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On the Bound State in Weakly Coupled $\lambda(\varphi^6-\varphi^4)_2$

J. Dimock* ** and J.-P. Eckmann

Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

Abstract. We consider the $\lambda(\varphi^6-\varphi^4)$ quantum field theory in two space-time dimensions. Using the Bethe-Salpeter equation, we show that there is a unique two particle bound state if the coupling constant $\lambda>0$ is sufficiently small. If m is the mass of single particles then the bound state mass is given by

$$\varkappa_B(\lambda)=2m\left(1-\frac{9}{8}\left(\frac{\lambda}{m^2}\right)^2+\mathcal{O}(\lambda^3)\right).$$

1. The Bound State Problem

We consider relativistic scalar boson quantum field theories in two dimensional space-time with polynomial interactions and we discuss some properties of bound states below the two particle threshold. For the model with interaction polynomial $P(\varphi)=\lambda(\varphi^6-\varphi^4)$, coupling constant $\lambda>0$ and bare mass m_0, bound states are known to exist if λ/m_0^2 is sufficiently small. This result is implicit in the combination of the two papers [4] and [7]. In the first paper, Glimm et al. argue that the $\lambda(\varphi^6-\varphi^4)$ model has mass spectrum above the one particle mass shell and below the two particle threshold. (They assumed that the physical mass $m=m(\lambda, m_0)$ has an asymptotic expansion as a function of λ near $\lambda=0$; this was subsequently proved in [2].) Secondly, Spencer and Zirilli, based on estimates by Spencer [6], showed that for any even P the mass operator has only discrete spectrum below $2m$, and that on each eigenspace of the mass operator the representation of the Poincaré group decomposes into a finite sum of irreducible representations. Thus the spectrum in question is interpreted as bound state particles.

In this paper we continue the study of the $\lambda(\varphi^6-\varphi^4)$ model and sharpen the above results. It is convenient (though not essential) to choose the bare mass $m_0=m_0(\lambda)$ such that the physical mass $m=m(\lambda, m_0(\lambda))$ is fixed [2]. Our main result is:

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* On leave from Departement of Mathematics, Suny at Buffalo, USA

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Theorem 1. *Given $m>0$, the $\lambda(\varphi^6-\varphi^4)$ model has, for sufficiently small coupling constant $\lambda>0$ exactly one bound state below the two particle threshold. The mass $\varkappa_B$ of this bound state is of the form*

$$\varkappa_B(\lambda)=2m\left(1-\frac{9}{8}\left(\frac{\lambda}{m^2}\right)^2+\mathcal{O}(\lambda^3)\right). \tag{1.1}$$

We remark that the theorem holds for any polynomial interaction of the form

$$P(\varphi)=\lambda\left(\sum_{n=3}^{N}a_{2n}\varphi^{2n}-\varphi^4\right),\qquad a_{2N}>0,\ N\geqq 3,$$

with the same constants in (1.1). The term $a_{2N}\varphi^{2N}$ ensures that the model exists, while $-\varphi^4$ gives an attractive force in first order and makes binding possible.

The proof of Theorem 1 is given at the end of Section 3. Here we outline it briefly. By considering the Bethe-Salpeter equation the problem of locating bound state masses is reduced to the solution of a non-linear eigenvalue problem on a certain Hilbert space of functions on $\mathbb{R}^2$. If one replaces the Bethe-Salpeter kernel by its lowest order term in λ (a point interaction) this problem can be solved explicitly and one finds that there is exactly one eigenvalue. By adapting the techniques of analytic perturbation theory to the nonlinear case at hand we show that the isolated eigenvalue persists when the higher order terms in the Bethe-Salpeter kernel are added. Spectrum away from this primary solution is ruled out in Section 3 by a variation of the technique that [7] use to rule out any bound states in $P(\varphi)=\lambda\left(\sum_{n=3}^{N}a_{2n}\varphi^{2n}+\varphi^4\right)$ models.

To fix the notation we now review the formulation of the Bethe-Salpeter equation as given in Spencer [6] and Spencer-Zirilli [7]. See also [8, 1, 3].

1. Let $\mathfrak{S}_{n,\lambda}$ be the n-point Schwinger function for a weakly coupled $P(\varphi)_2$ model with coupling constant λ [4]. The $\mathfrak{S}_{n,\lambda}$ are translation invariant real analytic functions except at coincident points where they have logarithmic singularities [3].

2. Define

$$D_\lambda(x_1,x_2,x_3,x_4)=\mathfrak{S}_{4,\lambda}(x_1,x_2,x_3,x_4)-\mathfrak{S}_{2,\lambda}(x_1,x_2)\mathfrak{S}_{2,\lambda}(x_3,x_4),$$
$$D_{0\lambda}(x_1,x_2,x_3,x_4)=\mathfrak{S}_{2,\lambda}(x_1,x_3)\mathfrak{S}_{2,\lambda}(x_2,x_4)+\mathfrak{S}_{2,\lambda}(x_1,x_4)\mathfrak{S}_{2,\lambda}(x_2,x_3).$$

These functions are the kernels of bounded symmetric operators on

$$L_2(\mathbb{R}^2)\otimes_s L_2(\mathbb{R}^2).$$

One further defines a bounded symmetric operator $K_\lambda^\wedge$ (essentially $K_\lambda^\wedge=D_\lambda^{-1}-D_{0\lambda}^{-1}$) so that the Bethe-Salpeter equation holds:

$$D_\lambda=D_{0\lambda}-D_{0\lambda}K_\lambda^\wedge D_\lambda.$$

3. Next the equation is transformed to momentum space and reduced to fixed total momentum. These operations are indicated by their action on the kernels of the operators. These kernels are functions of

$$\xi=x_1-x_2,\qquad \eta=x_3-x_4,\qquad \tau=\tfrac{1}{2}(x_1+x_2-x_3-x_4),$$

and we define

$$R_\lambda(k,p,q)=(2\pi)^{-3}\int e^{-i(k\tau+p\xi+q\eta)}D_\lambda(\tau,\xi,\eta)d\tau d\xi d\eta\,,$$
$$R_{0\lambda}(k,p,q)=(2\pi)^{-3}\int e^{-i(k\tau+p\xi+q\eta)}D_{0\lambda}(\tau,\xi,\eta)d\tau d\xi d\eta\,,$$
$$K_\lambda(k,p,q)=(2\pi)^{-1}\int e^{-i(k\tau+p\xi+q\eta)}K_\lambda\hat{}(\tau,\xi,\eta)d\tau d\xi d\eta\,.$$

Then the equation becomes

$$R_\lambda(k,p,q)=R_{0\lambda}(k,p,q)-\int R_{0\lambda}(k,p,p')K_\lambda(k,p',q')R_\lambda(k,q',q)dp'dq'\,,$$

which corresponds to an operator equation

$$R_\lambda(k)=R_{0\lambda}(k)-R_{0\lambda}(k)K_\lambda(k)R_\lambda(k)\,, \tag{1.2}$$

defined on $L_2^e(\mathbb{R}^2)$, the even subspace of $L_2(\mathbb{R}^2)$. [Note: Our definitions differ slightly from those of [7], e.g. $R_\lambda(k,p,q)=\text{const}\,R_\lambda^{SZ}(k,2p,2q)$.] We change notation to an energy variable $\varkappa$, and we write $R_\lambda(\varkappa)$ instead of $R_\lambda(k)$ for $k=(i\varkappa,0)$, etc. By the cluster expansion [4], $R_\lambda(\varkappa)$ is well defined for $\mathrm{Re}\,\varkappa$ small.

4. The fundamental result of Spencer [6] is that for λ sufficiently small, the kernel $K_\lambda(\varkappa,p,q)$ is analytic and bounded (uniformly in λ) in a region

$$\begin{aligned}&|\mathrm{Im}\,p_0|,|\mathrm{Im}\,q_0|\leqq\delta_0\,,\\&|\mathrm{Im}\,p_1|,|\mathrm{Im}\,q_1|\leqq\delta_1\,,\\&|\mathrm{Re}\,\varkappa|\leqq 4\delta_0\,,\end{aligned} \tag{1.3}$$

provided that $\delta_0+\delta_1<m$. We take $\delta_0=3m/4-\varepsilon$, $\delta_1=m/4-\varepsilon$.

5. Consider the Hardy space A_δ of functions analytic in $|\mathrm{Im}\,p_0|<\delta_0$, $|\mathrm{Im}\,p_1|<\delta_1$ and such that $f(p)=f(-p)$, with norm

$$\|f\|_{A_\delta}^2=\sup_{\substack{|\alpha_0|<\delta_0\\|\alpha_1|<\delta_1}}\int|w(p+i\alpha)f(p+i\alpha)|^2dp\,,$$

where $w(p)=(p^2+16m^2)^{-2/3}$. Using the analyticity of $K_\lambda(\varkappa)$ and the explicit form for $R_{0\lambda}(\varkappa)$, Spencer and Zirilli show that $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ extends from $L_2^e\cap A_\delta$ to A_δ and defines a compact operator there. Furthermore, $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ has an analytic continuation to $|\mathrm{Re}\,\varkappa|<2m$ (as compact operators). It follows by the analytic Fredholm theorem that $(1+K_\lambda(\varkappa)R_{0\lambda}(\varkappa))^{-1}$ is meromorphic in $|\mathrm{Re}\,\varkappa|<2m$.

6. Next the Bethe-Salpeter equation is realized on A_δ and extended to $|\mathrm{Re}\,\varkappa|<2m$. First note that $R_{0\lambda}(\varkappa)$ is analytic in this region, and that for $f,g\in L_2^e\cap A_\delta$,

$$(f,R_{0\lambda}(\varkappa)g)_2\leqq c(\varkappa)\|f\|_{A_\delta}\|g\|_{A_\delta}\,.$$

Thus $R_{0\lambda}(\varkappa)$ defines a bounded bilinear form on $A_\delta\times A_\delta$ and hence an operator in $\mathscr{L}(A_\delta,A_\delta^*)$, where A_δ^* is the dual of A_δ. We write

$$(f,R_{0\lambda}(\varkappa)g)_2=\langle f,R_{0\lambda}(\varkappa)g\rangle$$

where $\langle\,,\rangle$ is the pairing between A_δ and A_δ^*. Next let $|\mathrm{Re}\,\varkappa|$ be small and take g of the form $g=(1+K_\lambda(\varkappa)R_{0\lambda}(\varkappa))h$ with $h\in L_2^e\cap A_\delta$. Such functions are dense in A_δ and by the adjoint of Equation (1.2),

$$\begin{aligned}(f,R_\lambda(\varkappa)g)_2&=(f,(R_\lambda(\varkappa)+R_\lambda(\varkappa)K_\lambda(\varkappa)R_{0\lambda}(\varkappa))h)_2\\&=(f,R_{0\lambda}(\varkappa)h)_2\\&=\langle f,R_{0\lambda}(\varkappa)(1+K_\lambda(\varkappa)R_{0\lambda}(\varkappa))^{-1}g\rangle\,.\end{aligned}$$

It follows that $R_\lambda(\varkappa)$ defines an operator in $\mathscr{L}(A_\delta, A_\delta^*)$ such that $(f, R_\lambda(\varkappa)g)_2 = \langle f, R_\lambda(\varkappa)g\rangle$ and that

$$R_\lambda(\varkappa) = R_{0\lambda}(\varkappa)(1 + K_\lambda(\varkappa)R_{0\lambda}(\varkappa))^{-1}\,.$$

We see that $R_\lambda(\varkappa)$ has a meromorphic continuation to $|\mathrm{Re}\,\varkappa| < 2m$, by the analytic Fredholm theorem.

7. Let $dE(p)$ be the energy-momentum spectral measure for the field theory and let $f(p_0, p_1) = g(p_1)^\sim$ with $g \in C_0^\infty(\mathbb{R}^1)$. Then one has an identity of the form

$$\int \langle f, R_\lambda((ik_0, k_1))f\rangle h(k_1)dk_1$$
$$= \int h(p_1)\left(\frac{1}{p_0 - k_0} + \frac{1}{p_0 + k_0}\right) d(\theta(g), E(p)\theta(g))\,, \tag{1.4}$$

where

$$\theta(g) = \int \varphi(x)\varphi(-x)g(x)\Omega dx - (\Omega, \int \varphi(x)\varphi(-x)g(x)dx\Omega)\Omega\,,$$

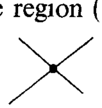
and $\varphi(x)$ is a time zero field. The identity allows one to conclude that any point in the mass spectrum in $(m, 2m)$ must be a pole of $R_\lambda(\varkappa)$, or equivalently a real value of $\varkappa$ such that $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ has eigenvalue -1. Here one uses the fact that vectors of the form Ω and $e^{-iPx}\theta(g)$ span the even subspace of the field theory up to energy $4m - \varepsilon$, $\varepsilon > 0$ [4]. (It is sufficient to consider the even subspace since the odd subspace has only single particle spectrum below $3m - \varepsilon$.)

2. The Eigenvalue Problem

Motivated by the previous discussion, we study the spectrum of $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ on the Hilbert space A_δ. For the $\lambda(\varphi^6 - \varphi^4)$ model we have [6],

$$K_\lambda(\varkappa) = -\lambda K^{(1)} + \lambda^2 K_\lambda^{(2)}(\varkappa)\,, \tag{2.1}$$

where $K_\lambda^{(2)}(\varkappa)$ is bounded in λ and has a kernel $K_\lambda^{(2)}(\varkappa, p, q)$ which is analytic in the region (1.3). The operator $K^{(1)}$ corresponds to the diagram



and has the kernel $K^{(1)}(p, q) = 3/\pi$. [This comes from the x-space kernel $K^{\sim(1)}(x_1, x_2, x_3, x_4) = 6\delta(x_1 - x_2)\delta(x_2 - x_3)\delta(x_3 - x_4)$.] We also decompose $R_{0\lambda}(\varkappa)$ which has a kernel given by

$$R_{0\lambda}(\varkappa, p, q) = 2(2\pi)S_\lambda^\sim\left(p - \frac{(i\varkappa, 0)}{2}\right)S_\lambda^\sim\left(p + \frac{(i\varkappa, 0)}{2}\right)\delta(p+q)$$
$$\equiv r_{0\lambda}(\varkappa, p)\delta(p+q)\,, \tag{2.2}$$

where S_λ is defined by $\mathfrak{S}_{2,\lambda}(x_1, x_2) = S_\lambda(x_1 - x_2)$. For $\lambda = 0$ this becomes

$$R_{00}(\varkappa, p, q) = 2(2\pi)^{-1}\left(\left(p - \frac{(i\varkappa, 0)}{2}\right)^2 + m^2\right)^{-1}$$
$$\cdot\left(\left(p + \frac{(i\varkappa, 0)}{2}\right)^2 + m^2\right)^{-1}\delta(p+q)$$
$$\equiv r_{00}(\varkappa, p)\delta(p+q)\,. \tag{2.3}$$

Then we define $R_{0\lambda}^{(2)}(\varkappa)$ (for $\lambda>0$) by

$$R_{0\lambda}(\varkappa)=R_{00}(\varkappa)+\lambda^2 R_{0\lambda}^{(2)}(\varkappa)\,. \tag{2.4}$$

We shall see that $R_{0\lambda}^{(2)}(\varkappa)$ is bounded as $\lambda\to 0$ so that this definition is appropriate. Collecting (2.1), (2.3) we write

$$K_\lambda(\varkappa)R_{0\lambda}(\varkappa)=-\lambda T^{(1)}(\varkappa)+\lambda^2 T_\lambda^{(2)}(\varkappa)\,, \tag{2.5}$$

where

$$T^{(1)}(\varkappa)=K^{(1)}R_{00}(\varkappa)\,,$$

and therefore

$$T_\lambda^{(2)}(\varkappa)=-\lambda K^{(1)}R_{0\lambda}^{(2)}(\varkappa)+K_\lambda^{(2)}(\varkappa)R_{00}(\varkappa) \\ +\lambda^2 K_\lambda^{(2)}(\varkappa)R_{0\lambda}^{(2)}(\varkappa)\,.$$

We proceed to study the operator $T^{(1)}(\varkappa)$.

Lemma 2.1. *For* $|\mathrm{Re}\,\varkappa|<2m$ *the operator* $T^{(1)}(\varkappa)$ *has rank one and it has the single non-zero eigenvalue*

$$\frac{12}{\pi}\,\frac{1}{(4m^2-\varkappa^2)^{1/2}}\,\frac{\arcsin(\varkappa/2m)}{\varkappa}\,.$$

Proof. By definition, we have for $\psi\in A_\delta$,

$$(T^{(1)}(\varkappa)\psi)(p)=\frac{3}{\pi}\int r_{00}(\varkappa,q)\psi(q)dq\,.$$

Thus the range of $T^{(1)}(\varkappa)$ is the constant functions (they are in A_δ), and therefore the only eigenfunction is $\psi=$ constant. The eigenvalue is $\frac{3}{\pi}r_{00}(\varkappa)$, where

$$\begin{aligned} r_{00}(\varkappa)&\equiv\int r_{00}(\varkappa,q)dq \\ &=4\int_0^\infty dq_1(q_1^2+4m^2)^{-1/2}(q_1^2+4m^2-\varkappa)^{-1} \\ &=4(4m^2-\varkappa^2)^{-1/2}\arcsin(\varkappa/2m)/\varkappa\,. \end{aligned} \tag{2.6}$$

The q_0 integral (first equality above) is done by a contour integral [7] and the q_1 integral is accomplished by the change of variables $x=q_1(q_1^2+4m^2)^{-1/2}$.

Because of point 7 of Section 1 we concentrate our interest on those values of $\varkappa$ for which $-\lambda T^{(1)}(\varkappa)$ has eigenvalue -1. This will turn out to be the correct second order approximation in λ to the bound state energy.

Lemma 2.2. *For* $\lambda>0$ *sufficiently small there exists a unique* $\varkappa=\varkappa^*(\lambda)$ *in* $(0,2m)$ *such that* $-\lambda T^{(1)}(\varkappa)$ *has eigenvalue* -1.

Proof. By Lemma 2.1, the unique eigenvalue of $-\lambda T^{(1)}(\varkappa)$ is $-\frac{3\lambda}{\pi}r_{00}(\varkappa)$. By the integral representation (2.6) for $r_{00}(\varkappa)$, the eigenvalue is monotone decreasing and unbounded as $\varkappa\to 2m$ and so the assertion follows.

Lemma 2.3. *Define $\varkappa^*(0)=2m$. Then $\varkappa^*(\lambda)$ is a C^∞ function on an interval $[0,\lambda_0)$ for sufficiently small $\lambda_0>0$ and has the asymptotic expansion*

$$\varkappa^*(\lambda)=2m(1-\tfrac{9}{8}(\lambda/m^2)^2+\mathcal{O}(\lambda^3)). \tag{2.7}$$

Proof. Define for $\lambda>0$ and σ small the function

$$F(\lambda,\sigma)=-\frac{3\lambda}{\pi}r_{00}\left(2m\left(1-\frac{9}{8}\left(\frac{\lambda}{m^2}\right)^2(1+\sigma)\right)\right). \tag{2.8}$$

F extends to a C^∞ function in a neighborhood of $(\lambda,\sigma)=(0,0)$. We note that $F(0,0)=-1$ and that $\partial_\sigma F(0,0)\neq 0$. By the implicit function theorem, there is a C^∞ function $\sigma^*(\lambda)$ such that $\sigma^*(0)=0$ and $F(\lambda,\sigma^*(\lambda))=-1$. By definition, we have for $\lambda>0$,

$$\varkappa^*(\lambda)=2m\left(1-\frac{9}{8}\left(\frac{\lambda}{m^2}\right)^2(1+\sigma^*(\lambda))\right), \tag{2.9}$$

and this identity extends to $\lambda=0$. Thus $\varkappa^*(\lambda)$ is C^∞ up to zero and since $|\sigma^*(\lambda)|=\mathcal{O}(\lambda)$, Equation (2.7) is proved. [Higher order coefficients $\partial_\lambda^n\varkappa^*(0)$ in the expansion for $\varkappa^*(\lambda)$ can be calculated from (2.9) and the expressions for $\partial_\lambda^n\sigma^*(0)$.]

As a prelude to the perturbation theory for the $\lambda^2T_\lambda^{(2)}(\varkappa)$ part of $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ we estimate the norms of the operators in question. The bounds of the next four lemmas hold in the region

$$\{\varkappa\,|\,|\mathrm{Re}\,\varkappa|<2m,\ |\mathrm{Im}\,\varkappa|<m\}. \tag{2.10}$$

Define $\Delta(\varkappa)=(4m^2-(\mathrm{Re}\,\varkappa)^2)^{-1/2}$.

Lemma 2.4. $\|T^{(1)}(\varkappa)\|=\|K^{(1)}R_{00}(\varkappa)\|\leqq\mathcal{O}(\Delta(\varkappa))$.

Since $K^{(1)}$ is constant (and hence bounded) the proof of this lemma is the same as the proof of Lemma 2.5 and we omit it.

Lemma 2.5. $\|K_\lambda^{(2)}(\varkappa)R_{00}(\varkappa)\|\leqq\mathcal{O}(\Delta(\varkappa))$.

Proof. We estimate the norm by the Hilbert-Schmidt norm. The latter is computed via a unitary transformation from A_δ to $L_2(\mathbb{R}^2,dx)$. As in [7] this gives

$$\begin{aligned}&\|K_\lambda^{(2)}(\varkappa)R_{00}(\varkappa)\|_{\mathrm{H.S.}}^2\\ &\quad\leqq\mathcal{O}(1)\int w(p)^2\left|\int K_\lambda^{(2)}(\varkappa,p+i\delta,q')r_{00}(\varkappa,q')\right.\\ &\quad\left.\cdot w^{-1}(q')B_\delta(q'-q)dq'\right|^2dp\,dq,\end{aligned} \tag{2.11}$$

where $B_\delta(q)=(q_0^2+\delta_0^2)^{-1}(q_1^2+\delta_1^2)^{-1}$. As noted in (1.3),

$$|K_\lambda^{(2)}(\varkappa,p+i\delta,q')|\leqq\mathcal{O}(1).$$

Using also $\int w(p)^{-2}w(q)^{-2}dp\,dq<\infty$ we obtain a bound on $\|K_\lambda^{(2)}(\varkappa)R_{00}(\varkappa)\|_{\mathrm{H.S.}}$ of the form

$$\mathcal{O}(1)\sup_q\int dq'|r_{00}(\varkappa,q')|w^{-1}(q')w^{-1}(q)B_\delta(q'-q). \tag{2.12}$$

This integral is bounded by considering the regions $|q'|<m$ and $|q'|\geqq m$ separately. In the first region, $|w^{-1}(q)w^{-1}(q')B_\delta(q-q')|=\mathcal{O}(1)$ so it suffices to bound

$\int |r_{00}(\varkappa, q')| dq'$. But by applying the the inequality $|ab| \leqq 2^{-1}(|a|^2+|b|^2)$ to the two factors in the definition (2.3) of $r_{00}(\varkappa, q')$ we see that

$$|r_{00}(\varkappa, q')| \leqq 2^{-1}\{r_{00}(\mathrm{Re}\,\varkappa, q'+2^{-1}(\mathrm{Im}\,\varkappa, 0)) \\ +r_{00}(\mathrm{Re}\,\varkappa, q'-2^{-1}(\mathrm{Im}\,\varkappa, 0))\}.$$

This gives

$$\int |r_{00}(\varkappa, q')| dq' \leqq r_{00}(\mathrm{Re}\,\varkappa) \leqq \mathcal{O}(\Delta(\varkappa)),$$

and hence a bound $\mathcal{O}(\Delta(\varkappa))$ for (2.12). In the region $|q'| \geqq m$, we use $|r_{00}(\varkappa, q')| \leqq \mathcal{O}(|q'|^{-4})$ to obtain the bound $\mathcal{O}(1)$ for (2.12). This completes the proof of Lemma 2.5.

Lemma 2.6. $\|K^{(1)}R_{0\lambda}^{(2)}(\varkappa)\| \leqq \mathcal{O}(\Delta(\varkappa))$.

Proof. We use the Lehmann spectral representation for the two point function

$$S_\lambda^{\sim}(p) = (2\pi)^{-1}(Z_\lambda^2(p^2+m^2)^{-1} + \int (p^2+a^2)^{-1} d\varrho_\lambda(a)), \tag{2.13}$$

where $\mathrm{supp}\,\varrho_\lambda$ is bounded away from m. First $R_{0\lambda}^{(2)}$ is expanded as

$$R_{0\lambda}^{(2)}(\varkappa) = \lambda^{-2}(R_{0\lambda}(\varkappa) - R_{00}(\varkappa)Z_\lambda^4) \\ +\lambda^{-2}(Z_\lambda^4-1)R_{00}(\varkappa). \tag{2.14}$$

The perturbation expansion for the field strength renormalization constant Z_λ is asymptotic [2] and one has $|1-Z_\lambda^2| \leqq \mathcal{O}(\lambda^2)$. Thus the second term in (2.14) contributes $\mathcal{O}(\Delta(\varkappa))$ to $\|K^{(1)}R_{0\lambda}^{(2)}(\varkappa)\|$, by Lemma 2.4. The first term in (2.14) acts as a multiplication by

$$\delta r_\lambda(\varkappa, p) = \lambda^{-2}\, 2(2\pi)^{-1} Z_\lambda^2 \left(\left(p+\frac{(i\varkappa, 0)}{2}\right)^2 + m^2\right)^{-1} \int \left(\left(p-\frac{(i\varkappa, 0)}{2}\right)^2 + a^2\right)^{-1} d\varrho_\lambda(a) \\ + \text{two similar terms}. \tag{2.15}$$

Following the proof of Lemma 2.5, it is sufficient to show

$$\int |\delta r_\lambda(\varkappa, q')| dq' \leqq \mathcal{O}(\Delta(\varkappa)), \\ |\delta r_\lambda(\varkappa, q')| \leqq \mathcal{O}(|q'|^{-4}), \qquad |q'| \geqq m,$$

to complete the proof. Since we have a canonical theory,

$$\int d\varrho_\lambda(a) = 1 - Z_\lambda^2 = \mathcal{O}(\lambda^2)$$

and thus the first bound follows from

$$\int dp \left|\left(p+\frac{(i\varkappa, 0)}{2}\right)^2 + m^2\right|^{-1} \left|\left(p-\frac{(i\varkappa, 0)}{2}\right)^2 + a^2\right|^{-1} \\ \leqq \int dp((p_0-\mathrm{Im}\,\varkappa)^2 + p_1^2 + 4m^2 - (\mathrm{Re}\,\varkappa)^2)^{-1} \\ \cdot((p_0+\mathrm{Im}\,\varkappa)^2 + p_1^2 + 4a^2 - (\mathrm{Re}\,\varkappa)^2)^{-1} \\ \leqq \int dp_0 dp_1 (p_1^2 + 4m^2 - (\mathrm{Re}\,\varkappa)^2)^{-1}((p_0+\mathrm{Im}\,\varkappa)^2+1)^{-1} \\ \leqq \mathcal{O}(\Delta(\varkappa)),$$

for all $a \in \mathrm{supp}\,\varrho_\lambda$. The two other terms in (2.15) are bounded in the same way. Finally the $\mathcal{O}(|p|^{-4})$ bound on $\delta r_\lambda(\varkappa, p)$ follows by inspection from (2.15). This completes the proof of Lemma 2.6.

In exactly the same way, using the boundedness of $K_\lambda^{(2)}(\varkappa, p, q)$ one shows

Lemma 2.7. $\|K_\lambda^{(2)}(\varkappa)R_{0\lambda}^{(2)}(\varkappa)\| \leqq \mathcal{O}(\Delta(\varkappa))$.

We now control the spectrum of $K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ through perturbation theory. Define

$$T_\lambda(\mu, \varkappa) = -\lambda T^{(1)}(\varkappa) + \mu T_\lambda^{(2)}(\varkappa),$$

so that

$$K_\lambda(\varkappa)R_{0\lambda}(\varkappa) = T_\lambda(\lambda^2, \varkappa).$$

For $|\mathrm{Re}\,\varkappa| < 2m$, $T_\lambda(\mu, \varkappa)$ is analytic in μ, $\varkappa$.

Lemma 2.8. *There exist constants $a>0$, $b>0$ such that for $\lambda>0$ sufficiently small and $|\mu|<a\lambda$, $|\varkappa-\varkappa^*(\lambda)|<b\lambda^2$, the spectrum of $T_\lambda(\mu, \varkappa)$ is contained in*

$$\{\zeta : |\zeta+1| \leqq \tfrac{1}{4} \quad \text{or} \quad |\zeta| < \tfrac{1}{4}\}.$$

Proof. We know that $T_\lambda(0, \varkappa^*(\lambda))$ has spectrum $\{0, -1\}$. The perturbation is

$$\begin{aligned}\delta T_\lambda(\mu, \varkappa) &= T_\lambda(\mu, \varkappa) - T_\lambda(0, \varkappa^*(\lambda)) \\ &= \lambda(T^{(1)}(\varkappa^*(\lambda)) - T^{(1)}(\varkappa)) + \mu T_\lambda^{(2)}(\varkappa).\end{aligned}$$

We estimate the norm of $\delta T_\lambda(\mu, \varkappa)$. First note that there is a constant c such that $|\varkappa-\varkappa^*(\lambda)|<c\lambda^2$ implies $|2m-\mathrm{Re}\,\varkappa| \geqq \mathcal{O}(1)\lambda^2$ and hence $|\Delta(\varkappa)| \leqq \mathcal{O}(\lambda^{-1})$. Thus in this region we have by Lemma 2.4,

$$\|\lambda T^{(1)}(\varkappa)\| \leqq \mathcal{O}(\lambda\Delta(\varkappa)) = \mathcal{O}(1). \tag{2.16}$$

Since $T^{(1)}(\varkappa)$ is an analytic family of operators we have for $|\varkappa-\varkappa^*(\lambda)| \leqq b\lambda^2$ and $b<\frac{1}{2}c$,

$$\begin{aligned}&\|\lambda(T^{(1)}(\varkappa^*(\lambda)) - T^{(1)}(\varkappa))\| \\ &= (2\pi)^{-1}|\varkappa-\varkappa^*(\lambda)| \left\| \int_{|\varkappa'-\varkappa^*(\lambda)|=c\lambda^2} \frac{\lambda T^{(1)}(\varkappa')d\varkappa'}{(\varkappa'-\varkappa^*(\lambda))(\varkappa'-\varkappa)} \right\| \\ &\leqq \mathcal{O}(1)b\lambda^2(\tfrac{1}{2}c\lambda^2)^{-1} \leqq b\mathcal{O}(1).\end{aligned}$$

For $|\varkappa-\varkappa^*(\lambda)| \leqq b\lambda^2$ and $|\mu| \leqq a\lambda$ we have by Lemma 2.5, 2.6, and 2.7

$$\|\mu T_\lambda^{(2)}(\varkappa)\| \leqq a\lambda\mathcal{O}(\Delta(\varkappa)) \leqq a\mathcal{O}(1).$$

Thus the overall bound is

$$\|\delta T_\lambda(\mu, \varkappa)\| \leqq (a+b)\mathcal{O}(1). \tag{2.17}$$

Next we estimate $\|(\zeta - T_\lambda(0, \varkappa^*(\lambda)))^{-1}\|$ for $\zeta \neq 0, -1$. Since $T_\lambda(0, \varkappa^*(\lambda))$ is a rank one operator, the estimate on this norm can be reduced to an estimate on the norm of a 2×2 matrix. After a short calculation one finds

$$\begin{aligned}&\|(\zeta - T_\lambda(0, \varkappa^*(\lambda)))^{-1}\| \\ &\leqq 2\max(|\zeta|^{-1}, |\zeta+1|^{-1}, |\zeta|^{-1}|\zeta+1|^{-1}\|T_\lambda(0, \varkappa^*(\lambda))\|).\end{aligned} \tag{2.18}$$

Finally, combining (2.16), (2.17), (2.18) we have for $|\zeta|^{-1}, |\zeta+1|^{-1}<4$,

$$\|\delta T_\lambda(\mu,\varkappa)\|\,\|(\zeta-T_\lambda(0,\varkappa^*(\lambda)))^{-1}\|\leqq(a+b)\mathcal{O}(1).$$

By choosing a and b sufficiently small this is less than one and so $(\zeta-T_\lambda(\mu,\varkappa))^{-1}$ exists as a Neumann series. Thus the spectrum of $T_\lambda(\mu,\varkappa)$ is contained in the complement of $|\zeta|^{-1}, |\zeta+1|^{-1}<4$. This completes the proof.

By using Lemma 2.8 and analytic perturbation theory [5], we conclude

Corollary 2.9. *There exist constants $a>0$, $b>0$ such that for $\lambda>0$ sufficiently small and $|\mu|\leqq a$, $|\varkappa-\varkappa^*(\lambda)|\leqq b\lambda^2$ the spectrum of $T_\lambda(\mu,\varkappa)$ consists of*

(a) *A simple eigenvalue $\alpha_\lambda(\mu,\varkappa)$, analytic in $\mu,\varkappa$ and satisfying $|\alpha_\lambda(\mu,\varkappa)+1|<\frac{1}{4}$.*

(b) *Other spectrum in $\{\zeta\,|\,|\zeta|<\frac{1}{4}\}$.*

Lemma 2.10. *For $\mu,\varkappa$ real, $\alpha_\lambda(\mu,\varkappa)$ is real.*

Proof. For $\mu,\varkappa$ real, $T_\lambda(\mu,\varkappa)$ commutes with complex conjugation. Thus both $\alpha_\lambda(\mu,\varkappa)$ and $\alpha_\lambda(\mu,\varkappa)^-$ are eigenvalues. This is only consistent with the uniqueness of α_λ if α_λ is real.

We now determine the critical value for $\varkappa$.

Lemma 2.11. *Let $\lambda>0$ be sufficiently small and $\mu,\varkappa$ real. For $|\mu|\leqq 2\lambda^2$, there exists a unique $\varkappa=\varkappa_\lambda(\mu)$ in $|\varkappa-\varkappa^*(\lambda)|\leqq\frac{1}{2}b\lambda^2$ such that $\alpha_\lambda(\mu,\varkappa_\lambda(\mu))=-1$.*

Proof. We start by bounding various derivatives of $\alpha_\lambda(\mu,\varkappa)$ by using contour integrals, with $\mu,\varkappa$ in the region $|\mu|\leqq 2\lambda^2$, $|\varkappa-\varkappa^*(\lambda)|\leqq\frac{1}{2}b\lambda^2$:

$$\begin{aligned}|\partial_\mu\alpha_\lambda(\mu,\varkappa)|&=(2\pi)^{-1}\left|\oint_{|\mu'|=a\lambda}d\mu'\,\frac{\alpha_\lambda(\mu',\varkappa)}{(\mu-\mu')^2}\right|\\&\leqq\mathcal{O}(\lambda^{-1}),\end{aligned}\tag{2.19}$$

$$\begin{aligned}|\partial_\mu\partial_\varkappa\alpha_\lambda(\mu,\varkappa)|&=(2\pi)^{-2}\left|\oint_{\substack{|\mu'|=a\lambda\\|\varkappa'-\varkappa^*(\lambda)|=b\lambda^2}}d\mu'd\varkappa'\,\frac{\alpha_\lambda(\mu',\varkappa')}{(\mu-\mu')^2(\varkappa-\varkappa')^2}\right|\\&\leqq\mathcal{O}(\lambda^{-3}).\end{aligned}\tag{2.20}$$

Thus in the same region

$$|\alpha_\lambda(\mu,\varkappa)-\alpha_\lambda(0,\varkappa)|\leqq\mathcal{O}(\lambda),$$

and in particular

$$|\alpha_\lambda(\mu,\varkappa^*(\lambda))+1|\leqq\mathcal{O}(\lambda).\tag{2.21}$$

On the other hand, by (2.20),

$$\begin{aligned}\partial_\varkappa\alpha_\lambda(\mu,\varkappa)&\leqq\partial_\varkappa\alpha_\lambda(0,\varkappa)+2\lambda^2\sup_{\mu'}|\partial_\mu\partial_\varkappa\alpha_\lambda(\mu',\varkappa)|\\&\leqq-c_1\lambda^{-2}+2\lambda^2\mathcal{O}(\lambda^{-3})\leqq-c_2\lambda^{-2},\end{aligned}\tag{2.22}$$

for λ sufficiently small. We have used $\alpha_\lambda(0,\varkappa)=-3\lambda\pi^{-1}r_{00}(\varkappa)$ and the bound

$$\partial_\varkappa r_{00}(\varkappa)\geqq c\Delta(\varkappa)^{-3}\geqq c\lambda^{-3},\quad c>0,\tag{2.23}$$

which may be proved starting with (2.6). Thus $\alpha_\lambda(\mu, \varkappa)$ is a decreasing function of $\varkappa$. Furthermore, for λ sufficiently small,

$$\begin{aligned}\alpha_\lambda(\mu, \varkappa^*(\lambda)+2^{-1}b\lambda^2) &= \alpha_\lambda(\mu, \varkappa^*(\lambda)) + \int_{\varkappa^*(\lambda)}^{\varkappa^*(\lambda)+\frac{1}{2}b\lambda^2} d\varkappa' \, \partial_\varkappa \alpha_\lambda(\mu, \varkappa') \\ &\leqq -1 + \mathcal{O}(\lambda) - 2^{-1}bc_2 < -1\,, \end{aligned} \tag{2.24}$$

by (2.21) and (2.22). Similarly, $\alpha_\lambda(\mu, \varkappa^*(\lambda)-2^{-1}b\lambda^2) > -1$. The existence and uniqueness follow.

Lemma 2.12. *The function $\varkappa_\lambda(\mu)$ is a real analytic function of μ for $|\mu| < 2\lambda^2$ and*

$$|\partial_\mu^n \varkappa_\lambda(\mu)| \leqq K_n \lambda^{2-n}\,.$$

Proof. The function $\varkappa_\lambda(\mu)$ solves $\alpha_\lambda(\mu, \varkappa_\lambda(\mu)) = -1$. Since $\alpha_\lambda(\mu, \varkappa)$ is analytic and $\partial_\varkappa \alpha_\lambda(\mu, \varkappa) \neq 0$ [by (2.22)], it follows from the implicit function theorem that $\varkappa_\lambda(\alpha)$ is analytic in a neighborhood of any μ and

$$\partial_\mu \varkappa_\lambda(\mu) = \{-\partial_\mu \alpha_\lambda(\mu, \varkappa)/\partial_\varkappa \alpha_\lambda(\mu, \varkappa)\}|_{\varkappa=\varkappa_\lambda(\mu)}\,.$$

We estimate the ∂_μ derivate by (2.19) and the $\partial_\varkappa$ derivative by (2.22) so that

$$|\partial_\mu \varkappa_\lambda(\mu)| \leqq \mathcal{O}(\lambda)\,. \tag{2.25}$$

The second derivative is

$$\begin{aligned}\partial_\mu^2 \varkappa_\lambda(\mu) = \{&(-\partial_\mu^2 \alpha_\lambda(\mu, \varkappa) - \partial_\mu \partial_\varkappa \alpha_\lambda(\mu, \varkappa)\partial_\mu \varkappa_\lambda(\mu))/\partial_\varkappa \alpha_\lambda(\mu, \varkappa) \\ &+ \partial_\mu \alpha_\lambda(\mu, \varkappa)(\partial_\mu \partial_\varkappa \alpha_\lambda(\mu, \varkappa) + \partial_\varkappa^2 \alpha_\lambda(\mu, \varkappa)\partial_\mu \varkappa_\lambda(\mu))/(\partial_\varkappa \alpha_\lambda(\mu, \varkappa))^2\}|_{\varkappa=\varkappa_\lambda(\mu)}\,.\end{aligned}$$

Estimating the derivatives by contour integrals (roughly $\partial_\mu \sim \mathcal{O}(\lambda^{-1})$, $\partial_\varkappa \sim \mathcal{O}(\lambda^{-2})$) gives $|\partial_\mu^2 \varkappa_\lambda(\mu)| \leqq \mathcal{O}(1)$. Continuing in this manner gives the general bound.

We now define (the bound state mass)

$$\varkappa_B(\lambda) = \varkappa_\lambda(\lambda^2)\,. \tag{2.26}$$

This is the unique $\varkappa$ in $|\varkappa^*(\lambda) - \varkappa| \leqq \frac{1}{2}b\lambda^2$ such that $T_\lambda(\lambda^2, \varkappa) = K_\lambda(\varkappa)R_{0\lambda}(\varkappa)$ has eigenvalue -1.

Lemma 2.13. *We have the expansion*

$$\varkappa_B(\lambda) = 2m(1 - \tfrac{9}{8}(\lambda/m^2)^2 + \mathcal{O}(\lambda^3))\,.$$

Proof.

$$\begin{aligned}|\varkappa_B(\lambda) - \varkappa^*(\lambda)| &= |\varkappa_\lambda(\lambda^2) - \varkappa_\lambda(0)| \\ &\leqq \lambda^2 \sup_{0 \leqq \mu \leqq \lambda^2} |\partial_\mu \varkappa_\lambda(\mu)| \\ &\leqq \mathcal{O}(\lambda^3)\,, \qquad \text{[by (2.25)]}\end{aligned}$$

The result now follows by Lemma 2.3.

Remark. By expanding $\varkappa_\lambda(\mu)$ up to n-th order we have

$$\varkappa_B(\lambda) = \varkappa_\lambda(0) + \lambda^2 \varkappa_\lambda'(0) + \ldots + \frac{1}{n!}\lambda^{2n}\varkappa_\lambda^{(n)}(0) + \mathcal{O}(\lambda^{n+3})\,.$$

This gives a (variable coefficient) asymptotic series for $\varkappa_B(\lambda)$ in which the k-th term is $\mathcal{O}(\lambda^{2-k}\lambda^{2k})=\mathcal{O}(\lambda^{k+2})$. We believe that this series can be rearranged to yield an asymptotic expansion for $\varkappa_B(\lambda)$ at $\lambda=0$, and we hope to come back to the details of this expansion in a further publication.

3. Absence of Poles

In this section we exclude poles of $R_\lambda(\varkappa)$ away from the pole established in Section 2 and we prove Theorem 1. The treatment follows closely that of Spencer and Zirilli [7, § 4]. Let $r_{0\lambda}(\varkappa)=\int r_{0\lambda}(\varkappa,p)dp$ and let $R_0'(\varkappa,p,q)$ be the bounded operator defined for $3\lambda\pi^{-1}r_{0\lambda}(\varkappa)\neq 1$ and $|\mathrm{Re}\,\varkappa|<2m$ by

$$R_{0\lambda}'(\varkappa,p,q)=r_{0\lambda}(\varkappa,p)\delta(p+q)+\frac{3\lambda\pi^{-1}}{1-3\lambda\pi^{-1}r_{0\lambda}(\varkappa)}r_{0\lambda}(\varkappa,p)r_{0\lambda}(\varkappa,q)\,. \tag{3.1}$$

By explicit computation one finds that

$$R_{0\lambda}'(\varkappa)=R_{0\lambda}(\varkappa)-R_{0\lambda}(\varkappa)(-\lambda K^{(1)})R_{0\lambda}'(\varkappa)\,. \tag{3.2}$$

One can show that $R_{0\lambda}'(\varkappa)$ extends to $A_\delta\times A_\delta$ and that (3.2) can be written in $\mathscr{L}(A_\delta,A_\delta^*)$ as

$$R_{0\lambda}'(\varkappa)=R_{0\lambda}(\varkappa)(1-\lambda K^{(1)}R_{0\lambda}(\varkappa))^{-1}\,. \tag{3.3}$$

Note that $\lambda K^{(1)}R_{0\lambda}(\varkappa)$ has the single non-zero eigenvalue $3\lambda\pi^{-1}r_{0\lambda}(\varkappa)$. Further one can show that $\lambda^2K_\lambda^{(2)}(\varkappa)R_{0\lambda}'(\varkappa)$ extends to an analytic compact operator valued function in $\mathscr{L}(A_\delta,A_\delta^*)$ and that

$$R_\lambda(\varkappa)=R_{0\lambda}'(\varkappa)(1+\lambda^2K_\lambda^{(2)}(\varkappa)R_{0\lambda}'(\varkappa))^{-1}\,, \tag{3.4}$$

except for a discrete set of $\varkappa$'s. We use this formula to exclude poles near the threshold at $2m$.

Lemma 3.1. *For λ sufficiently small $R_{0\lambda}'(\varkappa)$ has no poles in $2m-\lambda^{5/2}\leqq\varkappa<2m$.*

Proof. The only poles come when $3\pi\lambda^{-1}r_{0\lambda}(\varkappa)=1$. However by proceeding as in the proof of Lemma 2.6 we obtain $|r_{00}(\varkappa)-r_{0\lambda}(\varkappa)|\leqq\mathcal{O}(\lambda^2\Delta(\varkappa))$. Since $r_{00}(\varkappa)\geqq c_1\Delta(\varkappa)$ for some constant $c_1>0$, it follows that $r_{0\lambda}(\varkappa)\geqq c_2\Delta(\varkappa)$ for some $c_2>0$. For $2m-\lambda^{5/2}\leqq\varkappa<2m$, we have $\Delta(\varkappa)\geqq c_3\lambda^{-5/4}$, $c_3>0$, and hence for λ small

$$3\pi^{-1}\lambda r_{0\lambda}(\varkappa)\geqq c_4\lambda^{-1/4}\geqq 2\,. \tag{3.5}$$

Thus there is no pole.

Lemma 3.2. *For λ sufficiently small $R_\lambda(\varkappa)$ has no pole in $2m-\lambda^{5/2}\leqq\varkappa<2m$.*

Proof. By (3.4) and Lemma 3.1 it suffices to prove

$$\|\lambda^2K_\lambda^{(2)}(\varkappa)R_{0\lambda}'(\varkappa)\|\leqq\mathcal{O}(\lambda)\,. \tag{3.6}$$

Estimating the norm by the Hilbert-Schmidt norm this will follow from

$$|\int dpdq K_\lambda^{(2)}(\varkappa,p'+i\delta,p)R_{0\lambda}'(\varkappa,p,q)w^{-1}(q)B_\delta(q'-q)|$$
$$\leqq\mathcal{O}(\lambda^{-1}w(q'))\,.$$

This is of the form $|\int f(p)g(q)R'_{0\lambda}(\varkappa, p, q)dpdq|$ where

$$f(p)=K_\lambda^{(2)}(\varkappa, p'+i\delta, p),$$
$$g(q)=w^{-1}(q)B_\delta(q'-q).$$

Inserting (3.1) we have

$$\begin{aligned}
&\int f(p)g(q)R'_{0\lambda}(\varkappa, p, q)dpdq \\
&=\int (f(p)g(p)-f(0)g(0))r_{0\lambda}(\varkappa, p)dp \\
&\quad + f(0)g(0)\left\{r_{0\lambda}(\varkappa)+\frac{3\pi^{-1}\lambda r_{0\lambda}(\varkappa)^2}{1-3\pi^{-1}\lambda r_{0\lambda}(\varkappa)}\right\} \\
&\quad + \frac{3\pi^{-1}\lambda}{1-3\pi^{-1}\lambda r_{0\lambda}(\varkappa)}\int (f(p)-f(0))r_{0\lambda}(\varkappa, p)r_{0\lambda}(\varkappa)g(0)dp \\
&\quad + \frac{3\pi^{-1}\lambda}{1-3\pi^{-1}\lambda r_{0\lambda}(\varkappa)}\int f(p)r_{0\lambda}(\varkappa, p)r_{0\lambda}(\varkappa, q)(g(q)-g(0))dpdq \\
&=X_1+X_2+X_3+X_4.
\end{aligned}$$

Term X_1. Let $h(p)=f(p)g(p)$. [7] argue that it suffices to bound

$$\int r_{00}(\varkappa, p)(h(0, p_1)-h(0))dp_0dp_1.$$

The potentially singular part of this integral coming from $p=0$ is bounded by

$$\int_{|p_1|<2m} r_{00}(\varkappa, p)p_1^2\left|\int_0^1 h''(0, p_1\tau)(1-\tau)d\tau\right|, \tag{3.7}$$

[since $r_{00}(\varkappa, p)$ is even] which in turn is bounded uniformly in $\varkappa$ and is $\mathcal{O}(w(q'))$. The bound is uniform in λ because $K_\lambda^{(2)}(\varkappa, p', p)$ is analytic and uniformly bounded in the region (1.3) and so the second partial derivatives are also uniformly bounded. [The same bounds hold for $\varkappa$ complex with $r_{00}(\varkappa, p)$ replaced by $r_{00}(\mathrm{Re}\,\varkappa, p)$.]

Term X_2. Since $|f(0)|\leqq\mathcal{O}(1)$ and $|g(0)|\leqq\mathcal{O}(w(q'))$ it suffices to prove

$$\left|\frac{r_{0\lambda}(\varkappa)}{1-\left(\frac{3}{\pi}\right)\lambda r_{0\lambda}(\varkappa)}\right|\leqq\mathcal{O}(\lambda^{-1}). \tag{3.8}$$

This follows from (3.5) and the fact that for $x\geqq 2$, $|x(1-x)^{-1}|\leqq 2$. (Remark: a careful analysis shows that this bound also holds for complex $\varkappa$ away from the pole.)

Term X_3. We use the method of Term X_1 for the p integration, the bound of Term X_2 for the leading factor and $g(0)=\mathcal{O}(w(q'))$ to obtain the bound.

Term X_4. We write $f(p)=(f(p)-f(0))+f(0)$, apply the method of Term X_1 for $g(p)-g(0)$ and a variant of this method for the term coming from $(f(p)-f(0))$. For the leading factor we use the bound from Term X_2. The overall bound is $\mathcal{O}(w(q'))$. This completes the proof of Equation (3.6) and hence of Lemma 3.2.

Lemma 3.3. *For $\lambda>0$ sufficiently small $R_\lambda(\varkappa)$ has no pole in*

$$\varkappa^*(\lambda)+2^{-1}b\lambda^2\leqq\varkappa\leqq 2m-\lambda^{5/2},$$

or

$$\varkappa\leqq\varkappa^*(\lambda)-2^{-1}b\lambda^2.$$

Proof. As in Section 2 it is sufficient to show that $T_\lambda(\varkappa)=-\lambda T^{(1)}(\varkappa)+\lambda^2 T_\lambda^{(2)}(\varkappa)$ does not have eigenvalue -1. We know that $-\lambda T^{(1)}(\varkappa)$ has spectrum

$$\{3\lambda\pi^{-1}r_{00}(\varkappa),0\}$$

and hence as in (2.18)

$$\|(1-\lambda T^{(1)}(\varkappa))^{-1}\|$$
$$\leqq 2\max\left(1,\left|1-\frac{3\lambda}{\pi}r_{00}(\varkappa)\right|^{-1},\left|1-\frac{3\lambda}{\pi}r_{00}(\varkappa)\right|^{-1}\|\lambda T^{(1)}(\varkappa)\|\right).$$

By Lemma 2.4 we have for $\varkappa\leqq 2m-\lambda^{5/2}$,

$$\|\lambda T^{(1)}(\varkappa)\|\leqq\mathcal{O}(\lambda\Delta(\varkappa))\leqq\mathcal{O}(\lambda^{-1/4}).$$

By (2.23) there exists a constant $c>0$ such that for $|\varkappa-\varkappa^*(\lambda)|\geqq\frac{1}{2}b\lambda^2$,

$$\left|1-\frac{3\lambda}{\pi}r_{00}(\varkappa)\right|\geqq c.$$

Thus in the region of the lemma

$$\|(1-\lambda T^{(1)}(\varkappa))^{-1}\|\leqq\mathcal{O}(\lambda^{-1/4}).$$

On the other hand, by Lemma 2.5, 2.6, 2.7,

$$\|\lambda^2 T_\lambda^{(2)}(\varkappa)\|\leqq\mathcal{O}(\lambda^2\Delta(\varkappa))\leqq\mathcal{O}(\lambda^{3/4}).$$

The product of the last two norms is $\mathcal{O}(\lambda^{1/2})<1$ and so -1 is in the resolvent set of $T_\lambda(\varkappa)$.

Proof of Theorem 1. By Lemma 2.11 for $\lambda>0$ sufficiently small there is exactly one point $\varkappa_B(\lambda)$ in the interval $(\varkappa^*(\lambda)-\frac{1}{2}b\lambda^2,\varkappa^*(\lambda)+\frac{1}{2}b\lambda^2)$ where $R_\lambda(\varkappa)$ has a pole. By Lemmas 3.2 and 3.3 there are no other points in $(m,2m)$ which are poles. Thus any bound state must have mass $\varkappa_B(\lambda)$. Since bound states exist, there are bound states of mass $\varkappa_B(\lambda)$. Now consider the representation of the Poincaré group on the subspace of mass $\varkappa_B(\lambda)$. As explained in [7, Lemma 5.2], the representation is at most n times reducible where n is the multiplicity of the eigenvalue -1 of $K_\lambda(\varkappa_B(\lambda))R_{0\lambda}(\varkappa_B(\lambda))$. By Corollary 2.9, $n=1$. Hence the representation is irreducible, and there is exactly one bound state with mass $\varkappa_B(\lambda)$. Finally the expansion for $\varkappa_B(\lambda)$ is given in Lemma 2.13.

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Note Added in Proof

In a subsequent paper (to appear in Annals of Physics) we continue the study of two-body bound states in $\lambda P(\varphi)_2$ models. The results include a full asymptotic expansion for $\varkappa_B(\lambda)$.