

On the Calculation of Changes in the Earth's Inertia Tensor due to Faulting

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Summary

Recent interest has arisen in the possibility that changes in the Earth's inertia tensor accompanying earthquakes may provide the excitation for the 'Chandler' wobble of the rotation axis. We present a simple procedure for calculating these changes based on the elastic reciprocal theorem and Volterra's formula. In particular, if τ is the shear stress created in the slip direction on a prospective fault surface when a perfect, unfaulted sphere is steadily rotated at unit angular velocity about some axis, then the change in moment of inertia about that same axis is shown to be twice the work of τ when carried through the actual slip displacement which occurs in faulting. It is shown that changes of products of inertia may be computed in a similarly simple way. The method is applied to two homogeneous Earth models, the relation to previous treatments is discussed, and it is noted that inertia changes accompanying less-catastrophic mass movements may be calculated in a similar way. The development in the body of the paper is in the context of classical linear elasticity. An Appendix extends the reciprocal theorem and Volterra's formula, and hence the basis for similar inertia change calculations, to the linearized incremental deformation of self-gravitating, initially stressed elastic systems, such as real-Earth models.

Introduction

Our paper is concerned with calculating the changes in the inertia tensor of an elastic sphere due to mass displacements accompanying slip over an interior surface. The calculation is of interest to the theory of the Earth's rotation, in that sudden changes in the Earth's inertia accompanying earthquake faulting will alter the position and, more importantly, the subsequent motion of the rotation pole relative to an Earth-bound observer. In particular, Mansinha and Smylie (1967) and Smylie and Mansinha (1968) have re-awakened interest in earthquakes as a possible excitation source for the 'Chandler' wobble of the rotation pole.

Ben-Menahem & Israel (1970), basing their study on the earlier work of Ben-Menahem & Singh (1968), have presented a solution based on classical linear elasticity for the inertia changes due to faulting in a non-gravitating, homogeneous and isotropic sphere. Also, Smylie & Mansinha (1971a) and Dahlen (1971a) have separately extended the calculations to the faulting of 'real Earth models' of self-gravitating spheres, with large initial hydrostatic stresses, radially inhomogeneous properties, and fluid cores. These were done according to the linearized theory of

small deformations from an initially stressed state. (There are some important differences in the core models and core-mantle boundary conditions in the two cases (see the discussion by Dahlen 1971b, and reply by Smylie & Mansinha 1971b), but these need not directly concern us here.)

Our purpose is to show that there is a simpler way to the final results. This involves a direct application of the elastic reciprocal theorem and requires, for calculation of the inertia changes, that one know only the stress distribution which results when the elastic earth model in question is steadily rotated about some central axis. This brings a remarkable unity to the theory of the Earth's wobble, because this very same elasticity solution is (at least in principle) a pre-requisite to deriving the differential equations of the wobble, for which the inertia changes are thought to act as an excitation source.

Here we refer to the elastic bulging of the Earth about its rotation axis and the consequent effects on the equations of the wobble, including the alteration of the 10 month rigid-body period to the 14 month Chandler period (see, for example, Munk & MacDonald 1960). Such steady rotation solutions, from which our procedure allows calculation of inertia changes in faulting, have been given by Chree (see Love 1927, art. 174-5) for a homogeneous non-gravitating sphere or spherical shell, by Love (art. 177-179) for a homogeneous but incompressible, self-gravitating sphere and by Takeuchi (1950) for a radially non-homogeneous real-Earth model, with a fluid core, incorporating self-gravitation and large initial hydrostatic stresses.

The development in the body of the paper is in the context of classical linear elasticity. However, both Dahlen (1971a) and Smylie & Mansinha (1971a) have shown that the reciprocal theorem and Volterra's formula can be extended to the linearized incremental deformation of self-gravitating elastic systems in states of large initial hydrostatic stress. We give an elementary derivation of this extension of the reciprocal theorem and Volterra's formula in the Appendix, providing also their generalization to cases of non-hydrostatic initial stress states. With these developments, the procedures given here for calculating the inertia changes are applicable as well to real-Earth models.

Reciprocal theorem

Consider a spherically symmetric distributed mass M , bounded by an outer spherical surface and containing an internal surface Σ on which slip displacements are to be prescribed. The sides of Σ are denoted Σ^+ , Σ^- and these have opposite unit outward normals \mathbf{v}^+ , \mathbf{v}^- [directed so that \mathbf{v}^- points from the (-) to (+) side]. Let $\boldsymbol{\sigma}$ and \mathbf{u} denote the additional stress and displacement fields induced when relative slip displacements $\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ are prescribed on Σ in such a way that no external force layer need be applied to maintain equilibrium; i.e. $(\mathbf{v} \cdot \boldsymbol{\sigma})^+ + (\mathbf{v} \cdot \boldsymbol{\sigma})^- = 0$. The changes in the inertia tensor may be calculated as suitable weighted averages of \mathbf{u} .

Now let \mathbf{f}^* be an arbitrary self-equilibrating field of body force per unit mass, and let $\boldsymbol{\sigma}^*$ and \mathbf{u}^* be the stress and displacement fields which result when this is applied to the same spherically symmetric distributed mass M , but in the absence of any slip on Σ (so that both $\boldsymbol{\sigma}^*$ and \mathbf{u}^* are continuous across Σ).

We have thus defined two elastic fields $\boldsymbol{\sigma}$, \mathbf{u} and $\boldsymbol{\sigma}^*$, \mathbf{u}^* in the region bounded by the outer spherical surface and by the internal surface Σ . Both leave the outer surface traction free, and the former has zero associated body force. Hence, by the reciprocal theorem,

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm + \int_{\Sigma} [(\mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot \mathbf{u})^+ + (\mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot \mathbf{u})^-] d\Sigma = \int_{\Sigma} [(\mathbf{v} \cdot \boldsymbol{\sigma} \cdot \mathbf{u}^*)^+ + (\mathbf{v} \cdot \boldsymbol{\sigma} \cdot \mathbf{u}^*)^-] d\Sigma \quad (1)$$

where we write $\mathbf{v} \cdot \boldsymbol{\sigma}$, $\mathbf{v} \cdot \boldsymbol{\sigma}^*$ for the surface forces of the two fields.

But since \mathbf{u}^* is continuous across Σ and $(\mathbf{v} \cdot \boldsymbol{\sigma})^+ + (\mathbf{v} \cdot \boldsymbol{\sigma})^- = 0$, the integral on the right side vanishes. Further, since $\boldsymbol{\sigma}^*$ is continuous, the integrand on the left side may be written $-\mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot (\mathbf{u}^+ - \mathbf{u}^-) = -\mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot \Delta \mathbf{u}$ where $\mathbf{v} \equiv \mathbf{v}^- = -\mathbf{v}^+$.

Thus

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm = \int_{\Sigma} \mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot \Delta \mathbf{u} d\Sigma. \tag{2}$$

If we now define \mathbf{s} as a unit vector in the slip direction at every point of Σ , so that $\Delta \mathbf{u} = \mathbf{s} \Delta u$ where Δu is the slip magnitude, this becomes

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm = \int_{\Sigma} \tau^* \Delta u d\Sigma. \tag{3}$$

Here $\tau^* = \mathbf{v} \cdot \boldsymbol{\sigma}^* \cdot \mathbf{s}$ is the shear stress induced on Σ by \mathbf{f}^* . This last equation is Volterra's formula, at least when \mathbf{f}^* is identified as a unit point force in some direction so that the left side is the component of \mathbf{u} in the same direction. It is seen in the Appendix that the same equation applies in the case of large initial stresses, with τ^* defined as in equation (A10).

Inertia changes

The inertia tensor of a distributed mass M is

$$\mathbf{J} = \int_M [\mathbf{I}(\mathbf{x}, \mathbf{x}) - \mathbf{x}\mathbf{x}] dm \tag{4}$$

where \mathbf{x} is the position vector of the mass element dm , and the moment of inertia about an axis having the orientation of a unit vector \mathbf{n} is

$$J^{(n)} = \mathbf{n} \cdot \mathbf{J} \cdot \mathbf{n} = \int_M \xi^2 dm \tag{5}$$

where ξ is the perpendicular distance from the axis to dm . Thus, when the displacements \mathbf{u} are given to the mass elements of the sphere by slip on Σ we have a change in moment of inertia

$$\Delta J^{(n)} = \mathbf{n} \cdot \Delta \mathbf{J} \cdot \mathbf{n} = 2 \int_M \xi u_{\xi} dm \tag{6}$$

to first order, where u_{ξ} is the displacement component in the radially outward direction from the axis.

Now, to see how this inertia change is calculated from the steady rotation solution, let us suppose that an unslipped sphere is spun at unit angular velocity about the axis \mathbf{n} . This creates a centrifugal body force having radial component equal to ξ with all other components vanishing. Thus $\Delta J^{(n)}$ is simply twice the work done by this body force on the displacements due to slip.

We therefore conclude, in view of Volterra's formula (equation (3)) that the change in moment of inertia is given by

$$\Delta J^{(n)} = 2 \int_{\Sigma} \tau^{(n)} \Delta u d\Sigma \tag{7}$$

where $\tau^{(n)}$ is the shear stress on the fault surface induced by steadily rotating the sphere at a unit angular velocity about the axis \mathbf{n} .

The same formula may also be applied to the off-diagonal components of ΔJ (i.e. the products of inertia) by recalling from the laws of tensor transformation in a plane that off-diagonal components are expressible as differences between normal components at $\pm 45^\circ$. That is, if (x_1, x_2, x_3) are the cartesian reference axes and if directions \mathbf{n} and \mathbf{m} are defined by

$$\sqrt{2} \mathbf{n} = (1, 1, 0), \sqrt{2} \mathbf{m} = (1, -1, 0)$$

then by direct calculation one has

$$\Delta J^{(n)} - \Delta J^{(m)} = \mathbf{n} \cdot \Delta \mathbf{J} \cdot \mathbf{n} - \mathbf{m} \cdot \Delta \mathbf{J} \cdot \mathbf{m} = 2\Delta J_{12}.$$

Thus ΔJ_{12} is the work done on the slip surface by the difference between the shear stress which results from rotating at unit angular speed about the axis \mathbf{n} and that which results from rotating at unit angular speed about the axis \mathbf{m} .

This is the way in which the elastic stress distribution for steady rotation allows a complete determination of the changes in all components of the inertia tensor. A somewhat more systematic procedure from the point of view of calculations is as follows: Let us write the force field for steady rotation at unit speed about the axis \mathbf{n} in the form

$$\mathbf{f}^{(n)} = (\mathbf{n} \cdot \mathbf{n}) \mathbf{x} - \mathbf{n}(\mathbf{n} \cdot \mathbf{x}) = \nabla(\frac{1}{3}n^2 r^2 - \frac{1}{2}\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \tag{8}$$

where $n^2 = \mathbf{n} \cdot \mathbf{n}$, $r^2 = \mathbf{x} \cdot \mathbf{x}$ and

$$\mathbf{N} = \mathbf{m}\mathbf{m} - \frac{1}{3}n^2 \mathbf{I}. \tag{9}$$

(Here it is convenient *not* to take advantage of the fact that \mathbf{n} is a unit vector.) The first term in the force potential of (8) results in a purely radial force whereas the second is readily shown to be a spherical harmonic potential of degree 2.

Now, this force field is a linear function of n^2 and of components of the symmetric, deviatoric tensor \mathbf{N} . Hence, by elastic linearity, the resulting stress field $\sigma^{(n)}$ and the associated shear stress $\tau^{(n)}$ must be linear in these same variables, and may therefore be expressed in the forms

$$\sigma^{(n)} = n_i n_j \sigma^{(ij)}, \tau^{(n)} = n_i n_j \tau^{(ij)} \tag{10}$$

(summation on repeated indices) where the coefficients $\sigma^{(ij)}$ and $\tau^{(ij)}$, which we agree to choose so as to be symmetrical in i and j , are functions of position in space or, for the latter, of position on Σ only, and are completely independent of the direction chosen for \mathbf{n} .

If we insert the latter of equations (10) into equation (7), we have in cartesian component form

$$\Delta J^{(n)} = n_i n_j \Delta J_{ij} = 2 \int_{\Sigma} n_i n_j \tau^{(ij)} \Delta u d\Sigma. \tag{11}$$

Since this must hold for any choice of \mathbf{n} , we therefore have the general result

$$\Delta J_{ij} = 2 \int_{\Sigma} \tau^{(ij)} \Delta u d\Sigma. \tag{12}$$

Further, since $\sigma^{(ij)}$ will be a smooth function of position over size scales very much larger than typical fault dimensions, one will generally not have to integrate, but may instead write

$$\Delta J_{ij} = 2\tau^{(ij)} \Delta u \Sigma \tag{13}$$

where here Δu is interpreted as the area average slip displacement and $\tau^{(ij)}$ is evaluated at a representative point of Σ .

Hence we have reduced the calculation of ΔJ_{ij} to the determination of the symmetricized coefficient $\tau^{(ij)}$ of the product $n_i n_j$ in an expression for the shear stress $\tau^{(n)}$ induced on Σ by steady rotation at unit speed about the axis \mathbf{n} . Alternatively, by directly identifying the symmetricized coefficient of $n_i n_j$ in equation (8) for $\mathbf{f}^{(n)}$, it can be shown that $\tau^{(ij)}$ is simply the shear stress induced by the force field.

$$\mathbf{f}^{(ij)} = \nabla \left[\frac{1}{3} r^2 \delta_{ij} - \frac{1}{2} (x_i x_j - \frac{1}{3} r^2 \delta_{ij}) \right] \tag{14}$$

where, as in equation (8), the first term results in a radial force and the second is a spherical harmonic potential of degree 2.

It may be also noted that since only $n^2 = n_1^2 + n_2^2 + n_3^2$ enters the first term of the force potential in equation (8), that term can affect only the computation of the on-diagonal components of ΔJ . The off-diagonal components, which are of interest for the Chandler wobble, depend only the 2nd degree spherical harmonic term. Indeed, Ben-Menahem and Israel (1970), Dahlen (1971a), and Smylie and Mansinha (1971a) have noted that off-diagonal components are similarly dependent only on the 2nd degree spherical harmonic terms in their analyses.

Application to homogeneous Earth models

We here calculate the inertia change coefficients $\tau^{(ij)}$ from two simple, homogeneous earth models. First, consider the model studied by Ben-Menahem & Israel (1970) of a homogeneous, isotropic, non-gravitating sphere of radius a . From Love (1927), the displacement corresponding to the force field $\mathbf{f}^{(n)}$ of equation (8) is

$$\begin{aligned} \mathbf{u}^{(n)} = & \frac{\rho a^2 n^2}{15(\lambda + 2\mu)} \left(\frac{5\lambda + 6\mu}{3\lambda + 2\mu} - \frac{r^2}{a^2} \right) \mathbf{x} + \frac{\rho}{28(\lambda + 2\mu)} \nabla [r^2 (\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x})] \\ & + \frac{\rho(7\lambda + 6\mu)(5\lambda + 7\mu)}{28\mu(\lambda + 2\mu)(19\lambda + 14\mu)} \left[r^2 \nabla (\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) - 2 \frac{2\lambda + 7\mu}{5\lambda + 7\mu} (\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \mathbf{x} \right] \\ & - \frac{\rho(4\lambda + 3\mu) a^2}{2\mu(19\lambda + 14\mu)} \nabla (\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \end{aligned} \tag{15}$$

where λ and μ are the Lamé constants. Love's presentation has been followed directly, the first term corresponding to the purely radial force and the last three to the 2nd degree spherical harmonic.

The associated stress state is

$$\begin{aligned} \boldsymbol{\sigma}^{(n)} = & \lambda \mathbf{I} (\nabla \cdot \mathbf{u}^{(n)}) + \mu [(\nabla \mathbf{u}^{(n)}) + (\nabla \mathbf{u}^{(n)})^T] \\ = & \frac{4\rho\mu n^2}{15(\lambda + 2\mu)} \left[a^2 \mathbf{I} + \frac{5\lambda + 2\mu}{4\mu} (a^2 - r^2) \mathbf{I} - \mathbf{x}\mathbf{x} \right] \\ & - \frac{\rho(3\lambda + 2\mu)}{19\lambda + 14\mu} \left[(\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \mathbf{I} - \mathbf{x}(\mathbf{N} \cdot \mathbf{x}) - (\mathbf{N} \cdot \mathbf{x}) \mathbf{x} \right. \\ & \left. + a^2 \mathbf{N} + \frac{5\lambda + 4\mu}{3\lambda + 2\mu} (a^2 - r^2) \mathbf{N} \right] \end{aligned} \tag{16}$$

and from this we may compute the shear stress $\tau^{(n)} = \mathbf{v} \cdot \boldsymbol{\sigma}^{(n)} \cdot \mathbf{s}$ at a point on the fault plane as

$$\begin{aligned} \tau^{(n)} = & -\frac{4\rho\mu r^2}{15(\lambda+2\mu)} v_r s_r n^2 + \frac{\rho(3\lambda+2\mu)}{19\lambda+14\mu} \left\{ r^2 v_r (\mathbf{e} \cdot \mathbf{N} \cdot \mathbf{s}) \right. \\ & \left. + r^2 s_r (\mathbf{e} \cdot \mathbf{N} \cdot \mathbf{v}) - \left[a^2 + \frac{5\lambda+4\mu}{3\lambda+2\mu} (a^2 - r^2) \right] (\mathbf{v} \cdot \mathbf{N} \cdot \mathbf{s}) \right\}. \end{aligned} \quad (17)$$

Here we have noted that $\mathbf{v} \cdot \mathbf{s} = 0$, have written the position vector \mathbf{x} as er so that \mathbf{e} is a radial unit vector, and have introduced the notations $v_r = \mathbf{e} \cdot \mathbf{v}$ and $s_r = \mathbf{e} \cdot \mathbf{s}$ for the radial components of \mathbf{v} and \mathbf{s} .

To obtain the inertia change coefficients $\tau^{(ij)}$, we simply write $\tau^{(n)}$ in the form of equation (10) by expressing n^2 and \mathbf{N} in component forms and identifying the symmetricized coefficients of $n_i n_j$. Thus, recalling that $\Delta J_{ij} = 2\tau^{(ij)} \Delta u \Sigma$ (at least when no integration is necessary) we find that this Earth model leads to the inertia changes

$$\begin{aligned} \frac{\Delta J_{ij}}{\Delta u \Sigma} = 2\tau^{(ij)} = & -\frac{4}{3}\rho r^2 v_r s_r \left[\frac{2\mu}{5(\lambda+2\mu)} + \frac{3\lambda+2\mu}{19\lambda+14\mu} \right] \delta_{ij} \\ & + \rho r^2 \frac{3\lambda+2\mu}{19\lambda+14\mu} [v_r (e_i s_j + e_j s_i) + s_r (e_i v_j + e_j v_i)] \\ & - \rho \frac{2(4\lambda+3\mu)a^2 - (5\lambda+4\mu)r^2}{19\lambda+14\mu} (v_i s_j + v_j s_i). \end{aligned} \quad (18)$$

The result is particularly simple for a strike-slip fault for then $v_r = s_r = 0$ and only the last term remains. ΔJ_{ij} is then tensorially distributed among its components in proportion to $(v_i s_j + v_j s_i)$, the latter having the form of a pure shear if considered as a strain. By referring our results to geographic and epicenter co-ordinates, we have checked that equation (18) is indeed in agreement with the corresponding result of Ben-Menahem & Israel (1970).

Now, to study the effect of self-gravitation in the simplest possible context, consider Love's (1927, art. 177) model of a homogeneous, isotropic, incompressible ($\lambda \rightarrow \infty$), self-gravitating sphere under an initial state of hydrostatic pressure. The first term in the force potential of equation (8), corresponding to a purely radial force, induces zero displacement, while the remaining 2nd degree spherical harmonic causes displacement expressible from Love's solution as

$$\begin{aligned} \mathbf{u}^{(n)} = & \left(1 + \frac{2\rho g a}{19\mu} \right)^{-1} \left[\frac{5\rho}{76\mu} r^2 \nabla(\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \right. \\ & \left. - \frac{\rho}{19\mu} (\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \mathbf{x} - \frac{2\rho a^2}{19\mu} \nabla(\mathbf{x} \cdot \mathbf{N} \cdot \mathbf{x}) \right] \end{aligned} \quad (19)$$

where g is the gravitational acceleration at the Earth's surface. The bracketed term is the limit as $\lambda \rightarrow \infty$ of the displacement field given by equation (15), reflecting the well-known result of this model that displacements are simply reduced by the factor $(1 + 2\rho g a / 19\mu)^{-1}$ from those for the corresponding non-gravitating Earth model.

The hydrostatic part of the stress field $\sigma^{(n)}$ cannot be directly computed from $u^{(n)}$ because of incompressibility, but this is of no consequence since only the shear stress $\tau^{(n)}$ is of interest. In fact, from what has been noted above, it is clear that $\tau^{(n)}$ will simply be given by the factor $(1 + 2\rho ga/19\mu)^{-1}$ multiplied by the limit of equation (17) as $\lambda \rightarrow \infty$. Hence, by taking the same limit in equation (18), the inertia changes predicted from this model are

$$\begin{aligned} \frac{\Delta J_{ij}}{\Delta u \Sigma} = 2\tau^{(ij)} = & \left(1 + \frac{2\rho ga}{19\mu}\right)^{-1} \left\{ -\frac{4}{19} \rho r^2 v_r s_r \delta_{ij} \right. \\ & + \frac{3}{19} \rho r^2 [v_r(e_i s_j + e_j s_i) + s_r(e_i v_j + e_j v_i)] \\ & \left. - \rho \frac{8a^2 - 5r^2}{19} (v_i s_j + v_j s_i) \right\}. \end{aligned} \tag{20}$$

That is, the inertia changes are reduced by the factor $(1 + 2\rho ga/19\mu)^{-1}$ from those for the corresponding non-gravitating model.

To estimate the factor let us recall that $(1 + 19\mu/2\rho ga)^{-1}$ is the elastic Love number, according to this model, for the ratio of the tidal bulge of an elastic sphere to that of a fluid sphere. If, following Takeuchi's (1950) analysis of a real-Earth model, we take a value of 0.29 for the Love number, the factor by which the inertia changes are reduced through consideration of self-gravitation is 0.71.

Our procedure could also be applied to prediction of the inertia changes from a real-earth model by adopting Takeuchi's (1950) analysis for the stresses induced by steady rotation. We have not carried out the necessary numerical evaluations, although this is under investigation as a possible independent check of the conflicting results of Dahlen (1971a) and Smylie & Mansinha (1971a) (see Dahlen 1971b). Smylie and Mansinha report inertia changes from their real-Earth model which are about 3 times those obtained for the homogeneous, non-gravitating Earth model of Ben Menahem and Israel in the case of the 1964 Alaska earthquake, while Dahlen obtains somewhat smaller changes. Our simple assessment of self-gravitation effects suggests that it is the elastic inhomogeneity of the real Earth models which is most important in creating these differences.

Inertia changes in general mass re-distributions

The formalism outlined here for computing inertia changes in faulting can also be extended to include the inertia changes arising over longer time scales from sources such as creep deformations and progressive sliding along plate boundaries (although it is not clear that input data is known with sufficient accuracy to allow numerical evaluations). Let us suppose that during some given time period relative slip displacements Δu due to sudden faulting or progressive sliding occur over a number of surfaces, denoted collectively by Σ , within a sphere. If all associated deformation of the material outside these slip zones is elastic, equations (7) and (12) apply for the inertia changes during this time period, with the understanding that the integral is to be carried out over all slipping surfaces.

More generally, however, the material of the sphere will undergo both elastic and inelastic deformations. If ϵ is the total strain accumulated at some material point during this time period and σ the corresponding change in stress, we may define an elastic portion ϵ^e of the strain as that which would occur if the material had responded to the stress change σ in a purely elastic fashion. An inelastic (or plastic) portion of

the strain is then defined as $\epsilon^p = \epsilon - \epsilon^e$ corresponding to that part of the deformation due to creep and/or plastic flow. It is then straightforward to generalize the Reciprocal Theorem and obtain in place of (3)

$$\int_M \mathbf{f}^* \cdot \mathbf{u} \, dm = \int_{\Sigma} \tau^* \Delta u d\Sigma + \int_M \frac{1}{\rho} \sigma^* : \epsilon^p \, dm. \tag{21}$$

Hence, by following the same argument as presented earlier, the inertia change expressions of equation (7) and (12) become

$$\left. \begin{aligned} \Delta J^{(n)} &= 2 \int_{\Sigma} \tau^{(n)} \Delta u d\Sigma + 2 \int_M \frac{1}{\rho} \sigma^{(n)} : \epsilon^p \, dm \\ \Delta J_{ij} &= 2 \int_{\Sigma} \tau^{(ij)} \Delta u d\Sigma + 2 \int_M \frac{1}{\rho} \sigma^{(ij)} : \epsilon^p \, dm \end{aligned} \right\} \tag{22}$$

where $\sigma^{(n)}$, $\sigma^{(ij)}$, $\tau^{(n)}$, $\tau^{(ij)}$ are the stresses as defined earlier from the steady rotation solution.

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Appendix

The reciprocal theorem and Volterra's formula for incremental deformations of initially stressed, self-gravitating elastic systems

Dahlen (1971a) and Smylie & Mansinha (1971a) have shown that the elastic reciprocal theorem and Volterra's formula may be extended to treat the linearized incremental deformations of a self-gravitating, elastic-real-Earth model, with fluid core, in an initially stressed state of hydrostatic pressure. Their methods involve a detailed proof from the continuum field equations of equilibrium and gravitation in conjunction with an elastic incremental constitutive law. Our purpose here is to provide a simple derivation of the extension of the reciprocal theorem and Volterra's formula (and hence also of our method of computing inertia changes) to such self-gravitating elastic systems, under general initial stress states.

The argument parallels a derivation given in the context of elastic structural mechanics (Southwell 1936). It is simplest to begin in terms of the statics of a discrete particle system with elastic and gravitational forces of internal interaction between its respective parts.

Suppose that such a system is in equilibrium under a set of external generalized forces Q_1, Q_2, \dots, Q_n , and let q_1, q_2, \dots, q_n be the associated generalized displacements, defined so that $Q_i dq_i$ is a work increment. We measure the q 's from zero when the body is in an initial equilibrium state under a set of forces Q_i^0 . The work of the internal elastic and gravitational forces in any increment of deformation can be expressed as $-dU$ and $-dV$, where U is the elastic potential energy (or 'strain energy') and V is the gravitational potential energy. Hence, by the principle of virtual work,

$$Q_i dq_i - dU - dV = 0 \quad (\text{A1})$$

for any incremental change from one equilibrium state to another.

Now, since the equilibrium states are fully determined by the forces Q_i (at least in a sufficiently small neighbourhood of Q_i^0), U , V and q_i may be regarded as functions of the Q 's. Further, if we henceforth consider small incremental deformations from the initially stressed state, the q 's may be regarded as linear functions of the force differences $Q_i - Q_i^0$. Let q_i be the displacements produced by the force differences $Q_i - Q_i^0$ and q_i^* those by the force differences $Q_i^* - Q_i^0$.

Now suppose the forces are increased from Q_i^0 to Q_i by an increment $Q_i - Q_i^0$, causing the displacements q_i and then further increased by the increment $Q_i^* - Q_i^0$ (to a final set of forces $Q_i^* + Q_i - Q_i^0$), causing the further displacements q_i^* . From (A1) and the assumed linearity, the change in $U + V$ due to the first increment is

$$Q_i^0 q_i + \frac{1}{2}(Q_i - Q_i^0) q_i$$

and the further increase in the second increment is

$$Q_i q_i^* + \frac{1}{2}(Q_i^* - Q_i^0) q_i^*,$$

the first terms in each case being the work of the pre-existing forces. Hence, upon adding and rearranging terms, the total energy change is

$$\Delta(U + V) = Q_i^0(q_i + q_i^*) + \frac{1}{2}(Q_i - Q_i^0)q_i + \frac{1}{2}(Q_i^* - Q_i^0)q_i^* + (Q_i - Q_i^0)q_i^*. \quad (A2)$$

If instead the increment $Q_i^* - Q_i^0$ were first applied, the resulting energy change would differ only in that the last term would be $(Q_i^* - Q_i^0)q_i$. But since U and V are fully determined by the forces, the energy change cannot depend on the order of application of the force increments and we therefore conclude

$$(Q_i - Q_i^0)q_i^* = (Q_i^* - Q_i^0)q_i \quad (A3)$$

which is the desired extension of the reciprocal theorem.

The argument is extended at once to a continuous system such as a real-earth model. In this case $Q_i dq_i$ of (A1) is replaced by the work increment of distributed external body and surface forces, whereas V is the total gravitational potential energy and U the total strain energy, including that of the solid mantle and compressible fluid core.

It is then important to note that associated forces and displacements must be related in the work increment sense. Hence, if the surface integrals over Σ in equation (1) are viewed as being carried out relative to the position of Σ in the initial spherically symmetric reference state, the stress σ must be interpreted as the change in 'nominal' stress \mathbf{T} (defined so that $\mathbf{v} \cdot \mathbf{T} d\Sigma$ is the force acting on the area element of initial size $d\Sigma$ and with initial unit normal \mathbf{v}). Thus, equation (1) becomes

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm + \int_{\Sigma} \{[\mathbf{v} \cdot (\mathbf{T}^* - \mathbf{T}^0) \cdot \mathbf{u}]^+ + [\mathbf{v} \cdot (\mathbf{T}^* - \mathbf{T}^0) \cdot \mathbf{u}]^-\} d\Sigma \\ = \int_{\Sigma} \{[\mathbf{v} \cdot (\mathbf{T} - \mathbf{T}^0) \cdot \mathbf{u}^*]^+ + [\mathbf{v} \cdot (\mathbf{T} - \mathbf{T}^0) \cdot \mathbf{u}^*]^-\} d\Sigma. \quad (A4)$$

Since $(\mathbf{v} \cdot \mathbf{T}^0)^+$ and $(\mathbf{v} \cdot \mathbf{T}^*)^+$ are the negatives of their values on the $(-)$ side, the first surface integral becomes

$$- \int_{\Sigma} \mathbf{v} \cdot (\mathbf{T}^* - \mathbf{T}^0) \cdot \Delta \mathbf{u} d\Sigma$$

where again \mathbf{v} with no superscript denotes \mathbf{v}^- .

For the same reason, terms involving \mathbf{T}^0 may be deleted from the second integral. We *cannot*, however, assert that $(\mathbf{v} \cdot \mathbf{T})^+ + (\mathbf{v} \cdot \mathbf{T})^- = 0$ since these surface forces of the faulted state equilibrate one another at positions which are shifted by $\Delta \mathbf{u}$ relative to positions in the initial reference state. To simplify the second integral, let A denote the surface, after faulting, which was initially Σ . If \mathbf{z} denotes position in space, a material point at \mathbf{z} after faulting was initially at $\mathbf{x} = \mathbf{z} - \mathbf{u}$. If we consider an area element dA of the deformed fault surface, at position \mathbf{z} , equilibrium of the $(+)$ and $(-)$ forces on it is therefore expressed by

$$\left(\mathbf{v} \cdot \mathbf{T} \frac{d\Sigma}{dA}\right)_{\mathbf{x}=\mathbf{z}-\mathbf{u}^+}^+ + \left(\mathbf{v} \cdot \mathbf{T} \frac{d\Sigma}{dA}\right)_{\mathbf{x}=\mathbf{z}-\mathbf{u}^-}^- = 0. \quad (A5)$$

Using the symbols \mathbf{F}^+ , \mathbf{F}^- for these force intensities, the second integral may therefore

be written as

$$\begin{aligned} & \int_A \{ \mathbf{F}^+(\mathbf{z}) \cdot \mathbf{u}^*(\mathbf{z} - \mathbf{u}^+) + \mathbf{F}^-(\mathbf{z}) \cdot \mathbf{u}^*(\mathbf{z} - \mathbf{u}^-) \} dA(\mathbf{z}) \\ &= \int_A \{ \mathbf{F}^+ \cdot [\mathbf{u}^* - \mathbf{u}^+ \cdot \nabla \mathbf{u}^*] + \mathbf{F}^- \cdot [\mathbf{u}^* - \mathbf{u}^- \cdot \nabla \mathbf{u}^*] \} dA \\ &= \int_A \mathbf{F}^- \cdot [(\Delta \mathbf{u}) \cdot \nabla \mathbf{u}^*] dA = \int_{\Sigma} \mathbf{v} \cdot \mathbf{T}^0 \cdot (\nabla \mathbf{u}^*)^T \cdot \Delta \mathbf{u} d\Sigma. \end{aligned} \tag{A6}$$

Here in the second step we have written $\mathbf{u}^*(\mathbf{z} - \mathbf{u}^+) = \mathbf{u}^*(\mathbf{z}) - \mathbf{u}^+ \cdot \nabla \mathbf{u}^*$, in the third we have used continuity of \mathbf{u}^* and the result $\mathbf{F}^+ + \mathbf{F}^- = 0$, and in the fourth we have expressed $\mathbf{F}^- dA$ as $\mathbf{v} \cdot \mathbf{T}^0 d\Sigma$, the neglect of higher order terms in the second and fourth steps being consistent with a linearized theory for the incremental deformations.

Hence, the generalization of equation (2) is

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm = \int_{\Sigma} \mathbf{v} \cdot [\mathbf{T} - \mathbf{T}^0 + \mathbf{T}^0 \cdot (\nabla \mathbf{u}^*)^T] \cdot \Delta \mathbf{u} d\Sigma. \tag{A7}$$

Dahlen (1971a) and Smylie & Mansinha (1971a) have worked in terms of the change in true (or 'Cauchy') stress \mathbf{t} at a given material point. This is related to the nominal stress to first order in the displacement gradients by (e.g. Prager 1961, p. 202, equation (4.14)),

$$\mathbf{t} = \mathbf{T} - \mathbf{t}^0(\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u})^T \cdot \mathbf{t}^0 \tag{A8}$$

Note that $\mathbf{t}^0 = \mathbf{T}^0$ and that \mathbf{t} is always symmetric whereas \mathbf{T} is so only initially. Hence, if \mathbf{E} (in Dahlen's notation; $\boldsymbol{\tau}$ in that of Smylie and Mansinha) denotes the change in true stress at a given material point,

$$\mathbf{E} = \mathbf{T} - \mathbf{t}^0 - \mathbf{t}^0(\nabla \cdot \mathbf{u}) + (\nabla \mathbf{u})^T \cdot \mathbf{t}^0 \tag{A9}$$

and, writing $\Delta \mathbf{u} = \mathbf{s} \Delta u$ as before, equation (A7) becomes

$$\int_M \mathbf{f}^* \cdot \mathbf{u} dm = \int_{\Sigma} \boldsymbol{\tau}^* \Delta u d\Sigma, \tag{A10}$$

where

$$\boldsymbol{\tau}^* = \mathbf{v} \cdot [\mathbf{E}^* + \mathbf{t}^0(\nabla \cdot \mathbf{u}^*) - (\nabla \mathbf{u}^*)^T \cdot \mathbf{t}^0 + \mathbf{t}^0 \cdot (\nabla \mathbf{u}^*)^T] \cdot \mathbf{s}.$$

This rigorously generalizes Volterra's formula to circumstances of linearized incremental deformations of self-gravitating initially-stressed elastic bodies. In the case when the initial stress is hydrostatic, $\mathbf{t}^0 = -p^0 \mathbf{I}$ and all the additional terms dependent on $\nabla \mathbf{u}^*$ disappear, by cancellation and because $\mathbf{v} \cdot \mathbf{I} \cdot \mathbf{s} = 0$, so that $\boldsymbol{\tau}^* = \mathbf{v} \cdot \mathbf{E}^* \cdot \mathbf{s}$. This is the special version of (A10) derived by Dahlen and Smylie and Mansinha. The more general version allows calculation of earthquake displacement fields and (identifying \mathbf{E}^* and $\nabla \mathbf{u}^*$ as the stress increment and displacement gradient due to rotation at unit speed about a central axis) inertia changes in non-hydrostatically stressed Earth models.

The incremental linear elastic constitutive law to be used must, of course, be consistent with the existence of a strain energy function. This means that \mathbf{E} must be

given by an equation of the form (Prager 1961, equation (2.6), p. 206 and equation (4.13), p. 201)

$$\mathbf{E} = \mathbf{t}^0 \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \cdot \mathbf{t}^0 - \mathbf{t}^0 (\nabla \cdot \mathbf{u}) + \frac{1}{2} \mathcal{L} : [(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T]. \quad (\text{A11})$$

Here \mathcal{L}_{ijkl} is the second mixed partial derivative of the strain energy density, per unit initial volume, with respect to the Lagrangian strains ε_{ij} and ε_{kl} , evaluated in the initially stressed reference state when $\varepsilon = 0$, where $2\varepsilon = (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T + (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u})^T$. For an isotropic material, this may be expressed in the form

$$\mathbf{E} = \lambda \mathbf{I} (\nabla \cdot \mathbf{u}) + \mu [(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T] \quad (\text{A12})$$

as adopted by Dahlen and Smylie and Mansinha, only when the initial stress state is hydrostatic.