On the Bound-State Spectrum of A Nonrelativistic Particle in the Background of A Short-Ranged Linear Potential

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Abstract: The nonrelativistic problem of a particle immersed in a triangular potential well, set forth by N. A. Rao and B. A. Kagali, is revised. It is shown that these researchers misunderstood the full meaning of the potential and obtained a wrong quantization condition. By exploring the space inversion symmetry, this work presents the correct solution to this problem with potential applications in electronics in a simple and transparent way.

Keywords: Triangular Potential; Linear Potential; Airy Functions
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In a recent paper published in this Journal, Rao and Kagali [1] explored the one-dimensional nonrelativistic bound-state solutions of a particle immersed in a triangular potential well. In view of the mentioned significance in particle physics and exciting applications in solid state physics, it is of more than pedagogical interest to revise the problem. The present paper highlights that the authors of Ref. [1] misunderstood the full meaning of the novel potential and made a few erroneous calculations. Furthermore, the correct spectrum to the triangular potential well is presented in a simple way.

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Let us write the short-ranged linear potential well as

$$
\begin{aligned}
V(x) &= \frac{V_0}{L}\left(|x|-L\right)\left[\theta\left(x+L\right)-\theta\left(x-L\right)\right] \\
&= \begin{cases} \frac{V_0}{L}\left(|x|-L\right) & \text{for } |x|<L \\ \\ 0 & \text{for } |x|>L \end{cases}
\end{aligned} \tag{1}
$$

where $\theta\left(x\right)$ is the Heaviside function, $2L$ is the range of the potential and V_0 is its depth. Because $V\left(-x\right)=V\left(x\right)$, the Schrödinger equation

$$
\frac{d^2\psi(x)}{dx^2}+\frac{2m}{\hbar^2}\left[E-V\left(x\right)\right]\psi(x)=0 \tag{2}
$$

is invariant under space inversion ($x\to -x$) and so we can choose solutions with definite parities. In this circumstance it is enough to concentrate our attention on one side of the x-axis and use the continuity of $\psi(x)$ and $d\psi(x)/dx$ at the origin, inasmuch as $V(x)$ is finite. Hence, the two distinct classes of solutions can be discriminated by the behaviour of ψ and its first derivative at the origin: the homogeneous Neumann condition at the origin ($d\psi(x)/dx|_{x=0}=0$) for even parity solutions and the homogeneous Dirichlet condition ($\psi(0)=0$) for odd ones. We define

$$
\varepsilon=\frac{E}{\hbar^2/\left(2mL^2\right)},\quad v_0=\frac{V_0}{\hbar^2/\left(2mL^2\right)} \tag{3}
$$

and introduce the new variable

$$
z=\frac{v_0^{1/3}}{L}\left[|x|-L\left(1+\frac{\varepsilon}{v_0}\right)\right] \tag{4}
$$

so that, for $0<x<L$, the Schrödinger equation turns into the Airy differential equation

$$
\frac{d^2\psi(z)}{dz^2}-z\psi(z)=0 \tag{5}
$$

which has a general solution expressed as a linear superposition of the linearly independent oscillatory Airy functions $\mathrm{Ai}\left(z\right)$ and $\mathrm{Bi}\left(z\right)$ [2]

$$
\psi(z)=c_a\,\mathrm{Ai}\left(z\right)+c_b\,\mathrm{Bi}\left(z\right) \tag{6}
$$

Therefore,

$$
\begin{aligned}
&c_a\,\mathrm{Ai}'\left(z_0\right)+c_b\,\mathrm{Bi}'\left(z_0\right)=0 \quad \text{for even parity solutions} \\
&c_a\,\mathrm{Ai}\left(z_0\right)+c_b\,\mathrm{Bi}\left(z_0\right)=0 \quad \text{for odd parity solutions}
\end{aligned} \tag{7}
$$

where z_0 is the value of z at $x = 0$ and the prime means derivative with respect to z. For $x > L$, the evanescent free-particle solution (ψ must vanish as $x \to \infty$) is expressed as

$$\psi(x) = c\exp\left(-\frac{\sqrt{-\varepsilon}}{L}x\right) \tag{8}$$

where c is an arbitrary constant and $\varepsilon < 0$. The joining condition of ψ and its derivative at $x = L$ leads to

$$\begin{aligned} c_a\,\mathrm{Ai}\,(z_L) + c_b\,\mathrm{Bi}\,(z_L) &= c\,\exp\left(-\sqrt{-\varepsilon}\right) \\ c_a\,\mathrm{Ai}'\,(z_L) + c_b\,\mathrm{Bi}'\,(z_L) &= \alpha\,c\,\exp\left(-\sqrt{-\varepsilon}\right) \end{aligned} \tag{9}$$

with

$$\alpha = -\frac{\sqrt{-\varepsilon}}{v_0^{1/3}} \tag{10}$$

and

$$z_L = z_0 + v_0^{1/3} \tag{11}$$

Combining the top and bottom lines of (9) yields

$$\frac{c_a\,\mathrm{Ai}'\,(z_L) + c_b\,\mathrm{Bi}'\,(z_L)}{c_a\,\mathrm{Ai}\,(z_L) + c_b\,\mathrm{Bi}\,(z_L)} = \alpha \tag{12}$$

Hence, invoking the segregation between even and odd parity solutions expressed by (7), one finds

$$\frac{\mathrm{Ai}'\,(z_L) - \alpha\mathrm{Ai}\,(z_L)}{\mathrm{Bi}'\,(z_L) - \alpha\mathrm{Bi}\,(z_L)} = \begin{cases} \mathrm{Ai}'\,(z_0)\,/\mathrm{Bi}'\,(z_0) & \text{for even parity solutions} \\ \\ \mathrm{Ai}\,(z_0)\,/\mathrm{Bi}\,(z_0) & \text{for odd parity solutions} \end{cases} \tag{13}$$

By solving this quantization conditions one obtains the possible energy levels by inserting the allowed values of z_0 in (4), i.e.

$$\varepsilon = -v_0\left(1 + \frac{z_0}{v_0^{1/3}}\right) \tag{14}$$

Hence,

$$E = -V_0\left[1 + z_0\left(\frac{\hbar^2}{2mL^2V_0}\right)^{1/3}\right] \tag{15}$$

The numerical computation of z_0 can be done easily with a symbolic algebra program. The even (ψ_+) and odd (ψ_-) parity eigenfunctions on the entire x-axis can be written as

$$\begin{aligned} \psi_\pm\,(x) = \theta\,(+x)&\left\{\theta\,(L-x)\,[c_a\,\mathrm{Ai}\,(z) + c_b\,\mathrm{Bi}\,(z)] + \theta\,(x-L)\,c\,e^{-\sqrt{-\varepsilon}\,x/L}\right\} \\ \pm\theta\,(-x)&\left\{\theta\,(x+L)\,[c_a\,\mathrm{Ai}\,(z) + c_b\,\mathrm{Bi}\,(z)] + \theta\,(-x-L)\,c\,e^{+\sqrt{-\varepsilon}\,x/L}\right\} \end{aligned} \tag{16}$$

One can use (7) and the top (or bottom) line of (9) to write the three constants c_a, c_b and c in terms of just one of them. The remaining constant is to be determinate by normalization.

The set of eigenenergies is plotted in Fig. 1 as a function of v_0, and in Fig. 2 as a function of L. The spectra consist of a finite set of energy levels of alternate parities. Note that the number of bound states increases with v_0 and L, and that there is always at least one even parity bound-state solution no matter how weak or narrow the triangular potential is. Fig. 3 illustrates the behaviour of $\psi(x)$ for all the states corresponding to $L = 1$ and $v_0 = 20$. The normalization of the eigenfunctions was done by numerical computation using again a symbolic algebra program.

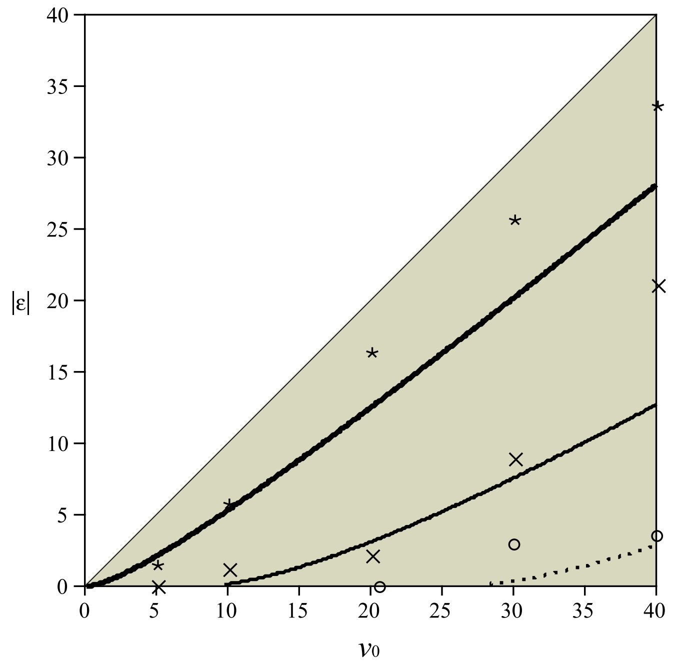

Fig. 1 Absolute values for the "eigenenergies" ($|\varepsilon|$) as a function of v_0 (*L is an arbitrary parameter*). The shaded area represents the lie zone for bound states ($0 < |\varepsilon| < v_0$). The thick line for the ground state, the thin line for the first-excited state and the dotted line for the second-excited state. The asterisks, crosses and circles stand for some values from Table I of Ref. [1] for the ground, the first-excited and the second-excited states, respectively.

A peculiar behaviour of the spectrum as $L \to 0$ can be taken into account by considering that Airy's functions have the power series expansions [2]

$$\mathrm{Ai}\,(z) = c_1 f(z) - c_2 g(z) \quad \text{and} \quad \mathrm{Bi}\,(z) = \sqrt{3}\,(c_1 f(z) + c_2 g(z)) \tag{17}$$

where

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{4}{6!}z^6 + \ldots \quad \text{and} \quad g(z) = z + \frac{2}{4!}z^4 + \ldots \tag{18}$$

with

$$\begin{aligned} c_1 &= \mathrm{Ai}\,(0) = \mathrm{Bi}\,(0)\,/\sqrt{3} = 3^{-2/3}/\Gamma\,(2/3) \\ c_2 &= -\mathrm{Ai}'\,(0) = \mathrm{Bi}'\,(0)\,/\sqrt{3} = 3^{-1/3}/\Gamma\,(1/3) \end{aligned} \tag{19}$$

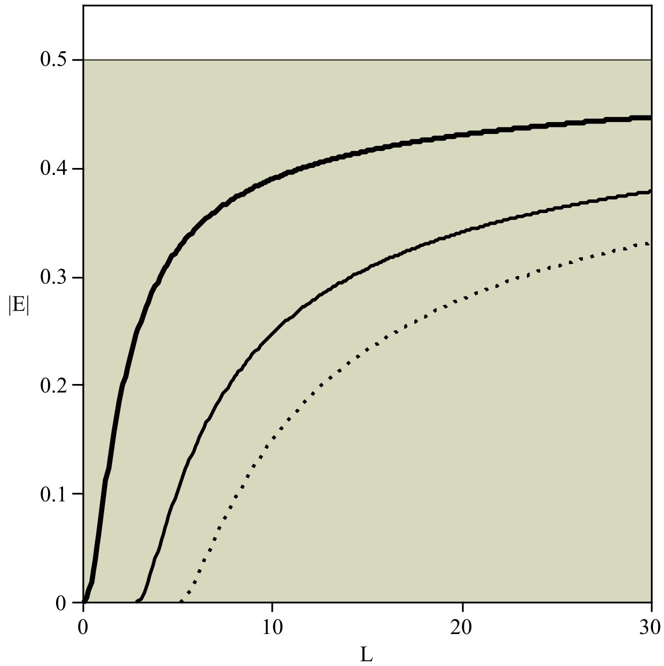

Fig. 2 Absolute values for the eigenenergies ($|E|$) as a function of L for the three lowest states with $V_0 = 0.5$ ($\hbar = m = 1$). The shaded area represents the lie zone for bound states ($0 < |E| < V_0$). The thick line for the ground state, the thin line for the first-excited state and the dotted line for the second-excited state.

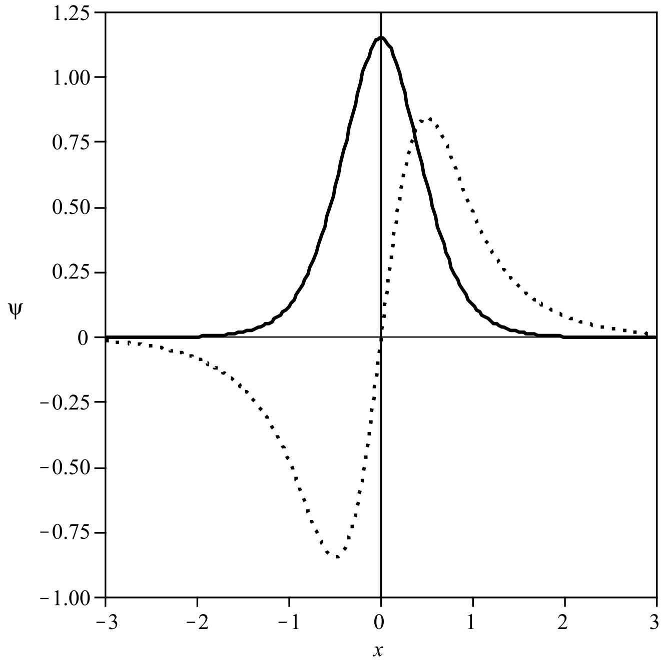

Fig. 3 ψ as a function of x for the ground-state (full line) and the first-excited state (dotted line), with $L = 1$ ($\hbar = m = 1$), $v_0 = 20$ and ε equal to -12.5029801 and -3.1015082 respectively.

To be specific, let us look at the case $V_0 = \lambda/L$, where λ is a positive constant. Then

$$z_0 \sim L^{1/3}, \quad z_L \sim L^{1/3}, \quad \alpha \sim L^{2/3}, \quad v_0 \sim L, \quad \varepsilon \sim L^2 \tag{20}$$

when L is taken to be a small number. When the series (17) are inserted in (13) and the like powers of L are collected one sees that the triangular potential does not acquiesce odd parity solutions for very small L. This can be concluded even in the lowest order. Nevertheless, the four-order approximation in z furnishes

$$z_L^2 = z_0^2 + 2\alpha \tag{21}$$

for even parity solutions, which combined with (11) gives

$$\varepsilon = -\frac{v_0^2}{4} \tag{22}$$

and

$$\psi(x) = c \exp\left(-\frac{v_0}{2L}|x|\right) \tag{23}$$

That is to say, the triangular potential only supports one bound-state solution. Of course! After all, the triangular potential goes over to the Dirac delta potential as $L \to 0$, that is $V(x) \to -\lambda\delta(x)$.

Comparison of our results (see Fig. 1) with Table I in Ref. [1] shows that the results fail to agree. The reason for this disagreement are a few mistakes in Ref. [1]. In Eqs. (5), (6) and (7) of Ref. [1] the authors should consider $|y|$ instead of y in the first change of variable. That *quid pro quo* propagates the error to the continuity conditions at the origin and makes the quantization condition wrong and too intricate.

A word should be said about the potential significance of the triangular well as a quark confining model. The short-ranged linear potential admits both bound states ($-v_0 < \varepsilon < 0$) and scattering states ($\varepsilon > 0$). Therefore, it is not a confining potential even though it is a binding one. A true confining potential, as one of those ones used in the phenomenological description of the quarkonium, should go to infinity as $|x| \to \infty$, even in a relativistic scheme.

Despite the pointed out drawbacks, the authors of the present work recognize that Rao and Kagali are high-spirited in pursuing such a simple problem never done before. A meritorious research apart from its potential applications in electronics. Of course, the investigation of the nonrelativistic scattering states as well as the extension to the relativistic domain are worthy.

Acknowledgments

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References

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