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# ON THE BOUNDEDNESS OF SINGULAR INTEGRALS WITH VARIABLE KERNELS

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**Abstract.** We prove the $L^p$ $(1 < p < \infty)$ estimates for the singular integrals with rough variable kernels. The $L^p$ boundedness of a class of modified directional Hilbert transforms is also given. As a consequence of this result, we get a good estimate for the singular integrals with rough odd kernels.

**1. Introduction.** In order to study the elliptic partial differential equations of order two with variable coefficients, Calderón and Zygmund [4] studied the $L^2$ boundedness of singular integrals $T$ with variable kernels. In 1980, Aguilera and Harboure [1] studied the $L^2$ boundedness of the associated maximal operator of singular integrals with variable kernels to study the problem of pointwise convergence of those singular integrals. In 2002, Tang and Yang [20] proved the $L^2$ boundedness of singular integrals with rough variable kernels associated to surfaces. In order to give a more precise statement, we first give some definitions.

DEFINITION 1. Let $K(x, y) : \boldsymbol{R}^n \times \boldsymbol{R}^n\backslash\{0\} \to \boldsymbol{R}$. Then $K(x, y)$ is said to be a *variable* C-Z *kernel* if

(a) $K(x, y)$ is positively homogeneous in $y$ of degree $-n$, namely, $K(x, \lambda y) = \lambda^{-n} K(x, y)$ for any $\lambda > 0$;

(b) $\displaystyle\int_{S^{n-1}} K(x, y')d\sigma(y') = 0$ for a.e. $x \in \boldsymbol{R}^n$.

Define the variable Calderón-Zygmund singular integral operator $T_\Phi$ associated to surface of the form $\{x = \Phi(|y|)y'\}$ by

$$T_\Phi(f)(x) = \text{p.v.} \int_{\boldsymbol{R}^n} K(x, y) f(x - \Phi(|y|)y')dy \tag{1.1}$$

for any $f \in C_0^\infty(\boldsymbol{R}^n)$. The truncated operator $T_\Phi^\varepsilon$ and the truncated maximal operator $T_\Phi^*$ are defined respectively by

$$T_\Phi^\varepsilon(f)(x) = \int_{\varepsilon<|y|<1/\varepsilon} K(x, y) f(x - \Phi(|y|)y')dy\,,$$
$$T_\Phi^*(f)(x) = \sup_{\varepsilon>0} |T_\Phi^\varepsilon(f)(x)|\,.$$

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These operators are considered by Fan and Pan [10] and Fan, Pan and Yang [11] in the case where $K(x, y) = K(y)$. In the variable kernel case, if we take $\Phi(|y|) = |y|$ for $y \in \boldsymbol{R}^n \setminus \{0\}$, then $T_\Phi = T$ is just the singular integral operator studied by Calderón and Zygmund [4]. Related to the work of Fan, Pan and Yang [11], Tang and Yang proved the following result.

THEOREM A ([20]). *Suppose $K(x, y)$ be a variable kernel as in Definition 1 and satisfies for some $q > 2(n-1)/n$*

$$\int_{S^{n-1}} |K(x, y')|^q d\sigma(y') \le C_1 \quad \text{for a.e. } x \in \boldsymbol{R}^n .$$

*Let $\Phi(t)$ be a nonnegative (or non-positive) $C^1$ function on $(0, \infty)$ satisfying $\Phi(t)/t = C_2 \Phi'(t)\varphi(t)$ for all $t \in (0, \infty)$ and suppose that*

(c) *$\Phi$ is strictly increasing (or decreasing);*

(d) *$\varphi$ is a monotonic and uniformly bounded function.*

*Then $T_\Phi^*$ is bounded on $L^2(\boldsymbol{R}^n)$ and $T_\Phi$ can be uniquely extended to be a bounded operator on $L^2(\boldsymbol{R}^n)$.*

REMARK 1. There is a defect in the proof of Theorem A for the $L^2$ boundedness of $T_\Phi^*$, i.e., the $L^2$ estimates for $III(f)$ in [20, p. 499]. This was given by using the following formula, which follows from the formula (2) in [21, p. 403]:

$$\int_0^\infty \frac{J_{n/2+m-1}(t)^2}{t^{n-1}} dt = \left(\frac{1}{2}\right)^{n-1} \frac{\Gamma(n-1)\Gamma(m)}{\Gamma(n/2)^2 \Gamma(n+m-1)} = O\left(\frac{1}{m^{n-1}}\right).$$

While in the proof of Theorem 1 in [20] this estimate of the integral played an essential role to guarantee the convergence of the sum in $m$, the wrong estimate $m^{-n}$ instead of $m^{1-n}$ was used there. It seems that it is not possible to obtain the $L^2$ boundedness of $T_\Phi^*$ by using their method (since one has $\sum_{m \ge 1} m^\delta m^{-n+1} m^{n-2} = +\infty$, for $0 < \delta < 1$). Hence we consider another method used by Duoandikoetxea etc., which even shows that we can get some $L^p$ estimates for $T_\Phi$ and $T_\Phi^*$.

REMARK 2. We note the following: If $g(t) \in C^1(0, \infty)$ is positive and decreasing (increasing) on $(0, \infty)$ and $g(t)/(tg'(t))$ is bounded on $(0, \infty)$, then $\lim_{t \to 0} g(t) = +\infty$ ($\lim_{t \to 0} g(t) = 0$) and $\lim_{t \to +\infty} g(t) = 0$ ($\lim_{t \to +\infty} g(t) = +\infty$), respectively. (See [23] for the proof.)

So, we should interpret

$$\text{p.v.} \int_{\boldsymbol{R}^n} K(x, y) f(x - \Phi(|y|)y') dy = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} K(x, y) f(x - \Phi(|y|)y') dy$$

in the case where $\Phi$ is increasing, and

$$\text{p.v.} \int_{\boldsymbol{R}^n} K(x, y) f(x - \Phi(|y|)y') dy = \lim_{\varepsilon \to \infty} \int_{|y| < \varepsilon} K(x, y) f(x - \Phi(|y|)y') dy$$

in the case where $\Phi$ is decreasing; cf. Lemma 2.6 (Section 2) and proofs of Theorem 1 (Section 3) and Theorem 2 (Section 4). Hence, it is adequate to define

$$\text{p.v.}\int_{\boldsymbol{R}^n} K(x,y)f(x-\Phi(|y|)y')dy = \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1/\varepsilon} K(x,y)f(x-\Phi(|y|)y')dy\,.$$

We also note that

$$T^*_\Phi f(x) \le 2\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} K(x,y)f(x-\Phi(|y|)y')dy\right|$$

in the case where $\Phi$ is increasing, and

$$T^*_\Phi f(x) \le 2\sup_{\varepsilon>0}\left|\int_{|y|<\varepsilon} K(x,y)f(x-\Phi(|y|)y')dy\right|$$

in the case where $\Phi$ is decreasing.

Now, in the $n$ ($n \ge 2$) dimensional case, if $K(x, y)$ is odd in $y$, using the rotation method as in Christ, Duoandikoetxea and Rubio de Francia [3, pp. 189–209], we see that the following theorem holds.

THEOREM 1. *Let $\Phi(t)$ be a nonnegative* (*or nonpositive*) *and monotonic $C^1$ function on* $(0,\infty)$ *such that $\Phi(t)/(t\Phi'(t))$ is bounded. Furthermore*, *assume $\Phi(t)$ satisfies one of the following conditions*:

(i) *$\varphi$ is monotonic*,

(ii) *$t\varphi'(t)$ is bounded*,

(iii) *$\Phi'(t)$ is monotonic on* $(0,\infty)$,

*where $\varphi(t) := \Phi(t)/(t\Phi'(t))$. Suppose $K(x, y)$ is odd in $y$ and satisfies*

$$\int_{S^{n-1}} |K(x,y')|^q d\sigma(y') \le C_1 \quad \textit{for a.e. } x\in \boldsymbol{R}^n\,.$$

*Then*, *$T_\Phi(f)(x)$ exists a.e. for any $f \in L^p(\boldsymbol{R}^n)$, and the operators $T_\Phi$ and $T^*_\Phi$ are bounded in $L^p(\boldsymbol{R}^n)$ provided that* $1 < p \le \max\{(n+1)/2, 2\}$ *and* $q > p'(n-1)/n$ $(n \ge 2)$, *where* $T^*_\Phi f = \sup_{\varepsilon>0}|T^\varepsilon_\Phi(f)|$.

For general kernel, we obtain only the following

THEOREM 2. *Let $n \ge 2$. Suppose $K(x, y)$ is a variable kernel homogeneous of degree $-n$ with respect to $y$ and satisfies $K(x,y') \in L^\infty(\boldsymbol{R}^n)\times L^q(S^{n-1})$ for some* $q > 2(n-1)/n$. *Let $\Phi(t)$ be as in Theorem* 1. *Then*, *$T_\Phi$ and $T^*_\Phi$ are bounded on $L^p(\boldsymbol{R}^n)$*, *if*

$$2-\frac{1-(2/\bar q-1)(n-1)}{n-(2/\bar q-1)(n-1)/2} < p < 2+(\bar q n/(n-1)-2)$$

*and* $p \le \max\{(n+1)/2, 2\}$, *where* $\bar q = \min\{2, q\}$.

REMARK 3. For an even kernel, we cannot modify the rotation method like as in Calderón and Zygmund [6] or Christ, Duoandikoetxea and Rubio de Francia [3, pp. 199–200]. In fact, we see p.v. $\int K(x,y)f(x-\Phi(|y|)y')dy = $ p.v. $\int K_1(x,y)f(x-y)dy$, where $K_1(x,y) = \varphi(\Phi^{-1}(|y|))K(x,y)$ (see the proof of Theorem 1). However, the kernel function

$K_1(x, y)$ is not homogeneous of degree $-n$ with respect to $y$, and so p.v. $\int K_1(x, u) R_j(y - u) du$ is not homogeneous of degree $-n$ with respect to $y$, where $R_j$ is the $j$-th Riesz transform kernel. Therefore, we cannot apply Theorem 1 to obtain the $L^p$ boundedness in the case where $K(x, y)$ is even with respect to $y$.

REMARK 4. There is no including relationship between Conditions (i) and (iii). There are also no including relationships between Conditions (i) and (ii), and between Conditions (ii) and (iii). These are seen in Examples 2 and 3 in [22], and Example 2.3 in [24].

REMARK 5. If $\Phi(t)$ is a positive and monotonic function on $(0, \infty)$ and $\Phi'(t)$ is monotonic, then the following (i) and (ii) are equivalent.

(i) $|\Phi(t)/(t\Phi'(t))| \le M$ $(0 < t < \infty)$ for some $0 < M < \infty$.

(ii) $\eta \le \max\{\Phi(2t)/\Phi(t),\ \Phi(t)/\Phi(2t)\}$ on $(0, \infty)$ for some $1 < \eta < \infty$.

This can be checked by elementary consideration, using convexity or concavity.

The condition $\eta \le \max\{\Phi(2t)/\Phi(t),\ \Phi(t)/\Phi(2t)\} \le L$ on $(0, \infty)$ for some $1 < \eta \le L < \infty$ and the monotonicity of $\Phi'(t)$ is used to prove $L^p$ boundedness of Marcinkiewicz integrals along surfaces with convolution type kernel by Al-Qassem [2].

We arrange our paper in the following way. In Section 2, we prepare some lemmas which will be used later. The proof of Theorem 1 can be found in Section 3. In Section 4, we give the proof of Theorem 2. In the last section, we prove the key lemma which appeared in Section 2.

Throughout this paper, the letter $C$ will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

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**2. Preliminary Lemmas.** We begin with recalling two known lemmas. The first lemma can be obtained by (2.19) in [19, p. 152] and Theorem 3.10 in [19, p. 158] (see also [20]).

LEMMA 2.1 ([19]). *Let $n \ge 2$, $k \ge 0$ and $P(y)$ be a spherical harmonics of degree $k$. Then*

$$\int_{S^{n-1}} P(y')e^{-ix\cdot y'} d\sigma(y') = (-i)^k (2\pi)^{n/2} \frac{J_{n/2+k-1}(|x|)}{|x|^{n/2-1}} P\left(\frac{x}{|x|}\right).$$

LEMMA 2.2 ([14]). *Suppose $\nu$ and $\lambda$ satisfy $\nu - \lambda > -1$, and $|\nu| > 1/2, \lambda \ge -1/2$ or $\nu > -1, \lambda \ge 0$. Then*

$$\left|\int_0^r \frac{J_\nu(t)}{t^\lambda} dt\right| \le \frac{C}{|\nu|^\lambda} \quad \textit{for} \quad 0 < r < \infty\,. \tag{2.1}$$

Next, we shall give several elementary considerations.

LEMMA 2.3. *Suppose $g(t)$ is positive and decreasing, and $g'(t)$ is increasing. Suppose further $-g(t)/(tg'(t)) \ge a$ for some $a > 0$. Then there exists $C > 1$ such that*

$$g(t) \le Cg(2t) \quad \textit{for} \ \ t > 0\,.$$

PROOF. We note that in this case $g'(t) < 0$. Since $g'(t)$ is increasing, $g(t)$ is convex. Hence we have for any $b > 1$

$$\frac{g(bt) - g(t)}{bt - t} \geq g'(t), \quad \text{i.e.,} \quad \frac{g(t) - g(bt)}{(b-1)t} \leq -g'(t).$$

This combined with $-g(t)/(tg'(t)) \geq a$ implies that

$$\frac{g(t) - g(bt)}{(b-1)t} \leq \frac{g(t)}{at}.$$

So, if we take $1 < b < 1 + a$, then we get

$$g(t) \leq \frac{1}{1 - (b-1)/a} g(bt).$$

Using this inequality $[\log 2/\log b] + 1$-times, we see that there exists $C > 1$ satisfying

$$g(t) \leq Cg(2t), \quad t > 0. \qquad \square$$

LEMMA 2.4. *Suppose $\Phi(t)$ is positive and decreasing, and $\Phi'(t)$ is increasing. Suppose further $-\Phi(t)/(t\Phi'(t)) \leq M$ for some $M > 0$. Then there exists $C > 1$ such that*

$$\Phi^{-1}(t) \leq C\Phi^{-1}(2t), \quad t > 0.$$

PROOF. From the assumption $-\Phi(t)/(t\Phi'(t)) \leq M$, it follows that

$$-\frac{\Phi^{-1}(t)}{t(\Phi^{-1}(t))'} = -\frac{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))}{\Phi(\Phi^{-1}(t))} \geq \frac{1}{M}.$$

On the other hand, since $\Phi(t)$ is positive and decreasing, and $\Phi'(t)$ is increasing, we have $\Phi^{-1}(t)$ is also positive and decreasing, and $(\Phi^{-1}(t))'$ is increasing.

Thus, $\Phi^{-1}(t)$ satisfies the assumption of Lemma 2.3, and hence there exists $C > 1$ such that

$$\Phi^{-1}(t) \leq C\Phi^{-1}(2t), \quad t > 0. \qquad \square$$

LEMMA 2.5. *Suppose $\Phi(t)$ is positive and increasing, and $\Phi'(t)$ is monotonic. Suppose further $\Phi(t)/(t\Phi'(t)) \leq M$ for some $M > 0$. Then, there exists $C > 1$ such that*

$$\Phi^{-1}(2t) \leq C\Phi^{-1}(t), \quad t > 0.$$

PROOF. (i) The case where $\Phi'(t)$ is increasing. Since in this case $\Phi^{-1}(t)$ is concave, we have

$$\frac{\Phi^{-1}(2t) - \Phi^{-1}(t)}{t} \leq (\Phi^{-1}(t))' = \frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))} \cdot \frac{\Phi^{-1}(t)}{t} \leq M\frac{\Phi^{-1}(t)}{t}.$$

Thus, we have

$$\Phi^{-1}(2t) \leq (1 + M)\Phi^{-1}(t), \quad t > 0.$$

(ii) The case where $\Phi'(t)$ is decreasing. From the assumption $\Phi(t)/(t\Phi'(t)) \leq M$, it follows that

$$\frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))} = \frac{\Phi(\Phi^{-1}(t))}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))} \leq M.$$

On the other hand, since $\Phi(t)$ is positive and increasing, and $\Phi'(t)$ is decreasing, we have $\Phi^{-1}(t)$ is increasing and convex. So, for any $0 < a < 1$, we have

$$\frac{\Phi^{-1}(t) - \Phi^{-1}(at)}{t - at} \le (\Phi^{-1}(t))' = \frac{1}{\Phi'(\Phi^{-1}(t))} = \frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))} \cdot \frac{\Phi^{-1}(t)}{t} \le M\frac{\Phi^{-1}(t)}{t}.$$

Thus, taking $0 < a < 1$ such that $1 - M(1-a) > 0$, we obtain

$$\Phi^{-1}(t) \le \frac{1}{1 - M(1-a)}\Phi^{-1}(at).$$

Using this inequality $[-1/\log_2 a] + 1$-times, we see that there exists $C > 1$ such that

$$\Phi^{-1}(t) \le C\Phi^{-1}\left(\frac{t}{2}\right), \quad t > 0. \qquad \square$$

The following lemma is used to give a reason why Remark 2 is well-grounded.

LEMMA 2.6. (i) *Let $\Phi(t)$ be a positive and non-decreasing $C^1$ function on* $(0, \infty)$, *such that $\varphi(t) = \Phi(t)/(t\Phi'(t)) \le M$ for some $M > 0$. Then, for $f \in L^p(\boldsymbol{R}^n)$, $1 \le p < \infty$, and for $\varepsilon > 0$ it holds that*

$$\int_{|y|>\varepsilon} \frac{|f(x - \Phi(|y|)y')|}{|y|^n}dy \le M\int_{|y|>\Phi(\varepsilon)} \frac{|f(x-y)|}{|y|^n}dy < +\infty.$$

(ii) *Let $\Phi(t)$ be a positive and non-increasing $C^1$ function on* $(0, \infty)$, *such that $-\varphi(t) = -\Phi(t)/(t\Phi'(t)) \le M$ for some $M > 0$. Then, for $f \in L^p(\boldsymbol{R}^n)$, $1 \le p < \infty$, and for $\varepsilon > 0$ it holds that*

$$\int_{|y|<\varepsilon} \frac{|f(x - \Phi(|y|)y')|}{|y|^n}dy \le M\int_{|y|>\Phi(\varepsilon)} \frac{|f(x-y)|}{|y|^n}dy < +\infty.$$

PROOF. (i) Setting $\Phi(r) = t$, we have

$$\begin{aligned}
\int_{|y|>\varepsilon} \frac{|f(x - \Phi(|y|)y')|}{|y|^n}dy &= \int_{S^{n-1}}\int_\varepsilon^\infty \frac{|f(x-\Phi(r)y')|}{r^n}r^{n-1}dr\,d\sigma(y') \\
&= \int_{S^{n-1}}\left(\int_{\Phi(\varepsilon)}^\infty \frac{|f(x-ty')|}{t}\frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))}dt\right)d\sigma(y') \\
&= \int_{S^{n-1}}\left(\int_{\Phi(\varepsilon)}^\infty \frac{|f(x-ty')|}{t}\varphi(\Phi^{-1}(t))dt\right)d\sigma(y') \\
&\le M\int_{S^{n-1}}\left(\int_{\Phi(\varepsilon)}^\infty \frac{|f(x-ty')|}{t}dt\right)d\sigma(y') = M\int_{|y|>\Phi(\varepsilon)}\frac{|f(x-y)|}{|y|^n}dy \\
&\le M\left(\int_{|y|>\Phi(\varepsilon)}\frac{dy}{|y|^{p'n}}\right)^{1/p'}\left(\int_{|y|>\Phi(\varepsilon)}|f(x-y)|^p\,dy\right)^{1/p} \le CM\|f\|_{L^p}.
\end{aligned}$$

(ii) Setting $\Phi(r) = t$, we have

$$\begin{aligned}
\int_{|y|<\varepsilon} \frac{|f(x-\Phi(|y|)y')|}{|y|^n}dy &= \int_{S^{n-1}} \int_0^{\varepsilon} \frac{|f(x-\Phi(r)y')|}{r^n} r^{n-1} dr d\sigma(y') \\
&= \int_{S^{n-1}} \left( \int_{\Phi(\varepsilon)}^{\infty} \frac{|f(x-ty')|}{t} \frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))} dt \right) d\sigma(y') \\
&= \int_{S^{n-1}} \left( \int_{\Phi(\varepsilon)}^{\infty} \frac{|f(x-ty')|}{t} \varphi(\Phi^{-1}(t)) dt \right) d\sigma(y') \\
&\leq M \int_{S^{n-1}} \left( \int_{\Phi(\varepsilon)}^{\infty} \frac{|f(x-ty')|}{t} dt \right) d\sigma(y') = M \int_{|y|>\Phi(\varepsilon)} \frac{|f(x-y)|}{|y|^n} dy \\
&\leq M \left( \int_{|y|>\Phi(\varepsilon)} \frac{dy}{|y|^{p'n}} \right)^{1/p'} \left( \int_{|y|>\Phi(\varepsilon)} |f(x-y)|^p \, dy \right)^{1/p} \leq CM \|f\|_{L^p} . \qquad \square
\end{aligned}$$

Finally, we state the following key estimate to prove Theorem 2, whose proof we postpone to Section 5.

LEMMA 2.7. *Let $\Phi(t)$ and $\varphi(t)$ be as in Theorem 2. Then for any $1 > \eta > 1/2 - n/2$ there exists $C > 0$ such that*

$$\left| \int_{h_1}^{h_2} \varphi\left( \Phi^{-1}\left( \frac{s}{|\xi|} \right) \right) \frac{J_{n/2+k-1}(s)}{s^{n/2+\eta}} ds \right| \leq \frac{C}{(n/2+k-1)^{n/2+\eta}} , \quad 0 \leq h_1 < h_2 \leq +\infty .$$

REMARK 6. Using the above lemma, we can prove $L^2$ boundedness of $T_\Phi$ directly, but we could not reduce that of $T_\Phi^*$ directly.

**3. Proof of Theorem 1.** To prove Theorem 1, we use the rotation method. So, we prepare two useful lemmas from the work by Christ, Duoandikoetxea and Rubio de Francia [3], which play essential roles in this paper. Given an operator $T$ on $L^p(\boldsymbol{R})$, $1 \leq p < \infty$, we define an associated operator $\tilde{T}$ by

$$\tilde{T} f(x,\theta) = T_\theta f(x) , \quad (x,\theta) \in \boldsymbol{R}^n \times S^{n-1} ,$$

where $T_\theta$ acts on $L^p(\boldsymbol{R}^n)$ by letting $T$ act on the variable $x \cdot \theta$ while leaving the remaining variables fixed.

We define the directional Hardy-Littlewood maximal operator $M_1$ by

$$M_1 f(x,\theta) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x-t\theta)| dt , \quad (x,\theta) \in \boldsymbol{R}^n \times S^{n-1} .$$

We state Lemma 4.1 and Theorem 1 in [3] as lemmas.

LEMMA 3.1. *Suppose that the inequality*

$$\|M_1 f\|_{L^p(L^{q_0})} \leq C \|f\|_{L^p}$$

*holds for some fixed* $p \geq 1$, *and* $q_0 > 1$. *If* $T$ *is an operator which is bounded in* $L^r(\boldsymbol{R}, w(x)dx)$ *for every Muckenhoupt* $A_r$*-weight* $w(x)$ *in* $\boldsymbol{R}$ $(1 < r < \infty)$, *then*

$$\|\tilde{T} f\|_{L^p(L^q)} \leq C_q \|f\|_{L^p}$$

*for the same* $p$ *as above and for all* $q < q_0$. *Here*

$$\|F\|_{L^p(L^q)} = \left(\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}} |F(x,\theta)|^q\, d\theta\right)^{p/q} dx\right)^{1/p}.$$

LEMMA 3.2. *If* $1 < p \leq \max\{(n+1)/2,\, 2\}$ *and* $q < p(n-1)/(n-p)$, *then* $M_1$ *is bounded from* $L^p(\boldsymbol{R}^n)$ *to* $L^p(L^q)$.

Now, we begin by giving a result on the following modified Hilbert transform.

PROPOSITION 3.1. *Let* $\phi$ *be a positive and bounded function defined on* $(0,\infty)$, *satisfying one of the following conditions*:

(i) $\phi(t)$ *is a monotonic function.*

(ii) $t\phi'(t)$ *is bounded.*

(iii) $\phi(t) = t/(\Phi^{-1}(t)\Phi'(\Phi^{-1}(t)))$, *where* $\Phi(t)$ *is a positive and monotonic function and its derivative* $\Phi'(t)$ *is a monotonic function.*

*Then the following two operators of one dimension*:

$$H_\phi(f)(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|} \frac{\phi(|y|)}{y} f(x-y)dy$$

*and*

$$H_\phi^*(f)(x) = \sup_{\varepsilon>0}\left|\int_{\varepsilon<|y|} \frac{\phi(|y|)}{y} f(x-y)dy\right|$$

*are bounded on* $L^p(w)$ *for any* $w \in A_p(\boldsymbol{R})$.

REMARK 7. As is known, in the one dimensional case, $H_\phi$ is not bounded even on $L^2(\boldsymbol{R})$ for $\phi(t) = \sin|t|$, cf. Namazi [15, p. 421].

PROOF OF PROPOSITION 3.1. Denote $\sigma_k(x) = (\chi_{[2^k,2^{k+1}]} - \chi_{[-2^{k+1},-2^k]}(x))(\phi(|x|)/|x|)$. Then

$$H_\phi(f) = \sum_{k=-\infty}^{\infty} \sigma_k * f\,,$$

and

$$\hat{\sigma}_k(\xi) = \int_{2^k}^{2^{k+1}} \frac{\phi(r)}{r} e^{-ir\xi} dr - \int_{2^k}^{2^{k+1}} \frac{\phi(r)}{r} e^{ir\xi} dr = -2i\int_{2^k}^{2^{k+1}} \frac{\phi(r)}{r} \sin r\xi\, dr\,.$$

Next, we shall give the estimates of $\hat{\sigma}_k(\xi)$ according to (i), (ii) and (iii).

Case (i). It is easy to see that if $\phi$ is monotonic, then by the second mean value theorem and the monotonicity of $\phi$, we immediately obtain

$$|\hat{\sigma}_k(\xi)| \leq \frac{C\|\phi\|_\infty}{|2^k\xi|}\,.$$

Case (ii). If $\|r\phi'(r)\|_\infty < \infty$, integrating by parts, we get

$$\hat{\sigma}_k(\xi) = -2i\left[\frac{\phi(r)}{r}\frac{\cos r\xi}{\xi}\right]_{2^k}^{2^{k+1}} + 2i\int_{2^k}^{2^{k+1}} \frac{r[\phi(r)]' - \phi(r)}{r^2}\frac{\cos r\xi}{\xi}\,dr\,.$$

Hence, we obtain

$$|\hat{\sigma}_k(\xi)| \le 4\frac{\|\phi\|_\infty}{|2^k\xi|} + \frac{2}{|2^k\xi|}(\|r[\phi(r)]'\|_\infty + \|\phi\|_\infty) \le 6\frac{\|\phi\|_\infty}{|2^k\xi|} + \frac{2\|r[\phi(r)]'\|_\infty}{|2^k\xi|}\,.$$

Case (iii).

$$\hat{\sigma}_k(\xi) = -2i\int_{2^k}^{2^{k+1}} \frac{\sin r\xi}{\Phi^{-1}(r)\Phi'(\Phi^{-1}(r))}\,dr\,.$$

If $\Phi$ and $\Phi'$ are increasing, then $\Phi^{-1}$ is also increasing and $1/(\Phi^{-1}(r)\Phi'(\Phi^{-1}(r)))$ is decreasing. So, by the second mean value theorem, we have for $2^k \le h_k \le 2^{k+1}$

$$\begin{aligned}|\hat{\sigma}_k(\xi)| &= 2\left|\int_{2^k}^{h_k} \sin r\xi\,dr\,\frac{1}{\Phi^{-1}(2^k)\Phi'(\Phi^{-1}(2^k))}\right| \\ &\le \frac{4}{|2^k\xi|}\frac{2^k}{\Phi^{-1}(2^k)\Phi'(\Phi^{-1}(2^k))} \le \frac{C\|\phi\|_\infty}{|2^k\xi|}\,.\end{aligned}$$

If $\Phi$ is increasing and $\Phi'$ is decreasing, then $1/\Phi^{-1}(r)$ is decreasing and $1/\Phi'(\Phi^{-1}(r))$ is increasing. Hence, using the second mean value theorem twice and Lemma 2.6, we get for some $2^k \le h_k \le h'_k \le 2^{k+1}$

$$\begin{aligned}|\hat{\sigma}_k(\xi)| &= 2\left|\int_{h_k}^{h'_k} \sin r\xi\,dr\,\frac{1}{\Phi^{-1}(2^k)\Phi'(\Phi^{-1}(2^{k+1}))}\right| \\ &\le \frac{C}{|\xi|}\frac{2^{k+1}}{\Phi^{-1}(2^{k+1})\Phi'(\Phi^{-1}(2^{k+1}))}\cdot\frac{1}{2^{k+1}} \le \frac{C\|\phi\|_\infty}{|2^k\xi|}\,.\end{aligned}$$

If $\Phi$ is decreasing, then the case that $\Phi'$ is decreasing does not occur. So we only need to consider the case that $\Phi'$ is increasing. Thus, $1/\Phi^{-1}(r)$ is increasing and $-1/\Phi'(\Phi^{-1}(r))$ is positive and decreasing. By Lemma 2.5, we get

$$|\hat{\sigma}_k(\xi)| = 2\left|\int_{h_k}^{h'_k} \sin r\xi\,dr\right|\left|\frac{1}{\Phi^{-1}(2^{k+1})\Phi'(\Phi^{-1}(2^k))}\right| \le \frac{C\|\phi\|_\infty}{|2^k\xi|}\,.$$

On the other hand, we see that

$$|\hat{\sigma}_k(\xi)| = 2\left|\int_{2^k}^{2^{k+1}} \frac{\phi(r)}{r}\sin r\xi\,dr\right| \le 2\int_{2^k}^{2^{k+1}} \frac{\phi(r)}{r}r|\xi|\,dr \le C\|\phi\|_\infty|2^{k+1}\xi|\,.$$

Thus, in all cases (i), (ii), (iii), we have proved

$$|\hat{\sigma}_k(\xi)| \le C\|\phi\|_\infty|2^{k+1}\xi| \quad \text{and} \quad |\hat{\sigma}_k(\xi)| \le \frac{C}{|2^k\xi|}\,. \tag{3.1}$$

Therefore, since $H_\phi(f) = \sum_{k=-\infty}^{\infty} \sigma_k * f$, by the same steps as in the proof of Corollary 4.2 in [9], one can show that $H_\phi$ and $H_\phi^*$ are bounded on $L^p(w)$ for any $w \in A_p(\boldsymbol{R})$. In fact, let $\{\psi_j\}_{j=-\infty}^{\infty}$ be a smooth partition of the unity in $\boldsymbol{R}_+ = (0, \infty)$ adapted to the intervals $[2^{-j}, 2^{-(j-1)}]$ such that

$$\psi_j \in C^1\,, \quad 0 \le \psi_j \le 1\,, \quad \sum_j \psi_j(t)^2 = 1\,,$$

$$\operatorname{supp} \psi_j \subset \{t;\ 2^{-(j+1)} \le t \le 2^{-(j-1)}\}\,, \quad \text{and} \quad |\psi_j'(t)| \le Ct^{-1}\,.$$

Also define the multiplier operator $S_j$ in $\boldsymbol{R}$ by $(S_j f)\hat{}\,(\xi) = \hat{f}(\xi)\psi_j(|\xi|)$. Then we have

$$H_\phi f = \sum_k \sigma_k * \Big(\sum_j S_{j+k} S_{j+k} f\Big) = \sum_j \Big(\sum_k S_{j+k}(\sigma_k * S_{j+k} f)\Big) =: \sum_j H_j f\,.$$

If $w \in A_2(\boldsymbol{R})$, we have, noting

$$|\sigma_k * g(x)| \le \int_{2^k \le |y-x| \le 2^{k+1}} \|\phi\|_\infty |g(y)|/|y-x| dy \le 4\|\phi\|_\infty Mg(x)\,,$$

and using the $A_p$ weighted boundedness of the Hardy-Littlewood maximal function and Littlewood-Paley decomposition operator (see [13]), that

$$\begin{aligned}\|H_j f\|_{L^2(w)} &\le C_1 \sum_k \|\sigma_k * S_{j+k} f\|_{L^2(w)} \le C_1 C_2 \sum_k \|M(S_{j+k} f)\|_{L^2(w)} \\ &\le C_1 C_2 C_3 \sum_k \|S_{j+k} f\|_{L^2(w)} \le C_1 C_2 C_3 C_4 \|f\|_{L^2(w)}\,.\end{aligned}$$

By the reverse Hölder inequality, we have

$$\|H_j f\|_{L^2(w^{1+\varepsilon})} \le C\|f\|_{L^2(w^{1+\varepsilon})}\,, \quad j \in \boldsymbol{Z}$$

for some $\varepsilon > 0$ (see [12]).

Now, using Estimate (3.1) for $\hat{\sigma}_k(\xi)$, we can show as in the proof of Theorem B in [9] that

$$\|H_j f\|_{L^2(\boldsymbol{R})} \le C 2^{-|j|} \|f\|_{L^2(\boldsymbol{R})}\,, \quad j \in \boldsymbol{Z}\,.$$

Hence, interpolating with change of measure, we obtain

$$\|H_j f\|_{L^2(w)} \le C 2^{-(\varepsilon/(1+\varepsilon))|j|} \|f\|_{L^2(w)}\,, \quad j \in \boldsymbol{Z}\,,$$

which implies

$$\|H_\phi f\|_{L^2(w)} \le C\|f\|_{L^2(w)}\,.$$

Thus by the extrapolation theorem for $A_p$ weights (see [12]), we have for every $w \in A_p(\boldsymbol{R})$, $1 < p < \infty$,

$$\|H_\phi f\|_{L^p(w)} \le C\|f\|_{L^p(w)}\,.$$

By adapting the proof of Theorem E in [9] and using the same arguments as above, we obtain the corresponding result for $H_\phi^*$. This completes the proof of Proposition 3.1. □

Using this Proposition 3.1 and Lemma 3.1, we obtain

PROPOSITION 3.2. *Let $\Phi(t)$ be as in Theorem* 1. *Then the following two directional operators of higher dimension*:

$$H_\varphi(f)(x,\theta) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|t|}\varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}$$

*and*

$$H_\varphi^*(f)(x,\theta) = \sup_{\varepsilon>0}\left|\int_{\varepsilon<|t|}\varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right|$$

*are bounded from $L^p(\boldsymbol{R}^n)$ to $L^p(L^q)$ for any $q < p(n-1)/(n-p)$ and* $1 < p \le \max\{2, (n+1)/2\}$.

PROOF. By Lemma 3.2, $M_1$ is bounded from $L^p(\boldsymbol{R}^n)$ to $L^p(L^q)$ for all $1 < p < \max(2, (n+1)/2)$ and $q < p(n-1)/(n-p)$.

Next, let $\phi(t) = \varphi(\Phi^{-1}(t))$. Then the positivity and boundedness of $\varphi$ imply those of $\phi$. In the case (i), since $\varphi$ and $\Phi$ are monotonic, $\phi$ is also monotonic. In the case (ii), we have

$$|t\phi'(t)| = |t(\varphi(\Phi^{-1}(t)))'| = |\Phi^{-1}(t)\varphi'(\Phi^{-1}(t))\varphi(\Phi^{-1}(t))| \le \|\varphi\|_\infty\|s\varphi'(s)\|_\infty < +\infty\,.$$

In the case (iii), since $\varphi(t) = \Phi(t)/(t\Phi'(t))$, we see that

$$\phi(t) = \varphi(\Phi^{-1}(t)) = \Phi(\Phi^{-1}(t))/(\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))) = t/(\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))\,.$$

Also $\Phi'(t)$ is monotonic by assumption.

Hence, applying Proposition 3.1 and then Lemma 3.1, we obtain the desired conclusion. □

Now, we are in a position to prove Theorem 1. We shall prove only the case where $\Phi(t)$ is nonnegative and nondecreasing, since the other cases are proved in a similar way. Then, under the condition of Theorem 1, we have $\lim_{\varepsilon\to 0}\Phi(\varepsilon) = 0$ and $\lim_{\varepsilon\to+\infty}\Phi(\varepsilon) = +\infty$. We first note

$$\begin{aligned}\int_{|y|>\varepsilon}\frac{K(x,y')f(x-\Phi(|y|)y')}{|y|^n}dy &= \int_{S^{n-1}}\int_{\varepsilon}^{\infty}\frac{K(x,y')f(x-\Phi(r)y')}{r^n}r^{n-1}drd\sigma(y')\\ &= \int_{S^{n-1}}\left(\int_{\Phi(\varepsilon)}^{\infty}\frac{K(x,y')f(x-ty')}{t}\frac{t}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))}dt\right)d\sigma(y')\\ &= \int_{S^{n-1}}\left(\int_{\Phi(\varepsilon)}^{\infty}\frac{K(x,y')f(x-ty')}{t}\varphi(\Phi^{-1}(t))dt\right)d\sigma(y')\\ &= \int_{|y|>\Phi(\varepsilon)}\varphi(\Phi^{-1}(|y|))K(x,y)f(x-y)dy\,.\end{aligned}$$

So, since $K(x, y)$ is odd in $y$ and $\Phi(t)$ is increasing, we obtain that

$$
\begin{aligned}
T_\Phi(f)(x) &= \text{p.v.} \int_{\boldsymbol{R}^n} K(x,y) f(x - \Phi(|y|)y')dy \\
&= \lim_{\varepsilon\to 0} \int_{|y|>\Phi(\varepsilon)} \varphi(\Phi^{-1}(|y|))K(x,y)f(x-y)dy \\
&= \lim_{\varepsilon\to 0} \int_{S^{n-1}} \int_{\Phi(\varepsilon)}^\infty \varphi(\Phi^{-1}(r))K(x,\theta)f(x-r\theta)\frac{dr}{r}d\sigma(\theta) \\
&= \lim_{\varepsilon\to 0} T_\Phi^\varepsilon(f)(x) \\
&= \lim_{\varepsilon\to 0} \frac{1}{2}\int_{S^{n-1}} \int_{\Phi(\varepsilon)}^\infty \varphi(\Phi^{-1}(r))K(x,\theta)(f(x-ry') - f(x+r\theta))\frac{dr}{r}d\sigma(\theta) \qquad (3.2)\\
&= \lim_{\varepsilon\to 0} \frac{1}{2}\int_{S^{n-1}} \int_{|s|>\Phi(\varepsilon)} \varphi(\Phi^{-1}(s))f(x-s\theta)\frac{dr}{r}K(x,\theta)d\sigma(\theta) \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \int_{S^{n-1}} K(x,\theta)\left(\int_{|t|>\Phi(\varepsilon)} \varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right)d\sigma(\theta) \\
&= \frac{1}{2}\lim_{\eta\to 0} \int_{S^{n-1}} K(x,\theta)\left(\int_{|t|>\eta} \varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right)d\sigma(\theta)\ .
\end{aligned}
$$

Also similarly as above, we get

$$
\begin{aligned}
T_\Phi^*(f)(x) &= \frac{1}{2}\sup_{\varepsilon>0}\left|\int_{S^{n-1}} K(x,\theta)\left(\int_{|t|>\Phi(\varepsilon)} \varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right)d\sigma(\theta)\right| \\
&\le \frac{1}{2}\int_{S^{n-1}} |K(x,\theta)| \sup_{\varepsilon>0}\left|\int_{|t|>\Phi(\varepsilon)} \varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right|d\sigma(\theta) \qquad (3.3)\\
&= \frac{1}{2}\int_{S^{n-1}} |K(x,\theta)| \sup_{\eta>0}\left|\int_{|t|>\eta} \varphi(\Phi^{-1}(|t|))f(x-t\theta)\frac{dt}{t}\right|d\sigma(\theta)\ .
\end{aligned}
$$

Note that $q > p'(n-1)/n$ implies $q' < p(n-1)/(n-p)$. Thus, applying Proposition 3.2, we have by (3.3)

$$
\begin{aligned}
\|T_\Phi^* f\|_{L^p} &\le \left(\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}}|K(x,\theta)|^q d\sigma(\theta)\right)^{p/q}\left(\int_{S^{n-1}}|H_\varphi^* f(x,\theta)|^{q'}d\sigma(\theta)\right)^{p/q'}dx\right)^{1/p} \\
&\le C_1\|H_\varphi^* f(x,\theta)\|_{L^p(L^{q'})} \le C\|f\|_{L^p}.
\end{aligned}
$$

Combining this with (3.2), we obtain that

$$
\begin{aligned}
\|T_\Phi f\|_{L^p} &\le \left(\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}}|K(x,\theta)|^q d\sigma(\theta)\right)^{p/q}\left(\int_{S^{n-1}}|H_\varphi f(x,\theta)|^{q'}d\sigma(\theta)\right)^{p/q'}dx\right)^{1/p} \\
&\le C_1\|H_\varphi f(x,\theta)\|_{L^p(L^{q'})} \le C\|f\|_{L^p}.
\end{aligned}
$$

This completes the proof of Theorem 1.

**4. Proof of Theorem 2.** In this section, we shall prove Theorem 2. Let $\mathcal{H}_k$ and $D_k$ be the space of surface spherical harmonics of degree $k$ on $S^{n-1}$ and its dimension. By the same argument as in [4], one can reduce the proof of Theorem 2 to the case as follows:

$$f \in C_0^\infty(\boldsymbol{R}^n) \text{ and } K(x,y) = \sum_{k\geq 1}\sum_{j=1}^{D_k} a_{k,j}(x)\frac{Y_{kj}(y')}{|y|^n} \text{ is a finite sum,}$$

where $\{Y_{kj}\}$, $k \geq 1$, $j = 1, 2, ..., D_k$, denotes the complete system of normalized surface spherical harmonics. Set

$$a_k(x) = \left(\sum_{j=1}^{D_k} |a_{k,j}(x)|^2\right)^{1/2} \quad \text{and} \quad b_{k,j}(x) = \frac{a_{k,j}(x)}{a_k(x)}.$$

Then we get

$$\sum_{j=1}^{D_k} b_{k,j}^2(x) = 1 \quad \text{and} \quad K(x,y) = \sum_{k\geq 1} a_k(x)\sum_{j=1}^{D_k} b_{k,j}(x)\frac{Y_{kj}(y')}{|y|^n}.$$

Note that if $2(n-1)/n < q_0 < 2$ and we take $0 < \delta < 1$ sufficiently close to 1, i.e., $(2/q_0 - 1)(n-1) < \delta < 1$, then by [7, p. 231] we have

$$\left(\sum_{k\geq 1} k^{-\delta} a_k^2(x)\right)^{1/2} \leq C\left(\int_{S^{n-1}} |K(x,y')|^{q_0}\, d\sigma(y')\right)^{1/q_0} = C.$$

If $q_0 = 2$, this inequality becomes trivial for $\delta = 0$. Moreover, let

$$\begin{aligned} T^j_{\Phi,k}(f)(x) &= \text{p.v.}\int_{\boldsymbol{R}^n} \frac{Y_{kj}(y')}{|y|^n} f(x - \Phi(|y|)y')dy \\ &= \text{p.v.}\int_{\boldsymbol{R}^n} \varphi(\Phi^{-1}(|y|))\frac{Y_{kj}(y')}{|y|^n} f(x-y)dy. \end{aligned}$$

Then

$$\begin{aligned} (T_\Phi(f)(x))^2 &= \left(\sum_{k\geq 1} a_k(x)\sum_{j=1}^{D_k} b_{k,j}(x) T^j_{\Phi,k}(f)(x)\right)^2 \\ &\leq \left(\sum_{k\geq 1} k^{-\delta}a_k^2(x)\right)\left(\sum_{k\geq 1} k^{\delta}\sum_{j=1}^{D_k}[T^j_{\Phi,k}(f)(x)]^2\right) \qquad (4.1) \\ &\leq C\left(\sum_{k\geq 1} k^{\delta}\sum_{j=1}^{D_k}[T^j_{\Phi,k}(f)(x)]^2\right). \end{aligned}$$

To prove Theorem 2 we prepare the following three lemmas:

LEMMA 4.1. *Let $0 < \eta < 1/2$. Then, for $T_{kj} = T^j_{\Phi,k}$, $k \in \boldsymbol{N}$, $j = 1, 2, \dots, D_k$, there exists $C > 0$ such that*

$$\|T_{kj} f\|_{L^2} \leq C\frac{1}{k^{n/2-\eta}}\|\hat{f}(\xi)Y_{kj}(\xi')\|_{L^2},$$

$$\|T^{**}_{kj} f\|_{L^2} \leq C\frac{1}{k^{n/2-\eta}}\|\hat{f}(\xi)Y_{kj}(\xi')\|_{L^2},$$

*where*

$$T_{kj}^{**}f(x)=\sup_{l\in\mathbf{Z}}\left|\int_{|y|>2^l}\varphi(\Phi^{-1}(|y|))\frac{Y_{kj}(y')}{|y|^n}f(x-y)dy\right|,$$

*and $C$ is independent of $j$, $k$ and $f$.*

PROOF. Setting

$$\sigma_l(x)=\varphi(\Phi^{-1}(|y|))\frac{Y_{km}(y')}{|y|^n}\chi_{[2^l,2^{l+1})}(|y|),\quad l\in\mathbf{Z}\,,k\in\mathbf{N},\ m=1,2,\dots,D_k\,,$$

we have

$$\begin{aligned}|\hat\sigma_l(\xi)|&=\left|\int\varphi(\Phi^{-1}(|y|))\frac{Y_{km}(y')}{|y|^n}\chi_{[2^l,2^{l+1})}(|y|)e^{-iy\cdot\xi}\,dy\right|\\&=C\left|\int_{2^l|\xi|}^{2^{l+1}|\xi|}\varphi\left(\Phi^{-1}\left(\frac{s}{|\xi|}\right)\right)\frac{J_{n/2+k-1}(s)}{s^{n/2}}\,ds\,Y_{km}(\xi')\right|\\&\le C\left|\int_{2^l|\xi|}^{2^{l+1}|\xi|}\varphi\left(\Phi^{-1}\left(\frac{s}{|\xi|}\right)\right)\frac{s^\eta J_{n/2+k-1}(s)}{s^{n/2+\eta}}\,ds\,Y_{km}(\xi')\right|\\&\le C\frac{(2^{l+1}|\xi|)^\eta}{(n/2+k-1)^{n/2+\eta}}|Y_{km}(\xi')|\le C\frac{(2^{l+1}|\xi|)^\eta}{k^{n/2+\eta}}|Y_{km}(\xi')|\ .\end{aligned}\tag{4.2}$$

Here in the last step we first used the second mean value theorem, then used Lemma 2.7;

$$\left|\int_{h_1}^{h_2}\varphi\left(\Phi^{-1}\left(\frac{s}{|\xi|}\right)\right)\frac{J_{n/2+k-1}(s)}{s^{n/2+\eta}}\,ds\right|\le\frac{C}{(n/2+k-1)^{n/2+\eta}}\,.$$

Similarly, we have

$$|\hat\sigma_l(\xi)|\le C\frac{(2^l|\xi|)^{-\eta}}{(n/2+k-1)^{n/2-\eta}}|Y_{km}(\xi')|\le C\frac{(2^l|\xi|)^{-\eta}}{k^{n/2-\eta}}|Y_{km}(\xi')|\,.\tag{4.3}$$

From (4.2) and (4.3) we have for the kernel function $K_{km}(x)=\varphi(\Phi^{-1}(|x|))(Y_{km}(x')/|x|^n)$

$$|\hat K_{km}(\xi)|\le\left(\sum_{2^l|\xi|\le1}|\hat\sigma_l(\xi)|+\sum_{2^l|\xi|>1}|\hat\sigma_l(\xi)|\right)\le\frac{C}{k^{n/2-\eta}}|Y_{km}(\xi')|\,.$$

Hence, by Plancherel's theorem, we obtain

$$\|T_{km}f\|_2\le C\frac{1}{k^{n/2-\eta}}\|\hat f(\xi)Y_{km}(\xi')\|_{L^2}\,.\tag{4.4}$$

Next, we treat the maximal operator $T_{km}^{**}f$. We denote $T_{km}$ by $T$. We modify the proof of Theorem E in [9]. We redefine $\psi$ as follows: Take $\psi\in\mathcal{S}(\mathbf{R}^n)$ such that $\psi(\xi)=1$ when $|\xi|<1$, $\psi(\xi)=0$ when $|\xi|\ge2$, and $0\le\psi(\xi)\le1$ when $\xi\in\mathbf{R}^n$; write $\psi_l(\xi)=\psi(2^l\xi)$ and $\hat\Psi_l(\xi)=\psi(2^l\xi)$. Setting

$$\sigma_l(x)=\varphi(\Phi^{-1}(|y|))\frac{Y_{km}(y')}{|y|^n}\chi_{[2^l,2^{l+1})}(|y|)$$

and

$$T_l f = \int_{2^l<|y|} \varphi(\Phi^{-1}(|y|)) \frac{Y_{km}(y')}{|y|^n} f(x-y)\,dy\,,$$

we decompose $T_l f = \sum_{j=l}^{\infty} \sigma_j * f(x)$ as

$$\begin{aligned} T_l f &= \Psi_l * \left( Tf - \sum_{j=-\infty}^{l-1} \sigma_j * f(x) \right) + (\delta - \Psi_l) * \sum_{j=l}^{\infty} \sigma_j * f(x) \\ &= \Psi_l * Tf - \Psi_l * \left( \sum_{j=-\infty}^{l-1} \sigma_j * f(x) \right) + (\delta - \Psi_l) * \sum_{j=l}^{\infty} \sigma_j * f(x)\,, \end{aligned}$$

where $\delta$ is the delta function at the origin. Set

$$Uf(x) = \sup_l \left| \Psi_l * \sum_{j=-\infty}^{l-1} \sigma_j * f(x) \right| \quad \text{and} \quad Vf(x) = \sup_l \left| (\delta - \Psi_l) * \sum_{j=l}^{\infty} \sigma_j * f(x) \right|.$$

Then we have

$$T_{km}^{**} f(x) \le \sup_l |\Psi_l * Tf(x)| + Uf(x) + Vf(x)\,. \tag{4.5}$$

We see

$$|\Psi_l * Tf(x)| \le C(Tf)^*(x) \ \text{ and so } \ \left\| \sup_l |\Psi_l * Tf(x)| \right\|_{L^p} \le C_p \|Tf(x)\|_{L^p}\,,$$

where $(Tf)^*$ is the Hardy-Littlewood maximal function of $Tf$, and hence by (4.4) we get

$$\left\| \sup_l |\Psi_l * Tf(x)| \right\|_{L^2} \le C_2 \frac{1}{k^{n/2-\eta}} \|\hat{f}(\xi) Y_{km}(\xi')\|_{L^2}\,. \tag{4.6}$$

Also, we have

$$\begin{aligned} Uf(x) &= \sup_l \left| \Psi_l * \sum_{j=-\infty}^{0} \sigma_{j+l-1} * f(x) \right| \\ &\le \sum_{j=-\infty}^{0} \sup_l \left| \Psi_l * \sigma_{j+l-1} * f(x) \right| =: \sum_{j=-\infty}^{0} U_j f(x)\,. \end{aligned}$$

Since $U_j f(x) \le (\sum_{l\in\mathbf{Z}} |\Psi_l * \sigma_{j+l-1} * f(x)|^2)^{1/2}$, we have

$$\begin{aligned} \int |U_j f(x)|^2\,dx &\le \sum_{l\in\mathbf{Z}} \int |\Psi_l * \sigma_{j+l-1} * f(x)|^2\,dx \\ &= C \sum_{l\in\mathbf{Z}} \int |\psi(2^l\xi) \hat{\sigma}_{j+l-1}(\xi) \hat{f}(\xi)|^2\,d\xi\,. \end{aligned}$$

Set $Y(\xi') = 1/k^{n/2-\eta} |Y_{km}(\xi')|$. Then, using (4.2) and the property of $\psi$, we get $|\psi(2^l\xi)\hat{\sigma}_{j+l-1}(\xi)| = 0\ (|2^l\xi| \ge 2)$ and $\le CY(\xi')|2^{j+l}\xi|^\eta \le CY(\xi')2^{\eta j}|2^l\xi|^\eta\ (|2^l\xi| \le 2)\,.$

Also, if $2^{-m-1} \le |\xi| < 2^{-m}$, we have $|2^l\xi| \ge 2$ for $l \ge m+2$, and hence $\psi(2^l\xi)\hat{\sigma}_{j+l-1}(\xi) = 0$ for $l \ge m+2$. Thus, if $2^{-m-1} \le |\xi| < 2^{-m}$, we get

$$\begin{aligned}\sum_{l\in\mathbf{Z}} |\psi(2^l\xi)\hat{\sigma}_{j+l-1}(\xi)|^2 &\le \sum_{l\le m+1} C^2Y(\xi')^2 2^{2j\eta}|2^l\xi|^{2\eta} \\ &= C^2Y(\xi')^2 2^{2j\eta}\sum_{l\le m+1} 2^{2(l-m)\eta}|2^m\xi|^{2\eta} \\ &\le \frac{C^2 2^{4\eta}}{2^{2\eta}-1}Y(\xi')^2 2^{2j\eta}\,.\end{aligned}$$

This implies $\|U_j f\|_2 \le C2^{j\eta}\|\hat{f}(\xi)Y(\xi')\|_2$. Thus we obtain

$$\|Uf\|_2 \le C\frac{1}{k^{n/2-\eta}}\|\hat{f}(\xi)Y_{km}(\xi')\|_{L^2}\,. \tag{4.7}$$

As for $Vf$, we get

$$\begin{aligned}Vf(x) &= \sup_l\left|(\delta-\Psi_l) * \sum_{j=0}^{\infty}\sigma_{j+l} * f(x)\right| \\ &\le \sum_{j=0}^{\infty}\sup_l|(\delta-\Psi_l) * \sigma_{j+l} * f(x)| =: \sum_{j=0}^{\infty} V_j f(x)\,.\end{aligned}$$

Since $V_j f(x) \le (\sum_{l\in\mathbf{Z}}|(\delta-\Psi_l) * \sigma_{j+l} * f(x)|^2)^{1/2}$, we have

$$\begin{aligned}\int|V_j f(x)|^2\,dx &\le \sum_{l\in\mathbf{Z}}\int|(\delta-\Psi_l) * \sigma_{j+l} * f(x)|^2\,dx \\ &= C\sum_{l\in\mathbf{Z}}\int|(1-\psi(2^l\xi))\hat{\sigma}_{j+l}(\xi)\hat{f}(\xi)|^2\,d\xi\,.\end{aligned}$$

Using (4.3) and the property of $\psi$, we see that

$$|(1-\psi(2^l\xi))\hat{\sigma}_{j+l}(\xi)|\begin{cases} = 0\,, & |2^l\xi| \le 1\,, \\ \le CY(\xi')|2^{j+l}\xi|^{-\eta} \le CY(\xi')2^{-\eta j}|2^l\xi|^{-\eta}\,, & |2^l\xi| > 1\,.\end{cases}$$

Also, if $2^{-m-1} \le |\xi| < 2^{-m}$, we have $|2^l\xi| \le 2^l/2^m \le 1$ for $l \le m$, and hence $(1-\psi(2^l\xi))\hat{\sigma}_{j+l}(\xi) = 0$ for $l \le m$. Thus, if $2^{-m-1} \le |\xi| < 2^{-m}$, we get

$$\begin{aligned}\sum_{l\in\mathbf{Z}}|(1-\psi(2^l\xi))\hat{\sigma}_{j+l}(\xi)|^2 &\le \sum_{l\ge m+1} C^2Y(\xi')^2 2^{-2j\eta}|2^l\xi|^{-2\eta} \\ &\le C^2Y(\xi')^2 2^{-2j\eta}\sum_{l\ge m+1} 2^{-2(l-m)\eta}|2^m\xi|^{-2\eta} \\ &\le C^2Y(\xi')^2 2^{-2j\eta}\left(\frac{2^{2\eta}}{2^{2\eta}-1}\right).\end{aligned}$$

This implies that $\|V_j f\|_2 \le C2^{-j\eta}\|\hat{f}(\xi)Y(\xi')\|_2$. Thus, we obtain

$$\|Vf\|_2 \le C\frac{1}{k^{n/2-\eta}}\|\hat{f}(\xi)Y_{km}(\xi')\|_2\,. \tag{4.8}$$

By Estimates (4.5)–(4.8), we obtain the desired estimate for $T_{km}^{**}f$. □

LEMMA 4.2. *Let $1 < p_0 < \infty$. Suppose sublinear operators $T_l$, $l = 1, 2, \ldots, m$, satisfy*

$$\left(\int \sum_{l=1}^{m} |T_l f(x)|^2 \, dx\right)^{1/2} \le C_1 \|f\|_2 ,$$

$$\|T_l f\|_{p_0} \le C_2 \|f\|_{p_0} , \quad l = 1, 2, \ldots, m .$$

*Then, if $1 < p_0 < p \le 2$,*

$$\left\| \left( \sum_{l=1}^{m} |T_l f(x)|^2 \right)^{1/2} \right\|_p \le C_1^{\theta} C_2^{1-\theta} m^{(1-\theta)/p_0} \|f\|_p ,$$

*and if $2 \le p < p_0 < \infty$,*

$$\left\| \left( \sum_{l=1}^{m} |T_l f(x)|^2 \right)^{1/2} \right\|_p \le C_1^{\theta} C_2^{1-\theta} m^{(1-\theta)/2} \|f\|_p ,$$

*where $1/p = \theta/2 + (1-\theta)/p_0$.*

PROOF. If $1 < p_0 < p \le 2$, using $p_0/2 < 1$, we have

$$\begin{aligned} \left(\int \left(\sum_{l=1}^{m} |T_l f(x)|^2\right)^{p_0/2} dx\right)^{1/p_0} &\le \left(\int \sum_{l=1}^{m} |T_l f(x)|^{p_0} \, dx\right)^{1/p_0} \\ &\le (m C_2^{p_0} \|f\|_{p_0}^{p_0})^{1/p_0} = m^{1/p_0} C_2 \|f\|_{p_0} . \end{aligned}$$

Hence, interpolating between 2 and $p_0$, we get

$$\left\| \left( \sum_{l=1}^{m} |T_l f(x)|^2 \right)^{1/2} \right\|_p \le C_1^{\theta} C_2^{1-\theta} m^{(1-\theta)/p_0} \|f\|_p .$$

If $2 \le p < p_0 < \infty$, we have by Mikowski's inequality

$$\begin{aligned} \left(\int \left(\sum_{l=1}^{m} |T_l f(x)|^2\right)^{p_0/2} dx\right)^{1/p_0} &\le \left(\sum_{l=1}^{m} \left(\int |T_l f(x)|^{p_0} \, dx\right)^{2/p_0}\right)^{1/2} \\ &\le (m C_2^{2} \|f\|_{p_0}^{2})^{1/2} = m^{1/2} C_2 \|f\|_{p_0} , \end{aligned}$$

and hence interpolation of sublinear operators between 2 and $p_0$ gives

$$\left\| \left( \sum_{l=1}^{m} |T_l f(x)|^2 \right)^{1/2} \right\|_p \le C_1^{\theta} C_2^{1-\theta} m^{(1-\theta)/2} \|f\|_p . \qquad \square$$

LEMMA 4.3. *Let $1 < p < \infty$. For $T_{kj} f(x) = \text{p.v.} \int \varphi(\Phi^{-1}(|y|))(Y_{kj}(y')/|y|^n) \, f(x-y) dy$ it follows that*

$$\|T_{kj} f\|_{L^p} \le C_p \|f\|_{L^p} ,$$

$$\|T_{kj}^{*} f\|_{L^p} \le C_p \|f\|_{L^p} .$$

PROOF. Noting $\|Y_{kj}\|_{L^2(S^{n-1})} = 1$, by Corollary 4.1 in [9], we have the desired conclusion. $\square$

Now, we are in a position to prove Theorem 2. We first prove the $L^p$ boundedness of $T_\Phi$, using Equation (4.1).

To prove our theorem, we consider two cases:

(i) The case $2 - (1 - (2/\bar{q} - 1)(n-1))/(n - (2/\bar{q} - 1)(n-1)/2) = (2n-1)/(n - (2/\bar{q} - 1)(n-1)/2) < p \le 2$. We first take $\delta > 0$ so that $(2/\bar{q} - 1)(n-1) < \delta < 1$ and $p > (2n-1)/(n - \delta/2)$, and then choose $p_0$ sufficiently near 1 and $\eta$ sufficiently near 0 so that $np\theta/2 - p\delta/2 - n + 2 - p\eta\theta = p(n - \delta/2) - 2n + 2 - np(1-\theta)(1 - 1/p_0) - p\eta\theta > 1$, where $1/p = \theta/2 + (1-\theta)/p_0$.

In this case, using Estimate (4.1), and noting $0 < p/2 < 1$, by Lemmas 4.1, 4.2 and 4.3, together with interpolation theorems, we have

$$\begin{aligned}\Big(\int |T_\Phi f(x)|^p\,dx\Big)^{1/p} &\le C\Big(\int \sum_{k=1}^{\infty} k^{p\delta/2}\Big(\sum_{j=1}^{D_k} |T_{kj} f(x)|^2\Big)^{p/2} dx\Big)^{1/p} \\ &\le C\Big(\sum_{k=1}^{\infty} k^{p\delta/2}\frac{D_k}{k^{(n/2-\eta)\theta p}}\Big)^{1/p}\|f\|_p \\ &\le C\Big(\sum_{k=1}^{\infty} k^{p\delta/2 - np\theta/2 + n - 2 + p\eta\theta}\Big)^{1/p}\|f\|_p \le C\|f\|_p\,.\end{aligned}$$

In the above, we have used the fact that $\sum_{j=1}^{D_k} |Y_{k,j}(\xi')|^2 = w^{-1}D_k \sim k^{n-2}$ (see [7, p. 255, (2.6)]), where $w$ denotes the area of $S^{n-1}$.

(ii) The case $2 < p < 2 + (\bar{q}n/(n-1) - 2) = \bar{q}n/(n-1)$. We first take $\delta > 0$ so that $(2/\bar{q} - 1)(n-1) < \delta < 1$ and $2n/p > n - 1 + \delta > n - 1 + (2/\bar{q} - 1)(n-1)$, and then choose $p_0$ sufficiently big and $\eta$ sufficiently near 0 so that $n\theta - \delta - n + 2 - 2\eta\theta = 2n/p - \delta - n + 2 + 2n(1-\theta)/p_0 - 2\eta\theta > 1$, where $1/p = \theta/2 + (1-\theta)/p_0$.

Since

$$|T_\Phi f(x)| \le C\Big(\sum_{k=1}^{\infty} k^{\delta} \sum_{j=1}^{D_k} |T_{kj} f(x)|^2\Big)^{1/2},$$

we have by Mikowski's inequality and Lemmas 4.1, 4.2 and 4.3,

$$\begin{aligned}\Big(\int |T_\Phi f(x)|^p\,dx\Big)^{1/p} &\le C\Big(\sum_{k=1}^{\infty} k^{\delta}\Big(\int\Big(\sum_{j=1}^{D_k} |T_{kj} f(x)|^2\Big)^{p/2} dx\Big)^{2/p}\Big)^{1/2} \\ &\le C\Big(\sum_{k=1}^{\infty} k^{\delta}\frac{D_k}{k^{2(n/2-\eta)\theta}}\Big)^{1/p}\|f\|_p \le C\Big(\sum_{k=1}^{\infty} k^{\delta - n\theta + n - 2 + 2\eta\theta}\Big)\|f\|_p \\ &\le C\|f\|_p\,.\end{aligned}$$

Thus we have proved the $L^p$ boundedness of $T_\Phi$.

Next, we shall prove the $L^p$ boundedness of $T_\Phi^*$. To prove it, we note that we have only to prove the $L^p$ boundedness of

$$T_\Phi^{**} f(x) = \sup_{k \in \boldsymbol{Z}} \left| \int_{|y|>2^k} \varphi(\Phi^{-1}(|y|)) K(x, y) f(x-y) dy \right| .$$

In fact, we see

$$\begin{aligned} \int_{2^l<|y|<2^{l+1}} \frac{|K(x, y')|}{|y|^n} |f(x-y)| dy &\le \int_{S^{n-1}} |K(x, y')| \int_{2^l}^{2^{l+1}} \frac{|f(x-ry')|}{r} dr d\sigma(y') \\ &\le 4 \int_{S^{n-1}} |K(x, y')| M_1 f(x, y') d\sigma(y') \le 4 \|K(x, \cdot)\|_{L^q(S^{n-1})} \|M_1 f(x, \cdot)\|_{L^{q'}(S^{n-1})} \\ &\le C \|M_1 f(x, \cdot)\|_{L^{q'}(S^{n-1})} , \end{aligned}$$

where $M_1 f(x, y')$ is the directional Hardy-Littlewood maximal function. From $2-(1-(2/q-1)(n-1))/(n-(2/q-1)(n-1)/2) < p \le 2$ it follows that $q' < p(n-1)/(n-p+(n-1)(1-p/2)) < p(n-1)/(n-p)$, and from $2 < p < 2+(qn/(n-1)-2)$ it follows $q' < p(n-1)/(n-p+(p-2)) < p(n-1)/(n-p)$. By assumption we have $p \le \max\{(n+1)/2,\ 2\}$. Hence, by Lemma 3.2, we obtain

$$\left( \int_{\boldsymbol{R}^n} (\|M_1 f(x, \cdot)\|_{L^{q'}(S^{n-1})})^p dx \right)^{1/p} \le C \|f\|_p ,$$

and hence

$$\left\| \sup_{l \in \boldsymbol{Z}} \int_{2^l<|y|<2^{l+1}} \varphi(\Phi^{-1}(|y|)) \frac{|K(x, y')|}{|y|^n} |f(x-y)| dy \right\|_p \le C \|\varphi\|_\infty \|f\|_p .$$

This implies our claim. Since, as in (4.1), we have

$$|T_\Phi^{**} f(x)| \le C \left( \sum_{k=1}^{\infty} k^\delta \sum_{j=1}^{D_k} |T_{kj}^{**} f(x)|^2 \right)^{1/2} ,$$

the above proof for the $L^p$ boundedness of $T_\Phi$ also works in this case. This completes the proof of Theorem 2.

**5. Proof of Lemma 2.7.** In this section, we will prove the key estimate, Lemma 2.7. First, we note that since, as is well-known, $J_\nu(s) = O(s^\nu)$ $(s \to 0)$ and $J_\nu(s) = O(s^{-1/2})$ $(s \to +\infty)$, we have

$$\int_0^\infty \left| \frac{J_{n/2+k-1}(s)}{s^{n/2+\eta}} \varphi\left( \Phi^{-1}\left( \frac{s}{|\xi|} \right) \right) \right| ds < +\infty \qquad \text{for} \quad \frac{1}{2} - \frac{n}{2} < \eta < k .$$

So, we have only to show Lemma 2.7 for $0 < h_1 < h_2 < \infty$. We also give only the proof in the case that $\Phi$ is positive, since the case for negative $\Phi$ is similar.

5.1. Case (i): $\varphi$ is monotonic. We assume that $\Phi(t)$ and $\varphi(t)$ are increasing, since the proof is similar for other cases. In this case $\varphi(t)$ is nonnegative, and $\Phi^{-1}(t)$ is increasing, and so $\varphi(\Phi^{-1}(t/|\xi|))$ is increasing. Let $\nu = n/2 + k - 1$, and $0 < h_1 < h_2$. Then, by the second mean value theorem, there exists $h'$ with $h_1 \le h' \le h_2$ such that

$$\int_{h_1}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) d\rho = \varphi\Big(\Phi^{-1}\Big(\frac{h_2}{|\xi|}\Big)\Big) \int_{h'}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} d\rho\,.$$

Hence for $1 > \eta > 1/2 - n/2$, by Lemma 2.2 there exists $C > 0$ such that

$$I := \left|\int_{h_1}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) d\rho\right| \le \frac{C\|\varphi\|_\infty}{(n/2+k-1)^{n/2+\eta}}\,.$$

5.2. Case (ii): $t\varphi'(t)$ is bounded. For $0 < h_1 < h_2 \le \nu$, since $J_\nu(\rho) > 0$ for $0 < \rho \le \nu$, we have by Lemma 2.2

$$I = \left|\int_{h_1}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) d\rho\right| \le \frac{C\|\varphi\|_\infty}{(n/2+k-1)^{n/2+\eta}}\,. \tag{5.1}$$

In the case where $h_2 \ge \nu$, we obtain

$$\begin{aligned} I \le &\left|\int_{h_1}^{\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) d\rho\right| \\ &+ \left|\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) d\rho\right| =: I_1 + I_2\,. \end{aligned} \tag{5.2}$$

For $I_1$, we know by (5.1) that if $h_1 < \nu$, $I_1 \le C\|\varphi\|_\infty/(n/2+k-1)^{n/2+\eta}$, and if $h_1 > \nu$, it is contained in the case $I_2$, which we will deal next.

For $h_2 \ge \nu$, take $\varepsilon > 0$ such that $n/2 + \eta - \varepsilon > 0$. Then, integrating by parts, we have

$$\begin{aligned} I_2 = \Bigg|&\Big(\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-\varepsilon}} d\rho\Big) \frac{\varphi(\Phi^{-1}(h_2/|\xi|))}{h_2{}^{\varepsilon}} - \int_{\nu}^{h_2} \Big(\int_{\nu}^{\rho} \frac{J_{n/2+k-1}(u)}{u^{n/2+\eta-\varepsilon}} du\Big) \\ &\times \Big\{\varphi'(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) \frac{\Phi^{-1}(\rho/|\xi|)}{\rho^{1+\varepsilon}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big) - \varepsilon \frac{1}{\rho^{1+\varepsilon}} \varphi\Big(\Phi^{-1}\Big(\frac{\rho}{|\xi|}\Big)\Big)\Big\} d\rho\Bigg|\,. \end{aligned}$$

Hence, by Lemma 2.2 we have

$$\begin{aligned} I_2 &\le \frac{C}{(n/2+k-1)^{n/2+\eta-\varepsilon}} \Big(\frac{\|\varphi\|_\infty}{\nu^\varepsilon} + \int_\nu^{h_2} \frac{d\rho}{\rho^{1+\varepsilon}} \{\|\varphi\|_\infty + \|t\varphi'(t)\|_\infty + \varepsilon\|\varphi\|_\infty\}\Big) \\ &\le \frac{C}{(n/2+k-1)^{n/2+\eta}} \{\|\varphi\|_\infty + \|t\varphi'(t)\|_\infty + \varepsilon\|\varphi\|_\infty\}\,. \end{aligned} \tag{5.3}$$

Thus, by (5.1), (5.2) and (5.3), we get

$$I \le \frac{C}{(n/2+k-1)^{n/2+\eta}} \{\|\varphi\|_\infty + \|t\varphi'(t)\|_\infty + \varepsilon\|\varphi\|_\infty\}\,.$$

5.3. Case (iii): $\Phi'(t)$ is monotonic on $(0, \infty)$. Since we have assumed $\Phi$ is positive, we only need to consider two cases, i.e., $\Phi$ is a positive and increasing function and $\Phi$ is a positive and decreasing function.

Case A. $\Phi$ is a positive and increasing function.

As in Case (ii), we have only to estimate

$$I_2 = \left|\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho\right| \tag{5.4}$$

for $h_2 > \nu$. For $I_2$, we consider the following two cases (A1) and (A2).

(A1) The case where $\Phi(t)$ is positive and increasing, and $\Phi'(t)$ is increasing.

Since $\Phi^{-1}(\rho/|\xi|)$ and $\Phi'(\Phi^{-1}(\rho/|\xi|))$ are positive and increasing, by Lemma 2.2 and the second mean value theorem we get for some $\nu \le h \le h_2$

$$\begin{aligned}
&\left|\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho\right| \\
&= \left|\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}} \frac{1}{\Phi^{-1}(\rho/|\xi|)\Phi'(\Phi^{-1}(\rho/|\xi|))} \frac{d\rho}{|\xi|}\right| \\
&= \left|\int_{\nu}^{h} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}} \frac{d\rho}{|\xi|} \frac{1}{\Phi^{-1}(\nu/|\xi|)\Phi'(\Phi^{-1}(\nu/|\xi|))}\right| \\
&= \left|\int_{\nu}^{h} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}} d\rho\right| \frac{\nu}{|\xi|\Phi^{-1}(\nu/|\xi|)\Phi'(\Phi^{-1}(\nu/|\xi|))} \times \frac{1}{\nu} \\
&\le \frac{C}{\nu^{n/2+\eta-1}} \|\varphi\|_\infty \times \frac{1}{\nu} = \frac{C}{(n/2+k-1)^{n/2+\eta}} \|\varphi\|_\infty .
\end{aligned} \tag{5.5}$$

(A2) The case where $\Phi(t)$ is positive and increasing, and $\Phi'(t)$ is decreasing.

As before, we have only to estimate

$$I_2 = \left|\int_{\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho\right|$$

for $h_2 > \nu$. In this case we proceed as follows.

We see $\Phi^{-1}(t)$ is increasing, and hence $\Phi'(t)$ is decreasing implies that $\Phi'(\Phi^{-1}(\rho))$ is positive and decreasing. Thus, we know that $1/\Phi'(\Phi^{-1}(\rho))$ is a positive and increasing function. Now, for any $h_2 > \nu$, it is easy to see that there exists a $j_0 \in \boldsymbol{N}$ such that $2^{j_0} \le h_2/\nu < 2^{j_0+1}$, and so we have

$$\begin{aligned}
I_2 \le \sum_{l=0}^{j_0} &\left|\int_{2^l\nu}^{2^{l+1}\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho\right| \\
&+ \left|\int_{2^{j_0}\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho\right| .
\end{aligned} \tag{5.6}$$

Noting the positivity and the monotonicity of $1/\Phi^{-1}(\rho)$ and $1/\Phi'(\Phi^{-1}(\rho))$, and using the second mean value theorem twice, we get

$$\begin{aligned}&\left|\int_{2^l\nu}^{2^{l+1}\nu}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}}\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right|\\&\quad=\left|\int_{2^l\nu}^{2^{l+1}\nu}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}}\frac{1}{\Phi^{-1}(\rho/|\xi|)\Phi'(\Phi^{-1}(\rho/|\xi|))}\frac{d\rho}{|\xi|}\right|\\&\quad=\frac{1}{|\Phi^{-1}(2^l\nu/|\xi|)\Phi'(\Phi^{-1}(2^{l+1}\nu/|\xi|))|}\frac{1}{|\xi|}\left|\int_{\eta'}^{\eta''}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}}d\rho\right|,\end{aligned}$$

where $\eta', \eta'' \in [2^l\nu, 2^{l+1}\nu]$. Hence, by Lemma 2.2 and Lemma 2.5, we get

$$\begin{aligned}&\left|\int_{2^l\nu}^{2^{l+1}\nu}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}}\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right|\\&\quad\le\left|\frac{\Phi^{-1}(2^{l+1}\nu/|\xi|)}{\Phi^{-1}(2^l\nu/|\xi|)}\frac{2^{l+1}\nu/|\xi|}{\Phi^{-1}(2^{l+1}\nu/|\xi|)\Phi'(\Phi^{-1}(2^{l+1}\nu/|\xi|))}\right|\frac{1}{2^{l+1}\nu}\frac{C}{\nu^{n/2+\eta-1}}\\&\quad\le\frac{C\|\varphi\|_\infty}{2^l\nu^{n/2+\eta}}\le\frac{C}{2^l\nu^{n/2+\eta}}\ .\end{aligned}\tag{5.7}$$

Since $1/\Phi'(\Phi^{-1}(\rho))$ is positive and increasing, we have as above for some $\eta', \eta'' \in [2^{j_0}\nu, h_2]$

$$\begin{aligned}&\left|\int_{2^{j_0}\nu}^{h_2}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}}\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right|\\&\quad=\frac{1}{|\Phi^{-1}(2^{j_0}\nu/|\xi|)\Phi'(\Phi^{-1}(h_2/|\xi|))|}\frac{1}{|\xi|}\left|\int_{\eta'}^{\eta''}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}}d\rho\right|\\&\quad\le C\frac{\Phi^{-1}(2^{j_0+1}\nu/|\xi|)}{\Phi^{-1}(2^{j_0}\nu/|\xi|)}\frac{\|\varphi\|_\infty}{2^{j_0+1}\nu^{n/2+\eta}}\le\frac{C}{2^{j_0}\nu^{n/2+\eta}}\ .\end{aligned}\tag{5.8}$$

Hence, by (5.6), (5.7) and (5.8), we get

$$I_2\le\sum_{l=0}^{j_0}\frac{C}{2^l\nu^{n/2+\eta}}+\frac{C}{2^l\nu^{n/2+\eta}}\le\frac{C}{(n/2+k-1)^{n/2+\eta}}\,.$$

Thus, we have proved that

$$I=\left|\int_{h_1}^{h_2}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2}}\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right|\le\frac{C}{(n/2+k-1)^{n/2+\eta}}\,.$$

Case B. $\Phi$ is a positive and decreasing function.

In this case, since $\Phi'(t)$ is monotonic and $\Phi(t)$ is decreasing, we see that only the case that $\Phi'(t)$ is increasing occurs. Otherwise, $\Phi(t)$ is concave and its graph traverses the $x$-axis. As before, we have only to estimate

$$I_2=\left|\int_{\nu}^{h_2}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}}\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right|$$

for $h_2 > \nu$. In this case we proceed as follows.

We see $\Phi^{-1}(t)$ is decreasing, and hence $\Phi'(t)$ is increasing implies that $-\Phi'(\Phi^{-1}(\rho/|\xi|))$ is positive and increasing. Thus, we know that $-1/\Phi'(\Phi^{-1}(\rho/|\xi|))$ is a positive and decreasing function. Now, for any $h_2 > \nu$, it is easy to see that there exists a $j_0 \in \boldsymbol{N}$ such that $2^{j_0} \le h_2/\nu < 2^{j_0+1}$, and so we have

$$\begin{aligned} I_2 \le& \sum_{l=0}^{j_0} \left| \int_{2^l\nu}^{2^{l+1}\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| \\ &+ \left| \int_{2^{j_0}\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| . \end{aligned} \tag{5.9}$$

Noting the positivity and the monotonicity of $1/\Phi^{-1}(\rho)$ and $-1/\Phi'(\Phi^{-1}(\rho))$, and using the second mean value theorem twice, we get

$$\begin{aligned} &\left| \int_{2^l\nu}^{2^{l+1}\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| \\ &= \left| \int_{2^l\nu}^{2^{l+1}\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}} \frac{1}{\Phi^{-1}(\rho/|\xi|)\Phi'(\Phi^{-1}(\rho/|\xi|))} \frac{d\rho}{|\xi|} \right| \\ &= \frac{1}{\left|\Phi^{-1}(2^{l+1}\nu/|\xi|)\Phi'(\Phi^{-1}(2^l\nu/|\xi|))\right|} \frac{1}{|\xi|} \left| \int_{\eta'}^{\eta''} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta-1}} d\rho \right| , \end{aligned}$$

where $\eta', \eta'' \in [2^l\nu, 2^{l+1}\nu]$. Hence, by Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned} &\left| \int_{2^l\nu}^{2^{l+1}\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| \\ &\le \left| \frac{\Phi^{-1}(2^l\nu/|\xi|)}{\Phi^{-1}(2^{l+1}\nu/|\xi|)} \frac{2^l\nu/|\xi|}{\Phi^{-1}(2^l\nu/|\xi|)\Phi'(\Phi^{-1}(2^l\nu/|\xi|))} \right| \frac{1}{2^l\nu} \frac{C}{\nu^{n/2+\eta-1}} \\ &\le C\|\varphi\|_\infty \frac{1}{2^l\nu^{n/2+\eta}} \le C\frac{1}{2^l\nu^{n/2+\eta}} . \end{aligned} \tag{5.10}$$

Since $\Phi^{-1}(t)$ is positive and decreasing, we have as above

$$\begin{aligned} \left| \int_{2^{j_0}\nu}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\eta}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| &\le C\frac{\Phi^{-1}(2^{j_0}\nu/|\xi|)}{\Phi^{-1}(h_2/|\xi|)} \frac{1}{2^{j_0}\nu^{n/2+\eta}} \\ &\le C\frac{\Phi^{-1}(2^{j_0}\nu/|\xi|)}{\Phi^{-1}(2^{j_0+1}\nu/|\xi|)} \frac{1}{2^{j_0}\nu^{n/2+\eta}} \le \frac{C}{2^{j_0}\nu^{n/2+\eta}} . \end{aligned} \tag{5.11}$$

Hence, by (5.9), (5.10) and (5.11), we get

$$I_2 \le \sum_{l=0}^{j_0} C\frac{1}{2^l\nu^{n/2+\eta}} \le \frac{C}{(n/2+k-1)^{n/2+\eta}} .$$

Thus, we have proved that

$$I = \left| \int_{h_1}^{h_2} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2}} \varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right) d\rho \right| \le \frac{C}{(n/2+k-1)^{n/2+\eta}} , \ 0 \le h_1 < h_2 \le \infty . \ \square$$

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