TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 335, Number 2, February 1993

ON THE BRAUER GROUP OF TORIC VARIETIES

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ABSTRACT. We compute the cohomological Brauer group of a normal toric variety whose singular locus has codimension less than or equal to 2 everywhere.

Associated to each algebraic variety X is the cohomological Brauer group $\mathrm{B}'(X) = \mathrm{tors}(\mathrm{H}^2(X, \mathrm{G}_m))$ which is the torsion subgroup of the second étale cohomology group of X with coefficients in the sheaf of units. Except in the easiest cases, calculations of this group are scarce. Toric varieties over an algebraically closed field of characteristic 0 provide a nontrivial class of higher dimensional varieties for which calculations of $\mathrm{B}'(X)$ can sometimes be made. These calculations are the purpose of this article.

Each toric variety X is determined by a combinatorical object Δ in real affine space called a fan. Tied into the structure of the fan are arithmetic properties of sublattices of free $\mathbb{Z}$-lattices. Our arguments therefore ultimately reduce questions about $\mathrm{B}'(X)$ to calculations with integer matrices.

In §1 we determine the Brauer group $\mathrm{B}(X) = \mathrm{B}'(X)$ of any nonsingular toric variety X (Theorem 1.1). This group is a direct sum of finitely many copies of finite cyclic groups and copies of $\mathbb{Q}/\mathbb{Z}$. The algebras generating this group are given explicitly as smash products of cyclic Galois extensions of X. In §2 we consider toric varieties whose singular locus has codimension at most 2 everywhere in X. Let T_N denote the torus identified with an open subset of X, $\mathrm{B}'(T_N/X)$ the elements in $\mathrm{B}'(X)$ split by T_N, and $\widetilde{X}$ a T_N-invariant desingularization of X. In Theorem 2.2 we construct an exact sequence $0 \to \mathrm{B}'(T_N/X) \to \mathrm{B}'(X) \to \mathrm{B}'(\widetilde{X}) \to 0$ which reduces the calculation of $\mathrm{B}'(X)$ to the calculation of $\mathrm{B}'(T_N/X)$. The hypotheses on X in §2 imply we can assume the associated fan Δ contains cones of dimension at most 2. Corresponding to each cone τ_i of dimension 2 is an irreducible closed subvariety $V_i = \mathrm{orb}\,\tau_i$ and an affine neighborhood U_{τ_i} of V_i which has a finite cyclic divisor class group $\mathrm{Cl}(U_{\tau_i})$. If Δ has 2-dimensional cones $\tau_1, \dots, \tau_m$, we construct an exact sequence (Theorem 2.3)

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{i=1}^{m} \mathrm{Cl}(U_{\tau_i}) \to \mathrm{B}'(T_N/X) \to 0.$$

Received by the editors May 1, 1990 and, in revised form, October 15, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13A20, 16A16; Secondary 14L32, 14F20.
The second author was supported in part by the NSF under grant DMS-8822944.

For each prime number p we find a subset $\tau_1, \dots, \tau_e$ (after a suitable relabelling) of $\{\tau_1, \dots, \tau_m\}$ such that $[\bigoplus_{i=1}^{e} \mathrm{Cl}(U_{\tau_i})]_p \cong [\mathrm{B}'(T_N/X)]_p$. We calculate the Brauer group of any toric surface (Corollary 2.9). In this case $\mathrm{B}'(T_N/X)$ is nontrivial when there is a cycle of divisors on X whose pairwise consecutive intersections are singular points on X whose local rings all have divisor class groups of order divisible by a common prime p. An analogous statement holds for X of higher dimension. We employ terminology and notation of [12] for toric varieties and [11] for étale cohomology.

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Following the notation terminology of [12] let $r > 0$ be an integer and $N = \mathbb{Z}^r$ a free abelian group of rank $= r$. Let Δ be a finite fan on $N_{\mathbb{R}}$ and $X = T_N \operatorname{emb}(\Delta)$ the associated toric variety containing the r-dimensional torus T_N as an open subset defined over the algebraically closed field k of characteristic 0. Let N' be the subgroup of N generated by $\bigcup_{\sigma \in \Delta} \sigma \cap N$. The basis theorem for finitely generated abelian groups gives a basis $n_1, \dots, n_r$ of N such that $N' = \mathbb{Z}a_1 n_1 \oplus \mathbb{Z}a_2 n_2 \oplus \cdots \oplus \mathbb{Z}a_r n_r$ where the a_i are nonnegative integers and $a_i | a_{i+1}$ for $1 \le i \le r-1$. Call $\{a_1, \dots, a_r\}$ the *set of invariant factors of* Δ (or $X = T_N \operatorname{emb}(\Delta)$). Let $\mathrm{B}(X)$ denote the Brauer group of Azumaya algebras on X and $\mathrm{B}'(X)$ the torsion subgroup of $\mathrm{H}^2(X, \mathrm{G}_m)$ the cohomological Brauer group of X. Our principal result of §1 is

Theorem 1.1. *If $X = T_N \operatorname{emb}(\Delta)$ is nonsingular and $a_1, \dots, a_r$ is the set of invariant factors of X, then*

$$\mathrm{B}(X) = \mathrm{B}'(X) \cong \bigoplus_{i=1}^{r-1} \operatorname{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}.$$

We list two special cases of Theorem 1.1.

Corollary 1.2. *If Δ contains a cone σ such that* $\dim \sigma \ge r-1$, *then* $\mathrm{B}(X) = (0)$.

Proof. Since X is nonsingular, [12, Theorem 1.10] implies there is a basis $n_1, \dots, n_r$ of N such that $\mathbb{R}_{\ge 0} n_1 + \cdots + \mathbb{R}_{\ge 0} n_{r-1} \subseteq \sigma$. Since each $n_i \in \sigma \cap N$, all of the invariant factors $a_i = 1$ for $1 \le i \le r-1$. So $\mathrm{B}(X) = (0)$ by Theorem 1.1. □

Corollary 1.3. $\mathrm{B}(X)$ *is finite if and only if* $\operatorname{Rank}_{\mathbb{Z}}(N') \ge r-1$. *In this case*

$$\mathrm{B}(X) \cong \bigoplus_{i=1}^{r-1} (\mathbb{Z}/a_i)^{r-i}.$$

The rest of this section is devoted to a proof of Theorem 1.1. From now on we assume $X = T_N \operatorname{emb}(\Delta)$ is nonsingular. Along the way we will obtain explicit information about the Azumaya algebras on X and show the Brauer group $\mathrm{B}(X)$ of Azumaya algebras on X is equal to the cohomological Brauer group $\mathrm{B}'(X) = \mathrm{H}^2(X, \mathrm{G}_m)$.

Let $\Gamma = \{0, \rho_1, \dots, \rho_n\}$ be the fan on $N_{\mathbb{R}}$ consisting of all cones in Δ of dimension ≤ 1 and let $U = T_N \operatorname{emb}(\Gamma)$. The open immersion $U \hookrightarrow X$ induces the isomorphisms of the next lemma.

Lemma 1.4. *For each positive integer ν,*

(a) $\mathrm{H}^1(X, \mathbb{Z}/\nu) \cong \mathrm{H}^1(U, \mathbb{Z}/\nu)$,

(b) $\mathrm{H}^2(X, \mathbb{Z}/\nu) \cong \mathrm{H}^2(U, \mathbb{Z}/\nu)$,
(c) $\mathrm{B}'(X) \cong \mathrm{B}'(U)$.

Proof. Let X, U be as above and let $Z = X - U$. Then part of the long exact sequence of cohomology with supports [11, Proposition 1.25] is

$$\mathrm{H}^1_Z(X, \mathbb{Z}/\nu) \to \mathrm{H}^1(X, \mathbb{Z}/\nu) \to \mathrm{H}^1(U, \mathbb{Z}/\nu) \to \mathrm{H}^2_Z(X, \mathbb{Z}/\nu)$$
$$\to \mathrm{H}^2(X, \mathbb{Z}/\nu) \to \mathrm{H}^2(U, \mathbb{Z}/\nu) \to \mathrm{H}^3_Z(X, \mathbb{Z}/\nu)$$

since the codimension of Z in X is ≥ 2, [11, Lemma 9.1, p. 268] implies $\mathrm{H}^s_Z(X, \mathbb{Z}/\nu) = (0)$ for $s < 4$, which proves (a) and (b) in our context. There is an exact sequence [3, Theorem 1.c]

$$0 \to \mathrm{H}^2(X, \mathrm{G}_m) \to \mathrm{H}^2(U, \mathrm{G}_m) \to \mathrm{H}^3_Z(X, \mu)$$

and $\mathrm{H}^3_Z(X, \mu) \cong \varinjlim \mathrm{H}^3_Z(X, \mathbb{Z}/\nu) = (0)$, which proves (c). □

Notice that $N' = \langle \bigcup_{\sigma\in\Delta} \sigma \cap N\rangle = \langle \bigcup_{i=1}^n \rho_i \cap N\rangle$ so as a consequence of Lemma 1.4(c) we can assume that $\Delta = \{0, \rho_1, \dots, \rho_n\}$ and $X = U$. We write $\rho_k = \mathbb{R}_{\geq 0}\eta_k$ where $\eta_k \in N$ and η_k is primitive (the GDC of the coordinates of η_k is 1). Let $n_1, \dots, n_r$ be a basis for N with $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ and $a_i | a_{i+1}$ for $1 \leq i \leq r$. ($\{a_1, \dots, a_r\}$ is the set of invariant factors of Δ.) Let $m_1, \dots, m_r$ be a dual basis for $M = \mathrm{Hom}(N, \mathbb{Z})$. Then $T_N = \operatorname{Spec} k[M]$. An element $\sum a_i m_i$ in M is usually identified with the Laurent monomial $x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}$ and $k[M]$ with $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$. Let ν be a positive integer and fix a primitive νth root of unity ζ. Given units α, β in $k[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$, the symbol algebra $(\alpha, \beta)_\nu$ is the associative k-algebra generated by elements u, v subject to the relations $u^\nu = \alpha$, $v^\nu = \beta$; and $uv = \zeta vu$. In what follows, we choose to identify $(x_i, x_j)_\nu$ as $(m_i, m_j)_\nu$ and work in $k(M)$. By [10, Theorem 6], ${}_\nu\mathrm{B}(T_N)$ is a free $\mathbb{Z}/\nu$-module with basis given by the set of symbol algebras $\{(m_i, m_j)_\nu\}_{1\leq i<j\leq r}$ for each $\nu \geq 2$. Since T_N is an open subset of X and X is nonsingular, $\mathrm{B}(X)$ is a subgroup of $\mathrm{B}(T_N)$ by restriction and our object is to identify this subgroup explicitly.

From [4, Corollary 1.4] there is an exact sequence

$$(1) \qquad 0 \to \mathrm{B}'(X) \to \mathrm{B}(T_N) \xrightarrow{a} \bigoplus_{i=1}^{n} \mathrm{H}^1(\operatorname{orb}\rho_i, \mathbb{Q}/\mathbb{Z})$$

where $\operatorname{orb}\rho_i$ is the T_N-invariant divisor on $X = T_N\operatorname{emb}(\Delta)$ corresponding to the face ρ_i of Δ. Given a symbol algebra $(\alpha, \beta)_\nu$ representing a class in $\mathrm{B}(T_N)$, the ramification map a agrees with the tame symbol (see the discussion following [4, Remark 1.7] and [14, Theorem 8, p. 155]). This means the kth coordinate of $a((\alpha, \beta)_\nu)$, the ramification of $(\alpha, \beta)_\nu$ along $\operatorname{orb}\rho_k$, is identified with a cyclic Galois extension of $\operatorname{orb}\rho_k$ of degree ν. Over the function field $K(\operatorname{orb}\rho_k)$ this extension is given by adjoining the νth root of $\alpha^{v_k(\beta)}/\beta^{v_k(\alpha)}$ where v_k is the valuation on $K(X)$ determined by the prime divisor $\operatorname{orb}\rho_k$.

From the remarks above, to determine the ramification of an arbitrary algebra Λ representing an element in $\mathrm{B}(T_N)$ along $\operatorname{orb}\rho_k$ it suffices to determine $K(\operatorname{orb}\rho_k)$ and $v_k(m_j)$ for each k, j. The following lemma is well known. We include its short proof for completeness and to fix notation.

Lemma 1.5. *Let* η_k *be the primitive vector in* $N \cap \rho_k$ *and* $\langle\ ,\ \rangle$ *the natural inner product from* $M \times N \to \mathbb{Z}$.

(a) $K(\operatorname{orb}\rho_k)$ *is the quotient field of* $k[\eta_k^{\perp}]$.

(b) $v_k(m) = \langle m, \eta_k\rangle$.

Proof. Since η_k is primitive there is a primitive $\mu_k \in M$ with $\langle \mu_k, \eta_k\rangle = 1$. Let $\eta_k^{\perp} = \{m \in M | \langle m, \eta_k\rangle = 0\}$. Then $M = \eta_k^{\perp} \oplus \mathbb{Z}\mu_k$ since

$$0 \to \eta_k^{\perp} \to M \xrightarrow{\varphi} \mathbb{Z} \to 0$$

splits, where $\varphi(m) = \langle m, \eta_k\rangle$. The affine coordinate ring of

$$U_{\rho_k} = T_N \operatorname{emb}\{0, \rho_k\}$$

is $k[\eta_k^{\perp}, \mu_k]$. Localizing X along $\operatorname{orb}\rho_k$ is equivalent to localizing U_{ρ_k} along $\operatorname{orb}\rho_k$. The prime ideal corresponding to $\operatorname{orb}\rho_k$ is the principal ideal in $k[\eta_k^{\perp}, \mu_k]$ generated by μ_k. Hence μ_k is a local parameter along $\operatorname{orb}\rho_k$. $K(\operatorname{orb}\rho_k)$ is the quotient field of $k[\eta_k^{\perp}, \mu_k]/(\mu_k)$ giving (a). The valuation v_k of any $m \in M$ is the μ_k-coordinate when m is written in terms of the decomposition $M = \eta_k^{\perp} \oplus \mathbb{Z}\mu_k$. Thus $v_k(m) = \langle m, \eta_k\rangle$. □

Keeping the notation above, define a homomorphism $\operatorname{ram}_{\operatorname{orb}\rho_k}\colon {}_{\nu}\mathrm{B}(T_N) \to M/\nu M$ by letting $\operatorname{ram}_{\operatorname{orb}\rho_k}(m_i, m_j)_{\nu} = \langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j + \nu M$ be the assignment on the basis for ${}_{\nu}\mathrm{B}(T_N)$, and extending by $\mathbb{Z}/\nu$-linearity.

Lemma 1.6. $(m_i, m_j)_{\nu}$ *is unramified along* $\operatorname{orb}\rho_k$ *if and only if*

$$\operatorname{ram}_{\operatorname{orb}\rho_k}(m_i, m_j)_{\nu} = 0.$$

Proof. The ramification of $(m_i, m_j)_{\nu}$ along $\operatorname{orb}\rho_k$ corresponds to the cyclic extension of the affine coordinate ring $k[\eta_k^{\perp}]$ of $\operatorname{orb}\rho_k$ obtained by adjoining the νth root of $v_k(m_j)m_i - v_k(m_i)m_j = \langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j$. (Note $\langle \langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j, \eta_k\rangle = 0$ so $\langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j \in \eta_k^{\perp}$.) Thus, $(m_i, m_j)_{\nu}$ is unramified along $\operatorname{orb}\rho_k$ if and only if $\langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j$ is a νth power in $k[\eta_k^{\perp}]$ if and only if $\langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j \in \nu M$ if and only if $\operatorname{ram}_{\operatorname{orb}\rho_k}(m_i, m_j)_{\nu} = 0$. □

Let Λ be any Azumaya algebra representing a class in ${}_{\nu}\mathrm{B}(T_N)$. We have seen Λ is equivalent to $\prod_{i<j}(m_i, m_j)_{\nu}^{e_{ij}}$ where $0 \le e_{ij} < \nu$. Moreover the class of Λ determines and is determined by the integers e_{ij}. Associate to the class represented by Λ in ${}_{\nu}\mathrm{B}(T_N)$ the matrix

$$M_{\Lambda} = \begin{bmatrix} 0 & e_{12} & e_{13} & \cdots & & e_{1r} \\ -e_{12} & 0 & e_{23} & \cdots & & e_{2r} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ -e_{1r} & -e_{2r} & & \cdots & -e_{r-1,r} & 0 \end{bmatrix}.$$

Lemma 1.7. (a) *The assignment* $\Lambda \to M_{\Lambda}$ *induces a monomorphism*

$$\phi\colon {}_{\nu}\mathrm{B}(T_N) \to \operatorname{Hom}_{\mathbb{Z}}(N, M/\nu M).$$

(b) Λ *is unramified along* $\operatorname{orb}\rho_k$ *if and only if* $M_{\Lambda} \cdot \eta_k = 0$.

Proof. (a) The matrix M_Λ defines the indicated homomorphism $\phi(\Lambda)$ by representing elements in N as column vectors with respect to the basis $n_1, \dots, n_r$; the elements in M as column vectors with respect to the dual basis $m_1, \dots, m_r$ and following left multiplication by M_Λ by reduction modulo νM. Since multiplication of symbols corresponds to addition of exponents modulo ν, it is clear that ϕ is a homomorphism. If $\phi(\Lambda) = M_\Lambda = 0$, then each $e_{ij} = 0$, so $\Lambda = 0$ in $\mathrm{B}(T_N)$. Thus ϕ is a monomorphism.

(b) Write $\eta_k = \sum_{i=1}^r \eta_{ki} n_i$ and let $\Lambda = \prod_{i<j}(m_i, m_j)_\nu^{e_{ij}}$. Then

$$\mathrm{ram}_{\mathrm{orb}\,\rho_k}(m_i, m_j)_\nu = \langle m_j, \eta_k\rangle m_i - \langle m_i, \eta_k\rangle m_j = \eta_{kj} m_i - \eta_{ki} m_j.$$

Hence

$$\begin{aligned}
\mathrm{ram}_{\mathrm{orb}\,\rho_k}(\Lambda) &= \mathrm{ram}_{\mathrm{orb}\,\rho_k}\left(\prod_{i<j}(m_i, m_j)_\nu^{e_{ij}}\right) \\
&= \sum_{i<j} e_{ij}\eta_{kj} m_i - e_{ij} m_j + \nu M \\
&= \sum_{i=1}^{r}\sum_{j=i+1}^{r} e_{ij}\eta_{kj} m_i - \sum_{i=1}^{r}\sum_{j=i+1}^{r} e_{ij}\eta_{ki} m_j + \nu M \\
&= \sum_{i=1}^{r}\sum_{j=i+1}^{r} e_{ij}\eta_{kj} m_i - \sum_{j=1}^{r}\sum_{i=j+1}^{r} e_{ji}\eta_{kj} m_i + \nu M \\
&= M_\Lambda \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix}.
\end{aligned}$$

As we observed in the proof of Lemma 1.6, $\mathrm{ram}_{\mathrm{orb}\,\rho_k}(\Lambda) = m + \nu M$ for some $m \in \eta_k^\perp$. The ramification of Λ along $\mathrm{orb}\,\rho_k$ is the cyclic extension of $k[\eta_k^\perp]$ obtained by adjoining the νth root of m and this extension is split (Λ is unramified along $\mathrm{orb}\,\rho_k$) if and only if $m \in \nu\eta_k^\perp$. Since $\eta_k^\perp$ is a direct summand of M, Λ is unramified along $\mathrm{orb}\,\rho_k$ if and only if $m \in \nu M$ if and only if

$$M_\Lambda \cdot \begin{bmatrix} \eta_{k1} \\ \vdots \\ \eta_{kr} \end{bmatrix} = 0. \quad \square$$

Theorem 1.8. *Let $X = T_N\,\mathrm{emb}(\Delta)$ be a nonsingular toric variety and $a_1, \dots, a_r$ the set of invariant factors of X. Then $\mathrm{B}'(X)$ is the subgroup of $\mathrm{B}(T_N)$ represented by algebra classes $\prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}}$ where $\nu_i | a_i$, $1 \le i \le r$.*

Proof. The exact sequence (1) and Lemma 1.7 imply ${}_\nu\mathrm{B}'(X)$ consists of those algebra classes Λ in ${}_\nu\mathrm{B}(T_N)$ such that $M_\Lambda \cdot \eta_k = 0$ for the primitive vector η_k on each 1-dimensional cone ρ_k in Δ $(1 \le k \le n)$. If N' is the subgroup of N generated by $\bigcup_{\sigma\in\Delta}\sigma \cap N$ then N' is generated by $\{\eta_k\}_{k=1}^n$ so Λ represents a class in ${}_\nu\mathrm{B}'(X)$ if and only if M_Λ vanishes on N'. For each $\nu > 0$ we have the commutative diagram with exact rows and columns

$$\begin{array}{ccccc} 0 & \longrightarrow & {}_\nu\mathrm{B}(T_N) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(N, M/\nu M) \\ & & \uparrow & & \uparrow \\ 0 & \longrightarrow & {}_\nu\mathrm{B}'(X) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(N/N', M/\nu M) \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array}$$

Taking the direct limit over all $\nu \geq 2$ gives a monomorphism

$$\mathrm{B}'(X) \to \mathrm{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}).$$

Let $n_1, \ldots, n_r$ be a basis for N such that $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ and $a_i|a_{i+1}$ for $1 \leq i \leq r-1$. That is $a_1, \ldots, a_r$ is the set of invariant factors for X. Then $\mathrm{Hom}_{\mathbb{Z}}(N/N', M \otimes \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\bigoplus(\mathbb{Z}n_i/\mathbb{Z}a_in_i), M \otimes \mathbb{Q}/\mathbb{Z})$. This means $\mathrm{B}'(X)$ is contained in the subgroup of $\mathrm{B}(T_N)$ of algebra classes $\prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}}$ where $0 < \nu_i$ and $\nu_i|a_i$, $1 \leq i \leq r$. Conversely, if $\nu_i|a_i$ and $\nu_i \geq 1$ then the matrix M_Λ for $(m_i, m_j)_{\nu_i}$ has a $+1$ in the ijth entry and a -1 in the jith entry. A typical element in N' is $x = \lambda_1a_1n_1 + \cdots + \lambda_ra_rn_r$ and $M_\Lambda \cdot x = \lambda_ja_jm_i - \lambda_ia_im_j \in a_iM$. Thus $(m_i, m_j)_{\nu_i}$ represents an element in $\mathrm{B}'(X)$. So $\mathrm{B}'(X) = \{\prod_{i<j}(m_i, m_j)_{\nu_i}^{e_{ij}} | 0 < \nu_i \text{ and } \nu_i|a_i\}$. □

Now it follows that $\mathrm{B}'(X) \cong \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{Z}/a_i, \mathbb{Q}/\mathbb{Z})^{r-i}$. To complete the proof of Theorem 1.1 it suffices to show $\mathrm{B}(X) = \mathrm{B}'(X)$. It suffices to find an Azumaya algebra Λ on X such that $K(X) \otimes \Lambda$ is equivalent to $(m_i, m_j)_{\nu_i}$ for each $\nu_i|a_i$.

Lemma 1.9. *Let X be as in Theorem* 1.8 *and let $N' = \langle \bigcup_{\sigma\in\Delta} \sigma \cap N\rangle$. Let $\nu \geq 2$ and let $M_\nu = \{m \in M \mid \langle m, n'\rangle \equiv 0 \pmod{\nu}$ for all $n' \in N'\}$. If (ν, a_i) is the greatest common divisor of ν and a_i, then*

$$\mathrm{H}^1(X, \mathbb{Z}/\nu) \cong M_\nu/\nu M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(\nu, a_i).$$

Proof. Restriction induces an embedding $\mathrm{H}^1(X, \mathbb{Z}/\nu) \to \mathrm{H}^1(T_N, \mathbb{Z}/\nu)$. The correspondence which assigns to each element $m \in M$ the cyclic extension of T_N obtained by adjoining the νth root of m induces an isomorphism $M/\nu M \cong \mathrm{H}^1(T_N, \mathbb{Z}/\nu)$. An element $m + \nu M$ corresponds to an element of $\mathrm{H}^1(X, \mathbb{Z}/\nu)$ if and only if $K(X)(m^{1/\nu})$ is unramified along $\mathrm{orb}\,\rho_k$ for $1 \leq k \leq n$ if and only if the restriction of m to $\mathrm{orb}\,\rho_k$ is a unit in the coordinate ring $k[\eta_k^\perp, \mu_k]$ of $\mathrm{orb}\,\rho_k$ if and only if $v_k(m) \equiv 0 \pmod{\nu}$ if and only if $\langle m, \eta_k\rangle \equiv 0 \pmod{\nu}$ where unexplained notation is as in Lemma 1.5. Thus

$$\begin{aligned} \mathrm{H}^1(X, \mathbb{Z}/\nu) &= \{m \in M \mid \langle m, \eta_k\rangle \equiv 0 \pmod{\nu}\ (1 \leq k \leq n)\} + \nu M \\ &= \{m \in M \mid \langle m, n'\rangle \equiv 0 \pmod{\nu} \text{ for all } n' \in N'\} + \nu M \\ &= M_\nu/\nu M. \end{aligned}$$

But $N' = \mathbb{Z}a_1n_1 \oplus \cdots \oplus \mathbb{Z}a_rn_r$ so it is easy to check that

$$M_\nu/\nu M \cong \bigoplus_{i=1}^{r} \mathbb{Z}\left(\frac{\nu}{(a_i, \nu)} m_i\right) / \mathbb{Z}(\nu m_i) \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(a_i, \nu). \quad \square$$

Lemma 1.10. *If X is as in Theorem* 1.8, *then* $\mathrm{B}(X) = \mathrm{B}'(X)$.

Proof. It suffices to show each $(m_i, m_j)_{\nu_i}$ is in the image of the cup product map $\mathrm{H}^1(X, \mathbb{Z}/\nu_i) \times \mathrm{H}^1(X, \mu_{\nu_i}) \to \mathrm{B}(X)$ when $\nu_i | a_i$, since cup products correspond to taking smash products of cyclic Galois extensions and thus are Azumaya algebras (e.g. [6]).

If $a_j = 0$, both $m_i, m_j \in M_{\nu_i}$ since $\nu_i/(a_i, \nu_i) = \nu_i/\nu_i = 1 = \nu_i/(0, \nu_i)$. If $a_j \neq 0$, then $(m_i, m_j)_{\nu_i} \sim (m_i, m_j)_{a_j}^{a_j/\nu_i} \sim ((a_j/\nu_i)m_i, m_j)_{a_j}$. But $a_j/(a_i, a_j) = a_j/a_i$ which divides a_j/ν_i since $\nu_i | a_i$. Thus $(a_j/\nu_i)m_i$ and m_j are both in M_{a_j} and $(m_i, m_j)_{\nu_i}$ is equivalent to an algebra in the image of the cup product map $\mathrm{H}^1(X, \mathbb{Z}/a_j) \times \mathrm{H}^1(X, \mu_{a_j}) \to {}_{a_j}\mathrm{B}(X)$. So $\mathrm{B}(X) = \mathrm{B}'(X)$. □

As a result of observations made so far, we can show the following proposition.

Proposition 1.11. *Let $\prod_{i<j}(m_i, m_j)_\nu^{e_{ij}}$ represent a class in* $\mathrm{B}(T_N)$ *of order ν. Let M_Λ be the matrix transformation in* $\mathrm{Hom}(N, M/\nu M)$ *defined in Lemma* 1.7 *and let t be the rank of* $\mathrm{kernel}(M_\Lambda)$. *Then there exists a direct summand P of M with* $\mathrm{rank}(P) = r - t$ *and an Azumaya algebra L over $k[P]$ with $\Lambda \cong k[M] \otimes_{k[P]} L$. No direct summand of M of smaller rank has this property.*

Proof. Find a basis $n_1, \dots, n_r$ of N such that $\ker(M_\Lambda) = \mathbb{Z}b_1 n_1 \oplus \mathbb{Z}b_2 n_2 \oplus \cdots \oplus \mathbb{Z}b_r n_r$ and $b_i | b_{i+1}$ for $1 \leq i \leq r-1$. Since $\ker(M_\Lambda)$ has rank t, $b_t \neq 0$ and $b_{t+i} = 0$ for $i \geq 1$.

Let P be a direct summand of M and assume Λ is obtained by extending an algebra over $k[P]$. Let $m'_1, \dots, m'_s$ be a basis for P and extend this basis to a basis for M. We can assume $\Lambda = \prod_{i<j\leq s}(m'_i, m'_j)_\nu^{e'_{ij}}$. If $n'_1, \dots, n'_r$ is the dual basis to $m'_1, \dots, m'_r$, then the matrix of the transformation M_Λ with respect to this new basis pair has a kernel which contains a direct summand of N of rank $r-s$. Therefore $t \geq r-s$ so $s \geq r-t$. Now let $m_1, \dots, m_r$ be a dual basis for M with respect to $n_1, \dots, n_r$. The matrix M_Λ with respect to this new basis is

$$M_\Lambda = \left[\begin{array}{c|cccc} 0 & & 0 & & \\ \hline & 0 & e_{t+1,t} & \cdots & e_{t+1,r} \\ 0 & -e_{t+1,t} & 0 & \cdots & \\ & \vdots & & & \vdots \\ & -e_{t+1,r} & & \cdots & 0 \end{array}\right].$$

So Λ is defined on the torus $k[m_{t+1}, \dots, m_r, -m_{t+1}, \dots, -m_r]$ and we can take $P = \langle m_{t+1}, \dots, m_r \rangle$. The rank of P is $r-t$. □

2

In this section we continue to let Δ be a finite fan on $N_\mathbb{R}$ and $X = T_N \operatorname{emb}(\Delta)$ the associated toric variety containing the r-dimensional torus T_N as an open subset. Assume Δ consists of cones of dimension ≤ 2 and let $\Delta(2) = \{\tau_1, \dots, \tau_m\}$. Let $U_i = U_{\tau_i}$, $V_i = V(\tau_i) = \operatorname{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. Then $X - V = T_N \operatorname{emb}(\Delta - \Delta(2))$ is nonsingular. In this situation our first lemma gives information about the étale cohomology groups of the affine open subsets U_i of X.

Lemma 2.1. (a) *For each* i *and each* $p \geq 0$, *we have a short exact sequence*

$$0 \to \mathrm{H}^p(U_i, \mathrm{G}_m) \to \mathrm{H}^p(U_i - V_i, \mathrm{G}_m) \to \mathrm{H}^{p+1}_{V_i}(U_i, \mathrm{G}_m) \to 0.$$

(b) $\mathrm{H}^p(U_i, \mathrm{G}_m) \cong \mathrm{H}^p(T_{r-2}, \mathrm{G}_m)$ *where* T_{r-2} *is a torus of dimension* $r-2$.

Proof. First we check that $\mathrm{H}^p(U_i, \mathrm{G}_m)$ is torsion for $p \geq 2$. For notational simplicity we suppress the subscript i from τ_i, U_i, and V_i. Now τ is a two-dimensional cone in $N_{\mathbb{R}}$. Let $\overline{\tau}$ be τ viewed as a two-dimensional cone in $\mathbb{R}\tau$. Then $U = U_{\overline{\tau}} \times T_{r-2}$ where T_{r-2} is an $(r-2)$-dimensional torus. Let R be the affine coordinate ring of $U_{\overline{\tau}}$, and R^h the henselization of R at the maximal ideal m corresponding to the closed point $\operatorname{orb}\overline{\tau}$. Let $R[X, X^{-1}]$ denote the affine coordinate ring of U and let $U^h = \operatorname{Spec} R^h[X, X^{-1}]$. Let $V^h = V \times U^h$. Then V^h is the closed set corresponding to $I = mR^h[X, X^{-1}]$. The completion of $R^h[X, X^{-1}]$ in the I-adic topology is $\widehat{R}[X, X^{-1}]$ where $\widehat{R}$ is the m-adic completion of R. By [13, p. 127], we see that $(R^h[X, X^{-1}], I)$ is a Hensel pair. By [5, p. 35] $\mathrm{Cl}(R^h[X, X^{-1}])$ embeds into $\mathrm{Cl}(\widehat{R}[X, X^{-1}])$. Since the singularity of U is given by a finite cyclic group action [12, p. 30], it is well known that $\mathrm{Cl}(\widehat{R}[X, X^{-1}]) = \mathrm{Cl}(\widehat{R})$ is also finite cyclic [2, Satz 2.11]. Thus $\mathrm{Cl}(U^h)$ is finite. The long exact sequences for the pairs $V \subseteq U$ and $V^h \subseteq U^h$ give the commutative diagram

$$\begin{array}{ccccccc}
\to \mathrm{H}^{p-1}(U-V, \mathrm{G}_m) & \to & \mathrm{H}^p_V(U, \mathrm{G}_m) & \to & \mathrm{H}^p(U, \mathrm{G}_m) & \to & \mathrm{H}^p(U-V, \mathrm{G}_m) \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\
\to \mathrm{H}^{p-1}(U^h - V^h, \mathrm{G}_m) & \to & \mathrm{H}^p_{V^h}(U^h, \mathrm{G}_m) & \to & \mathrm{H}^p(U^h, \mathrm{G}_m) & \to & \mathrm{H}^p(U^h - V^h, \mathrm{G}_m)
\end{array}$$

with exact rows. By excision $\mathrm{H}^p_V(U, \mathrm{G}_m) \cong \mathrm{H}^p_{V^h}(U^h, \mathrm{G}_m)$ [11, p. 92]. By [15] $\mathrm{H}^p(U^h, \mathrm{G}_m) \cong \mathrm{H}^p(V^h, \mathrm{G}_m) = \mathrm{H}^p((\operatorname{orb}\overline{\tau}) \times T_{r-2}, \mathrm{G}_m)$ which is torsion for $p \geq 2$ since T_{r-2} is smooth [7, p. 71]. Again by [7, p. 71] $\mathrm{H}^p(U^h - V^h, \mathrm{G}_m)$ and $\mathrm{H}^p(U-V, \mathrm{G}_m)$ are torsion for $p \geq 2$. But $\mathrm{H}^1(U^h - V^h, \mathrm{G}_m) = \mathrm{Pic}(U^h - V^h) = \mathrm{Cl}(U^h - V^h) = \mathrm{Cl}(U^h)$ is torsion. It now follows that $\mathrm{H}^p(U, \mathrm{G}_m)$ is torsion for $p \geq 2$.

The natural map $U \times A^1 \to U$ and Kummer theory induce the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{H}^{p-1}(U \times A^1, \mathrm{G}_m) \otimes \mathbb{Z}/n & \to & \mathrm{H}^p(U \times A^1, \mu_n) & \to & {}_n\mathrm{H}^p(U \times A^1, \mathrm{G}_m) & \to 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 \to & \mathrm{H}^{p-1}(U, \mathrm{G}_m) \otimes \mathbb{Z}/n & \to & \mathrm{H}^p(U, \mu_n) & \to & {}_n\mathrm{H}^p(U, \mathrm{G}_m) & \to 0
\end{array}$$

for all $p \geq 2$ and $n \geq 2$. By [11, p. 240] β is an isomorphism for $p \geq 2$. Since $\operatorname{Pic} U = 0 = \operatorname{Pic}(U \times A^1)$, α is an isomorphism for $p = 2$. Therefore, γ is an isomorphism for $p = 2$ and all $n \geq 2$. Taking the inductive limit over all n, we have $\mathrm{H}^2(U \times A^1, \mathrm{G}_m) \cong \mathrm{H}^2(U, \mathrm{G}_m)$. By induction on p we see that $\mathrm{H}^p(U \times A^1, \mathrm{G}_m) \cong \mathrm{H}^p(U, \mathrm{G}_m)$ for all $p \geq 2$.

We can give the coordinate ring $k[\mathscr{S}_\tau]$ of U a grading by the nonnegative integers such that the degree $= 0$ subring is the coordinate ring of T_{r-2}. Since $\mathrm{H}^p(U \times A^1, \mathrm{G}_m) = \mathrm{H}^p(U, \mathrm{G}_m)$, [8, Theorem 1.1] implies $\mathrm{H}^p(U, \mathrm{G}_m) \cong$

$H^p(T_{r-2}, G_m)$, which proves (b). We have a commutative diagram

$$\begin{array}{ccccc} H^p(T_{r-2}, G_m) & \cong & H^p(U, G_m) & \xrightarrow{\alpha} & H^p(U-V, G_m) \\ & \searrow^{\beta} & \downarrow{\gamma} & \swarrow^{\delta} & \\ & & H^p(T_r, G_m) & & \end{array}$$

where the maps β, γ, δ are induced from restriction. Since β is injective, α is injective and Lemma 2.1 now follows. □

Theorem 2.2. *Let Δ be a fan which consists of cones of dimension ≤ 2. Let Δ' be a nonsingular fan obtained from Δ by subdividing the two-dimensional faces of Δ and let $\widetilde{X} = T_N\,\mathrm{emb}(\Delta')$. Then the sequence $0 \to B'(T_N/X) \to B'(X) \to B'(\widetilde{X}) \to 0$ (with natural maps) is exact.*

Proof. Let $\pi: \widetilde{X} \to X$ be the desingularization resulting from the subdivision Δ' of Δ [12, Corollary 1.18] and let $\widetilde{U}_i = \pi^{-1}(U_i)$. From the long exact sequence of cohomology with supports, and the observation that V is a disjoint union of closed sets V_i (see pp. 92–93 of [11]) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to B'(T_N/X) \to & B'(X) & \to & B'(X-V) & \longrightarrow & \bigoplus_{i=1}^{m} H^3_{V_i}(U_i, G_m) \\ & \downarrow{\alpha} & & \downarrow{=} & & \downarrow{\beta} \\ 0 \to & B'(\widetilde{X}) & \to & B'(\widetilde{X} - \pi^{-1}(V)) & \to & \bigoplus_{i=1}^{m} H^3_{\pi^{-1}(V_i)}(\widetilde{U}_i, G_m) \end{array}$$

The second row is exact since $\widetilde{X}$ is nonsingular. First check β is injective. For each i, Lemma 2.1 yields the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B'(U_i) & \longrightarrow & B'(U_i - V_i) & \longrightarrow & H^3_{V_i}(U_i, G_m) & \longrightarrow & 0 \\ & & \downarrow{\alpha_i} & & \downarrow{=} & & \downarrow{\beta_i} & & \\ 0 & \longrightarrow & B'(\widetilde{U}_i) & \longrightarrow & B'(\widetilde{U}_i - \pi^{-1}(V_i)) & \longrightarrow & H^3_{\pi^{-1}(V_i)}(\widetilde{U}_i, G_m) & & \end{array}$$

Here $B'(U_i) = B'(U_{\overline{\tau}_i} \times T_{r-2}) = B(T_{r-2})$ by Lemma 2.1 and $B'(U_i - V_i) = H^2(U_i - V_i, G_m)$ since $U_i - V_i$ is nonsingular. If $\Delta'(\tau_i)$ is the fan whose cones are the cones of Δ' contained in τ_i, then $\Delta'(\tau_i)$ is a nonsingular fan whose one dimensional faces lie in a plane. The invariants for $\widetilde{U}_i = T_N\,\mathrm{emb}(\Delta'(\tau_i))$ are $\{1, 1, 0, \ldots, 0\}$ and Theorem 1.1 implies $B'(\widetilde{U}_i) = B'(\widetilde{U}_{\overline{\tau}_i} \times T_{r-2}) = B(T_{r-2})$ so α_i is an isomorphism. Since $\ker\beta_i = \operatorname{coker}\alpha_i$, β_i is injective so β is injective. But $\ker\beta = \operatorname{coker}\alpha$, so α is an epimorphism and the theorem follows. □

As a result of Theorem 2.2 and our analysis of the Brauer groups of nonsingular toric varieties in §1, we are left with the study of $B'(T_N/X)$.

Theorem 2.3. *Let Δ be a fan which consists of cones of dimension ≤ 2. Let $\Delta(2) = \{\tau_1, \ldots, \tau_m\}$. Let $X = T_N\,\mathrm{emb}(\Delta)$ and let $U_i = U_{\tau_i}$ be the open subsets* *of X associated to the τ_i. Then there is an exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{i=1}^{m} \mathrm{Cl}(U_i) \to \mathrm{B}'(T_N/X) \to 0.$$

Proof. Let $V_i = V(\tau_i) = \mathrm{orb}(\tau_i)$ and let $V = V_1 \cup \cdots \cup V_m$. From the long exact sequence of cohomology with supports in the closed set V we have (since V is the disjoint union of the closed sets V_i)

$$(1)\quad \begin{aligned} \cdots \to \mathrm{H}^1(X, \mathrm{G}_m) \to \mathrm{H}^1(X - V, \mathrm{G}_m) \to \bigoplus_{i=1}^{m} \mathrm{H}^2_{V_i}(X, \mathrm{G}_m) \\ \to \mathrm{H}^2(X, \mathrm{G}_m) \to \mathrm{H}^2(X - V, \mathrm{G}_m) \to \bigoplus_{i=1}^{m} \mathrm{H}^3_{V_i}(X, \mathrm{G}_m) \to \cdots. \end{aligned}$$

Since V has codimension 2 in X, and $X-V$ is nonsingular, $\mathrm{H}^1(X-V, \mathrm{G}_m) = \mathrm{Pic}(X - V) = \mathrm{Cl}(X - V) = \mathrm{Cl}(X)$. Since U_i is an open neighborhood of V_i, $\mathrm{H}^p_{V_i}(X, \mathrm{G}_m) = \mathrm{H}^p_{V_i}(U_i, \mathrm{G}_m)$ for all $p \geq 0$ [11, p. 93]. From Lemma 2.1 with $p = 1$ we get the exact sequences

$$0 \to \mathrm{Pic}\, U_i \to \mathrm{Cl}\, U_i \to \mathrm{H}^2_{V_i}(U_i, \mathrm{G}_m) \to 0 \qquad (1 \leq i \leq m).$$

Lemma 2.1(b) gives $\mathrm{Pic}\, U_i = \mathrm{Pic}\, T_{r-2} = 0$ so $\mathrm{Cl}(U_i) = \mathrm{H}^2_{V_i}(U_i, \mathrm{G}_m)$. Since τ_i is simplicial, $\mathrm{Cl}(U_i) = \mathrm{Pic}(U_i - V_i)$ is torsion [12, Proposition 2.1]. Since $X - V$ is nonsingular, $\mathrm{B}'(X - V) = \mathrm{H}^2(X - V, \mathrm{G}_m)$ and $\mathrm{B}'(X - V) \to \mathrm{B}(T_N)$ is injective. But $\mathrm{Pic}\, X \to \mathrm{Cl}(X)$ is injective [5]. With these identifications (1) reduces to the sequence of the theorem. □

Corollary 2.4. *In the context of Theorem* 2.3, *if* $\mathrm{rank}_{\mathbb{Z}}(N) = r \leq 3$ *and* $m \geq 1$, *then*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{i=1}^{m} \mathrm{Cl}(U_i) \to \mathrm{B}'(X) \to 0$$

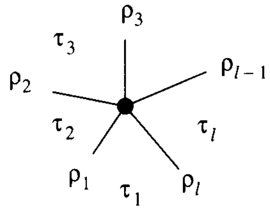
is exact.

Proof. We need to check $\mathrm{B}'(X) = \mathrm{B}'(T_N/X)$. $\mathrm{H}^2(U_i, \mathrm{G}_m) = \mathrm{H}^2(T_{r-2}, \mathrm{G}_m) = 0$ for $r \leq 3$ (Lemma 2.1 and [10]). From Lemma 2.1(a) we have (since $U_i - V_i$ is nonsingular) $\mathrm{B}'(U_i - V_i) = \mathrm{H}^2(U_i - V_i, \mathrm{G}_m) = \mathrm{H}^3_{V_i}(U_i, \mathrm{G}_m) = \mathrm{H}^3_{V_i}(X, \mathrm{G}_m)$ so (1) becomes

$$0 \to \mathrm{Pic}\, X \to \mathrm{Cl}(X) \to \bigoplus_{i=1}^{m} \mathrm{Cl}(U_i) \to \mathrm{B}'(X) \to \mathrm{B}'(X - V) \to \bigoplus_{i=1}^{m} \mathrm{B}'(U_i - V_i).$$

Since $X - V$ is nonsingular, restriction induces a monomorphism $\mathrm{B}'(X - V) \to \mathrm{B}'(U_i - V_i)$ for each i and the corollary follows. □

The object of the rest of this section is to give an algorithm for finding for each prime p a subset $\tau_1, \ldots, \tau_e$ of the two-dimensional faces of Δ such that $[\bigoplus_{i=1}^{e} \mathrm{Cl}(U_{\tau_i})]_p \cong [\mathrm{B}'(T_N/X)]_p$. (For G a finite abelian group, G_p is the Sylow p-subgroup.) In particular, the exact sequence of Theorem 2.3 is split-exact. To the fan Δ we associated a bipartite graph Γ. The vertex set of Γ is $\Delta(1) \cup \Delta(2) = \{\rho_1, \ldots, \rho_n\} \cup \{\tau_1, \ldots, \tau_m\}$ and there is an edge in Γ connecting ρ_j and τ_j if and only if ρ_j is a face of τ_i. If Y is the T_N-invariant divisor $X - T_N$ on X, then Γ is the graph associated to Y in the sense of [4]. A cycle Z in Γ (i.e., Z is homeomorphic to the unit circle) determines a finite set $\tau_1, \dots, \tau_l$ of two-dimensional cones and $\rho_1, \dots, \rho_l$ of one-dimensional faces of Δ configured as follows:



If Δ_Z is the subfan of Δ consisting of the cones $\{0, \rho_1, \dots, \rho_l, \tau_1, \dots, \tau_l\}$, we will show that the cohomological Brauer group of $T_N \operatorname{emb}(\Delta_Z)$ is cyclic of order the greatest common divisor of $\{|\operatorname{Cl} U_{\tau_i}|\}_{i=1}^{l}$. Of course, there may be many such cycles in Γ and the last step in the analysis is to choose for each prime p a list of cycles $\{Z_i\}_{i=1}^{e}$ and for each Z_i a face τ_i such that $[\bigoplus_{i=1}^{e} \operatorname{Cl}(U_{\tau_i})]_p \cong [\mathrm{B}'(T_N/X)]_p$.

We adopt the following notation: for each two-dimensional cone τ_i in Δ $(1 \le i \le m)$ let ρ_{i1} and ρ_{i2} be the one-dimensional faces of τ_i so $\tau_i = \rho_{i1} + \rho_{i2}$. We have observed $\operatorname{Cl}(X) = \operatorname{Cl}(X - V) = \operatorname{Pic}(X - V)$ and $\operatorname{Cl}(U_i) = \operatorname{Cl}(U_i - V_i) = \operatorname{Pic}(U_i - V_i)$. Now we want to present $\operatorname{Pic}(X - V)$ and $\operatorname{Pic}(U_i - V_i)$ in terms of support functions on the fan $\Delta - \{\tau_1, \dots, \tau_m\}$. If we let $\rho_1, \dots, \rho_n$ be the one-dimensional cones in Δ, then we can identify the support functions on $\Delta - \{\tau_1, \dots, \tau_m\}$ with the direct sum of copies of $\mathbb{Z}$ indexed by the ρ_i. If $\Delta_i = \{0, \rho_{i1}, \rho_{i2}\}$, then $U_i - V_i = T_N \operatorname{emb}(\Delta_i)$. It follows from [12, Corollary 2.5] that the sequences

$$
\begin{gathered}
M \to \bigoplus_{i=1}^{n} \mathbb{Z}\rho_i \to \operatorname{Cl}(X) \to 0, \\
M \xrightarrow{\beta} \mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2} \to \operatorname{Cl}(U_i) \to 0
\end{gathered}
\tag{2}
$$

are exact. Combining these sequences with the exact sequence of Theorem 2.3 gives a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
M & \longrightarrow & \bigoplus_{j=1}^{n} \mathbb{Z}\rho_j & \xrightarrow{\phi} & \operatorname{Cl}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varepsilon} & & \\
\bigoplus_{i=1}^{m} M & \xrightarrow{\beta} & \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2}) & \xrightarrow{\gamma} & \bigoplus_{i=1}^{m} \operatorname{Cl}(U_i) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \delta} & & \\
 & & & & \mathrm{B}'(T_N/X) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
$$

It is routine to check that $\operatorname{im}\alpha + \operatorname{im}\beta = \ker\delta\gamma$. As a result we have a fundamental exact sequence which we exploit for the remainder of this section: (in this sequence $\psi = \delta\gamma$),

$$(3)\qquad \left(\bigoplus_{j=1}^{n}\mathbb{Z}\rho_j\right) \oplus \left(\bigoplus_{i=1}^{m} M\right) \xrightarrow{\alpha+\beta} \bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2}) \xrightarrow{\psi} \mathrm{B}'(T_N/X) \to 0 .$$

Let Γ be the graph associated to Δ. Observe that Γ has $2m$ edges since each τ_i has exactly two one-dimensional faces ρ_{i1} and ρ_{i2}. The free abelian group $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})$ is called the edge space of Γ. If we write Γ as a union of its connected components Γ_i we get a corresponding decomposition of Δ into subfans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ whenever $i \neq j$. The decomposition of Δ gives an open cover of X where the elements in the open over are $T_N\,\mathrm{emb}(\Delta_i) = X_i$ and $X_i \cap X_j = T_N$ whenever $i \neq j$. With this notation we can prove

Proposition 2.5. *The natural map* $\mathrm{B}'(X) \to \bigoplus_i \mathrm{B}'(X_i)$ *induces an isomorphism* $\mathrm{B}'(T_N/X) \cong \bigoplus_i \mathrm{B}'(T_N/X_i)$.

Proof. Assume $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are fans with $\Delta_1 \cap \Delta_2 = \{0\}$. It is sufficient to prove $\mathrm{B}'(T_N/X) \cong \mathrm{B}'(T_N/X_1)\oplus\mathrm{B}'(T_N/X_2)$ where $X_i = T_N\,\mathrm{emb}(\Delta_i)$ $(i = 1, 2)$. Let $\Delta_1(1) = \{\rho_1, \dots, \rho_{n_1}\}$ and $\Delta_1(2) = \{\tau_1, \dots, \tau_{m_1}\}$ and $\Delta_2(1) = \{r_1, \dots, r_{n_2}\}$ and $\Delta_2(2) = \{t_1, \dots, t_{m_2}\}$. Also let $\tau_i = \rho_{i1} + \rho_{i2}$ and $t_j = r_{j1} + r_{j2}$ where $\rho_{ik} \in \Delta_1(1)$ and $r_{jl} \in \Delta_2(1)$. With respect to this decomposition the exact sequence (3) decomposes as

$$\left[\bigoplus_{j=1}^{n_1}\mathbb{Z}\rho_j \oplus \bigoplus_{i=1}^{m_1} M\right] \oplus \left[\bigoplus_{j=1}^{n_2}\mathbb{Z}r_j \oplus \bigoplus_{i=1}^{m_2} M\right] \xrightarrow{(\alpha_1+\beta_1)\oplus(\alpha_2+\beta_2)}$$

$$\left[\bigoplus_{i=1}^{m_1}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})\right] \oplus \left[\bigoplus_{i=1}^{m_2}(\mathbb{Z}r_{i1}\oplus\mathbb{Z}r_{i2})\right] \to \mathrm{B}'(T_N/X) \to 0 .$$

But $\operatorname{coker}(\alpha_1+\beta_1)\oplus(\alpha_2+\beta_2) = \mathrm{B}'(T_N/X_1)\oplus\mathrm{B}'(T_N/X_2)$ by (3) so $\mathrm{B}'(T_N/X) = \mathrm{B}'(T_N/X_1)\oplus\mathrm{B}'(T_N/X_2)$. □

Notice in Proposition 2.5 that if X_i corresponds to a connected component of Γ containing no two-dimensional faces τ_i as vertices, then $X_i = T_N\,\mathrm{emb}\{0, \rho\}$ for some one-dimensional cone ρ in Δ. In this case X_i is nonsingular and $\mathrm{B}'(T_N/X_i) = 0$. Thus, as a result of Proposition 2.5 we can assume Γ is connected and at least one vertex of Γ is a two-dimensional cone in Δ.

We now determine a matrix representation for the map $\alpha + \beta$ in (3). Let $\tau = \rho_1 + \rho_2 \in \Delta(2)$ and consider the map

$$M \xrightarrow{\beta} \mathbb{Z}\rho_1 \oplus \mathbb{Z}\rho_2 \to \mathrm{Cl}(U_\tau) \to 0$$

as in (2). Pick a basis $n_1, \dots, n_r$ for N and a dual basis $m_1, \dots, m_r$ for M. Let η_i be a primitive element in N with $\rho_i = \mathbb{R}_{\geq 0}\eta_i$. The matrix of β with respect to the basis pair $\{m_1, \dots, m_r\}$, $\{\rho_1, \rho_2\}$ is the $2 \times r$ matrix whose i, jth entry is $\langle m_j, \eta_i\rangle$. But $\langle m_j, \eta_i\rangle$ is the jth coordinate of η_i so we can write this matrix as $\binom{\eta_1}{\eta_2}$ where we think of η_i as a row vector. Therefore the map β in (3)

$$\bigoplus_{i=1}^{m} M \xrightarrow{\beta} \bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})$$

has a matrix representation which is a direct sum of $2\times r$ matrices $\binom{\eta_{i1}}{\eta_{i2}}$ where η_{i1} and η_{i2} are the primitive generators of ρ_{i1} and ρ_{i2} expressed with respect to the basis $\{n_1, \dots, n_r\}$. To determine the matrix for the homomorphism

$$\bigoplus_{j=1}^{n} \mathbb{Z}\rho_j \xrightarrow{\alpha} \bigoplus_{i=1}^{m} (\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$$

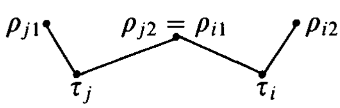
given in (3) we observe the jth column of this matrix is $\alpha(\rho_j)$. Thus the jth column has a 1 in the row determined by ρ_{ik} if $\rho_j = \rho_{ik}$. Otherwise this entry is 0. The matrix of the homomorphism $\alpha+\beta$ of (3) is then

$$Q = \left[\begin{array}{c|ccc} & \binom{\eta_{11}}{\eta_{12}} & & \\ \alpha(\rho_1)\cdots\alpha(\rho_n) & & \ddots & \\ & & & \binom{\eta_{m1}}{\eta_{m2}} \end{array}\right].$$

Note Q is an integral matrix with $2m$ rows and $n+rm$ columns, and we can identify $\operatorname{im}(\alpha+\beta)$ with the column space of Q. Since $\mathrm{B}'(T_N/X) \cong \operatorname{coker}(\alpha+\beta)$ from (3), calculating $\mathrm{B}'(T_N/X)$ is reduced to determining the column space of Q. Our first observation is a straightforward calculation:

$$Q \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \\ -I_r \\ \vdots \\ -I_r \end{bmatrix} = (0) \tag{4}$$

where I_r is the $r\times r$ identity matrix. Thus the last r columns of Q containing $\binom{\eta_{m1}}{\eta_{m2}}$ are linear combinations of the preceding columns. We now assume Γ is connected, and let T be a spanning tree for Γ. We observe that in Γ each vertex τ_i is joined by edges $\tau_i\text{–}\rho_{i1}$, $\tau_i\text{–}\rho_{i2}$ to vertices ρ_{i1}, ρ_{i2} so there are $2m$ edges in Γ. Since Γ is connected, there are $n+m-1$ edges in T [1].



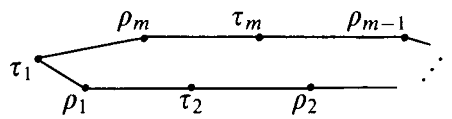
Thus, if $c_1, \dots, c_e$ denote the edges of Γ that are not in T, then $e = m-n+1$ and for each i at least one of $\tau_i\text{–}\rho_{i1}$, $\tau_i\text{–}\rho_{i2}$ is in T. By reindexing we can assume $c_1 = \tau_1\text{–}\rho_{12}, \dots, c_e = \tau_e\text{–}\rho_{e2}$. We identify c_j with the basis vector ρ_{j2} in the edge space $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. For $1 \le i \le m$ let n_{i1} be a primitive vector in N with $\mathbb{R}_{\ge 0} n_{i1} = \rho_{i1}$ and choose n_{i2} in N with $\{n_{i1}, n_{i2}\}$ a basis for $\mathbb{R}\tau_i \cap N$. We can extend $\{n_{i1}, n_{i2}\}$ to a basis $\{n_{i1}, n_{i2}, \dots, n_{ir}\}$ for N. With respect to this basis we can write $\eta_{i1} = n_{i1}$, $\eta_{i2} = a_i n_{i1} + b_i n_{i2}$ where the η_{ij} are as in Q. With respect to these basis choices for N and corresponding dual basis choices for M, and after deleting columns consisting of zeros, the matrix Q for $\alpha+\beta$ becomes

$$
Q = \begin{array}{l} \tau_1 - \rho_{11} \\ \tau_1 - \rho_{12} \\ \tau_i - \rho_{i1} \\ \tau_i - \rho_{i2} \\ \tau_m - \rho_{m1} \\ \tau_m - \rho_{m2} \end{array}
\overset{\begin{array}{ccccccc} \rho_1 & \rho_2 & \cdots & \rho_n & \alpha_1\beta_1 & \alpha_i\beta_i & \alpha_m\beta_m \end{array}}{\left(\begin{array}{c|ccc}
 & \left[\begin{smallmatrix} 1 & 0 \\ a_1 & b_1 \end{smallmatrix}\right] & & \\
\alpha(\rho_1) \quad \alpha(\rho_2)\cdots\alpha(\rho_n) & & \left[\begin{smallmatrix} 1 & 0 \\ a_i & b_i \end{smallmatrix}\right] & \\
 & & & \left[\begin{smallmatrix} 1 & 0 \\ a_m & b_m \end{smallmatrix}\right]
\end{array}\right)}
$$

where the first n columns span $\operatorname{im}\alpha$ and the last $2m$ columns span $\operatorname{im}\beta$. We checked in (4) above that the last 2 columns labeled α_m and β_m are linear combinations of the preceding $n + 2(m-1)$ columns. It follows from (2) that $|b_i| = |\operatorname{Cl}(U_{\tau_i})| = |\operatorname{Cl} U_i|$. The columns $\beta_1, \dots, \beta_e$ are $b_1c_1, \dots, b_ec_e$.

Theorem 2.6. *Let Δ be a fan on $N_{\mathbb{R}}$ and let $X = T_N \operatorname{emb}(\Delta)$. Assume all the cones in Δ have dimension ≤ 2. Assume the two-dimensional faces $\tau_1, \dots, \tau_m$ and one-dimensional faces $\rho_1, \dots, \rho_n$ of Δ can be ordered so that $\tau_i \cap \tau_{i+1} = \rho_i$ $(1 \leq i \leq m-1)$ and $\tau_m \cap \tau_1 = \rho_m$. Let b_i be the order of $\operatorname{Cl}(U_{\tau_i})$. Then $\mathrm{B}'(T_N/X)$ is cyclic of order $\gcd\{b_1, \dots, b_m\}$.*

Proof. Using Proposition 2.5 and the hypotheses, we can assume that the graph Γ is connected and consists of one cycle as shown:



We take the spanning tree T for Γ to be the graph obtained from Γ by deleting the edge $c_1 = \tau_{1-}\rho_m$. Let C be the matrix whose only column is c_1 and form the augmented matrix $[Q|C]$:

$$
\begin{array}{l}
 \\
\tau_1\text{-}\rho_1 \\ \tau_1\text{-}\rho_m \\ \tau_2\text{-}\rho_2 \\ \tau_2\text{-}\rho_1 \\ \tau_3\text{-}\rho_3 \\ \tau_3\text{-}\rho_2 \\ \vdots \\ \tau_{m-1}\rho_{m-1} \\ \tau_{m-1}\rho_{m-2} \\ \tau_m\text{-}\rho_m \\ \tau_m\text{-}\rho_{m-1}
\end{array}
\left[\begin{array}{ccccccc|cccccccccc|c}
\rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{m-2} & \rho_{m-1} & \rho_m & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots & \alpha_{m-1} & \beta_{m-1} & \alpha_m & \beta_m & & c_1 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & & & & & & & & & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & a_1 & b_1 & & & & & & & & & 1 \\
0 & 1 & 0 & \dots & 0 & 0 & 0 & & & 1 & 0 & & & & & & & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & & & a_2 & b_2 & & & & & & & 0 \\
0 & 0 & 1 & & \dots & & & & & & & & & & & & & \\
0 & 1 & 0 & & \dots & & & & & & & & & & & & & \\
 & & & & \vdots & & & & & & & & \vdots & & & & & \\
0 & 0 & 0 & \dots & 0 & 1 & 0 & & & & & & 1 & 0 & & & & 0 \\
0 & 0 & 0 & \dots & 1 & 0 & 0 & & & & & & a_{m-1} & b_{m-1} & & & & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & & & & & & & & 1 & 0 & & 0 \\
0 & 0 & 0 & \dots & 0 & 1 & 0 & & & & & & & & a_m & b_m & & 0
\end{array}\right]
$$

Here c_1 corresponds to the edge $\tau_1\text{-}\rho_m$ and $\bigoplus_{i=1}^m(\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ is the edge space of Γ where $\rho_0 = \rho_m$. We observed that $\mathrm{B}'(T_N/X)$ is the quotient of the edge space by the column space of Q. Let $[B|C]$ be the matrix whose columns are the columns labeled $\rho_1, \dots, \rho_m, \alpha_1, \alpha_2, \dots, \alpha_{m-1}, c_1$. We check that the columns of $[B|C]$ form a basis for the edge space by using column operations to reduce to a permutation matrix. Use column c_1 to eliminate a_1 from column α_1 and the 1 in the entry with row index $\tau_1\text{-}\rho_m$ and column index ρ_m. Then use the 1 in the new column α_1 to eliminate the 1 in the entry with row index $\tau_1\text{-}\rho_1$ and column index ρ_1. Use the new column ρ_1 to eliminate the a_2 in column α_2. Continue inductively, eliminating $a_3, \dots, a_{m-2}$ from columns indexed $\alpha_3, \dots, \alpha_{m-2}$ and the ones in the entries with row index $\tau_i\text{-}\rho_i$ and column index ρ_i, for $2 \le i \le m-2$. At the last step use the remaining 1 in column ρ_{m-2} to eliminate a_{m-1} in column α_{m-1}. Use 1 in the new column α_{m-1} to eliminate the 1 in the entry in row $\tau_{m-1}\text{-}\rho_{m-1}$, column ρ_{m-1}. The result is a matrix whose $\mathbb{Z}$-rank is $2m$, which shows that c_1 generates the quotient of the edge space $\bigoplus_{i=1}^m(\mathbb{Z}\rho_i \oplus \mathbb{Z}\rho_{i-1})$ $(\rho_0 = \rho_m)$ by the column space of Q. Recall that the last two columns of Q are a linear combination of the preceeding ones. Thus to calculate this quotient we simply project each of the columns $\beta_1, \dots, \beta_m$ on $\mathbb{Z}c_1$. These projections follow the recursive pattern:

Column vector	Projection on $\mathbb{Z}c_1$
$\beta_1 = b_1 c_1$	$b_1 c_1$
$\beta_2 = b_2(\rho_1 - (\alpha_1 - a_1 c_1))$	$b_2 a_1 c_1$
$\beta_3 = b_3(\rho_2 - (\alpha_2 - a_2(\rho_1 - (\alpha_1 - a_1 c_1))))$	$b_3 a_2 a_1 c_1$
$\vdots$	$\vdots$
β_m	$b_m a_{m-1} \cdots a_2 a_1 c_1$

The subgroup generated by the projections of the columns β_i on $\mathbb{Z}c_1$ is the subgroup generated by dc_1 where $d = \gcd\{b_1, b_2 a_1, \dots, b_m a_{m-1} \cdots a_2 a_1\}$. Since $\gcd(a_i, b_i) = 1$ for $1 \le i \le m$, we see $d = \gcd\{b_1, \dots, b_m\}$. But $|b_i|$ is the order of $\mathrm{Cl}(U_{\tau_i})$, so the theorem follows. $\square$

To extend Theorem 2.6 it is necessary to introduce some additional notation. Suppose the graph Γ we have associated to the fan Δ is connected and let T be a spanning tree for Γ. Since each vertex labeled by a two-dimensional face τ_i is connected by exactly two edges to vertices ρ_{i1} and ρ_{i2} corresponding to the one-dimensional faces of τ_i in Δ, each τ_i is a vertex in T. If $\Delta(2) = \{\tau_1, \dots, \tau_m\}$, designate τ_m as the root node for T. Let C be the matrix whose columns are $c_1, \dots, c_e$ and let $[Q|C]$ be the augmented matrix similar to that used in the proof of Theorem 2.6. Then $\tau_1, \dots, \tau_e$ are leaf nodes of T and $c_i = \tau_i\text{-}\rho_{i2}$ for $1 \le i \le e$. For $e < i < m$ relabel ρ_{i1} and ρ_{i2} if necessary so the edge $\tau_i\text{-}\rho_{i1}$ is closer to the root node τ_m than the edge $\tau_i\text{-}\rho_{i2}$. In our previous analysis this amounts to permuting the basis of the edge space $\bigoplus_{i=1}^m(\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$. This does not affect the columns labeled $\alpha_1, \beta_1, \dots, \alpha_e, \beta_e$ in Q. Let $[B|C]$ be the matrix obtained from $[Q|C]$ by deleting from Q the columns labeled $\beta_1, \dots, \beta_{m-1}, \beta_m, \alpha_m$. We note that the column space of B depends on the choice of τ_m.

Lemma 2.7. *The columns of* $[B|C]$ *form a basis for* $\bigoplus_{i=1}^m(\mathbb{Z}\rho_{i1} \oplus \mathbb{Z}\rho_{i2})$, *the edge space, for any choice of root node* τ_m.

Proof. If Γ is a tree, then $T = \Gamma$ and $e = 0$. In this case we need to show that the columns of B span the edge space. Each leaf node of Γ must be ρ_{i2} for some i since each τ_i is incident to two edges in $\Gamma = T$ and ρ_{i1} is closer to the root node τ_m than ρ_{i2}. We call the pair (τ_i, ρ_{i2}) a leaf node pair. If $i = m$, then it is possible for ρ_{i1} to be a leaf node. This is the only exception and will be treated in the basis step for our induction below. Assume (τ_i, ρ_{i2}) is a leaf node pair and $i \neq m$. In Δ, ρ_{i2} is a face of exactly one two-dimensional cone τ_i. Thus the column indexed by ρ_{i2} in B has exactly one nonzero entry which is a 1 in the row indexed $\tau_{i}\text{–}\rho_{i2}$ as indicated below.

$$\begin{array}{c} \\ \tau_i - \rho_{il} \end{array} \begin{array}{c} \begin{matrix} \rho_{il} & \rho_{i2} & \alpha_i \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & a_i \end{array}\right] \end{array}$$

Use the column indexed ρ_{i2} to eliminate a_i in the column α_i by an elementary column operation, then use the new column α_i to eliminate the 1 in the $\tau_{i}\text{–}\rho_{i1}$ entry of column ρ_{i1}. After these two steps are performed, we say we have pruned the leaf node pair from the tree $\Gamma = T$. The two columns indexed ρ_{i2} and α_i are now elementary basis vectors in our basis for the edge space and appear in no further column operations. After the columns indexed ρ_{i2} and α_i are deleted, the remaining matrix is the matrix we would associate to the fan Δ' obtained from Δ be deleting the cones ρ_{i2} and τ_i. Apply this leaf pruning algorithm iteratively to reduce to the case where Γ is the tree $\rho_{m1}\text{–}\tau_{m}\text{–}\rho_{m2}$. The matrix B for this tree is $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Thus our algorithm reduces the original matrix B, using elementary column operations, to a permutation matrix.

If Γ is not a tree, let $\tau_{1}\text{–}\rho_{12}, \dots, \tau_{e}\text{–}\rho_{e2}$ be the edges of Γ which are not in T. Fix i, $1 \leq i \leq e$. Since $\tau_{i}\text{–}\rho_{i1}$ is in T, it follows that τ_i is a leaf node of T. We can use the column indexed c_i and elementary column operations to eliminate the entry a_i from the column indexed α_i and the entry 1 in the column indexed ρ_{i2} and row indexed $\tau_{i}\text{–}\rho_{i2}$ in the matrix $[B|C]$. Use the new column α_i to eliminate the 1 in the $\tau_{i}\text{–}\rho_{i1}$ entry of column ρ_{i1}. Repeat this step for $i = 1, \dots, e$. Observe that the $2e \times 2e$ submatrix of $[B|C]$ whose columns are indexed α_i $(1 \leq i \leq e)$, c_i $(1 \leq i \leq e)$ and whose rows are indexed $\tau_{i}\text{–}\rho_{ij}$ $(1 \leq i \leq e$ and $1 \leq j \leq 2)$ has rank $2e$. If we delete this submatrix from $[B|C]$ (which corresponds to deleting $\tau_1, \dots, \tau_e$ from the fan Δ), the resulting matrix is the one we would associate to the fan $\Delta' = \Delta - \{\tau_1, \dots, \tau_e\}$. The graph of Δ' is a tree so the result follows from the first part of the proof. □

Corollary 2.8. *Let Δ be a fan on $\mathbb{R}^r$ whose cones all have dimension ≤ 2. If the graph Γ associated to Δ is a disjoint union of trees and $X = T_N \operatorname{emb}(\Delta)$, then $\mathrm{B}'(T_N/X) = 0$.*

Proof. By Proposition 2.5 we can assume Γ is connected, so the hypotheses imply Γ is a tree. By Lemma 2.7 the columns of B span the edge space of Γ. But the column space of B is contained in $\operatorname{im}(\alpha+\beta)$ in (3) $\mathrm{B}'(T_N/X) = 0$. □

Corollary 2.9. *Let Δ be a fan on $\mathbb{R}^2$ and $X = T_N \operatorname{emb}(\Delta)$ the associated toric surface.*

(a) *If $\Delta = \{0\}$, then $\mathrm{B}(X) \cong \mathbb{Q}/\mathbb{Z}$.*
(b) *If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$ (i.e., X is not complete), then $\mathrm{B}(X) = 0$.*
(c) *If $|\Delta| = \mathbb{R}^2$ (i.e., X is complete), $\Delta(1) = \{\rho_1, \dots, \rho_n\}$ and $N' = \langle \rho_1 \cap N, \dots, \rho_n \cap N \rangle$, then $\mathrm{B}(X) \cong N/N'$.*

Proof. Every toric surface is a normal projective surface [12] so by [9, Corollary 9] $B(X) = B'(X)$. If $\Delta = \{0\}$, then $X = T_N$ is nonsingular and since $r = 2$, $B(X) = \mathbb{Q}/\mathbb{Z}$ [10]. If $\Delta \neq \{0\}$ and $|\Delta| \neq \mathbb{R}^2$, then the graph Γ associated to Δ is a disjoint union of trees. By Proposition 2.5 we assume Γ is a tree and contains at least one two-dimensional cone τ. By Corollary 2.8 and Corollary 2.4, $B(X) = 0$. If $|\Delta| = \mathbb{R}^2$, then Γ is a cycle. Corollary 2.4 implies $B(X) = B(T_N/X)$. If $\Delta(2) = \{\tau_1, \dots, \tau_m\}$, then Theorem 2.6 implies $B(X)$ is cyclic of order $\gcd\{|\operatorname{Cl}(U_{\tau_i})|\}_{i=1}^m$. Let $\rho_1, \dots, \rho_m$ be the one-dimensional cones in Δ with $\tau_i = \rho_i + \rho_{i+1}$ $(1 \leq i \leq m)$ where $\rho_{m+1} = \rho_1$ and let $\rho_i = \mathbb{R}_{\geq 0}\eta_i$ for primitive vectors $\eta_1, \dots, \eta_m$ in N. Choose a basis n_1, n_2 for N with $n_1 = \eta_1$ and write $\eta_1 = a_i n_1 + b_i n_2$. Then $N/N' = N/\langle \eta_1, \dots, \eta_m\rangle$ is cyclic of order $\gcd\{b_1, \dots, b_m\}$. On the other hand for $(1 \leq i \leq m-1)$, $\operatorname{Cl}(U_{\tau_i})$ is cyclic of order $|\det\left[\begin{smallmatrix} a_i & a_{i+1} \\ b_i & b_{i+1}\end{smallmatrix}\right]|$ and $|\operatorname{Cl}(U_{\tau_m})| = |\det\left[\begin{smallmatrix} a_m & 1 \\ b_m & 0\end{smallmatrix}\right]|$ by (2). Since $\gcd(a_i, b_i) = 1$ for each i, an easy calculation shows $\gcd\{|\operatorname{Cl}(U_{\tau_i})|\}_{i=1}^m = \gcd\{b_1, \dots, b_m\}$. □

To determine the p-subgroups of $B'(T_N/X)$ for each prime number p, we introduce some additional notation and terminology. Let Γ be a finite edge-weighted graph such that to each edge E is associated the positive integer weight(E). Let v_p be the p-adic valuation on $\mathbb{Z}$ and set the *p-weight of* $E = \text{weight}_p(E) = v_p(\text{weight}(E))$. If Γ_1 is a subgraph of Γ, let $\text{weight}_p(\Gamma_1) = \sum \text{weight}_p(E)$ where the summation is over all edges E in Γ_1. A *p-maximal spanning tree for* Γ is a spanning tree T for Γ such that $\text{weight}_p(T)$ is maximal among the p-weights of all spanning trees. It is clear that every connected graph has a p-maximal spanning tree. Let T be a p-maximal spanning tree for Γ and let c denote an edge of Γ which is not an edge of T. Since T is a spanning tree, the subgraph Γ_1 of Γ obtained by adding the edge c to T contains a cycle Z which is unique since there is a unique path between any two vertices of the tree T. Suppose there is some edge E in Z with $\text{weight}_p(E) < \text{weight}_p(c)$. Then we could obtain a spanning tree of larger p-weight by deleting the edge E from Γ_1. This means that if T is a p-maximal spanning tree for Γ, c is an edge of Γ not in T and Z is the unique cycle in the graph $T \cup \{c\}$, then c is an edge of minimal p-weight in Z.

Let Γ be the (connected) graph we have associated to the fan Δ whose cones all have dimension ≤ 2. Assign weights to the edges $\tau_i - \rho_{ij}$ of Γ by setting $\text{weight}(\tau_i - \rho_{ij}) = b_i = |\operatorname{Cl}(U_{\tau_i})|$ (recall $\operatorname{Cl}(U_{\tau_i}) = \mathbb{Z}/b_i$ is cyclic from (2)). Let T be a p-maximal spanning tree for Γ. We have labeled the edges of Γ not in T as $\tau_1 - \rho_{12}, \tau_2 - \rho_{22}, \dots, \tau_e - \rho_{e2}$. We call the set of 2-dimensional cones $\{\tau_1, \dots, \tau_e\}$ in Δ a *p-minimal set of cones in* Δ. If Γ is not connected, then we can decompose Δ as a union of fans Δ_i with $\Delta_i \cap \Delta_j = \{0\}$ when $i \neq j$ and the graphs Γ_i associated to Δ_i are connected. We define a p-minimal set of cones in Δ to be the union of p-minimal sets of cones in each Δ_i.

Theorem 2.10. *Let Δ be a fan on $N_{\mathbb{R}}$ and assume every cone in Δ has dimension ≤ 2. Let $\{\tau_1, \dots, \tau_e\}$ be a p-minimal set of cones in Δ and let $|\operatorname{Cl}(U_{\tau_i})| = b_i$. If $X = T_N \operatorname{emb}(\Delta)$, then $B'(T_N/X)_p \cong [\bigoplus_{i=1}^e \mathbb{Z}/b_i]_p$. This isomorphism is induced by the epimorphism ψ of* (3).

Proof. By Proposition 2.5 and the discussion preceeding the theorem, we can assume the graph Γ associated to the fan Δ is connected. Let T be a p-maximal spanning tree for Γ. Continuing the analysis that was begun in the proof of Theorem 2.6 we consider the matrix $[Q|C]$ defined there. If $\psi\colon \bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})\to \mathrm{B}'(T_N/X)$ is the epimorphism given in (3) and $\widehat{\mathbb{Z}}_p$ is the p-adic integers, we have an epimorphism

$$\psi_p\colon \bigoplus_{i=1}^{m}(\widehat{\mathbb{Z}}_p\rho_{i1}\oplus\widehat{\mathbb{Z}}_p\rho_{i2})\to \mathrm{B}'(T_N/X)_p.$$

It follows from Lemma 2.7 that $\{c_1, \dots, c_e\}$ generates $\operatorname{coker}(\alpha+\beta)$ so $\{\psi_p(c_1), \dots, \psi_p(c_e)\}$ generates $\mathrm{B}'(T_N/X)_p$. We check

$$0=\langle\psi_p(c_j)\rangle\cap\langle\psi_p(c_1), \dots, \psi_p(c_{j-1}), \psi_p(c_{j+1}), \dots, \psi_p(c_e)\rangle$$

and $\psi_p(c_j)$ has order $p^{v_p(b_j)}$ for $1\le j\le e$ by identifying these elements in $\mathrm{B}'(T_N/X)$ with their corresponding preimages $c_j+\operatorname{image}(\alpha+\beta)\in\operatorname{coker}(\alpha+\beta)$ in (3) and then checking the corresponding statements in $\operatorname{coker}(\alpha+\beta)$.

Fix j and let π be a permutation of $\{1, \dots, m\}$ with π chosen so $\pi(1)=j$ where the edge $c_j=\tau_j\text{—}\rho_s$ and the cycle in $T\cup\{c_j\}$ is

$$\tau_j=\tau_{\pi(1)}\text{—}\rho_1\text{—}\tau_{\pi(2)}\text{—}\rho_2\text{—}\cdots\text{—}\tau_{\pi(s)}\text{—}\rho_s\text{—}\tau_{\pi(1)}.$$

Choose the vertex $\tau_{\pi(s)}$ as the root node for T. By Lemma 2.7 we know the columns of $[B|C]$ form a basis for $\bigoplus_{i=1}^{m}(\mathbb{Z}\rho_{i1}\oplus\mathbb{Z}\rho_{i2})$. The column space of B is a submodule of $\operatorname{image}(\alpha+\beta)$. Project the submodule $\widehat{\mathbb{Z}}_p\beta_1+\cdots+\widehat{\mathbb{Z}}_p\beta_m$ of $\operatorname{image}(\alpha+\beta)$ onto a $\widehat{\mathbb{Z}}_p$-submodule of the column space of C over $\widehat{\mathbb{Z}}_p$. Then $\operatorname{coker}(\alpha+\beta)_p$ is the quotient module.

If $s+1\le i\le m$ we check the projection of $\beta_{\pi(i)}$ on $\widehat{\mathbb{Z}}_pc_j$ is 0. The selection of $\tau_{\pi(s)}$ as the root node for T gives a partial order on the vertices of T. Let T_i be the subtree of T with root node $\rho_{\pi(i)2}$. This means the vertices v in T_i are those for which the unique path from v to $\tau_{\pi(s)}$ contains $\rho_{\pi(i)2}$. In the expression for $\beta_{\pi(i)}$ as a linear combination of the columns of $[B|C]$, the columns of C that appear are those c_k which when considered as edges of Γ are incident to some vertex in T_i (see the proof of Theorem 2.6). But neither τ_j nor ρ_s are in T_i since $i\ge s+1$ so the projection of $\beta_{\pi(i)}$ on c_j has coefficient $=0$.

If $1\le i\le s$ let the projection of $\beta_{\pi(i)}$ on $\bigoplus_{k=1}^{e}\widehat{\mathbb{Z}}_pc_k$ be $\sum_{k=1}^{e}d_{ki}c_k$. We say in the proof of Theorem 2.6 that $b_{\pi(i)}|d_{ji}$ (where $b_{\pi(i)}=|\operatorname{Cl}(U_{\pi(i)})|$). The projections of $\beta_{\pi(1)}, \dots, \beta_{\pi(m)}$ on $\bigoplus_{k=1}^{e}\widehat{\mathbb{Z}}_pc_k$ are the columns of the $e\times m$ matrix (d_{ki}). We have observed above that $d_{j(s+1)}=\cdots=d_{jm}=0$. The definitions of β_j and c_j imply $\beta_j=b_jc_j$ so $d_{k1}=0$ if $k\ne j$. Also $b_{\pi(i)}|d_{ji}$ for $2\le i\le s$ and we chose j with $v_p(b_j)=\min\{v_p(b_{\pi(1)}), \dots, v_p(b_{\pi(s)})\}$. Thus after elementary column operations over $\widehat{\mathbb{Z}}_p$ the column space of (d_{ki}) is equal to the column space of

$$\begin{bmatrix} 0 & d_{12} & \cdots & d_{1m}\\ \vdots & \vdots & & \vdots\\ 0 & d_{(j-1)2} & & \\ b_j & 0 & \cdots & 0\\ 0 & d_{(j+1)2} & & \\ \vdots & \vdots & & \vdots\\ 0 & d_{e2} & \cdots & d_{em}\end{bmatrix}$$

Therefore $c_1, \dots, c_e$ represents a basis for $\operatorname{coker}(\alpha + \beta)_p$, $\mathrm{B}'(T_N/X)_p = \langle \psi_p(c_1) \rangle \oplus \cdots \oplus \langle \psi_p(c_e) \rangle$, and $\langle \psi_p(c_j) \rangle$ is cyclic of order $p^{v_p(b_j)}$. □

Acknowledgment

This article was written while the first author visited at Florida Atlantic University. He thanks department chairman Jim Brewer for his hospitality.

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