ON THE BRAUER INDECOMPOSABILITY OF SCOTT MODULES

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Abstract. Let k be an algebraically closed field of prime characteristic p, and let P be a p-subgroup of a finite group G. We give sufficient conditions for the kG-Scott module $\mathrm{Sc}(G, P)$ with vertex P to remain indcomposable under the Brauer construction with respect to any subgroup of P. This generalizes similar results for the case where P is abelian. The background motivation for this note is the fact that the Brauer indecomposability of a p-permutation bimodule is a key step towards showing that the module under consideration induces a stable equivalence of Morita type, which then may possibly be lifted to a derived equivalence.

1. Introduction

Throughout this paper we denote by k an algebraically closed field of prime characteristic p. The Brauer construction with respect to a p-subgroup P of a finite group G sends a p-permutation kG-module M functorially to a p-permutation $kN_G(P)$-module $M(P)$; see e.g. [3, p.402] or [16, pp.91 and 219]. Following the terminology introduced in [9], the module M is called *Brauer indecomposable* if the $kC_G(Q)$-module $\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(M(Q))$ is indecomposable or zero for any p-subgroup Q of G. As mentioned in [9], the Brauer indecomposability of p-permutation modules is relevant for the gluing technique used for proving categorical equivalences between p-blocks of finite groups as in Broué's abelian defect group conjecture, see [10], [9] and [17]. For any subgroup H of G there is up to isomorphism a unique indecomposable direct summand of the permutation kG-module kG/H which has a Sylow p-subgroup of H as a vertex and the trivial kG-module as a quotient. This is called the *Scott kG-module with respect to H*, denoted by $\mathrm{Sc}(G, H)$. If P is a Sylow p-subgroup of H, then $\mathrm{Sc}(G, H) = \mathrm{Sc}(G, P)$ is, up to isomorphism, the unique indecomposable kG-module with P as a vertex, the trivial kP-module as a source, and the trivial kG-module as a quotient. See [14, Chap.4, §8] and [3] for more details on Scott modules. We first prove a criterion for the Brauer indecomposability of Scott modules in terms of the indecomposability of Scott modules of certain local subgroups.

Theorem 1.1. *Let P be a p-subgroup of a finite group G. Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system of G on P. Suppose that $\mathcal{F}$ is saturated, and that $\mathcal{F} = \mathcal{F}_P(N_G(P))$. Then $\mathrm{Sc}(G, P)$ is Brauer indecomposable if and only if*

$$\mathrm{Res}^{N_P(Q)C_G(Q)}_{C_G(Q)}(\mathrm{Sc}(N_P(Q)C_G(Q), N_P(Q))$$

is indecomposable for any subgroup Q of P.

It is shown in [9, Theorem 1.2] that if P is an abelian p-subgroup of G, and if the fusion system $\mathcal{F}_P(G)$ is saturated, then the kG-Scott module $\mathrm{Sc}(G,P)$ is Brauer indecomposable. The following result extends this in some cases to non-abelian P.

Theorem 1.2. *Let P be a p-subgroup of a finite group G. Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system of G on P. Suppose that $\mathcal{F}$ is saturated, and that $\mathcal{F} = \mathcal{F}_P(N_G(P))$. Suppose that, for every subgroup Q of P, at least one of the following holds:*
(a) *$N_P(Q) = QC_P(Q)$.*
(b) *$C_G(Q)$ is p-nilpotent.*
Then $\mathrm{Sc}(G,P)$ is Brauer indecomposable.

If P is a common subgroup of two groups G and H, we denote by ΔP the 'diagonal' subgroup $\Delta P = \{(u,u) \mid u \in P\}$ of $G \times H$.

Corollary 1.3. *Let G be a finite group and P a Sylow p-subgroup of G. Set $M = \mathrm{Sc}(G \times N_G(P), \Delta P)$. Suppose that, for every subgroup Q of P, at least one of the following holds:*
(a) *$N_P(Q) = QC_P(Q)$.*
(b) *$C_G(Q)$ is p-nilpotent.*
Then M is Brauer indecomposable.

Remark 1.4. For P abelian, this is Corollary 1.4 of [9], which follows also from [10, Theorem]. Examples of non-abelian p-groups to which the above applies are all groups of order p^3 and metacyclic p-groups of the form $M_{n+1}(p) \cong C_{p^n} \rtimes C_p$, see [6, p.190], where C_m denotes a cyclic group of order m, for any positive integer m. See the Example 3.2 below for a stable equivalence of Morita type which is constructed making use of Corollary 1.3.

The above results will be proved in section 3. We adopt the following notation and conventions. All modules over finite group algebras are assumed to be finitely generated unitary left modules. We write $H \leq G$ if H is a subgroup of a group G, and $H \trianglelefteq G$ if H is normal in G. The trivial kG-module will be denoted again by k. For G a group, H a subgroup of G, M a kG-module and N a kH-module, we write as usual $\mathrm{Res}^G_H(M)$ for the restriction of M from kG to kH and $\mathrm{Ind}^G_H(N)$ for the induction of N from kH to kG. For a subset S of G and an element $g \in G$, we write ${}^gS$ for gSg^{-1}, and for $h \in G$, we write ${}^gh = ghg^{-1}$. For $H, K \leq G$ we write $H \leq_G K$ when ${}^gH \leq K$ for an element $g \in G$. As mentioned before, given a p-subgroup P of a finite group G and a kG-module M, we write $M(P)$ for the Brauer construction with respect to P applied to M; see [4, p.402] or [16, pp.91 and 219]. We denote by $\mathcal{F}_P(G)$ the fusion system of G on P; that is, $\mathcal{F}_P(G)$ is the category whose objects are the subgroups of P and whose morphisms from Q to R are the group homomorphisms induced by conjugation by elements of G; see [2, Definition I.2.1] and [11, p.83]. If P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is *saturated*, see [2, Definition I.2.2]. If $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$, then the saturation of $\mathcal{F}_P(G)$ is equivalent to requiring that $N_G(P)/PC_G(P)$ has order prime to p. For any remaining notation and terminology, see the books of [14] and [16], and also [2] and [11] for fusion systems.

2. Lemmas

This section contains some technicalities needed for the proofs of the main results in the section. We start with a very brief review of some basic properties of Scott modules. Let G be a finite group, H a subgroup of G, and P a Sylow p-subgroup of H. Let M be a p-permutation kG-module. In particular, M has a k-basis X which is permuted by the action of P. By [3, §1] or [16, Proposition (27.6)], the image in $M(P)$ of the subset X^P of P-fixed points in X is a k-basis of $M(P)$, and we have a direct sum decomposition of $kN_G(P)$-modules $\mathrm{Res}^G_{N_G(P)}(M) = M(P) \oplus N$, where N is the span of the P-orbit sums of $X \setminus X^P$. For any subgroup Q of P we have $X^P \subseteq X^Q$. In particular, if $M(P) \neq \{0\}$, then $M(Q) \neq \{0\}$ for any subgroup Q of P. By [3, (1.3)], if M is an indecomposable p-permutation kG-module, then $M(P) \neq \{0\}$ if and only if P is contained in a vertex of M. By [3, (3.2) Theorem], if P is a vertex of M, then $M(P)$ is the Green correspondent of M. Frobenius' reciprocity implies that $\mathrm{Hom}_{kG}(\mathrm{Ind}^G_H(k), k) \cong \mathrm{Hom}_{kH}(k, k) \cong k$. Thus exactly one indecomposable direct summand of $\mathrm{Ind}^G_H(k)$ has a quotient isomorphic to the trivial kG-module. This summand is the *Scott module* $\mathrm{Sc}(G, H)$. Under the above isomorphism the identity map on k (viewed as a kP-module) corresponds to the unique kG-homomorphism $\eta : \mathrm{Ind}^G_H(k) \to k$ sending each $y \otimes 1_k$ to 1 for any $y \in G$. Thus the Scott module $\mathrm{Sc}(G, H)$ is, up to isomorphism, the unique indecomposable direct summand of $\mathrm{Ind}^G_H(k)$ which is not contained in $\ker(\eta)$. Applying the Brauer construction to η yields a non-zero map $\eta(P) : (\mathrm{Ind}^G_H(k))(P) \to k$, because the element $1 \otimes 1_k$ is a P-fixed element of the P-stable basis consisting of the elements $y \otimes 1_k$, with y running over a set of representatives of the cosets G/H in G. This shows in particular that $\mathrm{Sc}(G, H)$ has P as a vertex and therefore must coincide with $\mathrm{Sc}(G, P)$. We will use these facts without further reference. The following lemma is essentially a special case of a result of H. Kawai [8, Theorem 1.7].

Lemma 2.1. *Let G be a finite group, and let P and Q be p-subgroups of G such that $Q \leq P$. Suppose that for any $g \in G$ satisfying $Q \leq {}^gP$ we have $|N_{^gP}(Q)| \leq |N_P(Q)|$. Let M be an indecomposable p-permutation kG-module with vertex P. Set $H = N_G(Q)$. Then $\mathrm{Res}^G_H(M)$ has an indecomposable direct summand X satisfying $X(N_P(Q)) \neq \{0\}$, and any such summand has $N_P(Q)$ as a vertex. In particular, $\mathrm{Sc}(H, N_P(Q))$ is isomorphic to a direct summand of $\mathrm{Res}^G_H(\mathrm{Sc}(G, P))$ and of $(\mathrm{Sc}(G, P))(Q)$.*

Proof. We have $H \cap P = N_P(Q)$, and since $M(P)$ is non-zero, so is $M(H \cap P)$. Thus there is an indecomposable direct summand X of $\mathrm{Res}^G_H(M)$ such that $X(H \cap P) \neq \{0\}$. Let R be a vertex of X containing $H \cap P$. Since P is a vertex of M, it follows that M is isomorphic to a direct summand of $\mathrm{Ind}^G_P(k)$. The Mackey decomposition formula implies that X is isomorphic to a direct summand of

$$\mathrm{Res}^G_H(\mathrm{Ind}^G_P(k)) = \bigoplus_y \mathrm{Ind}^H_{H \cap {}^yP}(k) \ ,$$

where y runs over a set of representatives of the double cosets $H \backslash G/P$ in G. The indecomposability of X and the Krull-Schmidt theorem imply that there is $y \in G$ such that X is isomorphic to a direct summand of $\mathrm{Ind}^H_{H \cap {}^yP}(k)$. Then $H \cap {}^yP$ contains a vertex S of X. Since the vertices of an indecomposable module are conjugate, it follows that there is $h \in H$ such that $S = {}^hR$. The element h normalises Q, and hence $Q \leq S \leq H \cap {}^yP$. This implies $S \leq N_{^yP}(Q)$. The assumptions imply further that $|S| \leq |N_P(Q)| \leq |R|$. Since R and S are conjugate, they have the same order, whence $R = N_P(Q)$ is a vertex of X. For the second statement, suppose that $M = \mathrm{Sc}(G, P)$. That is, M is, up to isomorphism, the unique indecomposable direct summand of kG/P which is not in the kernel of the kG-homomorphism $kG/P \to k$ sending each coset yP to 1 where $y \in G$. As mentioned at the beginning of this section, the trivial coset P is a P-fixed point of the basis of kG/P consisting of the P-cosets in G, and hence applying the Brauer construction to a non-zero kG-homomorphism $M \to k$ yields a non-zero map $M(P) \to k$. Then also the map $M(R) \to k$ induced by a non-zero kG-homomorphism $M \to k$ is nonzero. It follows that $\mathrm{Res}^G_H(M)$ has an indecomposable direct summand X satisfying $X(R) \neq \{0\}$ such that there is a non-zero kH-homomorphism $X \to k$. By the first statement, R is a defect group of X. Thus $X \cong \mathrm{Sc}(H, R)$. This shows that $\mathrm{Sc}(H, R)$ is isomorphic to a direct summand of $\mathrm{Res}^G_{N_G(Q)}(M)$. Since R contains Q and Q is normal in H, it follows that Q acts trivially on $\mathrm{Sc}(H, Q)$, and thus $\mathrm{Sc}(H, Q)$ is isomorphic to a direct summand of $M(Q)$. $\square$

In fusion theoretic terminology, the hypothesis on the maximality of $|N_P(Q)|$ in the previous lemma is equivalent to requiring that Q is fully $\mathcal{F}$-normalised. If $\mathcal{F} = N_{\mathcal{F}}(P)$, then every subgroup of P is fully $\mathcal{F}$-normalised, which explains why this hypothesis is no longer needed in the second statemement of the next lemma. The proof of the first statement of the next lemma is essentially in [9, Theorem 1.2].

Lemma 2.2. *Let G be a finite group and P a p-subgroup of G. Set $\mathcal{F} = \mathcal{F}_P(G)$. Assume that $\mathcal{F}$ is saturated and that $\mathcal{F} = \mathcal{F}_P(N_G(P))$.*

(i) *Suppose that M is an indecomposable p-permutation kG-module with vertex P. Then for any subgroup Q of P and for any indecomposable direct summand X of $\mathrm{Res}^G_{N_G(Q)}(M)$ satisfying $X(Q) \neq \{0\}$, there is a vertex R of X such that $Q \leq R \leq P$.*

(ii) *For any subgroup Q of P, the module $\mathrm{Sc}(N_G(Q), N_P(Q))$ is a direct summand of $\mathrm{Res}^G_{N_G(Q)}(\mathrm{Sc}(G, P))$ and of $(\mathrm{Sc}(G, P))(Q)$.*

Proof. (i) Let X be an indecomposable direct summand of $\mathrm{Res}^G_{N_G(Q)}(M)$ such that $X(Q) \neq \{0\}$. There is a vertex R of X such that $Q \leq R$. Then $X(R) \neq \{0\}$, hence $M(R) \neq \{0\}$, and so R is contained in a vertex of M. Since the vertices of M are conjugate to P, it follows that there is $g \in G$ such that $Q \leq R \leq {}^gP$. Then ${}^{g^{-1}}Q \leq P$, which implies that the map ${}^{g^{-1}}Q \to Q$ sending u to ${}^gu$ is an isomorphism in the fusion system $\mathcal{F}$. The assumption $\mathcal{F} = \mathcal{F}_P(N_G(P))$ implies that there is an element $h \in N_G(P)$ such that $c = hg^{-1} \in C_G(Q)$. It follows that $Q = {}^cQ \leq {}^cR \leq {}^{cg}P = {}^hP = P$. Clearly ${}^cR$ is also a vertex of X, whence the statement.

(ii) Let $g \in G$ with $Q \leq {}^gP$. By the argument in the proof of (i), there is an element $h \in N_G(P)$ such that $c = hg^{-1} \in C_G(Q)$. Then $cg = h$ normalises P. Thus conjugation by c induces an isomorphism $N_{{}^gP}(Q) \cong N_P(Q)$; in particular, both groups have the same order. Therefore Lemma 2.1 implies the assertion. $\square$

3. Proof of the main result

Let G be a finite group and let M be an indecomposable p-permutation kG-module with vertex P. If Q is a p-subgroup of G which is not conjugate to a subgroup of P, then $M(Q) = \{0\}$. The property of $M(Q)$ being decomposable is invariant under conjugation of Q in G. Thus if M is not Brauer indecomposable, then there is a subgroup Q of P such that $M(Q)$ is decomposable as a $kC_G(Q)$-module. The key step towards proving the main results is the following lemma.

Lemma 3.1. *Let G be a finite group and P a p-subgroup of G. Set $\mathcal{F} = \mathcal{F}_P(G)$. Assume that $\mathcal{F} = \mathcal{F}_P(N_G(P))$ and that $\mathcal{F}$ is saturated. Set $M = \mathrm{Sc}(G, P)$. Suppose that M is not Brauer indecomposable. Let Q be a subgroup of maximal order in P such that $\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(M(Q))$ is decomposable. Then Q is a proper subgroup of P and setting $R = N_P(Q)$, we have*

$$\mathrm{Res}^{N_G(Q)}_{RC_G(Q)}(M(Q)) \cong \mathrm{Sc}(RC_G(Q), R)\ .$$

In particular, $\mathrm{Res}^{N_G(Q)}_{RC_G(Q)}(M(Q))$ is indecomposable with R as a vertex.

Proof. The $kC_G(P)$-module $M(P)$ is indecomposable by [9, Lemma 4.3(ii)] (this lemma requires the hypothesis on $\mathcal{F}$ being saturated). Thus Q is a proper subgroup of P, hence a proper subgroup of $R = N_P(Q)$, by [6, Chap.1, Theorem 2.11(ii)]. We first show that $M(Q)$ is indecomposable as a $kN_G(Q)$-module. By Lemma 2.2 (ii) we have $\mathrm{Res}^G_{N_G(Q)}(M) = \mathrm{Sc}(N_G(Q), R) \oplus X$ for some $kN_G(Q)$-module X, and $\mathrm{Sc}(N_G(Q), R)$ is isomorphic to a direct summand of $M(Q)$. We need to show that $X(Q) = \{0\}$. Arguing by contradiction, suppose that $X(Q) \neq \{0\}$. Then there exists an indecomposable direct summand Y of X such that $Y(Q) \neq \{0\}$. Since $\mathcal{F}$ is saturated and $N_{\mathcal{F}}(P) = \mathcal{F}$, it follows from Lemma 2.2(i) that Y has a vertex S such that $Q \le S \le P$. Then $S \le N_G(Q) \cap P = R$. Note that Q is not a vertex of M since $|Q| \neq |P|$. If $Q = S$, then Q is a vertex of Y, and hence Q is a vertex of M by the result of Burry-Carlson-Puig [14, Chap. 4, Theorem 4.6(ii)], a contradiction. Thus Q is a proper subgroup of S. Since Y is an indecomposable p-permutation $kN_G(Q)$-module with vertex S, we have $Y(S) \neq \{0\}$, and hence $X(S) \neq \{0\}$. Since R is a vertex of $\mathrm{Sc}(N_G(Q), R)$ and $S \le R$, it follows that $\Big(\mathrm{Sc}(N_G(Q), R)\Big)(S) \neq \{0\}$. We have

$$\Big(\mathrm{Res}^G_{N_G(Q)}(M)\Big)(S) \;=\; \Big(\mathrm{Sc}(N_G(Q), R)\Big)(S) \bigoplus X(S),$$

and both of the two direct summands of the right hand side are non-zero. This implies that $\Big(\mathrm{Res}^G_{N_G(Q)}(M)\Big)(S)$ is not indecomposable; in other words, $M(S)$ is not indecomposable as a $k(N_G(Q) \cap N_G(S))$-module. Since $C_G(S) \le C_G(Q) \cap N_G(S) \le N_G(Q) \cap N_G(S)$, it follows that $M(S)$ is not indecomposable as a $kC_G(S)$-module. But this contradicts the assumptions since $|P : S| < |P : Q|$. This shows that $X(Q) = \{0\}$, and hence that $M(Q)$ is indecomposable as a $kN_G(Q)$-module. Using Lemma 2.2 (ii), this shows that

$$M(Q) = \mathrm{Sc}(N_G(Q), R).$$

Set $L = RC_G(Q)$. Since $\mathcal{F} = \mathcal{F}_P(N_G(P))$, it follows that $N_G(Q) = (N_G(Q) \cap N_G(P))C_G(Q)$. The subgroup $N_G(P) \cap N_G(Q)$ normalises R, and hence L is a normal subgroup of $N_G(Q)$ and we have $N_G(Q) = (N_G(R) \cap N_G(Q))L$. In particular, L acts transitively on the set of $N_G(Q)$-conjugates of R. Since $M(Q)$ has R as a vertex and $R \le L$, there is an indecomposable kL-module V with vertex R such that $M(Q)$ is isomorphic to a direct summand of $\mathrm{Ind}^{N_G(Q)}_L(V)$. The Mackey formula, using that L is normal in $N_G(Q)$, implies that

$$\mathrm{Res}^{N_G(Q)}_L(M(Q)) = \bigoplus_x {}^xV$$

with x running over a subset E of $N_G(Q) \cap N_G(R)$. In particular, all indecomposable direct summands of $\mathrm{Res}^{N_G(Q)}_L(M(Q))$ have R as a vertex. Thus applying the

Brauer construction with respect to R sends every summand to a non-zero $kN_L(R)$-module. Therefore, if the set E has more than one element, then $M(Q)(R) = M(R)$ is decomposable as a $kN_L(R)$-module, hence also as a $kC_G(R)$-module. This contradicts the assumptions, and hence X consists of a single element, or equivalently, $\mathrm{Res}_L^{N_G(Q)}(M(Q))$ is indecomposable. Then necessarily $\mathrm{Res}_L^{N_G(Q)}(M(Q)) \cong \mathrm{Sc}(L, R)$, whence the result. □

Proof of Theorem 1.1. Set $M = \mathrm{Sc}(G, P)$. Suppose that M is Brauer indecomposable. Then $M(Q) = \mathrm{Sc}(N_G(Q), N_P(Q))$ by Lemma 2.2 (ii), and $M(Q)$ is indecomposable as a module for any subgroup of $N_G(Q)$ containing $C_G(Q)$. In particular, setting $M_Q = \mathrm{Sc}(N_P(Q)C_G(Q), N_P(Q))$, we have $M_Q \cong \mathrm{Res}^{N_G(Q)}_{N_P(Q)C_G(Q)}(M(Q))$. By the assumptions, the restriction to $kC_G(Q)$ of this module remains indecomposable. Suppose conversely that $\mathrm{Res}^{N_P(Q)C_G(Q)}_{C_G(Q)}(M_Q)$ remains indecomposable for all subgroups Q of P. Arguing by contradiction, suppose that M is not Brauer indecomposable. Let Q be a subgroup of maximal order of P such that $\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(M(Q))$ is decomposable. Set $R = N_P(Q)$. By Lemma 3.1, the $kRC_G(Q)$-module $\mathrm{Res}^{N_G(Q)}_{RC_G(Q)}(M(Q))$ is indecomposable with vertex R, hence isomorphic to $M_Q = \mathrm{Sc}(RC_G(Q), R)$ by Lemma 2.2 (ii). Thus $\mathrm{Res}^{RC_G(Q)}_{C_G(Q)}(M_Q)$ is decomposable, contradicting the assumptions. □

Proof of Theorem 1.2. Set $M = \mathrm{Sc}(G, P)$. Arguing by contradiction, let Q be a subgroup of maximal order in P such that $M(Q)$ is not indecomposable as a $kC_G(Q)$-module. Set $R = N_P(Q)$ and $L = RC_G(Q)$. It follows from Lemma 3.1 that Q is a proper subgroup of P, and that $\mathrm{Res}_L^{N_G(Q)}(M(Q))$ is indecomposable, with R as a vertex, hence isomorphic to $\mathrm{Sc}(L, R)$ by Lemma 2.2 (ii). By the construction of $M(Q)$, the group Q acts trivially on $M(Q)$.

Suppose first that hypothesis **(a)** holds; that is, $R = QC_P(Q)$. Then $L = QC_G(Q)$. Thus $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)}(M(Q))$ is indecomposable. Since Q acts trivially, it follows that $\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(M(Q))$ is indecomposable, a contradiction.

Thus hypothesis **(b)** holds; that is, $C_G(Q)$ is p-nilpotent. The indecomposable $kN_G(Q)$-module $M(Q) = \mathrm{Sc}(N_G(Q), R)$ is in the principal block as a $kN_G(Q)$-module, and its restriction to $L = RC_G(Q)$ remains indecomposable by the above. Hence we can assume that $O_{p'}(N_G(Q)) = 1$. Then also $O_{p'}(C_G(Q)) = 1$. This implies that $C_G(Q)$ is a p-group by (b). Hence the groups $C_G(Q)$, $C_G(R)$, $L = RC_G(Q)$, and $QC_G(Q)$ are all finite p-groups. Using that transitive permutation modules of finite p-groups are indecomposable, it follows that

$$\mathrm{Sc}(L, R) = \mathrm{Res}_L^{N_G(Q)}(M(Q)) \cong \mathrm{Ind}_R^L(k) .$$

The Mackey formula implies that

$$\begin{aligned} \mathrm{Res}^{N_G(Q)}_{QC_G(Q)}(M(Q)) &= \mathrm{Res}^L_{QC_G(Q)} \circ \mathrm{Res}_L^{N_G(Q)}(M(Q)) \\ &= \mathrm{Res}^L_{QC_G(Q)} \circ \mathrm{Ind}_R^L(k) \\ &= \mathrm{Ind}^{QC_G(Q)}_{QC_G(Q)\cap R}(k), \end{aligned}$$

since there is a single double coset here, and so only one term in the Mackey formula. This is again a transitive permutation module of the p-group $QC_G(Q)$, hence indecomposable. As before, since Q acts trivially on $M(Q)$, this implies that $\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(M(Q))$ is indecomposable. This concludes the proof. □

Proof of Corollary 1.3. Set $H = N_G(P)$. The fusion system of $G \times H$ on ΔP is equal to that of ΔH on ΔP, and this is saturated as P is a Sylow p-subgroup of H. Moreover, for Q a subgroup of P, we have $C_{G\times H}(\Delta Q) = C_G(Q) \times C_H(Q)$. Thus if $C_G(Q)$ is p-nilpotent, then so is $C_{G\times H}(\Delta Q)$. The result follows from Theorem 1.2. □

Example 3.2. Suppose that $p = 3$. Let G be a finite group. Assume that G has a Sylow 3-subgroup P such that $P \cong M_3(3)$, the extraspecial 3-group of order 27 of exponent 9. Set $H = N_G(P)$. Then the $k(G \times H)$-Scott module $M = \mathrm{Sc}(G \times H, \Delta P)$ induces a stable equivalence of Morita type between the principal blocks $B_0(kG)$ and $B_0(kH)$. This is trivial if G is 3-nilpotent because both blocks are isomorphic to kP in that case. If G is not 3-nilpotent, then $|N_G(P)/PC_G(P)| = 2$. Let Q be a non-trivial subgroup of P. It follows from Theorem 1.2, results of Hendren [7, Propositions 5.12 and 5.13] and the Z_3^*-theorem that $M(Q)$ induces a Morita equivalence between $B_0(kC_G(Q))$ and $B_0(kC_H(Q))$. Hence the gluing theorem [4, Theorem 6.3] implies that M induces a stable equivalence of Morita type between the principal blocks of kG and kH. Furthermore, by [4, Proposition 5.3], such a stable equivalence of Morita type implies the equality $\mathrm{k}(B_0(kG)) - \ell(B_0(kG)) = \mathrm{k}(B_0(kH)) - \ell(B_0(kH))$, where $\mathrm{k}(B_0(kG))$ and $\ell(B_0(kG))$ denote the number of ordinary and modular irreducible characters $B_0(kG)$, respectively, with the analogous notation for H instead of G. This yields a proof of a special case of a result of Hendren [7, Theorem 5.14]: if G is not 3-nilpotent, then $\mathrm{k}(B_0(kG)) - \ell(B_0(kG)) = 8$.

Remark 3.3. Let G be a finite group and P a Sylow p-subgroup of G. The Scott module $\mathrm{Sc}(G \times N_G(P), \Delta P)$ is the Green correspondent of the Scott module $\mathrm{Sc}(G \times G, \Delta P)$, which is isomorphic to the prinicipal block of kG viewed as a $k(G \times G)$-module. One might wonder how to generalise Corollary 1.3 to arbitrary blocks. Let b be a block of kG and let (P, e_P) be a maximal (G, b)-Brauer pair. Set $H = N_G(P, e_P)$. The $(G \times H)$-Green correspondent with vertex ΔP of the $k(G \times G)$-module kGb is of the form $M = kGf$ for some primitive idempotent f in $(kGb)^{\Delta H}$ satisfying $\mathrm{Br}_{\Delta P}(f)e_P \neq 0$ (see e.g. [1]). Note that (P, e_P) is also a maximal (H, e_P)-Brauer pair. For any subgroup Q of P denote by e_Q the unique block of $kC_G(Q)$ satisfying $(Q, e_Q) \leq (P, e_P)$ and by f_Q the unique block of $kC_H(Q)$ satisfying $(Q, e_Q) \leq (P, e_P)$. The 'obvious' generalisation of Corollary 1.3 would be the statement that the $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-bimodule $e_QM(\Delta Q)f_Q$ is indecomposable. This is, however, not the case in general. In order to construct an example for which this is not the case, we first translate this indecomposability to the source algebra level.

Let $j \in (kHe_P)^{\Delta P}$ be a source idempotent e_P as a block of kHe_P. Then $i = jf$ is a source idempotent of kGb (see e.g. [5, 4.10]). Thus multiplication by f induces an interior P-algebra homomorphism from $B = jkHj$ to $A = ikGi$. In particular, A can be viewed as an A-B-bimodule. Multiplication by a source idempotent, or more generally, by an almost source idempotent, is a Morita equivalence (cf. [15, 3.5] and [12, 4.1]). Moreover, the Brauer construction with respect to a fully $\mathcal{F}$-centralised subgroup Q of P sends the source idempotent i to the almost source idempotent $\mathrm{Br}_{\Delta Q}(i)$ in $kC_G(Q)e_Q$ (cf. [12, 4.5]). Through the appropriate Morita equivalences, the kGb-kHe_P-bimodule $M = kGf$ corresponds to the A-B-bimodule $iMj = ikGjf = A$, and the $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-bimodule $e_QM(\Delta Q)f_Q$ corresponds to the $A(\Delta Q)$-$B(\Delta Q)$-bimodule $A(\Delta Q)$. It follows that for Q a fully $\mathcal{F}$-centralised subgroup of P, the indecomposability of $e_QM(\Delta Q)f_Q$ is equivalent to the indecomposability of $A(\Delta Q)$ as an $A(\Delta Q)$-$B(\Delta Q)$-module.

We construct an example for which this fails. Suppose that p is odd. Let P be an extraspecial p-group of order p^3 of exponent p. Let Q be a subgroup of order p^2 in P; we have $C_P(Q) = Q$ and in particular, Q is fully centralised (even centric) with respect to any fusion system on P. Set $V = \mathrm{Inf}^P_{P/Q}(\Omega_{P/Q}(k))$. Thus $\dim_k(V) = p - 1$, and Q acts trivially on V. Setting $S = \mathrm{End}_k(V)$, it follows that $S = S^{\Delta Q} \cong S(\Delta Q)$. By the main result of Mazza in [13], there exists a nilpotent block of some finite group having a source algebra isomorphic to $A = S \otimes_k kP$. The Brauer correspondent of such a block has source algebra $B = kP$. We have $A(\Delta Q)$ $(S \otimes_k kP)(\Delta Q) \cong S \otimes_k kQ$ and $B(\Delta Q) = kQ$. Thus any primitive idempotent e in $S = S^{\Delta Q}$ determines a nontrivial direct bimodule summand $Se \otimes_k kQ$ of $A(\Delta Q)$, and hence $A(\Delta Q)$ is not indecomposable as an $A(\Delta Q)$-$B(\Delta Q)$-module.

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