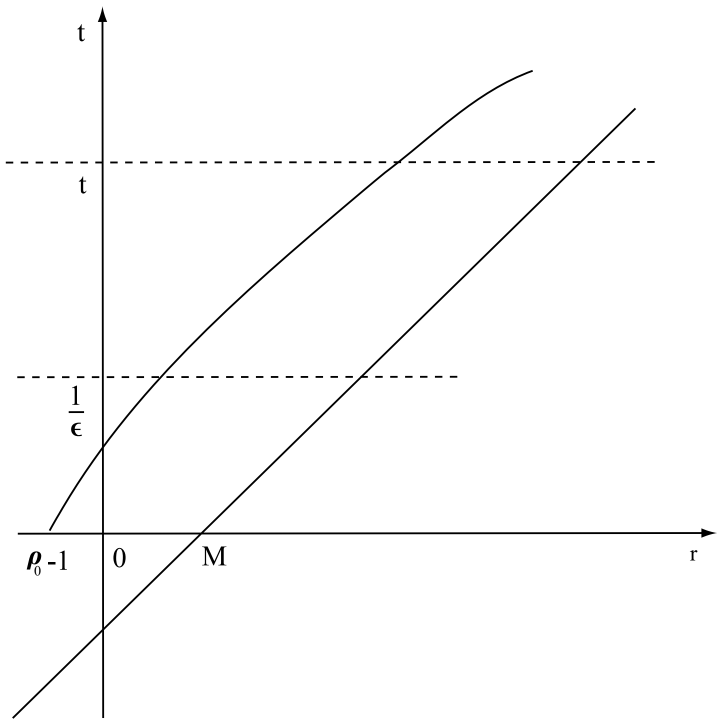

FIG. 1

For $c(u) = 1 + u + O(u^2)$, one knows from Section 2 that, for $\varepsilon\sqrt{1+T_b} = b < \tau_0$ with $b > 0$ a fixed constant, (1.1) has a C^∞-solution for $t \le T_b$. Choose $\varepsilon > 0$ so small that $\frac{1}{\varepsilon} < \frac{b^2}{\varepsilon^2} - 1$. Define the characteristic curve $\Gamma_\lambda^\pm$ by $\frac{dr}{dt} = \pm c(u(t,r))$ and let it pass through $(\lambda, 0)$ in the (r,t)-plane. Let D be the domain that is bounded by Γ_M^+, $\Gamma_{\rho_0-1}^+$, $\{t = 0\}$, and $\{t = T_b\}$ (see Figure 1), where ρ_0 is chosen in such a way that $F_0'(\rho_0) = \min_{\sigma \le M} F_0'(\sigma)$. Obviously, Γ_M^+ is the straight line $r = t + M$.

One now has

LEMMA 4.1. If (t,r), $(t',r') \in \Gamma_\mu^- \cap D$ $(\mu \in \mathbb{R})$ and $(t,r) \in \Gamma_\lambda^+$, $(t',r') \in \Gamma_{\lambda'}^+$, where $\lambda, \lambda' \in [\rho_0 - 1, M]$, then

$$|t - t'| \le C_b. \tag{4.6}$$

Proof. The equation $r = r(t)$ for Γ_λ^+ is

$$\begin{cases} \dfrac{dr(t)}{dt} = c(u(t, r(t))) \equiv c(t), \\ r(0) = \lambda, \end{cases}$$

which yields

$$r(t) - \lambda = \int_0^t (c(s) - 1)\, ds + t.$$

Because

$$|c(t)-1| \le C|u(t,r(t))| \le C_b\varepsilon(1+t)^{-\frac{1}{2}}(1+|t-r(t)|)^{\frac{1}{2}}, \quad 0\le \tau=\varepsilon\sqrt{1+t}\le b<\tau_0,$$

one has

$$\begin{aligned}|r(t)-t| &\le |\lambda| + C_b\varepsilon\int_0^t (1+s)^{-\frac{1}{2}}(1+|s-r(s)|)^{\frac{1}{2}}\,ds \\ &\le m_0 + C_b\varepsilon\int_0^t (1+s)^{-\frac{1}{2}}(1+|s-r(s)|)^{\frac{1}{2}}\,ds \\ &\le m_0 + C_b\varepsilon\int_0^t (1+s)^{-\frac{1}{2}}(1+|s-r(s)|)\,ds,\end{aligned} \tag{4.7}$$

where $m_0=\max\{|\rho_0-1|, M\}$.

Set $f(t)=1+m_0+C_b\varepsilon\int_0^t (1+s)^{-\frac{1}{2}}(1+|s-r(s)|)ds$. By (4.7), one then has

$$f'(t)\le C_b\varepsilon(1+t)^{-\frac{1}{2}}f(t), \quad f(0)=1+m_0.$$

This implies, for $\varepsilon\sqrt{1+t}\le b$,

$$0<f(t)\le (1+m_0)e^{C_b\varepsilon\sqrt{1+t}}\le C_b$$

and

$$|r(t)-t|\le C_b.$$

Therefore,

$$|t+M-r(t)|\le C_b, \tag{4.8}$$

which means that the horizontal width between $\Gamma^+_{\rho_0-1}$ and Γ^+_M in D is finite.

On the other hand, the equation $\widetilde{r}=\widetilde{r}(t)$ for Γ^-_μ is

$$\begin{cases}\dfrac{d\widetilde{r}(t)}{dt} = -c(u(t,\widetilde{r}(t)))\equiv -\tilde{c}(t),\\ \widetilde{r}(0)=\mu.\end{cases}$$

Let $\operatorname{supp} u\subset\{(t,r)\colon r<t+M\}$; then $|\tilde{r}(t)+t-\mu|=0$ for $t<t_*$ and $|\tilde{r}(t)+t-\mu|\le \int_{t_0}^t |\tilde{c}(s)+1|ds \le C_b\varepsilon\int_{t_*}^t (1+s)^{-\frac{1}{2}}(1+|\tilde{r}(s)-s|)^{\frac{1}{2}}ds$ for $t\ge t_*$, where (t_*, t_*+M) denotes the intersection of the curves Γ^-_μ and Γ^+_M.

Combining this with (4.8) yields

$$|\tilde{r}(t)+t-\mu|\le C_b\varepsilon\sqrt{1+t}\le C_b.$$

Thus, if $(t,r),(t',r')\in\Gamma^-_\mu\cap D$ ($\mu\in\mathbb{R}$) and $(t,r)\in\Gamma^+_\lambda$, $(t',r')\in\Gamma^+_{\lambda'}$, one then arrives at

$$|t-t'|\le \frac{1}{2}\big(|t+r-\mu|+|t'+r'-\mu|+|t-r-\lambda|+|t'-r'-\lambda'|+|\lambda-\lambda'|\big)\le C_b,$$

which finishes the proof of Lemma 4.1. □

For $t \le T_b$, define

$$A(t) = \sup_{1/\varepsilon \le s \le t} \int_{(s,r)\in D} |w_1(s,r)|\, dr,$$

$$B(t) = \sup_{\substack{1/\varepsilon \le s \le t \\ (s,r)\in D}} s^{\frac{1}{2}} |u(s,r)|,$$

$$C(t) = \sup_{\substack{1/\varepsilon \le s \le t \\ (s,r)\in D}} s |w_2(s,r)|.$$

Then one obtains

LEMMA 4.2. There exists a constant $E > 0$ such that, for small ε,

$$A\Big(\frac{1}{\varepsilon}\Big) \le \frac{E\varepsilon}{2}, \quad B\Big(\frac{1}{\varepsilon}\Big) \le E\varepsilon, \quad C\Big(\frac{1}{\varepsilon}\Big) \le E^2\varepsilon^2. \tag{4.9}$$

Proof. Since $w_1 = r^{\frac{1}{2}}\partial_t u - c(u) r^{\frac{1}{2}}\partial_r u - \frac{1}{2} r^{-\frac{1}{2}} c(u) u$, for $t \le \frac{b^2}{\varepsilon^2} - 1$, one has

$$|w_1(t,r)| \le C_b \varepsilon. \tag{4.10}$$

Thus, it follows from (4.8) and (4.10) that

$$\int_{(\frac{1}{\varepsilon},r)\in D} |w_1(s,r)|\, dr \le C_b\varepsilon. \tag{4.11}$$

Furthermore, because $|u(t,r)| \le C_b\varepsilon(1+t)^{-\frac{1}{2}}(1+|t-r|)^{\frac{1}{2}}$ for $t \le \frac{b^2}{\varepsilon^2} - 1$ together with (4.8), one has, for $(\frac{1}{\varepsilon}, r) \in D$,

$$\frac{1}{\varepsilon^{\frac{1}{2}}} \Big| u\Big(\frac{1}{\varepsilon}, r\Big)\Big| \le C_b \varepsilon. \tag{4.12}$$

Note that

$$\begin{aligned} w_2(t,r) &= r^{\frac{1}{2}}(\partial_t u + c(u)\partial_r u) + \frac{1}{2} r^{-\frac{1}{2}} c(u) u \\ &= r^{\frac{1}{2}} \frac{S+H}{t+r} u + r^{\frac{1}{2}}(c(u)-1)\partial_r u + \frac{1}{2} r^{-\frac{1}{2}} c(u) u, \end{aligned}$$

which implies that $|w_2(t,r)| \le C_b\varepsilon(1+t)^{-\frac{1}{2}}$ for $(t,r) \in D$ and $\varepsilon\sqrt{1+t} \le b$ in view of (3.9). Together with (4.3), this yields

$$|L_2 w_2| \le \frac{C_b \varepsilon^2}{1+t}. \tag{4.13}$$

For $w_2(t, t+M) = 0$ and (4.6), from (4.13) one then obtains

$$C\Big(\frac{1}{\varepsilon}\Big) \le C_b \varepsilon^2. \tag{4.14}$$

Collecting (4.11), (4.12) and (4.14) completes the proof of (4.9), where $E = 8(1 + C_b)$. □

Based on Lemma 4.2, we will use continuous induction to estimate the upper bound on T_ε when $c(u) = 1 + u + O(u^2)$. To this end, we assume that, for $0 \le t \le T' \le T_b$,

$$A(t) \le E\varepsilon, \qquad B(t) \le 2E\varepsilon, \qquad C(t) \le 3E^2\varepsilon^2. \tag{4.15}$$

We now establish

LEMMA 4.3. Under the hypothesis (4.15) and for ε sufficiently small, one has, for $\frac{1}{\varepsilon} \leq t \leq T'$,

$$A(t) \leq \frac{2}{3}E\varepsilon, \qquad B(t) \leq E\varepsilon, \qquad C(t) \leq \frac{5}{2}E^2\varepsilon^2. \tag{4.16}$$

Proof. First we estimate $A(t)$. By (4.4) and Green's formula, one has, for $\frac{1}{\varepsilon} \leq t \leq T'$,

$$\begin{aligned}
&\int_{(t,r)\in D} |w_1(t,r)|\, dr \\
\leq &\int_{(1/\varepsilon,r)\in D} |w_1(1/\varepsilon,r)|\, dr + \iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \Big| \frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u \\
&+ \frac{1}{2r^{\frac{3}{2}}}c'(u)c(u)u^2 - \frac{3}{4r}c'(u)uw_2 \Big| (s,r)\, dsdr \\
\leq &\frac{1}{2}E\varepsilon + \iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \Big| \frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u \\
&+ \frac{1}{2r^{\frac{3}{2}}}c'(u)c(u)u^2 - \frac{3}{4r}c'(u)uw_2 \Big| (s,r)\, dsdr.
\end{aligned} \tag{4.17}$$

By the inductive hypothesis (4.15), one has $|u(s,r)| \leq 2E\varepsilon s^{-\frac{1}{2}}$ for $\frac{1}{\varepsilon} \leq s \leq T'$ and $(s,r) \in D$. Note also that c is near 1 for small ε and $|r-s| \leq C_b$ holds for $s \geq 1/\varepsilon$. One then has $r \geq s/2$ and

$$\begin{aligned}
&\iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \Big| \frac{1}{4r}c'(u)uw_1 \Big|\, dsdr \\
\leq &\iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \frac{2E\varepsilon}{s^{\frac{3}{2}}} |w_1|(s,r)\, dsdr \\
= &2E\varepsilon \int_{1/\varepsilon}^{t} \frac{1}{s^{\frac{3}{2}}} ds \int_{(s,r)\in D} |w_1|(s,r)\, dr \\
&\leq 4E\varepsilon^{\frac{3}{2}} A(t) \leq 4E^2\varepsilon^{\frac{5}{2}}.
\end{aligned} \tag{4.18}$$

Similarly, one has

$$\iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \Big| \frac{1}{4r^{\frac{3}{2}}}c^2(u)u \Big|\, dsdr \leq 2E\varepsilon \iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \frac{1}{s^2}\, dsdr \leq C_b E\varepsilon^2, \tag{4.19}$$

$$\iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \Big| \frac{1}{2r^{\frac{3}{2}}}c'(u)c(u)u^2 \Big|\, dsdr \leq C_b E^2\varepsilon^2 \iint_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \frac{1}{s^{\frac{5}{2}}}\, dsdr \leq C_b E^2\varepsilon^{\frac{7}{2}}, \tag{4.20}$$

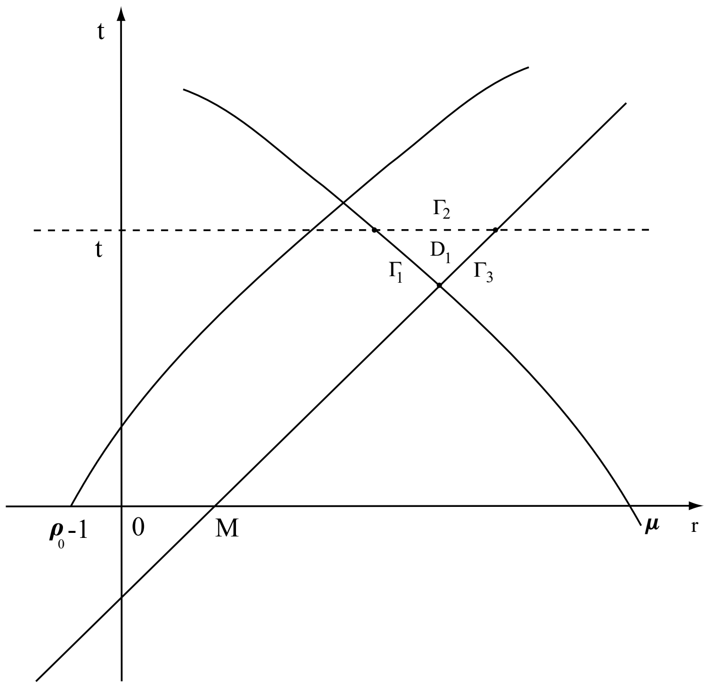

FIG. 2

and

$$\iint_{\substack{1/\varepsilon\le s\le t\\(s,r)\in D}}\left|\frac{3}{4r}c'(u)uw_2\right|(s,r)\,dsdr\le C_bE^2\varepsilon^2\iint_{\substack{1/\varepsilon\le s\le t\\(s,r)\in D}}\frac{1}{s^{\frac52}}\,dsdr\le C_bE^2\varepsilon^{\frac72}. \tag{4.21}$$

Substituting (4.18)–(4.21) into (4.17) yields

$$A(t)\le\frac12E\varepsilon+C_bE\varepsilon^2+C_bE^2\varepsilon^{\frac72},$$

which implies $A(t)\le\frac23E\varepsilon$ for sufficiently small ε.

Next we estimate $B(t)$. Note that u satisfies the equation

$$L_2u=L_2(r^{-\frac12}U)=r^{-\frac12}w_1+\frac{1}{2r^{\frac32}}c(u)u. \tag{4.22}$$

Integrate (4.22) along the characteristic curve Γ_μ^- which insects Γ_M^+ at the point (t',r'). When $t'\ge1/\varepsilon$, denote by D_1 the domain bounded by Γ_μ^-, the line $\{t=t'\}$, and Γ_M^+. Let Γ_1, Γ_2, and Γ_3 be the faces of the boundary (see Figure 2). By (4.4), one then has

$$\iint_{(s,r)\in D_1}\operatorname{sgn}w_1\left[\frac{1}{4r}c'(u)uw_1+\frac{1}{4r^{\frac32}}c^2(u)u+\frac{1}{2r^{\frac32}}c'(u)c(u)u^2-\frac{3}{4r}c'(u)uw_2\right](s,r)\,dsdr$$

$$=\left(\int_{\Gamma_1}+\int_{\Gamma_2}\right)|w_1|(dr-c\,dt),$$

which implies

$$\begin{aligned}
&\int_{\Gamma_1} |w_1|(dr - c\,ds)\\
&\leq \iint\limits_{(s,r)\in D_1} \left|\frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c'(u)c(u)u^2 - \frac{3}{4r}c'(u)uw_2\right|(s,r)\,dsdr\\
&\quad + \int_{\Gamma_2} |w_1|(dr - c\,dt)\\
&\leq \iint\limits_{\substack{1/\varepsilon\leq s\leq t\\(s,r)\in D}} \left|\frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c'(u)c(u)u^2\right.\\
&\quad \left.- \frac{3}{4r}c'(u)uw_2\right|(s,r)\,dsdr + \int_{(t,r)\in D_1} |w_1(t,r)|\,dr\\
&\leq \frac{1}{6}E\varepsilon + \frac{2}{3}E\varepsilon = \frac{5}{6}E\varepsilon,
\end{aligned}$$

the last inequality resulting the estimates (4.18)–(4.21). This yields

$$\begin{aligned}
&\int_{t'}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(s))^{\frac{1}{2}}}\,ds\\
&\leq \int_{t'}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(t))^{\frac{1}{2}}}\,ds \leq \frac{2}{t^{\frac{1}{2}}}\int_{t'}^{t} |w_1(s,r(s))|\,ds\\
&= \frac{2}{t^{\frac{1}{2}}}\int_{\Gamma_1} \frac{|w_1(s,r)|}{\sqrt{1+c^2}}\,ds = \frac{2}{t^{\frac{1}{2}}}\int_{\Gamma_1} \frac{|w_1|}{2c\sqrt{1+c^2}}(dr - c\,ds)\\
&\leq \frac{5}{6t^{\frac{1}{2}}}E\varepsilon.
\end{aligned} \tag{4.23}$$

Note also that $r(s) \geq \frac{s}{2} \geq \frac{1}{2\varepsilon}$. By (4.22), one then has

$$\begin{aligned}
|u(t,r)| &\leq \int_{t'}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(s))^{\frac{1}{2}}}\,ds + \int_{t'}^{t} \frac{|cu|(s,\tilde{r}(s))}{2(\tilde{r}(s))^{\frac{3}{2}}}\,ds\\
&\leq \frac{5}{6t^{\frac{1}{2}}}E\varepsilon + 3\varepsilon^{\frac{3}{2}}\int_{t'}^{t} |u(s,r(s))|\,ds.
\end{aligned}$$

By (4.6), one obtains

$$|u(t,\tilde{r}(t))| \leq \frac{5}{6t^{\frac{1}{2}}}E\varepsilon e^{C_b\varepsilon} \leq E\varepsilon t^{-\frac{1}{2}}.$$

One has $t \leq t' + |t - t'| \leq 2/\varepsilon$ for $t' \leq 1/\varepsilon$. Thus, $t^{\frac{1}{2}}|u(t,r)| \leq E\varepsilon$ for $(t,r) \in D$, and $B(t) \leq E\varepsilon$ follows.

Finally, we estimate $C(t)$. We rewrite (4.3) as

$$L_2 w_2 = a w_2 + b, \tag{4.24}$$

where

$$a = \frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)w_2 - \frac{c'(u)}{4r^{\frac{1}{2}}c(u)}\left(\frac{3}{r^{\frac{1}{2}}}c(u)u + 2w_1\right),$$
$$b = \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2 + \frac{3}{4}c'(u)uw_1.$$

Integrating (4.24) along Γ_μ^- as above, one obtains

$$|w_2(t,r)| \le \int_{t'}^{t} |aw_2 + b|(s,\tilde{r}(s))\,ds. \tag{4.25}$$

Noting that $t' \ge t - |t-t'| \ge t - C_b$, $|w_2(t,r)| \le \dfrac{3E^2\varepsilon^2}{t}$ and using (4.6), (4.8), and (4.23), by the choice of $E = 8(1+C_b)$ in Lemma 4.2, one arrives at

$$\begin{aligned}
&\int_{t'}^{t} |b(s,\tilde{r}(s))|\,ds \\
&\le \int_{t'}^{t} \frac{1}{4(\tilde{r}(s))^{\frac{3}{2}}}|c^2(u)u|(s,\tilde{r}(s))\,ds + \int_{t'}^{t} \frac{1}{2(\tilde{r}(s))^{\frac{3}{2}}}|c(u)c'(u)u^2|(s,\tilde{r}(s))\,ds \\
&+ \int_{t'}^{t} \frac{3}{4\tilde{r}(s)}|c'(u)uw_1|(s,\tilde{r}(s))\,ds \\
&\le \frac{1}{3}C_bE\varepsilon^2t^{-1} + \frac{2E^2\varepsilon^2}{t}\int_{t'}^{t}\frac{1}{s^{\frac{3}{2}}}\,ds + \frac{5E\varepsilon}{4t}\int_{t'}^{t}\frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(s))^{\frac{1}{2}}}\,ds \\
&\le \frac{1}{3}E^2\varepsilon^2t^{-1} + \frac{1}{2}E^2\varepsilon^2t^{-1} + \frac{1}{2}E^2\varepsilon^2t^{-1} \\
&\le \frac{5}{3}E^2\varepsilon^2t^{-1},
\end{aligned} \tag{4.26}$$

and similarly

$$\int_{t'}^{t} |aw_2|(s,r(s))\,ds \le \frac{2E^2\varepsilon^2}{t}\int_{t'}^{t}|a|(s,\tilde{r}(s))\,ds \le \frac{C_bE^3\varepsilon^3}{t}. \tag{4.27}$$

Substituting (4.26) and (4.27) into (4.25) yields

$$|w_2(t,r)| \le \int_{t'}^{t} |aw_2 + b|(s,\tilde{r}(s))\,ds \le \frac{5E^2\varepsilon^2}{2t},$$

which shows that $C(t) \le \dfrac{5E^2\varepsilon^2}{2}$. □

We will use Lemma A.4 to estimate the upper bound on the lifespan T_ε when $c(u) = 1 + u + O(u^2)$ in (1.1), based on Lemmas 4.2 and 4.3. More specifically, we will show that

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\sqrt{T_\varepsilon} \le -\frac{1}{2F_0'(\rho_0)} = \tau_0. \tag{4.28}$$

It follows from (4.2) that on the characteristic curve $\Gamma_{\rho_0}^+$, $w_1(t,r(t))$ satisfies

$$\frac{dw_1}{dt}(t,r(t)) = L_1w_1 = a_0(t)w_1^2 + a_1(t)w_1 + a_2(t), \tag{4.29}$$

where

$$\begin{aligned} a_0(t) &= \left(\frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)\right)(t,r(t)),\\ a_1(t) &= \left(\frac{c'(u)}{4r^{\frac{1}{2}}c(u)}\left(\frac{3}{r^{\frac{1}{2}}}c(u)u-2w_2\right)\right)(t,r(t)),\\ a_2(t) &= \left(\frac{1}{4r^{\frac{3}{2}}}c^2(u)u+\frac{1}{2r^{\frac{3}{2}}}c(u)c'(u)u^2-\frac{3}{4r}c'(u)uw_2\right)(t,r(t)). \end{aligned}$$

By (4.15), one has, for $\frac{1}{\varepsilon}\le t\le T_b$,

$$|a_1|\le\frac{5E\varepsilon}{t^{\frac{3}{2}}},\qquad |a_2|\le\frac{10E\varepsilon^{\frac{3}{2}}}{t^{\frac{3}{2}}},$$

which implies

$$\int_{1/\varepsilon}^{T_b}|a_1|\,ds\le 10E\varepsilon^{\frac{3}{2}},\qquad \int_{1/\varepsilon}^{T_b}|a_2|\,ds\le 20E^2\varepsilon^2. \tag{4.30}$$

This also yields

$$K=\left(\int_{1/\varepsilon}^{T_b}|a_2(t)|\,dt\right)\exp\left(\int_{1/\varepsilon}^{T_b}|a_1(t)|\,dt\right)=O(\varepsilon^2). \tag{4.31}$$

By the definition of u_a^I in (2.14), one has that $u_a^I(1/\varepsilon)=\varepsilon w_0(1/\varepsilon)$ on $\Gamma_{\rho_0}^+$. Moreover, it follows from Lemma 3.1 that

$$|Z^\alpha(u-u_a^I)|\le C_{\alpha b}\varepsilon^{\frac{3}{2}}(1+t)^{-\frac{1}{2}}(1+|t-r|)^{1/2}.$$

On the other hand, by [10, Theorem 6.2.1], one has

$$\left|\partial^\alpha Z^\beta\left(u_a^I(1/\varepsilon)-\left(\frac{\varepsilon}{r^{\frac{1}{2}}}F_0\right)(r(1/\varepsilon)-1/\varepsilon)\right)\right|\le C_{\alpha\beta}\varepsilon^{\frac{3}{2}}.$$

Therefore,

$$\begin{aligned} w_1(1/\varepsilon) &= (r^{\frac{1}{2}}\partial_t u)(1/\varepsilon)-\left[c\left(\frac{u}{2r^{\frac{1}{2}}}+r^{\frac{1}{2}}\partial_r u\right)\right](1/\varepsilon)\\ &= (r^{\frac{1}{2}}\partial_t u_a^I)(1/\varepsilon)-\left[c\left(\frac{u_a^I}{2r^{\frac{1}{2}}}+r^{\frac{1}{2}}\partial_r u_a^I\right)\right](1/\varepsilon)+O(\varepsilon^{3/2})\\ &= \varepsilon F_0'(r(1/\varepsilon)-1/\varepsilon))(-c(1/\varepsilon)-1)+O(\varepsilon^{3/2})\\ &= -2\varepsilon F_0'(r(1/\varepsilon)-1/\varepsilon))+O(\varepsilon^{3/2}). \end{aligned}$$

Note that one has $|r-t|\le C+|\rho_0|$ and $|u(t,r)|\le C_b\varepsilon(1+t)^{-\frac{1}{2}}$ on $\Gamma_{\rho_0}^+$. Hence,

$$|r(1/\varepsilon)-1/\varepsilon-\rho_0|\le\int_0^{1/\varepsilon}C_b\varepsilon(1+s)^{-\frac{1}{2}}\,ds=C_b\varepsilon\sqrt{1+\frac{1}{\varepsilon}}.$$

We now prove (4.28). By Lemma A.4 and (4.30) and (4.31), one has

$$\left(\int_{1/\varepsilon}^{T_b}\frac{1}{2(r(t))^{\frac{1}{2}}c(u)}c'(u)\,dt\right)\exp\left(-\int_{1/\varepsilon}^{T_b}|a_1(t)|\,dt\right)<(w_1(1/\varepsilon)-K)^{-1};$$

that is,

$$\left(\sqrt{T_b}-\sqrt{1+\frac{1}{\varepsilon}}\right)(1+O(\varepsilon))<(-2\varepsilon F_0'(\rho_0)+O(\varepsilon^{3/2}))^{-1}\exp(5E\varepsilon^2).$$

Thus,

$$\overline{\lim_{\varepsilon\to 0}}\varepsilon\sqrt{T_\varepsilon}\le -\frac{1}{2F_0'(\rho_0)}=\tau_0,$$

and (4.28) is proved.

Next we show that in the case $c(u)=1+u^2+O(u^2)$ in (1.1), the lifespan T_ε satisfies

$$\overline{\lim_{\varepsilon\to 0}}\varepsilon^2\ln T_\varepsilon\le \nu_0=-\frac{1}{2\min\limits_{\sigma}\{F_0(\sigma)F_0'(\sigma)\}}. \tag{4.32}$$

Although the proof is analogous to that of (4.28), for the readers' convenience, we provide the details.

Set $\tilde{T}_b=e^{\frac{b}{\varepsilon^2}}-1$, where $0<b<\nu_0$ is a fixed constant. As above, define $\tilde{\Gamma}^{\pm}_{\lambda}$ to be the characteristic curve given by $\dfrac{dr}{dt}=\pm c(u(t,r))$ and passing through the point $(\lambda,0)$. The domain $\tilde{D}$ is bounded by $\tilde{\Gamma}^+_M$, $\tilde{\Gamma}^+_{\tilde{\rho}_0-1}$, $\{t=0\}$, and $\{t=\tilde{T}_b\}$, where $\tilde{\rho}_0$ is chosen so that $F_0(\tilde{\rho}_0)F_0'(\tilde{\rho}_0)=\min\limits_{\sigma\le M}\{F_0(\sigma)F_0'(\sigma)\}$.

Similarly to Lemma 4.1, one has

Lemma 4.4. If $(t,r),(t',r')\in\tilde{\Gamma}^-_{\nu}\cap\tilde{D}$ $(\nu\in\mathbb{R})$ and $(t,r)\in\tilde{\Gamma}^+_{\lambda},(t',r')\in\tilde{\Gamma}^+_{\lambda'}$, where $\lambda,\lambda'\in[\tilde{\rho}_0-1,M]$, then

$$|t-t'|\le C_b. \tag{4.33}$$

Proof. For $\lambda\in[\tilde{\rho}_0-1,M]$, the equation $r=r(t)$ of $\tilde{\Gamma}^+_{\lambda}$ is

$$\begin{cases}\dfrac{dr(t)}{dt}=c(u(t,r(t)))\equiv c(t),\\ r(0)=\lambda.\end{cases}$$

Because $|c(t)-1|\le C_b|u(t,r(t))|^2\le C_b\varepsilon^2(1+t)^{-1}(1+|r-r(t)|)$ for $0<\tau=\varepsilon^2\ln(1+t)\le b<\nu_0$, one has

$$\begin{aligned}|r(t)-t|&\le|\lambda|+\int_0^t|c(u(s,r(s)))-1|\,ds\\&\le m_0+C_b\varepsilon^2\int_0^t(1+s)^{-1}(1+|r(s)-s|)\,ds,\end{aligned}$$

which implies $|r(t)-t|\le C_b$ for $t\le\tilde{T}_b$. The proof of Lemma 4.4 then concludes by an argument similar to that in the proof of Lemma 4.1. □

Define $A(t),B(t)$, and $C(t)$ as in Lemma 4.2. When $c(u)=1+u^2+O(u^3)$ in (1.1), one obtains the paralleling Lemmas 4.2 and 4.3:

Lemma 4.5. There exists a positive constant E such that, for small ε,

(*i*)

$$A\left(\frac{1}{\varepsilon}\right)\le\frac{E\varepsilon}{2},\quad B\left(\frac{1}{\varepsilon}\right)\le E\varepsilon,\quad C\left(\frac{1}{\varepsilon}\right)\le E^2\varepsilon^2. \tag{4.34}$$

(ii) If $0 \le t \le \tilde{T}_b$, then

$$A(t) \le E\varepsilon, \qquad B(t) \le 2E\varepsilon, \qquad C(t) \le 3E^2\varepsilon^2. \tag{4.35}$$

Proof. Since the proof is analogous to those of Lemmas 4.2 and 4.3, it is omitted. □

Next we prove (4.32). It follows from (4.2) that along the characteristic curve $\tilde{\Gamma}^+_{\tilde{\rho}_0}$, $w_1(t, r(t))$ satisfies

$$\frac{dw_1}{dt}(t, r(t)) = a_0(t)w_1^2 + a_1(t)w_1 + a_2(t), \tag{4.36}$$

where $a_0(t), a_1(t)$, and $a_2(t)$ are defined as in (4.29).

By (4.35), one has, for $\frac{1}{\varepsilon} \le t \le \tilde{T}_b$,

$$|a_1| \le \frac{5E\varepsilon^2}{t^2}, \qquad |a_2| \le \frac{10E\varepsilon^{\frac{3}{2}}}{t^{\frac{3}{2}}},$$

which implies

$$\int_{1/\varepsilon}^{\tilde{T}_b} |a_1|\, ds \le 10E\varepsilon^3, \qquad \int_{1/\varepsilon}^{\tilde{T}_b} |a_2|\, ds \le 20E^2\varepsilon^2 \tag{4.37}$$

and

$$K = \left(\int_{1/\varepsilon}^{\tilde{T}_b} |a_2(t)|\, dt\right) \exp\left(\int_{1/\varepsilon}^{\tilde{T}_b} |a_1(t)|\, dt\right) = O(\varepsilon^2). \tag{4.38}$$

By the definition of u_a^{II} in (2.31), one has that $u_a^{II}(1/\varepsilon) = \varepsilon w_0(1/\varepsilon)$ on $\tilde{\Gamma}^+_{\tilde{\rho}_0}$ holds true. Moreover, it follows from Lemma 3.2 that

$$|Z^\alpha(u - u_a^{II})| \le C_{\alpha b}\varepsilon^{\frac{3}{2}}|\ln\varepsilon|(1+t)^{-\frac{1}{2}}(1+|t-r|)^{1/2}. \tag{4.39}$$

On the other hand, by [10, Theorem 6.2.1] one has

$$\left|\partial^\alpha Z^\beta\left(u_a^{II}(1/\varepsilon) - \left(\frac{\varepsilon}{r^{\frac{1}{2}}}F_0\right)(r(1/\varepsilon) - 1/\varepsilon)\right)\right| \le C_{\alpha\beta}\varepsilon^{\frac{3}{2}}.$$

Therefore,

$$\begin{aligned}
w_1(1/\varepsilon) &= (r^{\frac{1}{2}}\partial_t u)(1/\varepsilon) - \left(c\left(\frac{u}{2r^{\frac{1}{2}}} + r^{\frac{1}{2}}\partial_r u\right)\right)(1/\varepsilon) \\
&= (r^{\frac{1}{2}}\partial_t u_a^{II})(1/\varepsilon) - \left(c\left(\frac{u_a^{II}}{2r^{\frac{1}{2}}} + r^{\frac{1}{2}}\partial_r u_a^{II}\right)\right)(1/\varepsilon) + O(\varepsilon^{3/2}) \\
&= \varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon)(-c(1/\varepsilon) - 1) + O(\varepsilon^{3/2}|\ln\varepsilon|) \\
&= -2\varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon) + O(\varepsilon^{3/2}|\ln\varepsilon|).
\end{aligned}$$

Note that $|r - t| \le C + |\tilde{\rho}_0|$ and $|u(t, r)| \le C_b\varepsilon(1+t)^{-\frac{1}{2}}$ on $\tilde{\Gamma}^+_{\tilde{\rho}_0}$. Hence,

$$|r(1/\varepsilon) - 1/\varepsilon - \tilde{\rho}_0| \le \int_0^{1/\varepsilon} C_b\varepsilon^2(1+s)^{-1}ds = C_b\varepsilon^2|\ln\varepsilon|.$$

Thus one has

$$w_1\left(\frac{1}{\varepsilon}\right) = -2\varepsilon F_0'(\tilde{\rho}_0) + O(\varepsilon^{3/2}|\ln\varepsilon|). \tag{4.40}$$

For later reference, we now provide properties of u when restricted to $\tilde{\Gamma}^+_{\tilde{\rho}_0}$.

By the definition of u_a^{II} in (2.31), one has, for $t \geq \dfrac{1}{\varepsilon}$,

$$u_a^{II}(t,x) = \varepsilon\Big(\chi(\varepsilon t)w_0(t,x) + (1-\chi(\varepsilon t))r^{-1/2}F_0(\tilde\rho_0)\Big) \quad \text{on } \tilde\Gamma^+_{\tilde\rho_0}. \tag{4.41}$$

Moreover, it follows from [10, Lemma 6.2.1] that

$$|\partial^\alpha Z^\beta(w_0(t,x) - r^{-1/2}F_0(\tilde\rho_0))| \leq C_{\alpha\beta}\varepsilon^{\frac12}(1+t)^{-1} \quad \text{on } \tilde\Gamma^+_{\tilde\rho_0}. \tag{4.42}$$

Substituting (4.42) into (4.41) yields, for $t \geq \dfrac{1}{\varepsilon}$,

$$u_a^{II} = \varepsilon r^{-1/2}F_0(\tilde\rho_0) + O(\varepsilon^{\frac32})(1+t)^{-1} \quad \text{on } \tilde\Gamma^+_{\tilde\rho_0}.$$

Together with (4.39), this implies, for $t \geq \dfrac{1}{\varepsilon}$,

$$u = \varepsilon r^{-1/2}F_0(\rho_0) + O(\varepsilon^{\frac32})|\ln\varepsilon|(1+t)^{-\frac12} + O(\varepsilon^{\frac32})(1+t)^{-1} \quad \text{on } \tilde\Gamma^+_{\tilde\rho_0}. \tag{4.43}$$

Relying on the preparations above, we now prove (4.32).

As $F_0(\tilde\rho_0)F_0'(\tilde\rho_0) < 0$, without loss of generality we can assume that $F_0(\tilde\rho_0) < 0$ and $F_0'(\tilde\rho_0) > 0$. One then has, for $t \geq \dfrac{1}{\varepsilon}$,

$$w_1\left(\frac{1}{\varepsilon}\right) < 0 \ \text{ and } \ a_0(t) = \frac{2u + O(u^2)}{2(r(t))^{1/2}c(u)} < 0 \quad \text{on } \tilde\Gamma^+_{\rho_0}.$$

Consider the equation for $\tilde w_1 = -w_1$. It follows from (4.36) that on the characteristic curve $\tilde\Gamma^+_{\tilde\rho_0}$, for $t > \dfrac{1}{\varepsilon}$,

$$\frac{d\tilde w_1}{dt}(t, r(t)) = -a_0(t)(\tilde w_1)^2 + a_1(t)\tilde w_1 - a_2(t),$$

where $\tilde w_1(\frac{1}{\varepsilon}) > 0$. By Lemma A.4, one has

$$\left(-\int_{1/\varepsilon}^{\tilde T_b} \frac{1}{2(r(t))^{\frac12}c(u)}c'(u)\,dt\right)\exp\left(-\int_{1/\varepsilon}^{\tilde T_b}|a_1(t)|\,dt\right) < (\tilde w_1(1/\varepsilon) - K)^{-1}.$$

From this, together with (4.37), (4.38), (4.40), (4.43), and $c'(u) = 2u + O(u^2)$, one arrives at

$$\varepsilon F_0(\tilde\rho_0)(\ln\tilde T_b - \ln(1/\varepsilon))(1+O(\varepsilon)) < (-2\varepsilon F_0'(\tilde\rho_0) + O(\varepsilon^{3/2}|\ln\varepsilon|))^{-1}\exp(10E\varepsilon^2),$$

which implies

$$\varlimsup_{\varepsilon\to 0}\varepsilon^2\ln T_\varepsilon \leq -\frac{1}{2F_0(\tilde\rho_0)F_0'(\tilde\rho_0)} = \nu_0.$$

Consequently, (4.32) is proved.

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, it follows from (3.22), (3.23), (4.28), and (4.32) that the lifespan T_ε satisfies

$$\lim_{\varepsilon\to 0}\varepsilon\sqrt{T_\varepsilon} = \tau_0 \quad \text{when } \ c(u) = 1 + u + O(u^2)$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^2\ln T_\varepsilon = \nu_0 \quad \text{when } \ c(u) = 1 + u^2 + O(u^3).$$

Thus, we have completed the proof of Theorem 1.1. ☐

Appendix A. Some useful lemmas.

LEMMA A.1. (*i*) For $\phi(t,r) \in C^1$,

$$|\partial\phi| \leq \frac{2}{1+|t-r|} \sum_{|\beta|=1} |Z^\beta \phi|. \tag{A.1}$$

(*ii*) Klainerman fields have the following expressions in (τ,σ) coordinates:

$$\begin{cases} \partial_t = -\partial_\sigma + \dfrac{\varepsilon}{2\sqrt{1+t}}\partial_\tau, \\ \partial_r = \partial_\sigma, \\ S = \sigma\partial_\sigma + \dfrac{\varepsilon t}{2\sqrt{1+t}}\partial_\tau, \\ H = -\sigma\partial_\sigma + \dfrac{\varepsilon r}{2\sqrt{1+t}}\partial_\tau. \end{cases}$$

Proof. (i) For $\partial_t = \dfrac{tS - rH}{t^2 - r^2}$ and $\partial_r = \dfrac{tH - rS}{t^2 - r^2}$,

$$(1+|t-r|)(|\partial_t\phi| + |\partial_r\phi|) \leq 2(|S\phi| + |H\phi| + |\partial_t\phi| + |\partial_r\phi|),$$

and (A.1) is proved.

(ii) This follows from a direct computation. □

LEMMA A.2. If $f(t,x) \in C^1(\mathbb{R}^+ \times \mathbb{R}^2)$ depends only on (t,r) and $\operatorname{supp} f \subseteq \{(t,x)\colon r \leq M+t\}$, then

$$\|(1+|t-r|)^{-1} f\|_{L^2} \leq C\|\partial_r f\|_{L^2}.$$

REMARK A.1. In the case of $x \in \mathbb{R}^3$, Lemma A.2 has been proved in [19].

Proof. Since $\operatorname{supp} f \subseteq \{r \leq M+t\}$,

$$f(t,r) = -\int_r^{M+t} \partial_r f(t,s)\, ds.$$

It follows that

$$\begin{aligned} |f(t,r)|^2 &\leq \left(\int_r^{M+t} |\partial_r f(t,s)|^2 (1+|t-s|)^{1/2}\, ds\right) \int_r^{M+t} (1+|t-s|)^{-1/2}\, ds \\ &\leq C\left(\int_r^{M+t} |\partial_r f(t,s)|^2 (1+|t-s|)^{1/2} ds\right)(1+|t-r|)^{1/2}. \end{aligned}$$

Thus,

$$
\begin{aligned}
&\int_0^{M+t} (1+|t-r|)^{-2}|f(t,r)|^2 r\,dr \\
&\le C\int_0^{M+t}\Big(\int_r^{M+t} |\partial_r f(t,s)|^2(1+|t-s|)^{1/2}\,ds\Big)(1+|t-r|)^{-3/2} r\,dr \\
&\le C\int_0^{M+t} |\partial_r f(t,s)|^2(1+|t-s|)^{1/2}\,ds\int_0^s (1+|t-r|)^{-3/2} r\,dr \\
&\le C\int_0^{M+t} |\partial_r f(t,s)|^2(1+|t-s|)^{1/2} s\,ds\int_0^s (1+|t-r|)^{-3/2}\,dr \\
&\le C\int_0^{M+t} |\partial_r f(t,s)|^2 s\,ds,
\end{aligned}
$$

and Lemma A.2 is proved. □

LEMMA A.3 (Generalized Gronwall inequality). Let $f \in C^1[0,\infty)$, $g, h \in C[0,\infty)$ be nonnegative and let

$$\frac{df^2(t)}{dt} \le f(t)g(t) + h(t)f^2(t).$$

Then

$$f(t) \le \Big(f(0) + \frac{1}{2}\int_0^t g(s)ds\Big)\exp\Big(\frac{1}{2}\int_0^t h(s)ds\Big).$$

LEMMA A.4 ([10, Lemma 1.3.2]). Let w be a solution in $[0,T]$ to the ordinary differential equation

$$\frac{dw}{dt} = a_0(t)w^2 + a_1(t)w + a_2(t)$$

with a_j continuous and $a_0 \ge 0$. Let

$$K = \Big(\int_0^T |a_2(t)|\,dt\Big)\exp\Big(\int_0^T |a_1(t)|\,dt\Big).$$

Then

$$\Big(\int_0^T a_0(t)\,dt\Big)\exp\Big(-\int_0^T |a_1(t)|\,dt\Big) < (w(0)-K)^{-1}$$

provided that $w(0) > K$.

LEMMA A.5 (Blowup of smooth solution to problem (2.37)). The smooth solution to (2.37) blows up in finite time if $F_0(\sigma) \not\equiv 0$.

Proof. Assume that (2.37) admits a global smooth solution. Due to $(F_0')^2(M) = (F_0')^2(-\infty) = 0$ and $(F_0')^2 \not\equiv 0$, one has

$$F_0'(\sigma)F_0''(\sigma) = \Big(\frac{1}{2}(F_0')(\sigma)\Big)' < 0 \ \text{ on some interval } I \subset (-\infty, M).$$

Without loss of generality, we can assume $F_0'(\sigma) < 0$ and $F_0''(\sigma) > 0$ on I.

Let $\Sigma = \{(\tau, \sigma(\tau, l)) \colon \tau \geq 0,\ l \in I\}$, where $\sigma(\tau, l)$ stands for the characteristics of (2.37) emanating from the point $(l, 0)$; i.e., $\sigma(\tau, l)$ satisfies

$$\begin{cases} \dfrac{d\sigma(\tau,l)}{d\tau} = G^2(\tau, \sigma(\tau, l)), \\ \sigma(0, l) = l. \end{cases} \tag{A.2}$$

Set $Q(\tau, l) = (\partial_\sigma G)(\tau, \sigma(\tau, l))$ and $G(\tau, l) = G(\tau, \sigma(\tau, l))$. It follows from the equation in (2.37) that

$$Q(\tau, l) = \frac{F_0'(l)}{1 + F_0'(l) \int_0^\tau G(t, l)\, dt} \tag{A.3}$$

and

$$(\partial_\sigma Q)(\tau, \sigma(\tau, l))\partial_l \sigma(\tau, l) = \frac{F_0''(l) - (F_0')^2(l) \int_0^\tau Q(t, l)\partial_l \sigma(t, l)\, dt}{(1 + F_0'(l) \int_0^\tau G(t, l)\, dt)^2}.$$

Therefore,

$$Q < 0 \text{ and } \partial_\sigma Q > 0 \text{ in } \Sigma. \tag{A.4}$$

Choose $l_i \in I$ $(i = 0, 1, 2)$ such that $l_0 < l_1 < l_2$ and denote $E_j = \displaystyle\int_{l_j}^{l_{j+1}} (F_0')^2(l)\, dl$ for $j = 0, 1$. It follows from the conservation of energy for problem (2.37) and (A.4) that, for $j = 0, 1$,

$$\begin{aligned} 0 < E_j &= \int_{\sigma(\tau, l_j)}^{\sigma(\tau, l_{j+1})} Q^2(\tau, s)\, ds \\ &\leq (-Q(\tau, l_j)) \int_{\sigma(\tau, l_j)}^{\sigma(\tau, l_{j+1})} (-Q)(\tau, s)\, ds = -Q(\tau, l_j)\big(G(\tau, l_j) - G(\tau, l_{j+1})\big), \end{aligned}$$

which yields

$$G(\tau, l_j) - G(\tau, l_{j+1}) \geq -\frac{E_j}{Q(\tau, l_j)}, \quad j = 0, 1. \tag{A.5}$$

By (A.2), one has that $\sigma(\tau, l) < M$ holds for $l \in I$ and all τ. Therefore,

$$\sum_{i=0}^{2} \int_0^\infty G^2(\tau, l_i)\, d\tau \leq 3M + \sum_{i=0}^{2} |l_i|,$$

which implies that there exists a sequence $\{\tau_k\} \subset [0, \infty)$ with $\tau_k \to \infty$ as $k \to \infty$ such that

$$G(\tau_k, l_i) \to 0 \quad \text{as } k \to \infty \text{ for } i = 0, 1, 2. \tag{A.6}$$

It then follows from (A.5) and (A.6) that

$$Q(\tau_k, l_j) \to -\infty \quad \text{as } k \to \infty \text{ for } j = 0, 1. \tag{A.7}$$

On the other hand, by (A.4), one has

$$\int_0^{\tau_k} G(t, l_0) dt > \int_0^{\tau_k} G(t, l_1) dt.$$

Together with (A.4) and (A.7), this yields as $k \to \infty$

$$-\frac{1}{F_0'(l_0)} \geq -\frac{1}{F_0'(l_1)}$$

and $F_0'(l_1) \le F_0'(l_0)$. The latter, however, contradicts the fact that $F_0'(l_0) < F_0'(l_1)$ holds due to $F_0''(\sigma) > 0$ in I and $l_0 < l_1$.

Thus, the proof of Lemma A.5 has been completed. □

References

[1] R. K. Agarwal and D. W. Halt, *A modified CUSP scheme in wave/particle split form for unstructured grid Euler flows,* Frontiers of Computational Fluid Dynamics (D. A. Caughey and M. M. Hafez), 1994.

[2] Serge Alinhac, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions,* English, with English and French summaries, Ann. of Math. (2) **149** (1999), no. 1, 97–127, DOI 10.2307/121020. MR1680539 (2000d:35147)

[3] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. **145** (2001), no. 3, 597–618, DOI 10.1007/s002220100165. MR1856402 (2002i:35127)

[4] Serge Alinhac, *An example of blowup at infinity for a quasilinear wave equation,* English, with English and French summaries, Autour de l'analyse microlocale, Astérisque **284** (2003), 1–91. MR2003417 (2005a:35197)

[5] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), no. 2, 267–282, DOI 10.1002/cpa.3160390205. MR820070 (87c:35111)

[6] Ding Bingbing and Yin Huicheng, *On the blowup of classical solutions to the 3-D pressure-gradient systems*, J. Differential Equations **252** (2012), no. 5, 3608–3629, DOI 10.1016/j.jde.2011.11.018. MR2876666 (2012k:35330)

[7] Robert T. Glassey, John K. Hunter, and Yuxi Zheng, *Singularities of a variational wave equation*, J. Differential Equations **129** (1996), no. 1, 49–78, DOI 10.1006/jdeq.1996.0111. MR1400796 (97f:35139)

[8] Paul Godin, *Lifespan of solutions of semilinear wave equations in two space dimensions*, Comm. Partial Differential Equations **18** (1993), no. 5-6, 895–916, DOI 10.1080/03605309308820955. MR1218523 (94g:35149)

[9] L. Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations,* Psuedodifferential Operators (Oberwolfach, 1986), Lecture Notes in Math. **1256**, Springer, Berlin, 1987. MR0897781 (88j:35024)

[10] L. Hörmander, *Lectures on nonlinear hyperbolic equations,* Mathematiques & Applications **26**, Springer, Berlin, 1997. MR1466700 (98e:35103)

[11] Akira Hoshiga, *The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions*, Hokkaido Math. J. **24** (1995), no. 3, 575–615. MR1357032 (96m:35213)

[12] Akira Hoshiga, *The existence of the global solutions to semilinear wave equations with a class of cubic nonlinearities in 2-dimensional space*, Hokkaido Math. J. **37** (2008), no. 4, 669–688. MR2474170 (2010j:35332)

[13] John K. Hunter and Ralph Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. **51** (1991), no. 6, 1498–1521, DOI 10.1137/0151075. MR1135995 (93a:76005)

[14] F. John, *Blow-up of radial solutions of* $u_{tt} = c^2(u_t)\Delta u$ *in three space dimensions,* English, with Portuguese summary, Mat. Apl. Comput. **4** (1985), no. 1, 3–18. MR808321 (87c:35114)

[15] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326. MR837683 (87h:35217)

[16] S. Klainerman, *Remarks on the global Sobolev inequalities in the Minkowski space* $\mathbf{R}^{n+1}$, Comm. Pure Appl. Math. **40** (1987), no. 1, 111–117, DOI 10.1002/cpa.3160400105. MR865359 (88a:46035)

[17] S. Klainerman and Gustavo Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Comm. Pure Appl. Math. **36** (1983), no. 1, 133–141, DOI 10.1002/cpa.3160360106. MR680085 (84a:35173)

[18] Li Ta-tsien and Chen Yun-mei, *Initial value problems for nonlinear wave equations*, Comm. Partial Differential Equations **13** (1988), no. 4, 383–422, DOI 10.1080/03605308808820547. MR920909 (89e:35102)

[19] Hans Lindblad, *On the lifespan of solutions of nonlinear wave equations with small initial data*, Comm. Pure Appl. Math. **43** (1990), no. 4, 445–472, DOI 10.1002/cpa.3160430403. MR1047332 (91i:35129)

[20] Hans Lindblad, *Global solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **45** (1992), no. 9, 1063–1096, DOI 10.1002/cpa.3160450902. MR1177476 (94a:35080)

[21] Hans Lindblad, *Global solutions of quasilinear wave equations*, Amer. J. Math. **130** (2008), no. 1, 115–157, DOI 10.1353/ajm.2008.0009. MR2382144 (2009b:58062)

[22] Jason Metcalfe and Christopher D. Sogge, *Global existence for high dimensional quasilinear wave equations exterior to star-shaped obstacles*, Discrete Contin. Dyn. Syst. **28** (2010), no. 4, 1589–1601, DOI 10.3934/dcds.2010.28.1589. MR2679723 (2011g:35257)

[23] R. A. Saxton, *Dynamic instability of the liquid crystal director*, Current progress in hyperbolic systems: Riemann problems and computations (Brunswick, ME, 1988), Contemp. Math., vol. 100, Amer. Math. Soc., Providence, RI, 1989, pp. 325–330, DOI 10.1090/conm/100/1033527. MR1033527 (90k:35246)

[24] Huicheng Yin, *Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data*, Nagoya Math. J. **175** (2004), 125–164. MR2085314 (2005f:35203)

[25] Huicheng Yin and Qingjiu Qiu, *The blowup of solutions for 3-D axisymmetric compressible Euler equations*, Nagoya Math. J. **154** (1999), 157–169. MR1689178 (2000g:35170)

[26] Ping Zhang and Yuxi Zheng, *On the second-order asymptotic equation of a variational wave equation*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 2, 483–509, DOI 10.1017/S0308210500001748. MR1899833 (2002m:35138)

[27] Ping Zhang and Yuxi Zheng, *Energy conservative solutions to a one-dimensional full variational wave system*, Comm. Pure Appl. Math. **65** (2012), no. 5, 683–726, DOI 10.1002/cpa.20380. MR2898888

[28] Yuxi Zheng, *Existence of solutions to the transonic pressure-gradient equations of the compressible Euler equations in elliptic regions*, Comm. Partial Differential Equations **22** (1997), no. 11-12, 1849–1868, DOI 10.1080/03605309708821323. MR1629498 (99c:35191)