GEORGIAN MATHEMATICAL JOURNAL: Vol. 5, No. 5, 1998, 401-414

ON THE BUCHSTABER SUBRING IN MSp_*

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Abstract. A formula is given to calculate the last n number of symplectic characteristic classes of the tensor product of the vector $Spin(3)$- and $Sp(n)$-bundles through its first $2n$ number of characteristic classes and through characteristic classes of $Sp(n)$-bundle. An application of this formula is given in symplectic cobordisms and in rings of symplectic cobordisms of generalized quaternion groups.

Introduction

Let $Spin(n)$ and $Sp(n)$ be classical Lie groups. The main result of this paper is the formula which, for any $Spin(3)$-bundle Λ and $Sp(n)$-bundle ζ^n, expresses the last n Pontrjagin characteristic classes of the symplectic bundle $\Lambda \otimes_R \zeta^n$ through its first $2n$ characteristic classes and through characteristic classes of the bundle ζ^n. This formula will be given in §1.

We obtain, as a corollary of the above-mentioned formula, some relations in the subring in symplectic cobordisms which is associated with universal Pontrjagin characteristic classes. This ring was introduced in [1] and shown to be generated by the coefficients of formal series given by the characteristic classes of the bundle $(\zeta_1 \otimes_H \zeta_2) \otimes_R \zeta_3$, where $\zeta_i \to BSp(1)$ are universal $Sp(1)$-bundles, and by Ray classes. In §2 some relations are established between the coefficients of these formal series. In particular, as we shall see, these relations generalize the formula $\varphi_i^4 = 0$ for Ray classes proved in [2].

In §3 some corollaries will be given for the rings of symplectic cobordisms of generalized quaternion groups.

§ 1. On Characteristic Classes of the Bundle $\Lambda \otimes_R \zeta^n$

The result of this section is

1991 *Mathematics Subject Classification.* 55N22, 55R12.
Key words and phrases. Symplectic cobordism, characteristic class, transfer.

Proposition 1.1. *For any $Spin(3)$-bundle Λ and $Sp(n)$-bundle ζ^n the formula*

$$pf_{3n-i}(\Lambda \otimes_R \zeta^n) = \sum_{k=0}^{i} pf_{n-k}(\zeta^n)\Delta_{2n-i+k}, \quad i = 0, \dots, n-1,$$

holds, where Δ_j, $j = n+1, \dots, 2n$, is the determinant of the $2n \times 2n$ matrix obtained from the matrix (a_{ql}),

$$a_{ql} = \begin{cases} pf_{q-l}(\zeta^n), & l < q \leq l+n, \\ 1, & q = l, \\ 0, & q < l \ \ or \ \ q > l+n \end{cases}$$

by replacing the j-column by the column

$$\begin{pmatrix} -pf_1(\zeta^n) + pf_1(\Lambda \otimes_R \zeta^n) \\ -pf_2(\zeta^n) + pf_2(\Lambda \otimes_R \zeta^n) \\ \vdots \\ -pf_n(\zeta^n) + pf_n(\Lambda \otimes_R \zeta^n) \\ \vdots \\ pf_{2n}(\Lambda \otimes_R \zeta^n) \end{pmatrix}.$$

Proof. Clearly, it is enough to prove the proposition for universal $Spin(3)$- and $Sp(n)$-bundles. Denote them also by Λ and ζ^n. The proof can be divided into two parts. As we shall see, the first part employs the standard technique of characteristic classes. The second part requires some calculations with transfer. The definition and properties of the transfer can be found in [3, 4, 5].

For the universal $Sp(1)$-bundle $\zeta \to BSp(1)$ we shall consider the real bundle $\zeta \otimes_H \zeta^*$, where ζ^* is the symplectic conjugate of ζ. Here the group $Sp(1)$ acts on $H = R^4$ by the conjugates, i.e., by $q(h) = qhq^{-1}$, $q \in Sp(1)$, $h \in H$. This action leaves the real numbers fixed and thus we have the section $\zeta \otimes_H \zeta^* = \Lambda + 1$ [1], where $\Lambda \to BSp(1) = BSpin(3)$ is the universal $Spin(3)$-bundle. The spherical bundle of the bundle Λ is $BU(1) \to BSp(1)$. The real projectivization of Λ coincides with the bundle $\pi : BN \to BSp(1)$ with the fiber RP^2, where N is the normalizer of the unitary group $U(1)$ in $Sp(1)$. Thus the canonical splitting

$$\pi^*(\Lambda) = \mu + \lambda$$

takes place, where μ is the plane and λ is the linear real bundle.

Let $\pi \times 1 : BN \times BSp(n) \to BSp(1) \times BSp(n)$. Then we have the bundle

$$(\pi \times 1)^*(\Lambda \otimes_R \zeta^n) = \mu \otimes_R \zeta^n + \lambda \otimes_R \zeta^n$$

to which we apply the Whitney formula for Pontrjagin's symplectic characteristic classes. We obtain

$$(\pi \times 1)^*\big(pf_1(\Lambda \otimes_R \zeta^n)\big) = pf_1(\mu \otimes_R \zeta^n) + pf_1(\lambda \otimes_R \zeta^n);$$
$$\vdots$$
$$(\pi \times 1)^*\big(pf_{2n}(\Lambda \otimes_R \zeta^n)\big) = pf_{2n}(\mu \otimes_R \zeta^n) +$$
$$+pf_{2n-1}(\mu \otimes_R \zeta^n)pf_1(\lambda \otimes_R \zeta^n) + \dots + pf_n(\mu \otimes_R \zeta^n)pf_n(\lambda \otimes_R \zeta^n);$$
$$(\pi \times 1)^*\big(pf_{2n+1}(\Lambda \otimes_R \zeta^n)\big) = pf_{2n}(\mu \otimes_R \zeta^n)pf_1(\lambda \otimes_R \zeta^n) +$$
$$+pf_{2n-1}(\mu \otimes_R \zeta^n)pf_2(\lambda \otimes_R \zeta^n) + \dots + pf_{n+1}(\mu \otimes_R \zeta^n)pf_n(\lambda \otimes_R \zeta^n);$$
$$\vdots$$
$$(\pi \times 1)^*\big(pf_{3n}(\Lambda \otimes_R \zeta^n)\big) = pf_{2n}(\mu \otimes_R \zeta^n)pf_n(\lambda \otimes_R \zeta^n).$$

It is obvious that from the first $2n$ equations we can express the characteristic classes of the bundle $\mu \otimes_R \zeta^n$ through the characteristic classes $(\pi \times 1)^*(pf_q(\Lambda \otimes_R \zeta^n))$, $q = 1, \dots, 2n$, and the characteristic classes of the bundle $\lambda \otimes_R \zeta^n$. By substituting these expressions of the classes $pf_j(\mu \otimes_R \zeta^n)$, $j = n+1, \dots, 2n$, into the other n equations we obtain the expressions of the characteristic classes $(\pi \times 1)^*(pf_{3n-i}(\Lambda \otimes_R \zeta^n))$, $i = 0, \dots, n-1$, through the classes $(\pi \times 1)^*(pf_q(\Lambda \otimes_R \zeta^n))$, $q = 1, \dots, 2n$, and through the classes of the bundle $\lambda \otimes_R \zeta^n$. Formally, the expressions for $pf_j(\mu \otimes_R \zeta^n)$ coincide with the determinant Δ_j from Proposition 1.1 if in the latter we replace the characteristic classes of the bundle ζ^n by the classes of the bundle $\lambda \otimes_R \zeta^n$, and the classes $pf_q(\Lambda \otimes_R \zeta^n)$ by the classes $(\pi \times 1)^*(pf_q(\Lambda \otimes_R \zeta^n))$, $q = 1, \dots, 2n$.

Thus we have

$$(\pi \times 1)^*\big(pf_{3n-i}(\Lambda \otimes_R \zeta^n)\big) =$$
$$= \sum_{k=0}^{i} pf_{n-k}(\lambda \otimes_R \zeta^n)\widetilde{\Delta}_{2n-i+k}, \quad i = 0, \dots, n-1, \tag{1.1}$$

where $\widetilde{\Delta}_j$, $j = n+1, \dots, 2n$, is the determinant of the $2n \times 2n$ matrix obtained from the matrix (b_{ql}),

$$b_{ql} = \begin{cases} pf_{q-l}(\lambda \otimes_R \zeta^n), & l < q \le l+n, \\ 1, & q = l, \\ 0, & q < l \ \text{ or } \ q > l+n, \end{cases}$$

by replacing the jth column by the column

$$\begin{pmatrix} -pf_1(\lambda\otimes_R\zeta^n)+(\pi\times 1)^*\big(pf_1(\Lambda\otimes_R\zeta^n)\big)\\ -pf_2(\lambda\otimes_R\zeta^n)+(\pi\times 1)^*\big(pf_2(\Lambda\otimes_R\zeta^n)\big)\\ \vdots\\ -pf_n(\lambda\otimes_R\zeta^n)+(\pi\times 1)^*\big(pf_n(\Lambda\otimes_R\zeta^n)\big)\\ \vdots\\ (\pi\times 1)^*\big(pf_{2n}(\Lambda\otimes_R\zeta^n)\big)\end{pmatrix}.$$

The second part of the proof of Proposition 1.1 is

Proposition 1.2. *Let $\tau(\pi\times 1)$ be the transfer mapping for the bundle $\pi\times 1$. Then*

$$\tau^*(\pi\times 1)\big(pf_i^k(\lambda\otimes_R\zeta^n)\cdot pf_j^l(\lambda\otimes_R\zeta^n)\big)=pf_i^k(\zeta^n)pf_j^l(\zeta^n)\tau^*(\pi)(1).$$

Proof. Let $\eta\to BZ_2$ be the universal $O(1)$-bundle. Since $MSp^*(BZ_2\times BSp(n))=MSp^*(BZ_2)[[pf_1(\zeta^n),\dots,pf_n(\zeta^n)]]$, the decomposition

$$pf_j(\eta\otimes_R\zeta^n)=pf_j(\zeta^n)+\sum_{i_1\dots i_n\geq 0}a^{(j)}_{i_1\dots i_n}pf_1^{i_1}(\zeta^n)\cdots pf_n^{i_n}(\zeta^n),\ \ j=1,\dots,n,$$

takes place for some $a^{(j)}_{i_1\dots i_n}\in\widetilde{MS}p^*(BZ_2)$.

Let now $f:BN\to BZ_2$ be the classifying mapping for the bundle $\lambda\to BN$. Consider the mapping $f\times 1:\ BN\times BSp(n)\to BZ_2\times BSp(n)$. Clearly, $(f\times 1)^*(\eta\otimes_R\zeta^n)=\lambda\otimes_R\zeta^n$, and in $MSp^*(BN\times BSp(n))$ we have

$$pf_j(\lambda\otimes_R\zeta^n)=pf_j(\zeta^n)+\sum_{i_1\dots i_n\geq 0}f^*(a^{(j)}_{i_1\dots i_n})pf_1^{i_1}(\zeta^n)\cdots pf_n^{i_n}(\zeta^n).$$

Thus $pf_i^k(\lambda\otimes_R\zeta^n)pf_j^l(\lambda\otimes_R\zeta^n)$ can be rewritten as

$$\begin{aligned} &pf_i^k(\lambda\otimes_R\zeta^n)pf_j^l(\lambda\otimes_R\zeta^n)=\\ &=pf_i^k(\zeta^n)pf_j^l(\zeta^n)+\sum_{i_1\dots i_n\geq 0}f^*(b^{(i,j,k,l)}_{i_1\dots i_n})pf_1^{i_1}(\zeta^n)\cdots pf_n^{i_n}(\zeta^n)\end{aligned}\tag{1.2}$$

for some $b^{(i,j,k,l)}_{i_1\dots i_n}\in\widetilde{MS}p^*(BZ_2)$. □

Next we apply the transfer mapping $\tau(\pi\times 1)$ to (1.2). Since $\tau(\pi\times 1)=\tau(\pi)\wedge\tau(1)=\tau(\pi)\wedge 1$ it is clear that Proposition 1.2 can be reduced to calculating the homomorphism $\tau^*(\pi)$. Thus we have to prove

Proposition 1.3. $\tau^*(\pi)f^*(a)=0\ \forall a\in\widetilde{MS}p^*(BZ_2)$.

This proposition is proved in [2]. Here we give an outline of another proof (compare with [2]).

Proof. For the real vector bundles Λ, $\zeta\otimes_H\zeta^* = \Lambda+1$, $\zeta_1\otimes_H\zeta_2^*$, $\zeta_1\otimes_H\zeta_2^*+1$, consider the associated spherical and projective bundles

$$S(\Lambda) = BU(1) \to BSp(1), \tag{1.3}$$
$$P(\Lambda) = BN \to BSp(1), \tag{1.4}$$
$$S(\Lambda+1) \to BSp(1), \tag{1.5}$$
$$P(\Lambda+1) \to BSp(1), \tag{1.6}$$
$$S(\zeta_1 \otimes_H \zeta_2^*) = BSp(1) \to BSp(1)^2, \tag{1.7}$$
$$P(\zeta_1 \otimes_H \zeta_2^*) \to BSp(1)^2, \tag{1.8}$$
$$S(\zeta_1 \otimes_H \zeta_2^* + 1) \to BSp(1)^2, \tag{1.9}$$
$$P(\zeta_1 \otimes_H \zeta_2^* + 1) \to BSp(1)^2. \tag{1.10}$$

Recall from the theory of spherical bundles [6] that the spherical bundle associated with the Whitney sum of the real vector bundle ξ over the space B and the trivial linear vector bundle over B is the suspension $\Sigma S(\xi)$ in the category of spaces over B. Therefore $S(\Lambda+1)$ is the suspension of $BU(1)$ over $BSp(1)$, i.e., the factor-space of the union

$$BU(1) \times I \cup BSp(1) \times \dot{I}, \quad I = [-1,1], \quad \dot{I} = \{-1,1\},$$

where the points (x,t) and (projection of x for (1.3), t), $x \in BU(1)$, $t \in \dot{I}$ are identified.

One can easily see that into the above suspension one can embed the disconnected union $BU(1)\times(-1,1)\cup BSp(1)\times\{-1\}\cup BSp(1)\times\{1\}$.

The involution that gives the projectivization $P(\Lambda+1)$ acts on $S(\Lambda+1)$ in the following manner: on I it changes the sign, on $BU(1)$ it coincides with the involution giving the projectivization $P(\Lambda)$ and leaves the points $BSp(1)$ motionless.

Thus into $P(\Lambda+1)$ we can embed the disconnected union

$$BSp(1) \cup P(\Lambda) \cup BU(1) \times (0,1).$$

Taking the above fact into account and applying the double co-set formula we split the transfer homomorphism of bundle (1.5) into three homomorphisms, of which the first one is the transfer homomorphism of bundle (1.3) with the minus sign, while the second and the third ones are induced by the identical mapping of $BSp(1)$ onto itself.

Furthermore, the transfer homomorphism of bundle (1.6) is the sum of three homomorphisms, of which the first one coincides with the transfer homomorphism of bundle (1.3) with the minus sign, the second one coincides with the transfer homomorphism of bundle (1.4), and the third one is induced by the identical mapping of $BSp(1)$ onto itself.

One can proceed in a similar manner in the case of bundles (1.9) and (1.10). Namely, $S(\zeta_1 \otimes_H \zeta_2^* + 1)$ is the suspension of $S(\zeta_1 \otimes_H \zeta_2^*)$ over $BSp(1)^2$, i.e., the factor-space of the union

$$S(\zeta_1 \otimes_H \zeta_2^*) \times I \cup BSp(1)^2 \times \dot{I},$$

where the points (x, t) and (projection of x by (1.7), t), $x \in BSp(1)$, $t \in \dot{I}$ are identified.

Thus we can embed the disconnected union

$$S(\zeta_1 \otimes_H \zeta_2^*) \cup BSp(1)^2 \times \{1\} \cup BSp(1)^2 \times \{-1\}$$

into $S(\zeta_1 \otimes_H \zeta_2^* + 1)$.

The involution acting on $S(\zeta_1 \otimes_H \zeta_2^* + 1)$ and giving the projectivization of (1.10) acts as follows: on I it changes the sign, on $Sp(1)^2$ it acts trivially, and on $S(\zeta_1 \otimes_H \zeta_2^*)$ it coincides with the involution giving the projectivization of (1.8). Thus into $P(\zeta_1 \otimes_H \zeta_2^* + 1)$ we can embed the disconnected union

$$P(\zeta_1 \otimes_H \zeta_2^*) \cup BSp(1)^2 \cup S(\zeta_1 \otimes_H \zeta_2^*) \times (0, 1).$$

Applying the double coset formula [7], we see that the transfer homomorphism of (1.9) is the sum of three homomorphisms, of which the first one coincides with the transfer homomorphism of bundle (1.7) with the minus sign, the second one is identical, and the third homomorphism is induced by permutation of the factors in $Sp(1)^2$.

Furthermore, the transfer homomorphism of bundle (1.10) is the sum of three homomorphisms, of which the first one corresponds to the transfer homomorphism of bundle (1.8), the second one corresponds to the transfer of bundle (1.7) but with the minus sign, and the third one is identical, since it is induced by the identical mapping of $BSp(1)^2$ onto itself.

Note that bundles (1.5) and (1.6) are the pullbacks of bundles (1.7) and (1.8), which enables us to apply the results of calculations for (1.7) and (1.8) to (1.5) and (1.6). Further, as shown in [2] for the proof of Proposition 1.3, it is sufficient to prove an analogous proposition for the projectivization $P(\zeta^2)$ of the universal $Sp(2)$-bundle ζ^2.

$P(\zeta^2)$ can be realized as BN, where N is the normalizer of $Sp(1)^2$ in $Sp(2)$ or as an orbit space $S^\infty \times BSp(1) \times BSp(1)/\mathit{involution}$, where the involution acts on an infinite sphere antipodally and on $BSp(1) \times BSp(1)$ by permuting the factors. Hence we have the obvious inclusion $i : BZ_2 \to BN$ and for the above $\pi : BN \to BZ_2$ the composition of i and π is the identity. Moreover, we have the bundle $p : BN \to BSp(2)$ with the fiber RP^4 with the Euler characteristic 1. Let tr be the reduced transfer

$E(BSp(2)) \to E(BSp(2)^+) \to E(BN^+) \to E(BN)$. It is sufficient to construct a splitting of the suspension spectrum $E(BN)$ of BN of the form $E(BZ_2) \vee E(BSp(2)) \vee X = E(BN)$ and to check that in terms of this splitting the projection $E(\pi)$ is the projection onto the first summand, and tr is the equivalence on the second term. First, we shall construct another splitting. Consider the Puppe cofibration sequence

$$BN \xrightarrow{\pi} BZ_2 \to BZ_2 \cup_\pi CBN \xrightarrow{\delta_\pi} \Sigma BN \xrightarrow{\Sigma\pi} SBZ_2$$

Since the composition πi is the identity, we have the splitting

$$\Sigma^k BZ_2 \vee \Sigma^{k-1}(BZ_2 \cup_\pi CBN) \xrightarrow{\Sigma^k i \vee \delta_\pi} \Sigma^k BN$$

for $k \geq 2$. Then we define $\rho : \Sigma^k BN \to \Sigma^k BZ_2 \vee \Sigma^k BSp(2)$ to be the composite

$$\Sigma^k BN \xrightarrow{\alpha} \Sigma^k BZ_2 \vee \Sigma^{k-1}(BZ_2 \cup_\pi CBN) \xrightarrow{1 \vee \Sigma^{k-1}\delta_\pi}$$
$$\Sigma^k BZ_2 \vee \Sigma^k BN \xrightarrow{1 \vee \Sigma^k p} \Sigma^k BZ_2 \vee \Sigma^k BSp(2).$$

where α is the homotopy inverse of $\Sigma^k i \vee \Sigma^k \delta_\pi$.

Consider the Puppe cofibration sequence for ρ and let Y be the cofiber of ρ. The composition of $\Sigma^k i \vee \Sigma^k tr$ and ρ induces an Id homomorphism in homologies. This follows from the homology structure of BZ_2 and $BSp(2)$ and obviously the composition of tr and $E(\pi)$ induces the zero homomorphism in homologies, i.e., $\operatorname{Im} tr_* \in \ker \pi_* = \operatorname{Im} \delta_{\rho_*}$. Therefore

$$\rho_*(i \vee tr)_*(a \oplus b) = a \oplus p_* tr_*(b) = a \oplus b$$

in the homologies of $BZ_2 \vee BSp(2)$ and the stable Whitehead lemma gives the self-homotopy equivalence of $\Sigma^{k+1}E(BZ_2) \vee \Sigma^{k+1}E(BSp(2))$ and the splitting

$$\Sigma^{k+1}E(Z_2) \vee \Sigma^{k+1}E(BSp(2)) \vee E(Y) = \Sigma^{k+1}E(BN).$$

After applying Σ^{-k-1}, we obtain the above splitting as needed. Now we can calculate the transfer homomorphism for (1.10). Next, applying the double co-set formula and the above arguments, we calculate the transfer homomorphism for (1.6). After that we repeat the procedure and calculate the transfer homomorphism for (1.4), which completes the proof of Proposition 1.3. □

We are now ready to complete the proof of Proposition 1.1. By Proposition 1.3 we have

$$\tau^*(\pi)\big(f^*(b^{(i,j,k,l)}_{i_1\dots i_n})\big) = 0,$$

where $b^{(i,j,k,l)}_{i_1\dots i_n} \in \widetilde{MSp}^*(BZ_2)$ from (1.2).

By these equalities we find from (1.2) that

$$(\pi\times 1)^*\big(pf_i^k(\lambda\otimes_R\zeta^n)pf_j^l(\lambda\otimes_R\zeta^n)\big)=\big(pf_i^k(\zeta^n)pf_j^l(\zeta^n)\big)\tau^*(\pi)(1),$$

which proves Proposition 1.2.

Next, by Proposition 1.2 we find from (1.1) that

$$\tau^*(\pi\times 1)\big((\pi\times 1)^*(pf_{3n-i}(\Lambda\otimes_R\zeta^n))\big)=$$
$$=\sum_{k=0}^{i}pf_{n-k}(\zeta^n)\tau^*(\pi\times 1)(\widetilde{\Delta}_{2n-i+k})=\sum_{k=0}^{i}pf_{n-k}(\zeta^n)\Delta_{2n-i+k})\tau^*(\pi)(1).$$

and, on the other hand,

$$\tau^*(\pi\times 1)\big((\pi\times 1)^*(pf_{3n-i}(\Lambda\otimes_R\zeta^n))\big)=$$
$$=pf_{3n-i}(\Lambda\otimes_R\zeta^n)\tau^*(\pi\times 1)(1)=pf_{3n-i}(\Lambda\otimes_R\zeta^n)\tau^*(\pi)(1),$$

Since $\tau^*(\pi)(1)$ is invertible, this completes the proof of Proposition 1.1. □

§ 2. On the Buchstaber Subring in MSp^*

In [1], V. M. Buchstaber introduced the subring in symplectic cobordisms associated with universal Pontrjagin characteristic classes. This ring is defined as a subring in symplectic cobordisms, consisting of the coefficients of formal series given by Pontrjagin characteristic classes of all symplectic bundles over the space $Y(q,n)$, where

$$Y(q,n)=S^q\wedge\Big(\prod^n BSp(1)\Big),\quad q\geq 0,\quad n\geq 1,$$

if we fix an additive isomorphism $MSp^*(Y(q,n))=MSp^*[\,[x_1,\dots,x_n]\,]$, where $x_i=pf_1(\zeta_i)$, $\zeta_i\to BSp(1)$ are the universal $Sp(1)$-bundles.

In the same paper, it was shown that this ring is generated by coefficients of the series $pf_j((\zeta_1\otimes_H\zeta_2^*)\otimes_R\zeta_3)$, $j=1,\dots,4$, from $MSp^*(\prod BSp(1))$ and by the coefficients of $pf_1(\eta\otimes_R\zeta)$ from $MSp^*(S^1\wedge BSp(1))$, i.e., by the Ray classes θ_i.

As one knows, of the Ray classes θ_i the classes θ_1, θ_{2i} are indecomposable and have order 2 [8]; $\theta_{2i-1}=0$, $i>1$ [9], so that most of the triple products of Ray classes are nonzero [10], [11].

We shall show that the formula $\theta_i^4=0\ \forall i\geq 1$ proved in [2] is a corollary of Proposition 1.1. Indeed, in a particular case with Λ as a universal $Spin(3)$-bundle and ζ^n as a trivial $Sp(1)$-bundle, Proposition 1.1 implies $pf_3(\Lambda\otimes_R H)=0$. But, as shown in [2], this formula is equivalent to the formula $\varphi_i=0$. Note also that in [2] the formula $\varphi_i\varphi_j\varphi_k\varphi_l=0\ \forall i,j,k,l\geq 1$ is shown, where $\varphi_i=\theta_{2i}$.

Now let us consider another particular case of Proposition 1.1. Let Λ be again the universal $Spin(3)$-bundle and ζ the universal $Sp(1)$-bundle. Then (1.1) takes the form

$$pf_3(\Lambda \otimes_R \zeta) = pf_1^3(\zeta) - pf_1^2(\zeta)pf_1(\Lambda \otimes_R \zeta) + pf_1(\zeta)pf_2(\Lambda \otimes_R \zeta). \quad (2.1)$$

Using the notation of §1, we write

$$pf_i(x, y) = pf_i\big((\zeta_1 \otimes_H \zeta_1^*) \otimes_R \zeta_2\big),$$
$$\Lambda + 1 = \zeta_1 \otimes_H \zeta_1^*, \quad \zeta_2 = \zeta, \quad x = pf_1(\zeta_1), \quad y = pf_1(\zeta_2).$$

We have

Corollary 2.1. $4y^3 - 3y^2 pf_1(x, y) + 2y pf_2(x, y) - pf_3(x, y) = 0.$

Proof. By the Whitney formula for characteristic classes we have

$$pf_1(x, y) = pf_1(\Lambda \otimes_R \zeta) + y,$$
$$pf_2(x, y) = pf_2(\Lambda \otimes_R \zeta) + y pf_1(\Lambda \otimes_R \zeta),$$
$$pf_3(x, y) = pf_3(\Lambda \otimes_R \zeta) + y pf_2(\Lambda \otimes_R \zeta).$$

Therefore

$$pf_3(\Lambda \otimes_R \zeta) = pf_3(x, y) - y pf_2(x, y) + y^2 pf_1(x, y) - y^3.$$

On the other hand, taking into account that by (2.1)

$$pf_3(\Lambda \otimes_R \zeta) = y^3 - y^2(pf_1(x, y) - y) + y\big(pf_2(x, y) - y(pf_1(x, y) - y)\big) =$$
$$= 3y^3 - 2y^2 pf_1(x, y) + y pf_2(x, y)$$

and equating these expansions for $pf_3(\Lambda \otimes_R \zeta)$, we obtain Corollary 2.1.

Clearly, the coefficients of series $pf_i(x, y)$ belong to the Buchstaber ring, since these series are obtained from the series

$$pf_i(x_1, x_2, x_3) = pf_i\big(\zeta_1 \otimes_H \zeta_2^*) \otimes_R \zeta_3\big)$$

if $x_1 = x_2 = x$ and $x_3 = y$.

On the other hand, the arguments used in proving Corollary 2.1 imply that the coefficients of the series $pf_i(x, y)$ are expressed through the coefficients of the series $pf_i(\Lambda \otimes_R \zeta)$ and vice versa. Thus, by equating the coefficients of the same monomes in the (2.1) or, with equal success, in Corollary 2.1 we obtain relations between the generatrices of the Buchstaber ring.

We identify $MU^*(BSp(1) \times BSp(1)$ with the subring $MU^*(BU(1) \times BU(1))$ by the monomorphism $(p \times p)^*$, where $p : BU(1) \to BSp(1)$ is the canonical bundle (it the spherical bundle $Spin(3)$ of the bundle $\Lambda \to BSp(1)$).

Let $\eta \to BU(1)$ be the universal $U(1)$-bundle. Since

$$p!(\Lambda) = \overline{\eta}_1^2 + R, \quad p!(\zeta) = \eta_2 + \overline{\eta}_2,$$

we have

$$\begin{aligned}(p \times p)!(\Lambda \otimes_R \zeta) &= (\eta_1^2 + \overline{\eta}_1^2 + \mathbb{C}) \otimes_C (\eta_2 + \overline{\eta}_2) = \\ &= (\eta_1^2 + \overline{\eta}_1^2) \otimes_C (\eta_2 + \overline{\eta}_2) + (\eta_2 + \overline{\eta}_2). \end{aligned} \tag{2.2}$$

Recall from [1] that for the homomorphism $\mu_{Sp}^U : MSp^*(\cdot) \to MU^*(\cdot)$ we have

$$\mu_{Sp}^U\big(pf_1((\eta_1 + \overline{\eta}_1) \otimes_C (\eta_2 + \overline{\eta}_2))\big) = \Theta_1\big(e(\eta_1 + \overline{\eta}_1), e(\eta_2 + \overline{\eta}_2)\big),$$
$$\mu_{Sp}^U\big(pf_2((\eta_1 + \overline{\eta}_1) \otimes_C (\eta_2 + \overline{\eta}_2))\big) = \Theta_2\big(e(\eta_1 + \overline{\eta}_1), e(\eta_2 + \overline{\eta}_2)\big),$$

where Θ_1, Θ_2 are the coefficients of a two-valued formal group in cobordisms, and e is Euler's class in complex cobordisms.

Therefore

$$\mu_{Sp}^U\big(pf_1(\eta_1^2 + \overline{\eta}_1^2)\big) = \mu_{Sp}^U\big(pf_1(\eta_1^2 + \overline{\eta}_1^2 + \mathbb{C})\big) =$$
$$= \mu_{Sp}^U\Big(pf_1\big((\eta_1 + \overline{\eta}_1) \otimes_C (\eta_1 + \overline{\eta}_1)\big)\Big) = \Theta_1\big(e(\eta_1 + \overline{\eta}_1), e(\eta_1 + \overline{\eta}_1)\big);$$

Further,

$$\mu_{Sp}^U\Big(pf_1\big((\eta_1^2 + \overline{\eta}_1^2) \otimes_C (\eta_2 + \overline{\eta}_2)\big)\Big) = \Theta_1\big(e(\eta_1^2 + \overline{\eta}_1^2), e(\eta_2 + \overline{\eta}_2)\big) =$$
$$= \Theta_1\big(\Theta_1(e(\eta_1 + \overline{\eta}_1), e(\eta_1 + \overline{\eta}_1)), e(\eta_1 + \overline{\eta}_2)\big),$$
$$\mu_{Sp}^U\Big(pf_2\big((\eta_1^2 + \overline{\eta}_1^2) \otimes_C (\eta_2 + \overline{\eta}_2)\big)\Big) =$$
$$= \Theta_2\big(\Theta_1(e(\eta_1 + \overline{\eta}_1), e(\eta_1 + \overline{\eta}_1)), e(\eta_1 + \overline{\eta}_2)\big).$$

By (2.2) we have

$$\mu_{Sp}^U(pf_1(\Lambda \otimes_R \zeta)) = \Theta_1\big(\Theta_1(x,x), y)\big) + y,$$
$$\mu_{Sp}^U(pf_2(\Lambda \otimes_R \zeta)) = \Theta_2\big(\Theta_1(x,x), y)\big) + \Theta_1\big(\Theta_1(x,x), y\big) \cdot y,$$
$$\mu_{Sp}^U(pf_3(\Lambda \otimes_R \zeta)) = \Theta_2\big(\Theta_1(x,x), y)\big) \cdot y.$$

Thus relation (2.1) in complex cobordisms takes the tautological form

$$y^3 - y^2\big(\Theta_1(\Theta_1(x,x), y)) + y\big) + y\big(\Theta_2(\Theta_1(x,x), y) + \Theta_1(x,x), y) \cdot y\big) =$$
$$= y\Theta_2\big(\Theta_1(x,x), y\big).$$

In symplectic cobordisms, relations obtained from (2.1) are not trivial in the dimensions where there are elements from the kernel $MSp^{4*} \to MU^{4*}$. □

§ 3. On Rings of Symplectic Cobordisms Classifying Spaces of Generalized Quaternion Groups

Let, as above, N be the normalizer of the unitary group $U(1)$ in the symplectic group $Sp(1)$ consisting of $U(1)$ and $jU(1)$, where j is the quaternion unit. Define, over BN, the real two-dimensional bundles $\eta^{(2m)}$ by the representation [2]

$$z \to z^{2m}, \quad j \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The real linear bundle $\lambda \to BN$ from §1 is defined by the representation $z \to 1$, $j \to -1$, where $z \in U(1)$, $m = 1, 2, \ldots$.

Let, further, H_m be a generalized quaternion group. For $m = 2$ the group $H_2 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a group of quaternions. In the general case H_m is defined as a subgroup in $Sp(1)$ generated by two elements

$$\alpha = \exp\left(\pi i / 2^{m-1}\right) \quad \text{and} \quad \beta = j.$$

Define, over BH_m, the real linear bundles $\eta_1^{(m)}$ and $\eta_2^{(m)}$ by the representations

$$\eta_1^{(m)} : \alpha \to 1, \quad \beta \to -1,$$
$$\eta_2^{(m)} : \alpha \to -1, \quad \beta \to 1.$$

Let, further, $\rho(K, L) : BK \to BL$ denote the mapping of the classifying spaces induced by the embedding of the Lie group K into L. We have

Proposition 3.1. (3.1.a) *The bundle $\rho(H_m, N)$ is the projectivization of the bundle $\eta^{(2k)}$, $k = 2^{m-2}$, and*

$$\rho^*(H_m, N)(\eta^{(2k)}) = \eta_2^{(m)} + \eta_1^{(m)} \otimes_R \eta_2^{(m)}, \quad \rho^*(H_m, N)(\lambda) = \eta_1^{(m)}.$$

(3.1.b) *For $k = 2^{n+1}$, $n \geq 1$, the real bundle $\eta^{(2)} + \eta^{(k)} \to BN$ is MSp^*-orientated.*

(3.1.c) *the real bundle $\eta^{(k)} + 3\lambda$ is MSp^*-orientated and has the zero symplectic Euler class.*

Proof. To prove (3.1.a), note that from the definition of the bundle $\eta^{(2m)}$ it follows that when N acts on the circle S^1 for the spherical bundle $S(\eta^{(2m)})$, the isotropy group is isomorphic to the group consisting of the elements $z = \pi i/m$ and j. When $m = 2^{n-1}$, this group coincides with H_n. Clearly, in that case $\rho(H_{n+1}, N)$ will be the projective bundle. The remainder of the proof follows from the definitions.

(3.1.b) This can be proved in two ways. Firstly, we can calculate Shtiefel-Whitney characteristic classes ω_i of the bundle $\eta^{(2)}+\eta^{(k)}$, $k=2^{n+1}$, using the result of [13]

$$H^*(BN,Z_2)=H^*(RP^2,Z_2)\otimes_{Z_2}H^*(BSp(1),Z_2).$$

This follows from the fact that the Serre spectral sequence for the bundle $BN\to BSp(1)$ with fiber RP^2 is trivial.

When the fiber RP^2 is embedded into BN, the bundle $\eta^{(k)}$ passes to $\nu+1$, where $\nu\to RP^2$ is a nontrivial linear bundle; the bundle $\eta^{(2)}$ passes to the tangent bundle $\tau\to RP^2$. But, as is well known, $\tau+1=3\nu$. The definitions of $\eta^{(2)}$ and $\eta^{(k)}$ imply that $\det(\eta^{(2)})=\det(\eta^{(k)})=\lambda$ and therefore

$$\omega_2(\eta^{(2)})=\omega_1^2(\eta^{(2)}),\quad \omega_1(\eta^{(k)})=\omega_1(\eta^{(2)}),\quad \omega_2(\eta^{(k)})=0.$$

Thus $\omega_1(\eta^{(2)}+\eta^{(k)})=\omega_2(\eta^{(2)}+\eta^{(k)})=0$ and hence $\eta^{(2)}+\eta^{(k)}$ is a $Spin(4)$-bundle. On the other hand, each $Spin(4)$ bundle is MSp-orientated. Indeed (see, e.g., [14]), for $Spin(4)$-bundle ψ, to the class of KO-orientation from $KO^4(T\psi)$, where $T\psi$ is the Thom space, there corresponds some symplectic bundle over $T\psi$ provided that the isomorphism $KO^4(T\psi)\approx KS^0(T\psi)$. The first symplectic Pontrjagin class can be taken as the class of MSp-orientation. This completes the proof.

The second way is as follows. Let $\xi\to BU(1)$ be the universal $U(1)$-bundle. Consider the 2-covering $BU(1)\to BN$ and bundles $\xi_!^{2i+1}\to BN$, the images of bundles ξ^{2i+1} for the Atiyah transfer of this covering [15]. The bundle $\xi_!$ is symplectic, since $\xi_!=\pi^*(\zeta)$, where as above $\pi: BN\to BSp(1)$ and $\zeta\to BSp(1)$ is the universal $Sp(1)$-bundle. We shall show that $\xi_!^{2i+1}$, $i\geq 1$, are also $Sp(1)$-bundles. Since $Sp(1)=SU(2)$, it is enough to show that $\xi_!^{2i+1}$ has the trivial determinant, i.e., that the first Chern class $c_1(\xi_!^{2i+1})=0$ in cohomologies with integral coefficients. This can be done immediately after writing the corresponding representations for $\xi_!^{2i+1}$ [12], but we shall do this in terms of transfer and characteristic Chern classes. For the transfer homomorphism τ^* of the covering $BU(1)\to BN$ we have the formula

$$\tau^*\big(c_1(\xi^{2i+1})\big)=c_1(1_!)+c_1(\xi_!^{2i+1})$$

in integral cohomologies. This follows from the corresponding formula for the case of complex cobordisms [16]. By virtue of equalities $c_1(\xi^{2i+1})=(2i+1)c_1(\xi)$ and $\tau^*(c_1(\xi))=c_1(1_!)$ we now have

$$\tau^*\big(c_1(\xi_!^{2i+1})\big)=(2i+1)\tau^*(c_1(\xi))=(2i+1)c_1(1_!)=c_1(1_!),$$

i.e., $c_1(\xi_!^{2i+1})=0$. Further,

$$\eta^{(2)}+\eta^{(k)}=\xi_!^{k/2+1}\otimes_H\xi_!^{k/2-1},$$

but the tensor product of $Sp(1)$-bundles over H is MSp-orientated, since $BSpin(4) = BSp(1)^2$. This proves 3.1.*b*.

(3.1.*c*) The bundle $\eta^{(k)} + 3\lambda$ is a difference between MSp-orientated bundles $\eta^{(2)} + \eta^{(k)} + 4\lambda$ and $\eta^{(2)} + \lambda$ and hence is MSp-orientated. On the other hand, this bundle is a $Spin(5)$-bundle and each $Spin(5)$-bundle has the zero symplectic Euler class [17]. □

Using this lemma, one can show some relations in rings of symplectic cobordisms of generalized quaternion groups. We have

Proposition 3.2. *The relation*

$$pf_1(\eta_1^{(m)} \otimes_R H) \cdot pf_1(\eta_2^{(m)} \otimes_R H) \cdot pf_1\big(\eta_1^{(m)} \otimes_R \eta_2^{(m)} \otimes_R H\big) = 0$$

holds in the ring $MSp^*(BH_m)$, $m \geq 2$.

Proof. For $m > 2$ we have

$$\rho^*(H_m, N)\big(\eta^{(m)} \otimes_R H + \lambda \otimes_R H\big) = \\ = \eta_1^{(m)} \otimes_R H + \eta_2^{(m)} \otimes_R H + \eta_1^{(m)} \otimes_R \eta_2^{(m)} \otimes_R H.$$

But the real bundle $\eta^{(m)} \otimes_R H + \lambda \otimes_R H$ coincides with the sum of the MSp-orientated bundles $\eta^{(m)} + 3\lambda$ and $3\eta^{(m)} + \lambda$. By Proposition (3.1.*c*) the first bundle has the zero symplectic Euler class. Therefore the Euler classes of both sides are equal to zero in the above-indicated splitting. This proves the proposition for $m > 2$.

For the case $m = 2$, i.e., for the group of quaternions, the above reasoning does not hold, since the bundle $\eta^{(2)} + 3\lambda$ is not MSp-orientated. But according to §1, for the universal $Spin(3)$-bundle $\Lambda \to BSp(1)$ the Euler class $e(\Lambda \otimes_R H) = pf_3(\Lambda \otimes_R H) = 0$. On the other hand, $\rho^*(H_2, Sp(1)) = \eta_1^{(2)} + \eta_2^{(2)} + \eta_1^{(2)} \otimes_R \eta_2^{(2)}$ so that the proposition is valid in this case too. □

Acknowledgment

The research described in this publication was made possible in part by Grant RVJ 200 from the International Science Foundation.

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(Received 15.12.1995)

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