The Annals of Probability
1984, Vol. 12, No. 2, 615–630

ON THE CADLAGUITY OF RANDOM MEASURES

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We consider finitely additive random measures taking independent values on disjoint Borel sets in R^k, and ask when such measures, restricted to some subclass $\mathscr{A}$ of closed Borel sets, possess versions which are "right continuous with left limits", in an appropriate sense. The answer involves a delicate relationship between the "Lévy measure" of the random measure and the size of $\mathscr{A}$, as measured via an entropy condition. Examples involving stable measures, Dudley's class $I(k, \alpha, M)$ of sets in R^k with α-times differentiable boundaries, and convex sets are considered as special cases, and an example given to show what can go wrong when the entropy of $\mathscr{A}$ is too large.

1. Introduction. Over the last few years there has been a resurgence of interest in the classical results of Lévy (1934, 1937) and Ito (1942) on the representation and structure of stochastic processes with independent increments. The research this has generated has followed two somewhat disjoint paths. One has involved the development of new constructions and representations of these processes. Examples can be found in the works of Ferguson and Klass (1972) and Le Page, Woodroofe and Zinn (1981), which develop representations for such processes based on sums of order statistics, thus shedding new light not only on the analytic structure of these processes, but also on their sample path behavior. The second path has involved extending the basic ideas of Lévy and Ito to stochastic processes defined on parameter spaces more general than the real line. Thus, for example, the sample path properties of random fields with independent increments over disjoint rectangles were studied in Adler *et al.* (1983).

It is a logical and conceptually simple step to progress from such random fields to random measures taking independent values on disjoint measurable sets. That is, if $\mathscr{B}^k$ denotes the Borel subsets of the unit cube $I_k = [0, 1]^k$ of R^k, and $(\Omega, \mathscr{F}, P)$ is a probability space, we wish to study mappings $X: \mathscr{B}^k \times \Omega \to (-\infty, \infty)$ satisfying

(1.1) For each $B \in \mathscr{B}^k$, $X(B, \cdot)$ is a random variable,

(1.2) Almost surely, $X(\cdot, \omega)$ is a *finitely* additive measure on $\mathscr{B}^k$,

(1.3) $B_1, \cdots, B_k$ pairwise disjoint implies $X(B_1, \cdot), \cdots, X(B_k, \cdot)$ independent.

As early as 1956, Prekopa studied the problem of the existence of such measures when, in place of (1.2), they were assumed to be *countably* additive. Kingman (1967) also studied the existence problem for the case of countably

Received January 1983; revised September 1983.
[1] Research supported in part by AFOSR grant no. 83-0068.
AMS 1980 *subject classifications.* Primary 60G17, 60J30; secondary 60G15.
Key words and phrases. Random measures, independent increments, cadlag, entropy, convex sets.

additive, *positive*, measures, which he showed must be purely atomic. When only finite additivity is demanded, the existence problem is far less delicate, as we shall show in the following section.

Despite prior interest in the existence problem, there has been no attempt, by any of the earlier authors, to investigate "sample path" properties of random measures. Indeed, even the simplest of questions, as to whether, or how, the classical cadlag (right continuity, left limits) properties of independent increment processes on the real line carry over to random measures has never been asked. Our aim here will be to tackle precisely this problem.

The paper is organized as follows. In the following section we shall define our problem and delineate our aims and tools. One of the main tools will turn out to be Dudley's (1978) "metric entropy with inclusion". In Section 3 we shall provide an example of a class of sets over which certain stable random measures fail to have left limits. This provides motivation for the problem of asking when cadlaguity occurs. (Remember that on the real line *all* independent increment processes, after the elimination of degenerate discontinuities, *always* possess cadlag versions.) In Section 4 we shall develop our main result, giving conditions under which stochastically continuous measures with homogeneous, independent increments possess cadlag versions. These conditions will be stated in terms of the size of the index set. Finally, in Section 5, we shall close with some comments of a general nature and a conjecture.

It seems worthwhile to point out at this stage why notions such as entropy need to be introduced here at all. In the case of Gaussian processes, entropy appears in discussing continuity of processes on the line, and so its appearance in the dicussion of continuity of measures (or set-indexed processes) is not surprising, (cf. Dudley, 1973). No such condition appears, however, in discussing independent increment processes on the line, and so its appearance here requires some justification. The justification stems from the fact that in moving from processes defined on R^1, or even R^k, $k > 1$, to random measures, we lose one of the main tools of that theory—that of the maximal inequality. It is exactly to replace this tool that entropy techniques need to be introduced. Their precise role in replacing the maximal inequality can be seen by comparing the proof of Theorem 4.2 below with that of the corresponding result for random fields, in Section 3 of Adler *et al.* (1983).

Finally, we wish to acknowledge the helpful comments of an associate editor and referee of an earlier version of this paper, who pointed out some home truths about the difficulties of defining general random measures that we had either overlooked or forgotten, as well as pointing out a few minor errors. Dick Dudley pointed out to us the existence of some early results of his that made the proof of Proposition 2.1 a simple matter.

2. Independent increment measures, cadlaguity and entropy. Let X be a random measure satisfying conditions (1.1)–(1.3). Thus, we emphasize, X is, for almost all ω, a finitely, but not necessarily countably, additive measure. Let us add one more restriction, that of *stochastic continuity*. That is, if d denotes the symmetric difference metric defined on $\mathscr{B}^k$ by $d(A, B) = \lambda(A \Delta B)$, where λ is

Lebesgue measure, then for any sequence $\{B_n\}$ of sets in $\mathscr{B}^k$ and all $\varepsilon > 0$ we have

$$P\{|X(B_n) - X(B)| > \varepsilon\} \to 0 \quad \text{whenever} \quad d(B_n, B) \to 0. \tag{2.1}$$

(For an alternative definition of stochastic continuity, in terms of a finer topology on Borel sets, see the remarks at the end of this section.) A stochastically continuous measure cannot have degenerate atoms. In general, we shall call a random measure satisfying (1.1)–(1.3) and (2.1) a *Lévy random measure.* (Note that this is not the same as Karr's (1978) usage of the same term.) It is reasonably straightforward to check (cf. Theorem 3.1 and equation (3.37) of Adler *et al.*, 1983) that if Lévy random measures actually exist, then the logarithm of their characteristic functions can be expressed, for each Borel A, by

$$\log\{E[e^{iuX(A)}]\} = \psi_A^{(1)}(u) + \psi_A^{(2)}(u), \tag{2.2}$$

where

$$\psi_A^{(1)}(u) = iuM(A) - \tfrac{1}{2}u^2V(A), \tag{2.3}$$

$$\psi_A^{(2)}(u) = \int_{|x|>1} [e^{iux} - 1]\nu(A, dx) + \int_{|x|\le 1} [e^{iux} - 1 - iux] \cdot \nu(A, dx), \tag{2.4}$$

and

(2.5) M is a real valued function on $\mathscr{B}^k$ continuous in the d metric;

(2.6) V is a finite, nonnegative measure on I_k, absolutely continuous with respect to Lebesgue measure;

(2.7) for all $A \in \mathscr{B}^k$, $\nu(A, \cdot)$ is a positive measure on R with $\nu(I_k, \{0\}) = 0$ and satisfying

$$\int_{|x|\le 1} x^2\nu(I_k, dx) < \infty, \quad \int_{|x|>1} \nu(I_k, dx) < \infty;$$

(2.8) for every Borel $B \subset R$ with positive distance to $\{0\}$, $\nu(\cdot, B)$ is a nonnegative, finite Borel measure on I_k, absolutely continuous with respect to Lebesgue measure.

It is clear from (2.3) that every Lévy random measure can be expressed as the sum of independent Gaussian and nonGaussian measures, the continuity properties of which can be investigated separately. Since the continuity properties of Gaussian measures have already been studied in depth (under the name of set-indexed processes; e.g. Dudley, 1973), we shall concentrate only on the non-Gaussian part in what follows.

Our first task must be to establish that set indexed processes with log characteristic function given by (2.4) actually exist, and moreover, have versions that are Lévy random measures. We shall prove

PROPOSITION 2.1. *There exists a set-indexed process with independent increments and log characteristic function* (2.4). *Moreover, this process has a version that is a finitely additive random measure on* I_k.

PROOF. We start with a slightly more general problem and then specialize. Let S be the vector space of bounded measurable functions with support in I_k, and let $\psi(f)$ be the functional on S defined by

$$\log[\psi(f)] \tag{2.9}$$
$$= \int_{\operatorname{supp} f} \left\{ \int_{|x|>1} [e^{ixf(t)} - 1]\nu(dt, dx) + \int_{|x|\le 1} [e^{ixf(t)} - 1 - ixf(t)]\nu(dt, dx) \right\},$$

where ν is as above. Calculations as in Section 3.4 of Gelfand and Vilenkin (1964) readily establish that ψ is positive definite. Furthermore, $\psi(0) = 1$ and ψ is obviously continuous in the topology of pointwise convergence of the f's. Consequently, by Theorem (1.4) of Dudley (1969), there exists a random linear functional, $\mathscr{L}$ say, on S, with log characteristic functional (2.9). Now use this $\mathscr{L}$ to define a finitely additive random measure X via the correspondence $X(A, \omega) = \mathscr{L}(I_A, \omega)$, where I_A is the indicator function of A. It is trivial to check, using the linearity of $\mathscr{L}$ and (2.9), that X has all the required properties, and the proof is complete.

Having established their existence in general, let us now look at some special cases of random measures. Note that hereafter, unless explicitly stated otherwise, we shall always assume that we are working with the measure version of our processes.

When the measure $\nu(A, B)$ of (2.4) is of the form $\lambda(A)F(B)$, where F is a measure on $R - \{0\}$ (whose distribution function we also denote by F) satisfying

$$\int_{|x|<1} x^2\, dF(x) < \infty, \quad \int_{|x|>1} dF(x) < \infty \tag{2.10}$$

then X has *stationary increments* in the sense that the distribution of $X(A)$ depends on A only through $\lambda(A)$. In the particular case that

$$F(dx) = \begin{cases} c_1 x^{-(1+\beta)}\, dx, & x > 0 \\ c_2 |x|^{-(1+\beta)}\, dx, & x < 0 \end{cases} \tag{2.11}$$

for some $\beta \in (0, 2)$ and finite constants c_1, c_2 we call X a *stable* Lévy measure with stationary increments and parameter β. If, furthermore, $c_1 = c_2 = c$ we say the measure is *symmetric*, and we can write the logarithm of the characteristic function of $X(A)$ as

$$\log E\{e^{iuX(A)}\} = -c\lambda(A)\,|u|^{\beta}. \tag{2.12}$$

Let us now move to a definition of "right continuity with left limits" for random measures. Let $\mathscr{A}$ be a collection of closed Borel subsets of I_k. Then we shall call a random measure *cadlag* on $\mathscr{A}$ if the following two conditions hold with probability one:

(2.13) For every $A \in \mathscr{A}$ and decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ in $\mathscr{A}$ for which $A \subseteq \bigcap_{n=1}^{\infty} A_n$ and $d(A_n, A) \to 0$ we have $X(A_n) \to X(A)$.

(2.14) For every increasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ in $\mathscr{A}$ the sequence $X(A_n)$ converges to a finite limit.

When $\mathscr{A}$ is the set of closed intervals in R^1 or rectangles in R^k this definition reduces to the usual one for the right continuity with left limits of random processes and fields. (Note that had we decided to work with open, rather than closed, sets, then in these simple cases our definition of cadlaguity would yield *left* continuity with *right* limits.) In these simple cases, a Lévy random measure always possesses a cadlag version, and, moreover, a cadlag modification (Adler *et al.*, 1983, Section 3 and Proposition 4.1). However, if the collection of sets $\mathscr{A}$ is too large then whether or not cadlag versions exist will depend on a delicate relationship between the size of $\mathscr{A}$ and the Lévy measure ν.

In order to measure the size of $\mathscr{A}$ we shall use Dudley's (1978) notion of *metric entropy with inclusion*, which we recall as:

DEFINITION 2.1. Let $\mathscr{A}$ be a collection of subsets of $\mathscr{B}^k$. For each $\varepsilon > 0$ let $N_I(\varepsilon) := N_I(\varepsilon, \mathscr{A})$ be the smallest n such that for some $A_1, \cdots, A_n \in \mathscr{B}^k$ (not necessarily elements of $\mathscr{A}$), for every $A \in \mathscr{A}$ there exist i, j with $A_i \subset A \subset A_j$ and $\lambda(A_j/A_i) < \varepsilon$. $H_I(\varepsilon) := H_I(\varepsilon, \mathscr{A}) = \log \mathrm{N}_I(\varepsilon, \mathscr{A})$ is called the metric entropy with inclusion (of $\mathscr{A}$).

This notion has previously been exploited by Dudley (1978) in investigating weak convergence of empirical measures, in the process of which the entropy with inclusion of a number of interesting classes of sets has been calculated.

For example, consider Dudley's class $I(k, \alpha, M)$ of sets in R^k where boundaries are given by functions from the sphere S^{k-1} into R^k with derivatives of order $\leq \alpha$, all bounded by M. (For a precise definiton of $I(k, \alpha, M)$, see Dudley, 1974.) For this class

$$H(\varepsilon, I(k, \alpha, M)) = O(1/\varepsilon)^{\eta} \quad \text{as} \quad \varepsilon \downarrow 0, \tag{2.15}$$

where

$$\eta = \begin{cases} (k-1)/\alpha & \text{if} \quad \alpha \geq 1, \\ (k-1)/(k\alpha - k + 1) & \text{if} \quad (k-1)/k < \alpha \leq 1. \end{cases} \tag{2.16}$$

If $\mathscr{A}$ is the set of convex sets in I_k, then (2.15) once again holds, but with $\eta = (k-1)/2$.

For Vapnik-Červonenkis classes (Vapnik and Červonenkis, 1971) Lemma 7.13 of Dudley (1978) states that

$$H(\varepsilon) \leq O(\log 1/\varepsilon). \tag{2.17}$$

Dudley (1978) gives a detailed discussion of Vapnik-Červonenkis classes and provides many examples, to which we refer the interested reader.

REMARK. In the definitions of both stochastic continuity and cadlaguity we relied on the symmetric difference metric to provide a notion of closeness for Borel sets. An entirely parallel theory could be developed by replacing this metric with the topology induced on the Borel sets by the pointwise convergence of indicator functions. Such a theory would differ from that presented here in only three respects; (i) There would be no need to restrict consideration to closed sets, (ii) The notion of stochastic continuity would be somewhat weaker, (iii) The arguments of Section 4 that suffice to prove cadlaguity when the symmetric difference metric is used would also suffice, with only the most trivial changes, to prove actual *continuity* under the topology of pointwise convergence of indicator functions.

3. A non-cadlag example. We commence by defining a class of sets, $\mathscr{A}_\gamma$, on R^2. Let A_0 be the unit square $[0, 1] \times [0, 1]$. Then given $A_0, \cdots, A_{n-1}$, let A_n be the closed rectangle of height 1 and breadth $2^{n(1-\gamma)}$ whose leftmost vertical edge is the rightmost vertical edge of A_{n-1}. Thus the sum of the breadths of all the rectangles is $\sum_{n=0}^{\infty} 2^{n(1-\gamma)}$, which will be finite if $1 < \gamma < 2$. Now divide each A_n into 2^n equally sized, closed, horizontal slices, $A_{n1}, \cdots, A_{n2^n}$. Each A_{nk} thus has area $2^{-n} \times 2^{n(1-\gamma)} = 2^{-n\gamma}$. The collection $\mathscr{A}_\gamma$ is now defined as all possible finite and infinite unions of the sets $\{A_{nk}\}$. We make two claims about $\mathscr{A}_\gamma$. The first relates to its size.

PROPOSITION 3.1. *If* $\gamma \in (1, 2)$ *then* $H_I(\varepsilon, \mathscr{A}_\gamma) = O(\varepsilon^{1/(1-\gamma)})$ *for every* $\varepsilon > 0$.

The second claim relates to the non-cadlag behavior of stable measures on $\mathscr{A}_\gamma$. Indeed, we prove something even more pathological, namely

PROPOSITION 3.2. *Let X be a stationary, symmetric stable random measure with parameter* β. *If* $\beta > \gamma$ *then* $\sup\{X(A):A \in \mathscr{A}_\gamma\} = \infty$ *with probability one.*

In fact, something stronger than the above proposition is true. A check of the proof will show that the fact that X is a measure on all Borel sets is never actually used, but only the fact that X is finitely additive on $\mathscr{A}_\gamma$. Consequently, this condition could replace the condition that X be a measure in the statement of the proposition.

The obvious import of the two propositions combined is that the cadlag property of a given measure can disappear if the entropy of its index set is increased too far. (We have yet to prove that stable measures are ever cadlag, but this will be shown in the following section.)

The condition in Proposition 3.1 that $\gamma > 1$ follows from the fact that for $\gamma < 1$ the sets in $\mathscr{A}_\gamma$ cannot be enclosed in a finite rectangle, but lie anywhere in the unbounded set $[0, 1] \times [0, \infty)$. In such a case the unboundedness of $\sup\{X(A)\}$ is uninteresting, and does not contradict the cadlag property. However, for $\gamma > 1$ the unboundedness of $\sup\{X(A)\}$ implies the existence of a sequence, and hence of an increasing sequence, of bounded closed sets $\{B_j\}$ for which $\{X(B_j)\}$ diverges, thus contradicting the cadlag requirement (2.14). We shall see later that when $\beta < 1$ a cadlag version will always exist.

Note that even for $\gamma > 1$ the sets of $\mathscr{A}_\gamma \notin I_2$, a condition we have generally demanded. A change of scale will, however, correct this. Indeed, all that really counts here is the boundedness of the sets of $\mathscr{A}_\gamma$.

PROOF OF PROPOSITION 3.1. Choose $\varepsilon = 2^{-m}$, $m > 1$. Set $S_\gamma = 2^\gamma/(2^\gamma - 2) = \lambda(\bigcup_{n,k} A_{nk})$, and define

$$M = M(\varepsilon) := 1 + [(m + \log_2 S_\gamma)/(\gamma - 1)]$$

where $[\cdot]$ denotes "integer part", and $\log_2$ is a base 2 logarithm. Then

$$\textstyle\sum_{n=M(\varepsilon)}^{\infty} 2^{n(1-\gamma)} \le 2^{-m} = \varepsilon,$$

and so if we write $A(\varepsilon)$ for the union $\bigcup_{n\ge M(\varepsilon)} \bigcup_{k=1}^{2^n} A_{nk}$ we have $\lambda(A(\varepsilon)) \le \varepsilon$. There are $1 + 2 + 2^2 + \cdots + 2^{M(\varepsilon)-1} = 2^{M(\varepsilon)} - 1$ sets A_{nk} with $n < M(\varepsilon)$. Take these sets together with $A(\varepsilon)$, and consider all the possible sets formed by taking unions of any number of these $2^{M(\varepsilon)}$ sets. Then we have a collection of $2^{2^{M(\varepsilon)}}$ sets from which, for any $A \in \mathscr{A}_\gamma$, we can always choose two, say A_O and A_I, such that $A_I \subseteq A \subseteq A_O$ and $d(A_O, A_I) = \lambda(A_O/A_I) < \varepsilon$. That is,

$$H_I(\varepsilon, \mathscr{A}_\gamma) \le \log(2^{2^{M(\varepsilon)}}) \le \text{const. } 2^{m/(\gamma-1)}.$$

Since $H_I(\varepsilon)$ is clearly monotone in ε, we can drop the requirement that $\varepsilon = 2^{-m}$ and so obtain

$$H_I(\varepsilon, \mathscr{A}_\gamma) \le O(\varepsilon^{1/(1-\gamma)}). \tag{3.1}$$

A little thought shows that one can do little better (in terms of order of magnitude) in choosing an approximating class of sets. (Consider merely trying to approximate from above the $2^{2^{(M(\varepsilon)-1)}}$ possible collections among $\{A_{M(\varepsilon)-1,k},\ k = 1, \cdots, 2^{M(\varepsilon)-1}\}$.) Thus the upper bound of (3.1) is, in order of magnitude, sharp and so the proposition is established.

PROOF OF PROPOSITION 3.2. The proof we shall use here was motivated by Dudley's (1979) proof that lower layers in R^2 are not GB classes.

Let A_{nj}; $j = 1, \cdots, 2^n$; $n = 1, 2, \cdots$ be the sets described above, and let $\Pi(A_{nj})$ be the projection of A_{nj} on the vertical axis. For each n, each point $p := (0, p)$, $0 \le p \le 1$, belongs to $\Pi(A_{n,j(n,p)})$ for some unique $j(n, p)$. For each such p and $M < +\infty$ the events $E_{np} := \{X(A_{n,j(n,p)}) > M/2^n\}$ are independent for $n = 1, 2, \cdots$. Furthermore, these probabilities have a uniform strictly positive lower bound, since

$$P\{X(A_{nk}) > M2^{-n}\} = P\left\{\frac{X(A_{nk})}{2^{-n\gamma/\beta}} > M2^{-n(1-\gamma/\beta)}\right\}$$

$$\ge P\left(\frac{X(A_{nk})}{[\lambda(A_{nk})]^{1/\beta}} > M\right) \quad (\text{since } \beta > \gamma)$$

and the last probability is independent of n and k (cf. (2.12)). Thus, for each p and M, an E_{np} occurs with probability one, and so, for each $p \in (0, 1)$, $n^* = n^*(p, \omega) = \inf\{n : \omega \in E_{np}\}$ ($\inf \phi = \infty$) is well defined, and finite for almost all ω. Define the indicator functions $I_n(p, \omega) = 1$ if $\omega \in E_{np}$ or 0 otherwise, and set $I_\infty(p, \omega) = 0$. Since the I_n are jointly measurable in (p, ω), Fubini's theorem implies that

$I_{n^*}(p, \omega) = 1$ for almost all p when $\omega \notin N$, where N is some ω-set of probability zero. Thus, for $\omega \notin N$, $n^*(p, \omega) < \infty$ for almost all p. Now fix $\omega \notin N$, write $\mathscr{A}(\omega)$ for the (countable or finite) collection of sets $\{A_p = A_{n^*, j(n^*, p)}: p \in (0, 1)$ and $n^*(p, \omega) < \infty\}$. Since almost all p belong to an interval of length $2^{-n^*(p,\omega)}$ which is the projection $\Pi(A_p)$ of an A_p in $\mathscr{A}(\omega)$ we have that $\sum_{A \in \mathscr{A}(\omega)} \lambda(\Pi(A)) = 1$. However, $X(A) > M\lambda(\Pi(A))$ for every $A \in \mathscr{A}(\omega)$, implying

$$\sum_{A \in \mathscr{A}(\omega)} X(A) > M.$$

Thus there exists a *finite* subset $\mathscr{A}'(\omega) \subset \mathscr{A}(\omega)$ such that

$$X(\cup_{\mathscr{A}'(\omega)} A) = \sum_{\mathscr{A}'(\omega)} X(A) > M,$$

by the finite additivity of X. Since M was arbitrary we have that X is almost surely unbounded on $\mathscr{A}_\gamma$, and the proof is complete.

4. Entropy-cadlag results. Let X be a Lévy random measure with Lévy measure $\nu(A, B)$. Let $\mathscr{A}$ be a class of closed Borel subsets of I_k with inclusion entropy $H(\varepsilon) = \log N_I(\varepsilon, \mathscr{A})$. We shall prove the following two results.

THEOREM 4.1. *If* $\int_{-1}^{1} |x| \nu(I_k, dx) < \infty$ *then X possesses a version cadlag on $\mathscr{A}$. (i.e. No entropy condition is required.)*

THEOREM 4.2. *Suppose* $\int_{-1}^{1} |x| \nu(I_k, dx) = \infty$. *Let X have stationary increments and set*

$$\mu_n = \int_{2^{-n} < |x| \le 2^{-n+1}} \nu(I_k, dx), \quad n = 1, 2, \cdots \tag{4.1}$$

$$v_n = \int_{2^{-n} < |x| \le 1} |x| \nu(I_k, dx), \quad n = 1, 2, \cdots. \tag{4.2}$$

If for some $\gamma > 1$

$$\sum_{n=1}^{\infty} \exp\{H(V_n^{-\gamma}) + \gamma \log V_n - V_n^{-2\gamma} \mu_n / 16\gamma\} < \infty, \tag{4.3}$$

then X possesses a version cadlag on $\mathscr{A}$.

Before we prove these results we need to clarify what we mean here by the term "random measure". As an example, take as $\mathscr{A}$ the convex sets in R^2, and suppose we have constructed an X cadlag on $\mathscr{A}$. The measure property of X is that $X(\cup_{i=1}^n A_i) = \sum_{i=1}^n X(A_i)$ for disjoint convex A_i and n finite. But $\cup_{i=1}^n A_i$ need not be in $\mathscr{A}$!. Consequently, throughout this section we shall take additivity to mean additivity when both the A_i and $\cup A_i$ belong to $\mathscr{A}$.

Let us now consider some examples. In particular, for stable measures we have

COROLLARY 4.1. *Let X be a stable Lévy measure with stationary increments and parameter $\beta \in (0, 2)$. Then X will possess a cadlag version if*

$$\beta < 1: \quad \textit{always}, \tag{4.4}$$

(4.5) $$\beta \geq 1: \quad \int_0^1 [H(x^2)]^\eta \, dx < \infty \quad \textit{for some} \quad \eta > \frac{\beta - 1}{2(2 - \beta)}$$

where $H(\varepsilon)$ is the inclusion entropy of the index set.

PROOF. By (2.11) the Lévy measure of a stable random measure has finite mean on $[-1, 1]$ whenever $\beta < 1$, so Theorem 4.1 trivially covers this case. The cases $\beta = 1$ and $\beta > 1$ follow from Theorem 4.2, as follows.

First, take $\beta > 1$ and note that there exist (β-dependent) constants C_1, C_2 and C_3 such that

$$\mu_n = C_1 2^{n\beta}, \quad C_2 2^{n(\beta-1)} \leq V_n \leq C_3 2^{n(\beta-1)},$$

(cf. (2.11)). Thus, by (4.3) a cadlag version will exist if for some $\gamma \in (1, \beta/2(\beta - 1))$

$$\textstyle\sum_{n=1}^\infty \exp\{H(C_3 2^{-n\gamma(\beta-1)}) - 2^{n(2\gamma+\beta(1-2\gamma))} C_1 C_2^{-2\gamma}/8\} < \infty.$$

A sufficient condition for this convergence to hold is that for some $\varepsilon > 0$, some $C_4 < \infty$ and all n large enough

$$H(C_3 2^{-n\gamma(\beta-1)}) < C_4 2^{n\gamma(\beta-1)/2\xi}$$

where

$$\xi = \frac{\frac{1}{2}\gamma(\beta - 1)}{2\gamma + \beta(1 - 2\gamma) - \varepsilon}.$$

This, in turn will be satisfied if

$$\textstyle\sum_{n=1}^\infty [H(C_3 2^{-n\gamma(\beta-1)})]^\xi \cdot 2^{-n\gamma(\beta-1)/2} < \infty,$$

which itself follows from

$$\int_0^1 y^{-(1/2)} H^\xi(y) \, dy < \infty.$$

This is equivalent to

$$\int_0^1 H^\xi(x^2) \, dx < \infty.$$

Now send $\varepsilon \to 0$ and $\gamma \to 1$ to obtain (4.5) as the requisite condition, thus proving the corollary for $\beta > 1$.

When $\beta = 1$, simply use the facts that

$$\mu_n = O(2^n), \quad V_n = O(n),$$

and apply the argument as before.

It is now a trivial matter to apply (4.5), (2.15) and (2.16) to obtain

COROLLARY 4.2. *Let X be a stationary stable measure with parameter $\beta \in (1, 2)$. Let $I(k, \alpha, M)$ be the sets described in Section 2. Then X has a cadlag* *version on $I(k, \alpha, M)$ if*

$$\alpha > 1: \quad \beta < (2\alpha + k - 1)/(\alpha + k - 1)$$

$$(k-1)/k \le \alpha \le 1: \quad \beta < (2k\alpha - k + 1)/(k\alpha).$$

Furthermore, X will have a version cadlag on the convex sets in I_k if

$$\beta < (3 + k)/(1 + k). \tag{4.6}$$

Our final example, relating to Vapnik-Červonenkis classes, is

COROLLARY 4.3. *A stationary stable measure always has a version cadlag on any given Vapnik-Červonenkis class.*

The proof of this follows readily from (2.17) and Corollary 4.1.

We now turn to the proofs of Theorems 4.1 and 4.2, and shall tackle the more difficult one first.

PROOF OF THEOREM 4.2. To prove the existence of a cadlag version, we shall simply construct one. The construction goes back in spirit to Ito (1969), although we shall adopt the approach of Adler *et al.* (1983) to which we shall occasionally refer to avoid repetitive detail.

We shall require some notation. Let $\bar{\nu}$ be the measure on $I_k \times [0, \infty)$ defined by

$$\bar{\nu}(A, B) = \nu(A, B) + \nu(A, -B) \tag{4.7}$$

for all $A \in \mathscr{B}^k$ and Borel $B \subset [0, \infty)$. Set $\varepsilon_n = 2^{-n}$, $n = 0, 1, \cdots$ and define

$$D_0 = (1, \infty), \quad D_n = (\varepsilon_n, \varepsilon_{n-1}], \quad n = 1, 2, \cdots. \tag{4.8}$$

Define sequences of measures $\{\nu_n\}$ and probability measures $\{F_n\}$, $n \ge 0$, by

$$\nu_n(A, B) = \nu(A, B \cap [D_n \cup (-D_n)]), \quad n = 0, 1, \cdots, \tag{4.9}$$

$$F_n(B) = \begin{cases} [\nu_n(I_k, B)]/\mu_n, & \mu_n > 0, \\ 0 & \mu_n = 0, \end{cases} \tag{4.10}$$

where μ_n was defined at (4.1). Furthermore, let Λ be the measure defined by

$$\Lambda(dx) = |x| \, \bar{\nu}(I_k, dx), \tag{4.11}$$

and set $V(x) = \int_x^1 \Lambda(dx)$. Finally, for $n \ge 1$ set

$$v_n = \int_{D_n} \Lambda(dx), \quad V_n = V(n) = \textstyle\sum_{j=1}^n v_n. \tag{4.12}$$

Now let $\{\xi_n\}$, $n \ge 0$, be a sequence of independent Poisson random variables with parameters μ_n. Furthermore for each $n \ge 0$ let

$$(T_k^n, J_k^n), \quad k = 1, 2, \cdots$$

be a sequence of independent, (mutually and of the ξ_n), identically distributed random vectors with joint distribution

(4.13) $$P\{T_k^n \in A,\ J_k^n \in B\} = \lambda(A)F_n(B)$$

for every Borel $A \subset I_k$ and $B \subset R$. Denote the mean of J_k^n by θ_n:

(4.14) $$\theta_n = E\{J_k^n\} = \int xF_n(dx) = \int_{\varepsilon_n < |x| \le \varepsilon_{n-1}} \frac{x\nu(I_k, dx)}{\mu_n},$$

and write σ_n^2 for the variance of J_k^n. Note that the following simple inequalities hold:

(4.15) $$|\theta_n| \le \varepsilon_{n-1}, \quad \sigma_n^2 \le \varepsilon_{n-1}^2, \quad |J_k^n| < \varepsilon_{n-1} \quad \text{a.s.},$$

and

(4.16) $$|J_k^n - \theta_n| \le 2\varepsilon_{n-1} \quad \text{a.s.},$$

where we interpret ε_{-1} as ∞.

We can now define a sequence of independent *compound Poisson* random measures X^n by

(4.17) $$X^n(A, \omega) := \sum_{k=1}^{\xi_n(\omega)} J_k^n(\omega) I_A(T_k^n(\omega))$$

for all Borel $A \subset I_k$, with I_A the indicator function of A. It is easy to check (cf. Adler *et al*, (1983), page 14) that each X^n has stationary and independent increments, and log characteristic function

$$\int (e^{iux} - 1)\nu_n(A, dx).$$

Furthermore, $E\{X^n(A)\} = \int x\nu_n(A, dx)$, so that it follows from (2.5) that a version of X is given by

(4.18) $$\tilde{X}(A, \omega) := X^0(A, \omega) + \sum_{n=1}^{\infty} \{X^n(A, \omega) - E[X^n(A, \omega)]\},$$

as long as we can prove the convergence of this sum. Indeed we shall prove both the convergence and that $\tilde{X}$ is a.s. cadlag.

To do this, we commence by noting that since $E\{\xi_n\} = \mu_n < \infty$, for every $n \ge 0$ (cf. (2.8)) we have that each X^n has only an a.s. finite number of atoms. Furthermore, since $E\{X^n(A)\}$ is clearly continuous in A, it follows that each $X^n(\cdot, \omega) - EX^n(\cdot)$ is a cadlag measure over closed Borel sets in I_k and so, a fortiori, over $\mathscr{A}$. Thus the sum (4.18) will also be a cadlag measure (in the restricted sense described above) over $\mathscr{A}$ if we can find a summable sequence $\{\alpha_n\}$ and an a.s. finite random variable $\eta(\omega)$ for which

$$\sup_{A \in \mathscr{A}}\{|X^n(A, \omega) - E[X^n(A)]|\} \le \alpha_n \quad \text{for all} \quad n \ge \eta(\omega).$$

(Note, however, that this cadlag (restricted sense) measure is not necessarily extendable to a measure on all Borel sets.) To avoid measurability problems we shall bound the above supremum by a measurable function (cf. (4.26)), say B_n. By Borel-Cantelli, cadlaguity will follow if the probabilities $P\{B_n > \alpha_n\}$ are summable. The remainder of the proof consists in showing that the conditions of the theorem guarantee that this is in fact the case.

We shall require two inequalities. The first is Bernstein's inequality (Bennett, 1962, Hoeffding, 1963) which states that for i.i.d. X_i with $|X_i - E(X_i)| < M$ and variance σ^2,

(4.19) $$P\{|\sum_{i=1}^n (X_i - EX_i)| > x\} \le 2\exp\{-x^2/2n\sigma^2(1 + Mx/3n\sigma^2)\}.$$

The second, which is undoubtedly known but for which we could not find a reference is

LEMMA 4.1. *Let ξ be a Poisson variable with mean λ. Then for all $\delta > 0$*

(4.20) $$P\{|\xi - \lambda| > \delta\lambda\} < 2\exp\{-\lambda[(1+\delta)\log(1+\delta) - \delta] + \log(1+\delta^{-1})\}.$$

Furthermore,

(4.21) $$(1+\delta)\log(1+\delta) - \delta \ge \delta^2/4 \quad \textit{for} \quad 0 \le \delta \le 1,$$

(4.22) $$(1+\delta)\log(1+\delta) - \delta \ge 1/4 \quad \textit{for} \quad \delta > 1.$$

PROOF. Set $d = \lambda\delta$. Then the proof proceeds in two parts, corresponding to $\delta < 1$ and $\delta > 1$.

(a) $\delta < 1$: First, note

$$\begin{aligned} P\{\xi < \lambda - \delta\lambda\} &= \textstyle\sum_{k<\lambda-d} e^{-\lambda}\lambda^k/k! \\ &\le e^{-\lambda}(\lambda/(\lambda-d))^{\lambda-d}\textstyle\sum_{k<j-d}(\lambda-d)^k/k! \le e^{-d}(\lambda/(\lambda-d))^{\lambda-d} \\ &= \exp\{-\lambda[\delta + (1-\delta)\log(1-\delta)]\}. \end{aligned}$$

For the second part of this bound note first that, for all $n \ge 2$, $n! \ge n^n e^{-n+1}$, since

$$\log(n!) = \textstyle\sum_{j=2}^n \log j \ge \displaystyle\int_1^n \log x\, dx = n\log n - n + 1.$$

Thus, writing $[x]$ for the integer part of x, we obtain

$$\begin{aligned} P\{\xi > \lambda + \lambda\delta\} &= \frac{\sum_{k>\lambda+d} e^{-\lambda}\lambda^k}{k} \le \frac{e^{-\lambda}\lambda^{[\lambda+d+1]}}{[\lambda+d+1]!}\,\frac{\sum_{k>0}\lambda^k}{(\lambda+d)^k} \\ &\le e^{-\lambda}\lambda^{[\lambda+d+1]}\left(\frac{\lambda+d}{\lambda}\right)([\lambda+d+1])^{-[\lambda+d+1]}e^{[\lambda+d+1]-1} \\ &\le e^d\left(\frac{\lambda+d}{d}\right)\left(\frac{\lambda}{\lambda+d}\right)^{\lambda+d} \\ &= \exp\{-\lambda[(1+\delta)\log(1+\delta) - \delta] + \log(1+\delta^{-1})\}. \end{aligned}$$

It is simple to check that, for $\delta < 1$,

$$\delta + (1-\delta)\log(1-\delta) > (1+\delta)\log(1+\delta) - \delta,$$

so that (4.20) is established for $\delta < 1$.

(b) $\delta > 1$: When $\delta > 1$, $|\xi - \lambda| > \delta\lambda \Leftrightarrow \xi - \lambda > \delta\lambda$, (since $\xi > 0$) in which case the second part of part (a) can be applied to establish (4.20).

Relationships (4.21) and (4.22) follow from elementary calculations.

We now return to the proof of the theorem. Choose $\gamma > 1$, and define the sequences $\{\alpha_n\}$ and $\{\delta_n\}$, $n \geq 1$, by

$$\alpha_n = 8V_n^{-\gamma}v_n, \quad \delta_n = V_n^{-\gamma}. \tag{4.23}$$

We claim that $\{\alpha_n\}$ is summable, since

$$\sum_{n=1}^{\infty} \alpha_n = 8 \sum_{n=1}^{\infty} v_n V_n^{-\gamma} = 8 \sum_{n=1}^{\infty} (V_n - V_{n-1}) V_n^{-\gamma}$$
$$< 8 \sum_{n=1}^{\infty} \int_{V_{n-1}}^{V_n} x^{-\gamma}\, dx = 8 \int_{V_0}^{\infty} x^{-\gamma}\, dx,$$

since $V_n \to \infty$ under the hypothesis of the theorem. The integral is finite as long as $\gamma > 1$ and $V_0 > 0$. If $V_0 = 0$, we simply take the $\{\alpha_n\}$ and $\{X^n\}$ sequences for $n \geq \min\{k: V_k > 0\}$, and nothing in the proof or theorem changes.

For each $n \geq 1$ let $\mathscr{A}_n$ denote a collection of Borel subsets of I_k such that for any $A \in \mathscr{A}$ there exist $A_{n,I}$ and $A_{n,O}$ in $\mathscr{A}_n$ satisfying

$$A_{n,I} \subseteq A \subseteq A_{n,O}, \quad \lambda(A_{n,O}/A_{n,I}) \leq \delta_n.$$

Recall that $\mathscr{A}_n$ can be chosen with $\#\mathscr{A}_n = N(\delta_n, \mathscr{A}) = \exp\{H(\delta_n)\}$. Setting, for each $A \in \mathscr{A}$,

$$S^n(A) := X^n(A) - E\{X^n(A)\} = X^n(A) - \lambda(A)\mu_n\theta_n, \tag{4.24}$$

we have

$$S^n(A) = S^n(A_{n,O}) - S^n(A_{n,O}\backslash A) = S^n(A_{n,I}) + S^n(A\backslash A_{n,I}).$$

Thus

$$\begin{aligned} |S^n(A)| &\leq \tfrac{1}{2}\{|S^n(A_{n,O})| + |S^n(A_{n,I})|\} \\ &\quad + \tfrac{1}{2}\{|S^n(A_{n,O}\backslash A)| + |S^n(A\backslash A_{n,I})|\}. \end{aligned} \tag{4.25}$$

From (4.17) and (4.24) we trivially obtain

$$S^n(A) = \sum_{k=1}^{\xi_n} (J_k^n - \theta_n) I_A(T_k^n) + [\eta_n(A) - \lambda(A)\mu_n]\theta_n$$

and also

$$|S^n(A)| < \sum_{k=1}^{\xi_n} |J_k^n| I_A(T_k^n) + \lambda(A)\mu_n|\theta_n|,$$

where $\eta_n(A) := \sum_{k=1}^{\xi_n} I_A(T_k^n)$ is a simple Poisson random measure. Substituting this into (4.25) we now have

$$\begin{aligned} &\sup_{\mathscr{A}} |S^n(A)| \\ &\quad \leq \sup_{\mathscr{A}_n} \Big|\sum_{k=1}^{\xi_n} (J_k^n - \theta_n) I_A(T_k^n)\Big| + \sup_{\mathscr{A}_n} |\eta_n(A) - \lambda(A)\mu_n| \cdot |\theta_n| \\ &\quad + \tfrac{1}{2}\sup\Big\{\sum_{k=1}^{\xi_n} |J_k^n| I_{A\backslash A'}(T_k^n) : A' \subset A,\ \lambda(A\backslash A') < \delta_n,\ A, A' \in \mathscr{A}_n\Big\} \\ &\quad + \tfrac{1}{2}\mu_n\delta_n|\theta_n|. \end{aligned} \tag{4.26}$$

Recall that we need to show that $P\{\sup_{\mathscr{A}} |S^n(A)|$ ⅝ α_n! is summable. We break the remainder of the proof into three parts, corresponding to the three stochastic terms on the right hand side of (4.25).

(i) We commence with

$$P\{\sup_{\mathscr{A}_n} | \sum_{k=0}^{\xi_n} (J_k^n - \theta_n) I_A(T_k^n) | > \alpha_n/4 - \tfrac{1}{2}\mu_n\delta_n |\theta_n|\}$$

(4.27)

$$\leq \exp\{H(\delta_n)\} P\{| \sum_{k=1}^{\xi_n} (J_k^n - \theta_n) | > 2^{-n}\mu_n\delta_n\} \quad \text{(by (4.15), (4.23))}$$

To bound this expression, consider the cases $\xi_n > \gamma\mu_n$ and $\xi_n < \gamma\mu_n$ separately. For the latter case we have

$$P\{| \sum_{k=1}^{\xi_n} (J_k^n - \theta_n) | > 2^{-n}\mu_n\delta_n;\ \xi_n < \gamma\mu_n\}$$

$$= \sum_{m=0}^{\gamma\mu_n} P\{| \sum_{k=1}^{m} (J_k^n - \theta_n) | > 2^{-n}\mu_n\delta_n\} e^{-\mu_n}(\mu_n)^m/m!$$

$$\leq \sum_{m=0}^{\gamma\mu_n} \frac{e^{-\mu_n}\mu_n^m}{m!} \cdot 2 \exp\left\{\frac{-2^{-2n}\mu_n^2\delta_n^2}{2m\sigma_n^2(1 + 4 \cdot 2^{-2n}\mu_n\delta_n/3m\sigma_n^2)}\right\}$$

by Bernstein's inequality (4.19). A simple rearrangement of the above yields

$$2 \sum_{m=0}^{\gamma\mu_n} \frac{e^{-\mu_n}\mu_n^m}{m!} \exp\left\{\frac{-\mu_n\delta_n}{8/3 + (2^{2n+1}m\sigma_n^2/\mu_n\delta_n)}\right\}.$$

Noting that $(m/\mu_n) < \gamma$, and applying inequalities (4.15) and (4.16) we easily bound this by

$$2 \exp\{-\mu_n\delta_n/8(1/3 + \gamma\delta_n^{-1})\}.$$

Taking n large enough so that $\delta_n < 3(1 - \gamma)$ (this is possible since $\delta_n \to 0$ as $n \to \infty$) we bound the above by

$$2 \exp\{-\mu_n\delta_n^2/8\}.$$

Applying Lemma 4.1 to the case $\xi_n > \gamma\mu_n$ and combining the two bounds yields an upper bound for (4.27) of the form

$$2 \exp\{H(\delta_n) - \mu_n\delta_n^2/8\} + C \cdot \exp\{H(\delta_n) - \mu_n/4\}$$

for a finite constant C. This is clearly summable under the conditions of the theorem, and thus so is (4.27).

(ii) Now consider

$$P\{\sup_{\mathscr{A}_n} | \eta_n(A) - \lambda(A)\mu_n | \cdot | \theta_n | > \alpha_n/4\}$$

(4.28)

$$\leq \exp\{H(\delta_n)\} \cdot \sup_{\mathscr{A}_n} P\{| \eta_n(A) - \lambda(A)\mu_n | > 2^{n-1}\alpha_n/4\} \quad \text{by (4.15)}$$

$$\leq \exp\{H(\delta_n)\} \cdot P\{| \xi_\eta - \mu_n | > \delta_n\mu_n\}.$$

Once again, it is straightforward to apply Lemma 4.1 to bound the above and establish the summability of (4.26) under (4.3).

(iii) Finally, let B_n be an arbitrary (but fixed) set in $\mathscr{B}^k$ with $\lambda(B_n) = \delta_n$. Then the probability that the third supremum of (4.26) is greater than $\alpha_n/2$ is clearly no greater than

$$\exp\{2H(\delta_n)\} \cdot P\{\eta_n(B_n) > 2\mu_n\delta_n\} \quad \text{by (4.15)}$$
$$\le C[\exp\{H(\delta_n) - \mu_n\delta_n/8\}]^2$$

by Lemma 4.1. Clearly, this is also summable under (4.3), since $V_n \to \infty$ implies $\delta_n^2 < \delta_n$ for large enough n.

Collecting parts (i)–(iii) and applying them to (4.26) now completes the proof of Theorem 4.2.

We can thus turn to the

PROOF OF THEOREM 4.1. Theorem 4.1 is an immediate consequence of Theorem 5.4 of Adler *et al.* (1983), which establishes the existence of cadlag Lévy processes with the paths of finite total variation. The corresponding Lévy measure is easily seen to be both cadlag and *countably* additive.

5. A comment and a conjecture. We now wish to make one comment concerning the sharpness of the results of the preceding sections, and to conjecture what form improvements to these results may take. Take Corollary 4.1 as an example. There it is shown that if for some $\varepsilon > 0$

$$\int_0^1 [H(x^2)]^{(\beta-1)/2(2-\beta)+\varepsilon}\, dx < \infty \tag{5.1}$$

for a stable measure with parameter $\beta \in (1, 2)$, then a cadlag version exists. It is known, (Dudley, 1973) that if $\beta = 2$ (the Brownian sheet case), (5.1) can be replaced by

$$\int_0^1 [H(x^2)]^{1/2}\, dx < \infty, \tag{5.2}$$

in which case the measure is *continuous*. One would hope that sending $\beta \to 2$ in (5.1) would yield (5.2). The fact that this is clearly not the case raises doubts as to the sharpness of (5.1).

In fact, we conjecture that (5.1) should be replaced by

$$\int_0^1 [H(x^2)]^{(\beta-1)/2}\, dx < \infty, \tag{5.3}$$

but can see no way of proving this. In fact, if (5.3) could be shown to be sufficient for the existence of a cadlag version, then the results of Section 3 would indicate that is is also a reasonably sharp condition.

Note Added in Proof. Pyke and Bass (*Z. Wahrsch. verw. Gebiete*, to appear) have recently obtained a result stronger than our Theorem 4.2 that, modulo minor technical differences, implies the above conjecture. Their approach is very close to ours, the primary difference lying in the replacement of Bernstein's inequality with a sharper inequality specifically tailored to infinitely divisible distributions. The use of the sharper inequality yields results which, according to our "counter-example" of Section 3, are essentially best possible.

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