On the calculation of the transmission line parameters for long tubes using the method of multiple scales

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The present paper deals with the classical problem of linear sound propagation in tubes with isothermal walls. The perturbation technique of the method of multiple scales in combination with matched asymptotic expansions is applied to derive the first-order solutions and, in addition, the second-order solutions representing the correction due to boundary layer attenuation. The propagation length is assumed to be so large that in order to obtain asymptotic solutions which extend over the whole spatial range the first-order corrections to the classical attenuation rates of the different modes come into play as well. Starting with the case of the characteristic wavelength being large compared to the characteristic dimension of the duct, the analysis is then extended to the case where both of these quantities are of the same order of magnitude. Furthermore, the transmission line parameters and the transfer functions relating the sound pressures at the ends of the duct to the axial velocities are calculated. [DOI: 10.1121/1.1639323]

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I. INTRODUCTION

The subject of linear sound propagation in rigid tubes with isothermal walls has been attracting considerable interest over the years. The ''exact'' solution for tubes of circular cross-sectional shapes derived by Kirchhoff[1] (see also Rayleigh,[2] pp. 319–328) accounts for the effects of shear viscosity and heat conduction on the attenuation of sound waves. Later Zwikker and Kosten[3] (pp. 25–40) and independently also Iberall,[4] Daniels,[5] and Kraak[6] introduced an approximate theory based on the so-called *low reduced frequency assumptions* that enabled the simplification of the basic equations such that the transmission line parameters could be given in closed form. A thorough discussion of the applicability of this approach including a comparison with numerical solutions of Kirchhoff's general dispersion equations is presented in the 1975 paper by Tijdeman.[7] More recently, Stinson[8] considered an alternative treatment of the problem applying simplifying approximations to the equations that make up the Kirchhoff solution, rather than reducing the governing equations, and showed the equivalence of both approaches. These investigations then provided the basis for developing a general procedure applicable to tubes of arbitrary cross-sectional shape. Similar calculations were also carried out by Kergormard.[9]

From the point of view of a perturbation analysis the low reduced frequency assumptions can be interpreted as defining two scaling parameters that relate the most relevant geometrical scales, i.e., the wavelength, the characteristic diameter of the duct, and the thickness of the acoustic boundary layer, to each other: The spatial range consumed by the boundary layer as well as the diameter are presupposed to be small compared to the wavelength. Further simplifications are then possible assuming the boundary layer to be either small or large relative to the tube diameter, which introduces a third scaling parameter and thus two different ordering relationships for the two other parameters. In the following, these cases will be called the *high* and the *low frequency limit*, respectively. If, however, such additional constraints are imposed *a posteriori* on the solutions given by Zwikker and Kosten in order to derive approximate series expansions of the transmission line parameters with respect to that third ratio (see, e.g., Keefe's results[10] for the cylindrical tube), it will remain unclear whether the resulting expressions are the correct asymptotic solutions one would obtain if the two different ordering relationships were applied to the basic equations themselves.

Moreover, the length of the duct might become so large that the exponentially growing effects arising from viscosity and heat conduction in the boundary layer do not only affect the second-order terms of the sound pressure but also the leading order terms. The present study is motivated by the observation that sound propagation in tubes of this type has not yet been systematically studied. Thus, one of its primary aims is to derive the asymptotically correct solutions for the involved field quantities including the second-order terms that extend over a considerably large spatial range, suggesting the application of the *method of multiple scales*, as presented, e.g., in Nayfeh[11] or Crighton *et al.*[12] (pp. 209–232). By the removal of secular terms, the extra freedom such an approach introduces can be exploited to increase the range of validity of the asymptotic expansions. Since the following calculations proceed from the assumption that the boundary layer is small compared to the tube diameter (high frequency limit), the changes in lateral direction will be analyzed using the perturbation technique of the *method of matched asymptotic expansions*. A further goal to be pursued is the derivation of the asymptotically correct expressions for the transmission line parameters and the transfer functions of a long tube up to the second-order terms. The investigations are structured as follows.

As far as the diameter to wavelength ratio is concerned,

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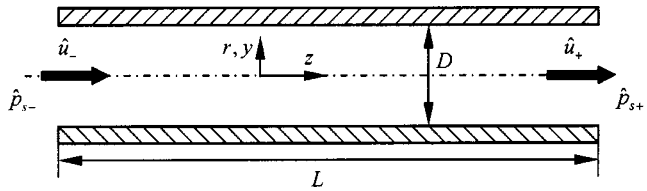

FIG. 1. Sketch and notation of an acoustical four pole.

the first part of the analysis deals with long wavelengths, in accordance with the low reduced frequency assumptions. Results for the first-order and the second-order terms of the sound pressure, the velocity components, and the transfer matrix in the case of long circular tubes will be presented. Similar solutions derived for the case of long rectangular and slit-shaped tubes are found in the appendices; these are, however, valid only to leading order. In the second part, beginning with Sec. V, the long wavelength assumption is then replaced with the condition that the characteristic wavelength and the diameter of the tube are of the same order of magnitude. To demonstrate the utility of the procedure, the transfer functions up to the second order of a long circular tube will again be derived. Since in this case the occurrence of higher order modes has to be taken into account as well, the study will confine itself to the case of axisymmetric flow. It should be mentioned that in a similar investigation concerning the sound propagation in a slit-shaped waveguide carried out by Anderson and Vaidya[13] the authors pointed out that the application of the method of multiple scales to the linear problem requires several observations suggested by results obtained from the so-called classical analysis that poses the boundary value problem as an eigenvalue problem, and would fail otherwise. However, as it will turn out in the following investigation, here such difficulties are not encountered.

II. PROBLEM FORMULATION

A. Transmission line parameters for acoustical four poles

For the time being, assume that the driving frequency is sufficiently low that only the fundamental mode is able to propagate in a tube of length L regarded as a transmission line (see Fig. 1). Furthermore, let $\hat{Z}$ and $\hat{Y}$ be the series impedance and shunt admittance per unit length along the z axis. The sound pressure $p_s(z,t)=\Re(\hat{p}_s(z)e^{j\omega t})$ and the volume flow $u(z,t)=\Re(\hat{u}(z)e^{j\omega t})$ are then given by

$$\frac{d\hat{p}_s}{dz}=-\hat{Z}\hat{u}, \quad \frac{d\hat{u}}{dz}=-\hat{Y}\hat{p}_s. \tag{1}$$

Hence, the four-pole transfer matrix $\hat{\mathbf{A}}$ can be formulated as

$$\hat{\mathbf{A}}=\begin{bmatrix} \cosh(\hat{\Gamma}L) & \hat{Z}_c \sinh(\hat{\Gamma}L) \\ \frac{1}{\hat{Z}_c}\sinh(\hat{\Gamma}L) & \cosh(\hat{\Gamma}L) \end{bmatrix}, \tag{2}$$

where

$$\hat{Z}_c=\sqrt{\frac{\hat{Z}}{\hat{Y}}}, \quad \hat{\Gamma}=\sqrt{\hat{Z}\hat{Y}} \tag{3}$$

are the characteristic impedance and the propagation parameter of the duct. The values of the sound pressure and the volume flow at the entrance of the tube, in the following denoted by $\hat{p}_{s-}$ and $\hat{u}_-$, respectively, can then simply be calculated from the values $\hat{p}_{s+}$ and $\hat{u}_+$ at the end of the tube by employing the relationship

$$\begin{pmatrix} \hat{p}_{s-} \\ \hat{u}_- \end{pmatrix}=\hat{\mathbf{A}}\begin{pmatrix} \hat{p}_{s+} \\ \hat{u}_+ \end{pmatrix}. \tag{4}$$

B. Low reduced frequency and low Mach number assumptions

In order to be able to calculate the transmission line parameters entering the transfer matrix from the basic equations, i.e., the two- or three-dimensional Navier–Stokes equations, the energy equation, the continuity equation, and the equation of state for a perfect gas, the following so-called low reduced frequency assumptions (see, e.g., Tijdeman[7]) are adopted:

$$\mathrm{Re}=\frac{c_0\lambda\rho_0}{\mu_0}\gg 1, \quad l=\frac{D}{\lambda}\ll 1. \tag{5}$$

Here, the quantities c_0, λ, ρ_0, μ_0, D, and the parameters Re and l denote the speed of sound, the characteristic wavelength, the density of the fluid, the dynamic shear viscosity, the characteristic dimension of the cross section, the acoustic Reynolds number, and the reduced frequency, which is of the order of the Helmholtz number $\mathrm{He}=\omega D/(2c_0)$. By the subscript 0, quantities evaluated at the equilibrium reference state are indicated.

Since the Prandtl number

$$\mathrm{Pr}=\frac{\mu_0 C_p}{\kappa_0}=O(1), \tag{6}$$

where C_p and κ_0 represent the specific heat at constant pressure and the thermal conductivity, is of order $O(1)$ for a wide class of fluid (e.g., Pr≈0.7 in case of air), the thermal as well as the viscous boundary layer thicknesses are given by (see, e.g., Morse and Ingard,[14] p. 286)

$$\delta\sim\sqrt{\frac{\mu_0\lambda}{c_0\rho_0}}=\frac{\lambda}{\sqrt{\mathrm{Re}}}. \tag{7}$$

Consequently, from the first restriction in Eq. (5) it follows that δ is small compared to the characteristic wavelength, ensuring that the flow is not dominated by viscous effects; it is easily verified that $\mathrm{Re}\gg 1$ holds for the complete range of audible and even a wide range of ultrasonic frequencies, provided that the fluid is air. This, together with the long wavelength assumption $l\ll 1$ stated in Eq. (5), then guarantees that only a single mode propagates over large distances relative to the tube diameter and a simplified, one-dimensional formulation of the problem as in Eq. (1) can be derived.

In the following, it will furthermore be assumed that the Mach number

$$M=\frac{W}{c_0}\ll 1, \tag{8}$$

with W as a characteristic particle velocity in longitudinal direction, is small as well, so that a linear analysis of the problem is possible.

C. Ducts with circular cross sections and isothermal walls

By imposing the low reduced frequency assumptions ($D=2R$), evaluation of the linearized basic equations yields (see Zwikker and Kosten,[3] pp. 25–40, but also Refs. 4–7, 10)

$$\hat{Z}=\frac{j\omega\rho_0}{S\left(1-2\sqrt{\frac{\mu_0}{-j\omega\rho_0}}\frac{\hat{G}\left(R\sqrt{\frac{-j\omega\rho_0}{\mu_0}}\right)}{R}\right)},$$

$$\hat{Y}=\frac{j\omega S\left[1+2(\gamma-1)\sqrt{\frac{\kappa_0}{-j\omega\rho_0 C_p}}\frac{\hat{G}\left(R\sqrt{\frac{-j\omega\rho_0 C_p}{\kappa_0}}\right)}{R}\right]}{\gamma p_0}, \tag{9}$$

where

$$\hat{G}(\hat{\xi})=\frac{\mathrm{J}_1(\hat{\xi})}{\mathrm{J}_0(\hat{\xi})} \tag{10}$$

and the cross-sectional area $S=R^2\pi$. The quantities p_0 and $\gamma=C_p/C_v$ are the equilibrium pressure and the ratio of the specific heats. Worth mentioning is the fact that Eq. (9) can also be derived by averaging the expressions for the velocity in the direction of the tube axis and the sound pressure given in Morse and Ingard[14] (pp. 519–522) over the cross section. These solutions were obtained from an axisymmetric two-dimensional analysis by using assumptions very similar to those Stinson's generalized theory for tubes of arbitrary cross-sectional shape is based on (see Ref. 8 and Appendix A).

High and low frequency limits can now be defined as

$$\text{LFL:}\quad l^2\,\mathrm{Re}\sim\left(\frac{R}{\delta}\right)^2\sim\frac{\omega\rho_0 R^2}{\mu_0}=\mathrm{St}^2\ll 1,$$
$$\text{HFL:}\quad \mathrm{St}^2\gg 1. \tag{11}$$

Here, the quantity St denotes the so-called Stokes number, its inverse is sometimes referred to as shear wave number Sh. As mentioned earlier, $\mathrm{Pr}=O(1)$ and therefore the LFL and HFL can easily be deduced from Eq. (9) by applying a power series expansion with respect to St and St^{-1}, respectively. In connection with the HFL, it should be noted that in the limit as $\mathrm{St}\to\infty$,

$$\hat{G}(\hat{\xi})\to -j, \tag{12}$$

as $\hat{\xi}$ is proportional to $\mathrm{St}\sqrt{-j}$.

Consequently, the expressions (9) reduce to

$$\text{LFL:}\quad \hat{Z}=\frac{8\mu_0}{R^2 S}\left(1+j\frac{\omega\rho_0 R^2}{6\mu_0}+O(\mathrm{St}^4)\right),$$
$$\hat{Y}=\frac{\omega S}{p_0}\left(j+\frac{\gamma-1}{\gamma}\frac{\omega\rho_0 R^2}{8\mu_0}\frac{\mu_0 C_p}{\kappa_0}+O(\mathrm{St}^4)\right), \tag{13}$$

which corresponds to Rayleigh's *narrow* tube solution (Ref. 2, p. 327), if terms of $O(\mathrm{St}^2)$ are neglected as well, and

$$\text{HFL:}\quad \hat{Z}=\frac{\omega\rho_0}{S}\left[j+(1+j)\sqrt{\frac{2\mu_0}{\omega\rho_0 R^2}}+O(\mathrm{St}^{-2})\right],$$
$$\hat{Y}=\frac{\omega S}{\gamma p_0}\left[j+(1+j)(\gamma-1)\sqrt{\frac{2\mu_0}{\omega\rho_0 R^2}\frac{\kappa_0}{\mu_0 C_p}}+O(\mathrm{St}^{-2})\right], \tag{14}$$

which is in accordance with Kirchhoff's *wide* tube solution.[1] Simplified expressions for the limiting values of $\hat{Z}$ can also be found in the book by Beranek[15] (pp. 135–138). Equivalent results for the HFL of the series impedance as well as the shunt admittance in case of rectangular or slit-shaped cross sections are given in Appendices A and B.

A completely different method of finding the LFL and HFL would be an asymptotic analysis of the basic equations themselves, with M, Re^{-1}, l, and either St or St^{-1} used as (small) perturbation parameters. In the case of the HFL such an approach then necessitates separate investigations of the acoustic motion in the core region and in the boundary layer, since the scaling of the terms in the basic equations changes completely, depending on which region is under consideration. Such a so-called matched asymptotic analysis, which again reproduces the solutions (14), is presented, e.g., in Makarov and Vatrushina[16] as well as in Qi *et al.*[17] However, if the evolution of waves over distances of the order $O(\mathrm{St}\,\lambda)$ is taken into consideration, evaluation of the transfer matrix $\hat{\mathbf{A}}$ as defined in Eq. (2) using $\hat{Z}$ and $\hat{Y}$ from Eq. (14) will produce results which are valid only to leading order. This is a direct consequence of the fact that the exponential terms of the order $O(e^{(\mathrm{St}^{-1}\lambda^{-1}L)})$ contained in the transfer matrix will then become order $O(1)$ quantities. In other words, in order to calculate asymptotically correct expressions for the leading order terms and the correction terms [of order $O(\mathrm{St}^{-1})$] of the quantities $p_s(z,t)$ and $u(z,t)$, the above-presented results for the HFL have to be based on the assumption that the propagation length L of the acoustic waves is comparable to the wavelength.

The aim of the following investigations is thus twofold: First, to show that the application of the method of multiple scales (MMS) in the HFL together with a matched asymptotic analysis leads to analytical solutions for the sound pressure and the volume flow *including* the second-order terms that are uniformly valid over a considerably larger spatial range than that constituted by the wavelengths and, second, to derive the asymptotically correct expressions for the coefficients of the transfer matrix $\hat{\mathbf{A}}$. As will furthermore be shown in Sec. V, the MMS can even be applied if the condition $l\ll 1$ is relaxed such that the reduced frequency is assumed to be of order $O(1)$ and, consequently, the excitation of higher order modes can no longer be disregarded.

III. BASIC EQUATIONS

A natural nondimensionalization of the governing equations involves the wavelength λ, the radius R, as well as the equilibrium quantities p_0, ρ_0, c_0, and μ_0 introduced above. Nondimensional variables are then constructed from

$$z^* = \frac{z}{\lambda}, \quad L^* = \frac{L}{\lambda}, \quad r^* = \frac{r}{R}, \quad t^* = \frac{tc_0}{\lambda}, \quad \omega^* = \frac{\omega\lambda}{c_0},$$

$$v_z^* = \frac{v_z}{c_0}, \quad v_r^* = \frac{v_r}{c_0}, \quad u^* = \frac{u}{c_0 S}, \quad p^* = \frac{p}{\gamma p_0}, \quad \rho^* = \frac{\rho}{\rho_0}, \tag{15}$$

$$\vartheta^* = \frac{\vartheta}{\vartheta_0}, \quad Z^* = \frac{Zc_0\lambda S}{\gamma p_0}, \quad Y^* = \frac{Y\gamma p_0 \lambda}{c_0 S}, \quad \eta^* = \frac{\eta_0}{\mu_0}.$$

Here v_z, v_r, p, ϑ, and η denote, respectively, the velocities in axial and radial direction, the fluid pressure, the temperature, and the bulk viscosity. Furthermore, the scaling parameters

$$\epsilon = a\frac{R}{\lambda}, \quad \alpha = d\frac{\lambda}{R\sqrt{\mathrm{Re}}} \tag{16}$$

are introduced where a and d are arbitrary constants of order $O(1)$, which, together with Eqs. (5) and (11), leads to the relationships $\epsilon \sim l \sim \mathrm{He}$ and $\alpha \sim \mathrm{St}^{-1}$.

In the following analysis it will be assumed that the variations of the thermal conductivity and the dynamic viscosities are so small that these quantities can be regarded as constant, i.e., $\kappa = \kappa_0$, $\mu = \mu_0$, and $\eta = \eta_0$. However, it should be emphasized that due to the assumption of a very small Mach number [see Eq. (8) as well as Eq. (26)] the results derived in the following would remain unchanged even if the commonly used approximative power laws $\kappa = \kappa_0(\vartheta/\vartheta_0)^\beta$ and $\mu = \mu_0(\vartheta/\vartheta_0)^\beta$, where the coefficient $\beta = O(1)$, were adopted.

Since the fluid is presupposed to be a perfect gas, the equilibrium sound speed c_0 equals $(\gamma p_0/\rho_0)^{1/2}$ and $\vartheta_0 = c_0^2/((\gamma-1)C_p)$. The two-dimensional Navier–Stokes equations in cylindrical coordinates for axisymmetric flow, the continuity equation, the energy equation, and the equation of state then read

$$\rho\frac{\partial v_z}{\partial t} + \rho v_z \frac{\partial v_z}{\partial z} + \frac{a}{\epsilon}\rho v_r \frac{\partial v_z}{\partial r} - \frac{\epsilon^2\alpha^2}{a^2 d^2}\left(\frac{4}{3}+\eta\right)\frac{\partial^2 v_z}{\partial z^2} - \frac{\alpha^2}{d^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{\epsilon\alpha^2}{ad^2}\left(\frac{1}{3}+\eta\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial z}\right) + \frac{\partial p}{\partial z} = 0, \tag{17}$$

$$\rho\frac{\partial v_r}{\partial t} + \rho v_z \frac{\partial v_r}{\partial z} + \frac{a}{\epsilon}\rho v_r \frac{\partial v_r}{\partial r} - \frac{\epsilon^2\alpha^2}{a^2 d^2}\frac{\partial^2 v_r}{\partial z^2} - \frac{\alpha^2}{d^2}\left(\frac{4}{3}+\eta\right) \times \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2}\right] - \frac{\epsilon\alpha^2}{ad^2}\left(\frac{1}{3}+\eta\right)\frac{\partial^2 v_z}{\partial z \partial r} + \frac{a}{\epsilon}\frac{\partial p}{\partial r} = 0, \tag{18}$$

$$\frac{\partial \rho}{\partial t} + v_z\frac{\partial \rho}{\partial z} + \frac{a}{\epsilon}v_r\frac{\partial \rho}{\partial r} + \rho\frac{\partial v_z}{\partial z} + \frac{a}{\epsilon}\rho\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0, \tag{19}$$

$$\rho\frac{\partial \vartheta}{\partial t} + \rho v_z\frac{\partial \vartheta}{\partial z} + \frac{a}{\epsilon}\rho v_r\frac{\partial \vartheta}{\partial r} - (\gamma-1)\left(\frac{\partial p}{\partial t} + v_z\frac{\partial p}{\partial z} + \frac{a}{\epsilon}v_r\frac{\partial p}{\partial r}\right) - \frac{\epsilon^2\alpha^2}{a^2d^2\,\mathrm{Pr}}\frac{\partial^2\vartheta}{\partial z^2} - \frac{\alpha^2}{d^2\,\mathrm{Pr}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta}{\partial r}\right) - \frac{1}{a^2d^2}(\gamma-1)\Phi = 0, \tag{20}$$

$$\vartheta\rho = \gamma p. \tag{21}$$

Here, the superscripts * introduced to indicate nondimensional quantities have already been omitted. The quantity Φ,

$$\Phi \sim \epsilon^2\alpha^2 \times \max\left(\left(\frac{\partial v_z}{\partial z}\right)^2, \frac{a^2}{\epsilon^2}\left(\frac{\partial v_z}{\partial r}\right)^2, H\left(\frac{\partial v_r}{\partial z}\right)^2, \frac{a^2}{\epsilon^2}\left(\frac{\partial v_r}{\partial r}\right)^2, \frac{a}{\epsilon}\frac{\partial v_z}{\partial z}\frac{\partial v_r}{\partial r}, \frac{a}{\epsilon}\frac{\partial v_z}{\partial r}\frac{\partial v_r}{\partial z}\right), \tag{22}$$

is the so-called dissipation function, which will turn out to be negligibly small as well.

Equations (17)–(21) will be solved subject to the boundary conditions at the tube wall

$$r=1: \quad v_z = v_r = 0, \quad \vartheta = 1, \tag{23}$$

and the symmetry conditions at the center of the tube

$$r=0: \quad v_r = \frac{\partial v_z}{\partial r} = \frac{\partial p}{\partial r} = \frac{\partial \vartheta}{\partial r} = 0. \tag{24}$$

IV. APPLICATION OF THE MMS

Utilizing the parameters introduced in Eq. (16), the HFL can now simply be defined as

$$\epsilon \ll 1, \quad \alpha \ll 1 \tag{25}$$

since then $\mathrm{Re}^{-1} \sim \epsilon^2\alpha^2 \ll 1$, $l \sim \epsilon \ll 1$, and $\mathrm{St}^{-1} \sim \delta/R \sim \alpha \ll 1$, as required by conditions (5) and (11). In order to avoid nonlinear effects entering the first- and second-order terms, the Mach number is assumed to be of any order $O(\epsilon^i\alpha^j)$ such that

$$M \ll \epsilon^m\alpha^n, \quad m+n=2 \tag{26}$$

holds, suggesting the use of M, α, and ϵ as perturbation parameters for an asymptotic analysis. Hence, the velocity components and the relevant thermodynamic quantities are expressed in the form:

$$v_z = M(v_{z1} + \epsilon v_{z\epsilon} + \alpha v_{z\alpha} + \epsilon^2 v_{z\epsilon^2} + \epsilon\alpha v_{z\epsilon\alpha} + \alpha^2 v_{z\alpha^2} + \cdots) + M^2(\cdots) + \cdots,$$
$$v_r = M(v_{r1} + \epsilon v_{r\epsilon} + \alpha v_{r\alpha} + \epsilon^2 v_{r\epsilon^2} + \epsilon\alpha v_{r\epsilon\alpha} + \alpha^2 v_{r\alpha^2} + \cdots) + M^2(\cdots) + \cdots,$$
$$p = \frac{1}{\gamma} + M(p_1 + \epsilon p_\epsilon + \alpha p_\alpha + \epsilon^2 p_{\epsilon^2} + \epsilon\alpha p_{\epsilon\alpha} + \alpha^2 p_{\alpha^2} + \cdots) + M^2(\cdots) + \cdots, \tag{27}$$
$$\rho = 1 + M(\rho_1 + \epsilon\rho_\epsilon + \alpha\rho_\alpha + \epsilon^2\rho_{\epsilon^2} + \epsilon\alpha\rho_{\epsilon\alpha} + \alpha^2\rho_{\alpha^2} + \cdots) + M^2(\cdots) + \cdots,$$
$$\vartheta = 1 + M(\vartheta_1 + \epsilon\vartheta_\epsilon + \alpha\vartheta_\alpha + \epsilon^2\vartheta_{\epsilon^2} + \epsilon\alpha\vartheta_{\epsilon\alpha} + \alpha^2\vartheta_{\alpha^2} + \cdots) + M^2(\cdots) + \cdots.$$

The sound pressure is then given by $p_s = p - 1/\gamma$.

The investigation of the most significant features of acoustic waves emerging over spatial ranges in longitudinal direction of the orders $O(1)$ and $O(\alpha^{-1})$ requires at least the introduction of the length scales z and, additionally, $z_1 = \alpha z$. However, in order to resolve the changes of the second-order terms, e.g., $v_{z\alpha}$ and $v_{z\epsilon}$, over long distances as well, a third scale $z_2 = \alpha^2 z$ according to

$$\frac{\partial}{\partial z} \to \frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial z_1} + \alpha^2\frac{\partial}{\partial z_2} \tag{28}$$

has to be used, because otherwise the generation of secular terms could not be avoided. Here it should be noted that the application of additional length scales proportional to powers of ϵ is not necessary since the solutions will retain their validity even for z as large as ϵ^{-1} or ϵ^{-2}. Results of a simplified study in the case of linear waves in tubes with a *rectangular* cross section that is restricted to the leading order terms and thus involves only the two length scales z and z_1 are presented in the Appendices A and B.

Substitution of the expressions (27) into the system (17)–(21) leads to a set of equations which is valid in the entire width of the tube except the boundary region and, therefore, is called the *outer* expansion. Close to the tube wall, where the stretched lateral coordinate

$$s = \frac{1-r}{\alpha} \tag{29}$$

is of $O(1)$, viscosity and heat conduction play an important role; these effects have to be accounted for by a separate investigation of the boundary layer. Consequently, for the *inner* expansion, the coordinate r then has to be replaced with $1-\alpha s$ and, furthermore,

$$\frac{\partial}{\partial r} \to -\frac{1}{\alpha}\frac{\partial}{\partial s}. \tag{30}$$

Please note that in the inner expansion, the density terms will be denoted by Ψ, whereas all other inner quantities will be written in capital letters, e.g., $V_{z\epsilon}$. In order to match the quantities arising from the two expansions, Van Dyke's *asymptotic matching principle* (see, e.g., Ref. 12, pp. 173–179) will be applied.

Since the resulting equations, more precisely, those that are relevant to solving the problem, will turn out to be linear, a Fourier transform with respect to the time could be used. Equivalently, each unsteady perturbation w or W is decomposed into a steady modal amplitude and a time-harmonic function such that

$$w(z,z_1,z_2,r,t) = \Re(\hat{w}(z,z_1,z_2,r)e^{j\omega t}),$$
$$W(z,z_1,z_2,s,t) = \Re(\hat{W}(z,z_1,z_2,s)e^{j\omega t}). \tag{31}$$

Evaluation of the continuity equation (19) together with conditions (23) and (24) then yields

$$\frac{\partial\hat{V}_{r1}}{\partial s} = \frac{\partial\hat{V}_{r\epsilon}}{\partial s} = \frac{\partial\hat{V}_{r\alpha}}{\partial s} = \frac{\partial\hat{V}_{r\epsilon^2}}{\partial s} = 0 \Rightarrow \hat{V}_{r1} = \hat{V}_{r\epsilon} = \hat{V}_{r\alpha} = \hat{V}_{r\epsilon^2} = 0, \tag{32}$$

$$j\omega\hat{\Psi}_1 + \frac{\partial\hat{V}_{z1}}{\partial z} - a\frac{\partial\hat{V}_{r\epsilon\alpha}}{\partial s} = 0, \tag{33}$$

$$j\omega\hat{\Psi}_\epsilon + \frac{\partial\hat{V}_{z\epsilon}}{\partial z} - a\frac{\partial\hat{V}_{r\epsilon^2\alpha}}{\partial s} = 0, \tag{34}$$

$$j\omega\hat{\Psi}_\alpha + \frac{\partial\hat{V}_{z\alpha}}{\partial z} + \frac{\partial\hat{V}_{z1}}{\partial z_1} - a\frac{\partial\hat{V}_{r\epsilon\alpha^2}}{\partial s} + a\hat{V}_{r\epsilon\alpha} = 0, \tag{35}$$

and, furthermore,

$$\frac{\partial}{\partial r}(r\hat{v}_{r1}) = \frac{\partial}{\partial r}(r\hat{v}_{r\alpha}) = 0 \Rightarrow \hat{v}_{r1} = \hat{v}_{r\alpha} = 0, \tag{36}$$

$$j\omega\hat{\rho}_1 + \frac{\partial\hat{v}_{z1}}{\partial z} + a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r\epsilon}) = 0, \tag{37}$$

$$j\omega\hat{\rho}_\epsilon + \frac{\partial\hat{v}_{z\epsilon}}{\partial z} + a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r\epsilon^2}) = 0, \tag{38}$$

$$j\omega\hat{\rho}_\alpha + \frac{\partial\hat{v}_{z\alpha}}{\partial z} + \frac{\partial\hat{v}_{z1}}{\partial z_1} + a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r\epsilon\alpha}) = 0, \tag{39}$$

$$j\omega\hat{\rho}_{\epsilon\alpha} + \frac{\partial\hat{v}_{z\epsilon\alpha}}{\partial z} + \frac{\partial\hat{v}_{z\epsilon}}{\partial z_1} + a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r\epsilon^2\alpha}) = 0, \tag{40}$$

$$j\omega\hat{\rho}_{\alpha^2} + \frac{\partial\hat{v}_{z\alpha^2}}{\partial z} + \frac{\partial\hat{v}_{z\alpha}}{\partial z_1} + \frac{\partial\hat{v}_{z1}}{\partial z_2} + a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r\epsilon\alpha^2}) = 0. \tag{41}$$

As a consequence of Eqs. (32) and (36), the inner and the outer expansion of the Navier–Stokes equation (18) governing the radial motion give

$$\frac{\partial\hat{P}_1}{\partial s} = \frac{\partial\hat{P}_\epsilon}{\partial s} = \frac{\partial\hat{P}_\alpha}{\partial s} = \frac{\partial\hat{P}_{\epsilon\alpha}}{\partial s} = \frac{\partial\hat{P}_{\alpha^2}}{\partial s} = 0, \tag{42}$$

$$\frac{\partial\hat{p}_1}{\partial r} = \frac{\partial\hat{p}_\epsilon}{\partial r} = \frac{\partial\hat{p}_\alpha}{\partial r} = \frac{\partial\hat{p}_{\epsilon\alpha}}{\partial r} = \frac{\partial\hat{p}_{\alpha^2}}{\partial r} = 0. \tag{43}$$

This agrees with the expectation that the pressure in the boundary layer is set by the pressure fluctuations in the core region, which, due to the long wavelength assumption, are independent of the lateral coordinate r.

The set of equations following from the inner expansion of Eq. (21) reads

$$\hat{\Psi}_1=\gamma\hat{P}_1-\hat{\Theta}_1,\quad \hat{\Psi}_\epsilon=\gamma\hat{P}_\epsilon-\hat{\Theta}_\epsilon,\quad \hat{\Psi}_\alpha=\gamma\hat{P}_\alpha-\hat{\Theta}_\alpha,$$
$$\hat{\Psi}_{\epsilon\alpha}=\gamma\hat{P}_{\epsilon\alpha}-\hat{\Theta}_{\epsilon\alpha},\quad \hat{\Psi}_{\alpha^2}=\gamma\hat{P}_{\alpha^2}-\hat{\Theta}_{\alpha^2}, \tag{44}$$

which holds for the outer expansion as well.

Substituting the expansions (27) into the Navier–Stokes equation for the axial direction (17) and using Eqs. (32) and (36), one obtains for the inner quantities

$$j\omega\hat{V}_{z1}-\frac{1}{d^2}\frac{\partial^2\hat{V}_{z1}}{\partial s^2}+\frac{\partial\hat{P}_1}{\partial z}=0, \tag{45}$$

$$j\omega\hat{V}_{z\epsilon}-\frac{1}{d^2}\frac{\partial^2\hat{V}_{z\epsilon}}{\partial s^2}+\frac{\partial\hat{P}_\epsilon}{\partial z}=0, \tag{46}$$

$$j\omega\hat{V}_{z\alpha}-\frac{1}{d^2}\frac{\partial^2\hat{V}_{z\alpha}}{\partial s^2}+\frac{1}{d^2}\frac{\partial\hat{V}_{z1}}{\partial s}+\frac{\partial\hat{P}_\alpha}{\partial z}+\frac{\partial\hat{P}_1}{\partial z_1}=0, \tag{47}$$

and for the outer quantities

$$j\omega\hat{v}_{z1}+\frac{\partial\hat{p}_1}{\partial z}=0, \tag{48}$$

$$j\omega\hat{v}_{z\epsilon}+\frac{\partial\hat{p}_\epsilon}{\partial z}=0, \tag{49}$$

$$j\omega\hat{v}_{z\alpha}+\frac{\partial\hat{p}_\alpha}{\partial z}+\frac{\partial\hat{p}_1}{\partial z_1}=0, \tag{50}$$

$$j\omega\hat{v}_{z\epsilon\alpha}+\frac{\partial\hat{p}_{\epsilon\alpha}}{\partial z}+\frac{\partial\hat{p}_\epsilon}{\partial z_1}=0, \tag{51}$$

$$j\omega\hat{v}_{z\alpha^2}+\frac{\partial\hat{p}_{\alpha^2}}{\partial z}+\frac{\partial\hat{p}_\alpha}{\partial z_1}+\frac{\partial\hat{p}_1}{\partial z_2}=0, \tag{52}$$

where in the last Eq. (52) the relationship $\partial\hat{v}_{z1}/\partial r=0$ resulting from Eqs. (43) and (48) has already been applied; the other expansion terms of $\hat{v}_z$ appearing in Eqs. (49)–(52) also turn out to be independent of the radial coordinate r.

Hence, the dissipation function Φ defined in Eq. (22) is of the order $O(M^2\epsilon^2\alpha^2)$ in the core region and $O(M^2)$ in the boundary layer, and the expansions of the energy equation (21) take the form

$$j\omega\hat{\Theta}_1-j\omega(\gamma-1)\hat{P}_1-\frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2\hat{\Theta}_1}{\partial s^2}=0, \tag{53}$$

$$j\omega\hat{\Theta}_\epsilon-j\omega(\gamma-1)\hat{P}_\epsilon-\frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2\hat{\Theta}_\epsilon}{\partial s^2}=0, \tag{54}$$

$$j\omega\hat{\Theta}_\alpha-j\omega(\gamma-1)\hat{P}_\alpha-\frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2\hat{\Theta}_\alpha}{\partial s^2}+\frac{1}{d^2\,\mathrm{Pr}}\frac{\partial\hat{\Theta}_1}{\partial s}=0, \tag{55}$$

$$\hat{\vartheta}_1=(\gamma-1)\hat{p}_1,\quad \hat{\vartheta}_\epsilon=(\gamma-1)\hat{p}_\epsilon,\quad \hat{\vartheta}_\alpha=(\gamma-1)\hat{p}_\alpha,$$
$$\hat{\vartheta}_{\epsilon\alpha}=(\gamma-1)\hat{p}_{\epsilon\alpha},\quad \hat{\vartheta}_{\alpha^2}=(\gamma-1)\hat{p}_{\alpha^2}. \tag{56}$$

Upon comparison with the expressions (44) this shows that

$$\hat{\rho}_1=\hat{p}_1,\quad \hat{\rho}_\epsilon=\hat{p}_\epsilon,\quad \hat{\rho}_\alpha=\hat{p}_\alpha,\quad \hat{\rho}_{\epsilon\alpha}=\hat{p}_{\epsilon\alpha},$$
$$\hat{\rho}_{\alpha^2}=\hat{p}_{\alpha^2}. \tag{57}$$

Solving systems (37)–(41) and (48)–(52) for the velocity components in radial direction subject to condition (24) gives

$$\hat{v}_{r\epsilon}=r\hat{f}_\epsilon(z,z_1,z_2),\quad \hat{v}_{r\epsilon^2}=r\hat{f}_{\epsilon^2}(z,z_1,z_2),$$
$$\hat{v}_{r\epsilon\alpha}=r\hat{f}_{\epsilon\alpha}(z,z_1,z_2), \tag{58}$$
$$\hat{v}_{r\epsilon^2\alpha}=r\hat{f}_{\epsilon^2\alpha}(z,z_1,z_2),\quad \hat{v}_{r\epsilon\alpha^2}=r\hat{f}_{\epsilon\alpha^2}(z,z_1,z_2).$$

Equation (32) in combination with Van Dyke's matching principle then implies that $\hat{f}_\epsilon=\hat{f}_{\epsilon^2}=0$. Thus, the wave equations resulting from Eqs. (37), (38), (48), and (49), which determine the evolution of the pressure fluctuations $\hat{p}_1=\hat{P}_1$, $\hat{p}_\epsilon=\hat{P}_\epsilon$ over distances of the order $O(1)$ (i.e., distances comparable to the wavelengths), are

$$\omega^2\hat{p}_1+\frac{\partial^2\hat{p}_1}{\partial z^2}=0,\quad \omega^2\hat{p}_\epsilon+\frac{\partial^2\hat{p}_\epsilon}{\partial z^2}=0. \tag{59}$$

Additionally, by applying the matching rules to the expressions for $\hat{V}_{z1}$ and $\hat{\Theta}_1$ derived from Eqs. (45), (53), and the boundary condition (23), one obtains

$$\hat{V}_{z1}=\hat{v}_{z1}(1-e^{-(1+j)ds\sqrt{(\omega/2)}}),$$
$$\hat{\Theta}_1=\hat{p}_1(\gamma-1)(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}). \tag{60}$$

Precisely the same functional dependence on the boundary layer coordinate s is valid for $\hat{V}_{z\epsilon}$ and $\hat{\Theta}_\epsilon$ as well. As a consequence, Eqs. (33) and (34) can be solved to give

$$\hat{V}_{r\epsilon\alpha}=\hat{p}_1\frac{1+j}{ad}\sqrt{\frac{\omega}{2}}\left[\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})+(1-e^{-(1+j)ds\sqrt{(\omega/2)}})\right],$$
$$\hat{V}_{r\epsilon^2\alpha}=\hat{p}_\epsilon\frac{1+j}{ad}\sqrt{\frac{\omega}{2}}\left[\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})+(1-e^{-(1+j)ds\sqrt{(\omega/2)}})\right]. \tag{61}$$

Carrying out the matching procedure with the expansion terms of $\hat{v}_r$ given by Eq. (58) leads to

$$\hat{f}_{\epsilon\alpha}(z,z_1,z_2)=\hat{p}_1\frac{j\omega}{a}\hat{F},\quad \hat{f}_{\epsilon^2\alpha}(z,z_1,z_2)=\hat{p}_\epsilon\frac{j\omega}{a}\hat{F}, \tag{62}$$

where

$$\hat{F}=\frac{1-j}{d\sqrt{2\omega}}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right), \tag{63}$$

and it then follows from Eqs. (39), (40), (50), and (51) that

$$\omega^2\hat{p}_\alpha+\frac{\partial^2\hat{p}_\alpha}{\partial z^2}=-2\frac{\partial^2\hat{p}_1}{\partial z\partial z_1}-2\omega^2\hat{p}_1\hat{F},$$
$$\omega^2\hat{p}_{\epsilon\alpha}+\frac{\partial^2\hat{p}_{\epsilon\alpha}}{\partial z^2}=-2\frac{\partial^2\hat{p}_\epsilon}{\partial z\partial z_1}-2\omega^2\hat{p}_\epsilon\hat{F}. \tag{64}$$

Inspection of the relationships (64) shows that in order to rule out secular solutions of the form $\hat{p}_\alpha\propto z$, $\hat{p}_{\epsilon\alpha}\propto z$, the resonant forcing terms on the right-hand sides must be annihilated, generating the wave equations

$$\omega^2\hat{p}_\alpha+\frac{\partial^2\hat{p}_\alpha}{\partial z^2}=0,\quad \omega^2\hat{p}_{\epsilon\alpha}+\frac{\partial^2\hat{p}_{\epsilon\alpha}}{\partial z^2}=0,$$
$$\frac{\partial^2\hat{p}_1}{\partial z\partial z_1}+\omega^2\hat{p}_1\hat{F}=0,\quad \frac{\partial^2\hat{p}_\epsilon}{\partial z\partial z_1}+\omega^2\hat{p}_\epsilon\hat{F}=0, \tag{65}$$

which govern the propagation of the pressure perturbations $\hat{p}_\alpha$ and $\hat{p}_{\epsilon\alpha}$ over distances of order $O(1)$ and, respectively, the propagation of the pressure perturbations $\hat{p}_1$ and $\hat{p}_\epsilon$ over distances of order $O(\alpha^{-1})$. The imaginary part $\Im(-\omega\hat{F})$ is thus identified as the leading order decay rate of the sound pressure due to boundary layer attenuation.[1] Worth mentioning is the fact that the equation for $\hat{p}_1$ represents the multiple scales equivalent to the model equation that was derived by Pierce[18] (pp. 531–534) from a variational principle.

Proceeding in very much the same way as before, Eqs. (47) and (55) are solved for the axial velocity and the temperature in the boundary layer:

$$\hat{V}_{z\alpha}=\hat{v}_{z\alpha}(1-e^{-(1+j)ds\sqrt{(\omega/2)}})-\hat{v}_{z1}\frac{s}{2}e^{-(1+j)ds\sqrt{(\omega/2)}},$$
$$\hat{\Theta}_\alpha=\hat{p}_\alpha(\gamma-1)(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})$$
$$-\hat{p}_1(\gamma-1)\frac{s}{2}e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}. \tag{66}$$

Substituting these expressions together with the relationships for $\hat{\Psi}_\alpha$, $\hat{V}_{z1}$, and $\hat{V}_{r\epsilon\alpha}$ given in Eqs. (44), (60), and (61), respectively, into Eq. (35), the following result for the inner quantity $\hat{V}_{r\epsilon\alpha^2}$ can be derived:

$$\hat{V}_{r\epsilon\alpha^2}=\hat{p}_\alpha\frac{1+j}{ad}\sqrt{\frac{\omega}{2}}\left[\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})\right.$$
$$\left.+(1-e^{-(1+j)ds\sqrt{(\omega/2)}})\right]-\hat{p}_1\frac{\gamma-1}{2ad^2\,\mathrm{Pr}}$$
$$\times\left[1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}+(1+j)ds\sqrt{\frac{\omega\,\mathrm{Pr}}{2}}\right.$$
$$\left.\times e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}\right]-\hat{p}_1\frac{1}{2ad^2}\left[1-e^{-(1+j)ds\sqrt{(\omega/2)}}\right.$$
$$\left.+(1+j)ds\sqrt{\frac{\omega}{2}}e^{-(1+j)ds\sqrt{(\omega/2)}}\right]+\hat{p}_1\frac{1+j}{ad}\sqrt{2\omega}\hat{F}$$
$$\times(1-e^{-(1+j)ds\sqrt{(\omega/2)}})-\hat{p}_1\frac{j\omega}{a}\hat{F}s. \tag{67}$$

When the matching procedure is again applied to the inner and outer radial velocity components, one then finds that

$$\hat{f}_{\epsilon\alpha^2}(z,z_1,z_2)=\frac{j\omega}{a}(\hat{p}_\alpha\hat{F}+\hat{p}_1\hat{G}), \tag{68}$$

with

$$\hat{G}=\frac{j}{2d^2\omega}\left(1+\frac{\gamma-1}{\mathrm{Pr}}\right)+\frac{1-j}{d}\sqrt{\frac{2}{\omega}}\hat{F}$$
$$=-\frac{j}{2d^2\omega}\left(3-\frac{\gamma-1}{\mathrm{Pr}}+4\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right). \tag{69}$$

After substituting $\hat{v}_{r\epsilon\alpha^2}$ from Eqs. (58) and (68) into Eq. (41), Eq. (52) can be recast into

$$\omega^2\hat{p}_{\alpha^2}+\frac{\partial^2\hat{p}_{\alpha^2}}{\partial z^2}=-2\frac{\partial^2\hat{p}_1}{\partial z\partial z_2}-2\frac{\partial^2\hat{p}_\alpha}{\partial z\partial z_1}+\omega^2(\hat{p}_1\hat{F}^2$$
$$-2\hat{p}_1\hat{G}-2\hat{p}_\alpha\hat{F}), \tag{70}$$

where the right-hand side is identified as resonant forcing, since it would involve secular terms in $\hat{p}_{\alpha^2}$. This implies that the quantity $\hat{p}_\alpha$ has to satisfy the solvability condition

$$\frac{\partial^2\hat{p}_\alpha}{\partial z\partial z_1}+\omega^2\hat{p}_\alpha\hat{F}=-\frac{\partial^2\hat{p}_1}{\partial z\partial z_2}-\omega^2\hat{p}_1\left(\hat{G}-\frac{\hat{F}^2}{2}\right). \tag{71}$$

In turn, unless this right-hand side is annihilated, it would inevitably lead to $\hat{p}_\alpha$ being proportional to z_1. Thus, the resulting equations read

$$\omega^2\hat{p}_{\alpha^2}+\frac{\partial^2\hat{p}_{\alpha^2}}{\partial z^2}=0,\quad \frac{\partial^2\hat{p}_\alpha}{\partial z\partial z_1}+\omega^2\hat{p}_\alpha\hat{F}=0,$$
$$\frac{\partial^2\hat{p}_1}{\partial z\partial z_2}+\omega^2\hat{p}_1\hat{H}=0. \tag{72}$$

Here

$$\hat{H}=\hat{G}-\frac{\hat{F}^2}{2}=-\frac{j}{d^2\omega}\left[1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-\frac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right], \tag{73}$$

with $\Im(-\omega\hat{H})$ being the correction term to the attenuation rate $\Im(-\omega\hat{F})$.

The solutions of the wave equations (59), (65), and (72) can then be written as

$$\hat{p}_1=\hat{c}_{11}e^{-j\omega(z+\hat{F}z_1+\hat{H}z_2)}+\hat{c}_{21}e^{j\omega(z+\hat{F}z_1+\hat{H}z_2)},$$
$$\hat{p}_\epsilon=\hat{c}_{1\epsilon}(z_2)e^{-j\omega(z+\hat{F}z_1)}+\hat{c}_{2\epsilon}(z_2)e^{j\omega(z+\hat{F}z_1)},$$
$$\hat{p}_\alpha=\hat{c}_{1\alpha}(z_2)e^{-j\omega(z+\hat{F}z_1)}+\hat{c}_{2\alpha}(z_2)e^{j\omega(z+\hat{F}z_1)}, \tag{74}$$
$$\hat{p}_{\epsilon\alpha}=\hat{c}_{1\epsilon\alpha}(z_1,z_2)e^{-j\omega z}+\hat{c}_{2\epsilon\alpha}(z_1,z_2)e^{j\omega z},$$
$$\hat{p}_{\alpha^2}=\hat{c}_{1\alpha^2}(z_1,z_2)e^{-j\omega z}+\hat{c}_{2\alpha^2}(z_1,z_2)e^{j\omega z}.$$

A. Results

The goal pursued in the study presented here is to derive the asymptotically correct expressions for the series impedance $\hat{Z}$ and the shunt admittance $\hat{Y}$, which were introduced in Eq. (1), or, equivalently, the expressions for tube parameters $\hat{Z}_c$ and $\hat{\Gamma}$ defined in Eq. (3). In order to calculate the volume flow $\hat{u}$, the axial Navier–Stokes equation (17) is averaged over the cross-sectional area and expanded with respect to the perturbation parameters ϵ and α, giving

$$j\omega\hat{\bar{v}}_{z1}+\frac{\partial\hat{p}_1}{\partial z}=0,$$

$$j\omega\hat{\bar{v}}_{z\epsilon}+\frac{\partial\hat{p}_\epsilon}{\partial z}=0,$$

$$j\omega\hat{\bar{v}}_{z\alpha}+\frac{2}{d^2}\frac{\partial\hat{V}_{z1}}{\partial s}\bigg|_{s=0}+\frac{\partial\hat{p}_\alpha}{\partial z}+\frac{\partial\hat{p}_1}{\partial z_1}=0, \tag{75}$$

$$j\omega\hat{\bar{v}}_{z\epsilon\alpha}+\frac{2}{d^2}\frac{\partial\hat{V}_{z\epsilon}}{\partial s}\bigg|_{s=0}+\frac{\partial\hat{p}_{\epsilon\alpha}}{\partial z}+\frac{\partial\hat{p}_\epsilon}{\partial z_1}=0,$$

$$j\omega\hat{\bar{v}}_{z\alpha^2}+\frac{2}{d^2}\frac{\partial\hat{V}_{z\alpha}}{\partial s}\bigg|_{s=0}+\frac{\partial\hat{p}_{\alpha^2}}{\partial z}+\frac{\partial\hat{p}_\alpha}{\partial z_1}+\frac{\partial\hat{p}_1}{\partial z_2}=0.$$

Furthermore, as shown in Eq. (28), the multiple scales technique applied here requires the derivatives with respect to z appearing in the definitions of $\hat{Z}$ and $\hat{Y}$ to be replaced with derivatives with respect to the three length scales z, z_1, and z_2. Using the wave equations (59), (65), and (72) governing the sound pressure and the expansion terms of the volume flow $\hat{u}=\hat{\bar{v}}_z$ that can be deduced from Eq. (75), the following expressions for the series impedance and the shunt admittance are obtained:

$$\begin{aligned}\hat{Z}&=-\frac{1}{\hat{u}}\left(\frac{d}{dz}+\alpha\frac{d}{dz_1}+\alpha^2\frac{d}{dz_2}\right)\hat{p}_s\\&=j\omega+\alpha\frac{1+j}{d}\sqrt{2\omega}+\alpha^2\frac{3}{d^2}+O(\epsilon\alpha^2,\alpha^3),\\\hat{Y}&=-\frac{1}{\hat{p}_s}\left(\frac{d}{dz}+\alpha\frac{d}{dz_1}+\alpha^2\frac{d}{dz_2}\right)\hat{u}\\&=j\omega+\alpha\frac{1+j}{d}(\gamma-1)\sqrt{\frac{2\omega}{\mathrm{Pr}}}-\alpha^2\frac{\gamma-1}{d^2\,\mathrm{Pr}}+O(\epsilon\alpha^2,\alpha^3).\end{aligned} \tag{76}$$

The expansions for tube parameters $\hat{Z}_c$ and $\hat{\Gamma}$ then read

$$\begin{aligned}\hat{Z}_c&=1+\alpha\frac{1-j}{d\sqrt{2\omega}}\left(1-\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)-\alpha^2\frac{j}{2d^2\omega}\left(2-2\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right.\\&\quad\left.-\frac{5\gamma-3\gamma^2-2}{\mathrm{Pr}}\right)+\cdots,\\\hat{\Gamma}&=j\omega+\alpha j\omega\hat{F}+\alpha^2 j\omega\hat{H}+\cdots\\&=j\omega+\alpha\frac{1+j}{d}\sqrt{\frac{\omega}{2}}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)+\alpha^2\frac{1}{d^2}\left[1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right.\\&\quad\left.\times\left(1-\frac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right]+\cdots.\end{aligned} \tag{77}$$

It should be emphasized that the expression for the characteristic impedance $\hat{Z}_c$ could have been truncated after the

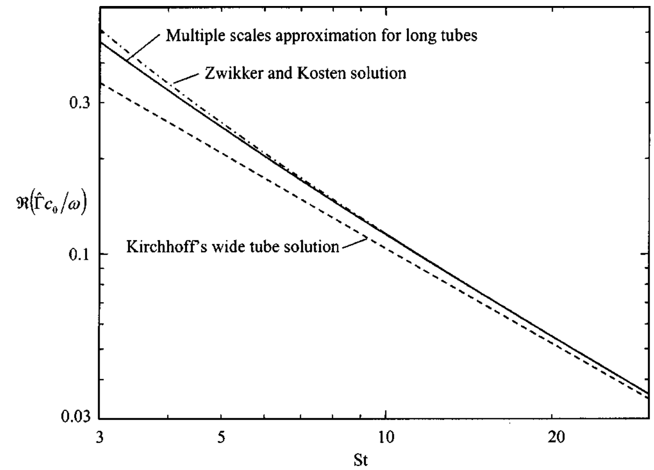

FIG. 2. Graphs of $\Re(\hat{\Gamma}c_0/\omega)$ as functions of St in double logarithmic scale; Pr=0.707, γ=1.402 (air at 300 K); St$\gg$1: $\Re(\hat{\Gamma}c_0/\omega)=O(\mathrm{St}^{-1})$.

second term as well, since the investigation is confined to the first- and second-order terms of the sound pressure and the volume flow. However, as far as the propagation parameter is concerned, all three expansion terms have to be included, because the length of the tube might be much larger than the wavelengths, i.e., even of the order $O(\alpha^{-1})$.

The corresponding results for the dimensional transmission line parameters are presented in Appendix C. Interestingly, Eq. (C2) completely conforms to the results given by Keefe,[10] which were derived by high order series expansions of the Zwikker and Kosten solutions (9) with respect to the (small) inverse of the Stokes number. Comparison with the expressions (14) clearly shows that the terms of the order $O(\mathrm{St}^{-2})$ appearing in Eq. (C2) must not be neglected if the effects emerging over distances $L\gg\lambda$ are to be incorporated into the HFL of the series impedance and the shunt admittance. In addition to providing precise solutions for the sound pressure, the velocity components, and the other thermodynamic quantities, the study presented here thus extends the validity of the Zwikker and Kosten approach to the case of sound propagation in long tubes in the limit of large Stokes numbers (HFL), provided that the low reduced frequency assumptions (5) hold. In order to demonstrate this equivalence graphically, the different solutions for the real part of the propagation parameter resulting from Eqs. (9), (14), and (C2) are depicted in Fig. 2.

B. Example

The results summarized so far are sufficient to evaluate the transmission line parameters entering the four-pole transfer matrix $\hat{\mathbf{A}}$ defined in Eq. (2) for a long circular duct with isothermal walls. Exemplarily, the total load impedance $\hat{Z}_t=\hat{p}_{s-}/\hat{u}_-$ of a tube radiating into open space will be calculated. The parameters α and ϵ are required to be small, in order to comply with the requirements the multiple scales analysis elaborated in Sec. IV is based on. Furthermore, the tube length L shall be so large that the nondimensional quantity

$$L_1=\alpha L \tag{78}$$

is of order $O(1)$. These conditions can be satisfied if, e.g., a thin tube of dimensional length 10 cm and radius 0.5 mm is considered, the fluid is assumed to be air (at 300 K) and the characteristic dimensional wavelength λ equals 7 mm, which corresponds to a driving frequency f of approximately 50 000 Hz and leads to $\epsilon/a=0.071\ll 1$, $\alpha/d=0.036\ll 1$, $L=14.286\gg 1$. Then one may choose $\epsilon=\alpha=0.071$, $a=1$, and $d=2.008$, resulting in $L_1=1.020=O(1)$.

Since the condition $\epsilon\ll 1$ holds, the radiation impedance can be expanded into

$$\hat{Z}_e=\epsilon\hat{Z}_{e\epsilon}+\epsilon^2\hat{Z}_{e\epsilon^2}+\cdots. \tag{79}$$

In the case of radiation of sound from a circular duct with an infinite flange, the leading order term $\hat{Z}_{e\epsilon}$ is a pure imaginary and given by

$$\hat{Z}_{e\epsilon}=L_e j\omega, \tag{80}$$

where $L_e=0.8217/a$ is the so-called quasistatic end correction. The validity of this result will be proved in Sec. IV C.

Substituting the solutions from Eq. (77) and the relationship $\hat{p}_{s+}/\hat{u}_+=\hat{Z}_e$ into Eq. (4), a series expansion of the resulting total impedance with respect to ϵ and α then yields (see Appendix C for the dimensional form of this result)

$$\begin{aligned}\hat{Z}_t=\frac{\hat{p}_{s-}}{\hat{u}_-}&=\tanh\left(j\omega L_1\left(\frac{1}{\alpha}+\hat{F}\right)\right)+\epsilon\frac{\hat{Z}_{e\epsilon}}{\cosh\left(j\omega L_1\left(\frac{1}{\alpha}+\hat{F}\right)\right)^2}\\&+\alpha\left[\frac{1-j}{d\sqrt{2\omega}}\left(1-\frac{\gamma-1}{\sqrt{\Pr}}\right)\tanh\left(j\omega L_1\left(\frac{1}{\alpha}+\hat{F}\right)\right)\right.\\&\left.+\frac{j\omega L_1\hat{H}}{\cosh\left(j\omega L_1\left(\frac{1}{\alpha}+\hat{F}\right)\right)^2}\right]+\cdots\\&=\tanh\left(L_1\left[\frac{j\omega}{\alpha}+\frac{1+j}{d}\sqrt{\frac{\omega}{2}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\right]\right)\\&+\alpha\left[\frac{1-j}{d\sqrt{2\omega}}\left(1-\frac{\gamma-1}{\sqrt{\Pr}}\right)\right.\\&\times\tanh\left(L_1\left[\frac{j\omega}{\alpha}+\frac{1+j}{d}\sqrt{\frac{\omega}{2}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\right]\right)\\&\left.+\frac{\frac{\epsilon}{\alpha}\hat{Z}_{e\epsilon}+\frac{L_1}{d^2}\left[1+\frac{\gamma-1}{\sqrt{\Pr}}\left(1-\frac{\gamma}{2\sqrt{\Pr}}\right)\right]}{\cosh\left(L_1\left[\frac{j\omega}{\alpha}+\frac{1+j}{d}\sqrt{\frac{\omega}{2}}\left(1+\frac{\gamma-1}{\sqrt{\Pr}}\right)\right]\right)^2}\right]+\cdots,\end{aligned} \tag{81}$$

which reveals the fact that in the case of $\epsilon\sim\alpha$, the effects resulting from the radiation at the end of the tube enter the expansion of the total impedance at the same order as the

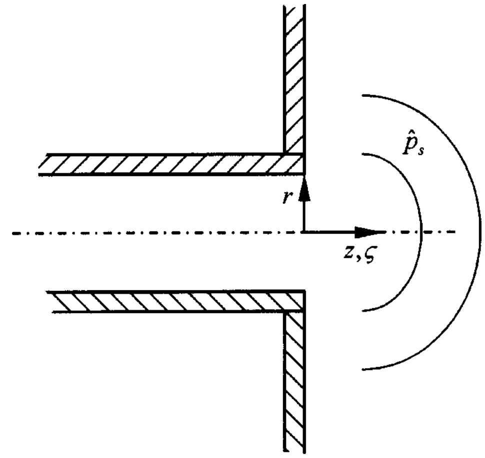

FIG. 3. Sketch and notation of a duct radiating into half-space.

viscothermal effects accumulating over the considerably large length of the tube. It should be emphasized that the arguments of the functions $1/\cosh(\cdot)$ and $\tanh(\cdot)$ always contain a real part of order $O(1)$, since L_1 is assumed to be of order $O(1)$. Thus, the moduli of these terms will remain $O(1)$ quantities as well, even if the dimensional tube length assumes a value close to $(n+1/2)c_0/(2f)$, where n is an integer of $O(\alpha^{-1})$, i.e., $L_1\approx\alpha(n+1/2)\pi/\omega$. This is in contrast to the well-known resonance phenomenon occurring in shorter tubes, where thermal and viscous effects contribute much less to the total load impedance.

C. End correction for circular tubes with an infinite flange

The radiation impedance at the flanged opening will excite higher-order modes in the backward propagating wave. However, due to the long wavelength assumption $l\sim\epsilon\ll 1$ these modes have cut-off frequencies well above the driving frequency $\omega/(2\pi)$ and, consequently, die out rapidly within a spatial range comparable to the radius of the cross section, leaving only the lowest mode to propagate over any longer distance along the tube (see Ref. 14, p. 499). In order to resolve the details of the flow close to the exit of the duct, a separate perturbation analysis is therefore necessary: For convenience, the origin of the axial coordinate is set to the position of the opening as shown in Fig. 3. Similarly to the inner expansion for the boundary layer, a stretched inner coordinate for the end region

$$\zeta=\frac{z}{\epsilon},\quad \frac{\partial}{\partial z}\rightarrow\frac{1}{\epsilon}\frac{\partial}{\partial\zeta} \tag{82}$$

can then be introduced. The quantities arising from this second inner expansion will be denoted by the subscript e.

Substitution of the expressions (27) into the basic equations (17)–(21) in the same manner as was done in Sec. IV leads to

$$\frac{\partial\hat{V}_{re1}}{\partial s}=0\Rightarrow\hat{V}_{re1}=0, \tag{83}$$

$$\frac{\partial\hat{v}_{ze1}}{\partial\zeta}+a\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{re1})=0, \tag{84}$$

$$\frac{\partial \hat{P}_{e1}}{\partial s} = \frac{\partial \hat{P}_{e\epsilon}}{\partial s} = \frac{\partial \hat{P}_{e\alpha}}{\partial s} = 0, \tag{85}$$

$$\frac{\partial \hat{p}_{e1}}{\partial r} = \frac{\partial \hat{p}_{e\alpha}}{\partial r} = 0, \quad j\omega \hat{v}_{re1} + a\frac{\partial \hat{p}_{e\epsilon}}{\partial r} = 0, \tag{86}$$

$$\hat{\Psi}_{e1} = \gamma \hat{P}_{e1} - \hat{\Theta}_{e1}, \quad \hat{\Psi}_{e\epsilon} = \gamma \hat{P}_{e\epsilon} - \hat{\Theta}_{e\epsilon},$$

$$\hat{\Psi}_{e\alpha} = \gamma \hat{P}_{e\alpha} - \hat{\Theta}_{e\alpha}, \tag{87}$$

$$\hat{\rho}_{e1} = \gamma \hat{p}_{e1} - \hat{\vartheta}_{e1}, \quad \hat{\rho}_{e\epsilon} = \gamma \hat{p}_{e\epsilon} - \hat{\vartheta}_{e\epsilon},$$

$$\hat{\rho}_{e\alpha} = \gamma \hat{p}_{e\alpha} - \hat{\vartheta}_{e\alpha}, \tag{88}$$

$$\frac{\partial \hat{P}_{e1}}{\partial \zeta} = \frac{\partial \hat{P}_{e\alpha}}{\partial \zeta} = 0, \quad j\omega \hat{V}_{ze1} - \frac{1}{d^2}\frac{\partial^2 \hat{V}_{ze1}}{\partial s^2} + \frac{\partial \hat{P}_{e\epsilon}}{\partial \zeta} = 0, \tag{89}$$

$$\frac{\partial \hat{p}_{e1}}{\partial \zeta} = \frac{\partial \hat{p}_{e\alpha}}{\partial \zeta} = 0, \quad j\omega \hat{v}_{ze1} + \frac{\partial \hat{p}_{e\epsilon}}{\partial \zeta} = 0, \tag{90}$$

$$j\omega \hat{\Theta}_{e1} - j\omega(\gamma-1)\hat{P}_{e1} - \frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2 \hat{\Theta}_{e1}}{\partial s^2} = 0,$$

$$j\omega \hat{\Theta}_{e\epsilon} - j\omega(\gamma-1)\hat{P}_{e\epsilon} - \frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2 \hat{\Theta}_{e\epsilon}}{\partial s^2} = 0, \tag{91}$$

$$j\omega \hat{\Theta}_{e\alpha} - j\omega(\gamma-1)\hat{P}_{e\alpha} - \frac{1}{d^2\,\mathrm{Pr}}\frac{\partial^2 \hat{\Theta}_{e\alpha}}{\partial s^2} = 0,$$

$$\hat{\vartheta}_{e1} = (\gamma-1)\hat{p}_{e1}, \quad \hat{\vartheta}_{e\epsilon} = (\gamma-1)\hat{p}_{e\epsilon},$$

$$\hat{\vartheta}_{e\alpha} = (\gamma-1)\hat{p}_{e\alpha}, \tag{92}$$

showing that the leading order term of the sound pressure and the correction term $\hat{p}_{e\alpha}$ do not change at all in the end region $\zeta \leqslant 0$. Obviously, the second-order term $\hat{p}_{e\epsilon}$ satisfies the Laplace equation

$$\zeta \leqslant 0: \quad \frac{\partial^2 \hat{p}_{e\epsilon}}{\partial \zeta^2} + a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \hat{p}_{e\epsilon}}{\partial r} \right) = 0, \tag{93}$$

subject to the boundary condition

$$r = 1: \quad \frac{\partial \hat{p}_{e\epsilon}}{\partial r} = 0. \tag{94}$$

In the region outside the tube where the dimensional axial and the dimensional radial coordinate are comparable to the wavelength, i.e., $z = O(1)$ and $r = O(1/\epsilon)$, the expansion terms of the pressure fluctuations are determined by a set of Helmholtz equations. However, close to the mouth where $\zeta = O(1)$ and $r = O(1)$, the Helmholtz equation for $\hat{p}_{e\epsilon}$ reduces to the Laplace equation so that Eq. (93) turns out to hold equally well for $\zeta > 0$. Moreover, the quantities $\hat{p}_{e1}$ and $\hat{p}_{e\alpha}$ are found to fulfill Eqs. (86) and (90) outside the tube too. Since the acoustic waves spread hemispherically, the pressure perturbation $\hat{p}_{e\epsilon}$ in the region $\zeta > 0$ thus is related to the axial velocity in the opening by

$$\zeta > 0: \quad \hat{p}_{e\epsilon}(\zeta, r) = j\omega \frac{1}{2a\pi}\int_0^1 r_+ \hat{v}_{ze1}(r_+) \int_0^{2\pi} \frac{1}{\sqrt{\zeta^2 + r^2 + r_+^2 - 2rr_+\cos(\beta)}}\, d\beta\, dr_+$$
$$= j\omega \frac{1}{a}\int_0^1 r_+ \hat{v}_{ze1}(r_+) \int_0^\infty e^{-\zeta\tau} \mathrm{J}_0(\tau r)\mathrm{J}_0(\tau r_+)\, d\tau\, dr_+, \tag{95}$$

with $\hat{v}_{ze1}(r_+)$ given by the relationship in Eq. (90), whereas one obtains for the other expansion terms

$$\zeta > 0: \quad \hat{p}_{e1}(\zeta, r) = 0, \quad \hat{p}_{e\alpha}(\zeta, r) = 0. \tag{96}$$

Evaluation of Eqs. (86) and (90) renders Eq. (96) valid for $\zeta \leqslant 0$ as well. As a consequence, Van Dyke's matching rules applied to the outer solutions (74) give

$$\hat{c}_{11} = -\hat{c}_{21}, \quad \hat{c}_{1\alpha}(0) = -\hat{c}_{2\alpha}(0) \tag{97}$$

and, furthermore,

$$\zeta \to -\infty: \quad \hat{p}_{e\epsilon} \to \hat{c}_{1\epsilon}(0) + \hat{c}_{2\epsilon}(0) - 2j\omega \hat{c}_{11}\zeta. \tag{98}$$

Interestingly, Eqs. (93), (94), (98), together with the boundary condition that can be deduced from Eq. (95) by taking the limit as $\zeta \to 0+$, turn out to constitute precisely the same system of equations that was solved by Rayleigh[2] (pp. 487–491, see also Ref. 19) in order to calculate the so-called quasistatic end correction

$$L_e = \frac{\hat{c}_{1\epsilon}(0) + \hat{c}_{2\epsilon}(0)}{2j\omega \hat{c}_{11}}. \tag{99}$$

A summary of this investigation can also be found in the paper by Howe.[20] Using a variational approach based on trial functions, Rayleigh obtained an approximate value of the end correction as $L_e = 0.8242/a$. In addition, Daniell[21] provided a solution bounded by the narrow range $0.82141 < aL_e < 0.82168$. More recently, other authors, e.g., Norris and Sheng,[22] calculated the quasistatic as well as the dynamic reflection of sound from the end of a flanged pipe by implementing a rational function approximation with the Bessel functions used as basis functions. In these studies, more accurate numerical results for the end correction in the limit of zero frequency are presented which are all very close to the value $L_e = 0.8217/a$ also presented in Sec. IV B.

Finally, introducing the relationship (99) in combination with condition (97) into the expressions for the sound pressure (74) and the volume flow (75) and performing a series expansion of $\hat{Z}_e = \hat{p}_{s+}/\hat{u}_+$ with respect to the perturbation parameters ϵ and α yields the result given by Eq. (80).

V. EXTENSION: THE CASE $l = O(1)$

The following part will deal with the propagation of sound waves in cylindrical ducts, proceeding from the assumption that the characteristic wavelength is comparable to the diameter of the tube. In this case the HFL is defined as

$$\epsilon = 1, \quad \alpha \ll 1, \tag{100}$$

leading to $\mathrm{Re}^{-1} \sim \alpha^2 \ll 1$, $l \sim \mathrm{He} = O(1)$, and $\mathrm{St}^{-1} \sim \delta/R \sim \alpha \ll 1$.

As a consequence, the inner and outer expansions of the basic equations (17)–(21) subject to conditions (23) and (24) are carried out with respect to the remaining perturbation parameter α only. One then obtains for the continuity equation:

$$\frac{\partial \hat{V}_{r1}}{\partial s} = 0 \Rightarrow \hat{V}_{r1} = 0, \tag{101}$$

$$j\omega \hat{\Psi}_1 + \frac{\partial \hat{V}_{z1}}{\partial z} - a \frac{\partial \hat{V}_{r\alpha}}{\partial s} = 0, \tag{102}$$

$$j\omega \hat{\Psi}_\alpha + \frac{\partial \hat{V}_{z\alpha}}{\partial z} + \frac{\partial \hat{V}_{z1}}{\partial z_1} - a \frac{\partial \hat{V}_{r\alpha^2}}{\partial s} + a \hat{V}_{r\alpha} = 0, \tag{103}$$

$$j\omega \hat{\rho}_1 + \frac{\partial \hat{v}_{z1}}{\partial z} + a \frac{1}{r} \frac{\partial}{\partial r}(r \hat{v}_{r1}) = 0, \tag{104}$$

$$j\omega \hat{\rho}_\alpha + \frac{\partial \hat{v}_{z\alpha}}{\partial z} + \frac{\partial \hat{v}_{z1}}{\partial z_1} + a \frac{1}{r} \frac{\partial}{\partial r}(r \hat{v}_{r\alpha}) = 0, \tag{105}$$

$$j\omega \hat{\rho}_{\alpha^2} + \frac{\partial \hat{v}_{z\alpha^2}}{\partial z} + \frac{\partial \hat{v}_{z\alpha}}{\partial z_1} + \frac{\partial \hat{v}_{z1}}{\partial z_2} + a \frac{1}{r} \frac{\partial}{\partial r}(r \hat{v}_{r\alpha^2}) = 0, \tag{106}$$

for the radial Navier–Stokes equation:

$$\frac{\partial \hat{P}_1}{\partial s} = \frac{\partial \hat{P}_\alpha}{\partial s} = 0, \tag{107}$$

$$j\omega \hat{V}_{r\alpha} - a \frac{\partial \hat{P}_{\alpha^2}}{\partial s} = 0, \tag{108}$$

$$j\omega \hat{v}_{r1} + a \frac{\partial \hat{p}_1}{\partial r} = 0, \quad j\omega \hat{v}_{r\alpha} + a \frac{\partial \hat{p}_\alpha}{\partial r} = 0, \tag{109}$$

$$j\omega \hat{v}_{r\alpha^2} - \frac{1}{a^2 d^2} \frac{\partial^2 v_{r1}}{\partial z^2} - \frac{1}{d^2}\left(\frac{4}{3} + \eta\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_{r1}}{\partial r}\right) - \frac{v_{r1}}{r^2}\right] - \frac{1}{a d^2}\left(\frac{1}{3} + \eta\right)\frac{\partial^2 v_{z1}}{\partial z \partial r} + a \frac{\partial \hat{p}_{\alpha^2}}{\partial r} = 0, \tag{110}$$

for the equation of state:

$$\hat{\Psi}_1 = \gamma \hat{P}_1 - \hat{\Theta}_1, \quad \hat{\Psi}_\alpha = \gamma \hat{P}_\alpha - \hat{\Theta}_\alpha, \quad \hat{\Psi}_{\alpha^2} = \gamma \hat{P}_{\alpha^2} - \hat{\Theta}_{\alpha^2}, \tag{111}$$

which holds for the outer expansion as well, furthermore, for the axial Navier–Stokes equation:

$$j\omega \hat{V}_{z1} - \frac{1}{d^2} \frac{\partial^2 \hat{V}_{z1}}{\partial s^2} + \frac{\partial \hat{P}_1}{\partial z} = 0, \tag{112}$$

$$j\omega \hat{V}_{z\alpha} - \frac{1}{d^2} \frac{\partial^2 \hat{V}_{z\alpha}}{\partial s^2} + \frac{1}{d^2} \frac{\partial \hat{V}_{z1}}{\partial s} + \frac{\partial \hat{P}_\alpha}{\partial z} + \frac{\partial \hat{P}_1}{\partial z_1} = 0, \tag{113}$$

$$j\omega \hat{v}_{z1} + \frac{\partial \hat{p}_1}{\partial z} = 0, \tag{114}$$

$$j\omega \hat{v}_{z\alpha} + \frac{\partial \hat{p}_\alpha}{\partial z} + \frac{\partial \hat{p}_1}{\partial z_1} = 0, \tag{115}$$

$$j\omega \hat{v}_{z\alpha^2} - \frac{1}{a^2 d^2}\left(\frac{4}{3} + \eta\right)\frac{\partial^2 v_{z1}}{\partial z^2} - \frac{1}{d^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_{z1}}{\partial r}\right) - \frac{1}{a d^2}\left(\frac{1}{3} + \eta\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_{r1}}{\partial z}\right) + \frac{\partial \hat{p}_{\alpha^2}}{\partial z} + \frac{\partial \hat{p}_\alpha}{\partial z_1} + \frac{\partial \hat{p}_1}{\partial z_2} = 0, \tag{116}$$

and, finally, for the energy equation:

$$j\omega \hat{\Theta}_1 - j\omega(\gamma - 1)\hat{P}_1 - \frac{1}{d^2\,\mathrm{Pr}} \frac{\partial^2 \hat{\Theta}_1}{\partial s^2} = 0, \tag{117}$$

$$j\omega \hat{\Theta}_\alpha - j\omega(\gamma - 1)\hat{P}_\alpha - \frac{1}{d^2\,\mathrm{Pr}} \frac{\partial^2 \hat{\Theta}_\alpha}{\partial s^2} + \frac{1}{d^2\,\mathrm{Pr}} \frac{\partial \hat{\Theta}_1}{\partial s} = 0, \tag{118}$$

$$\hat{\vartheta}_1 = (\gamma - 1)\hat{p}_1, \quad \hat{\vartheta}_\alpha = (\gamma - 1)\hat{p}_\alpha, \tag{119}$$

$$j\omega \hat{\vartheta}_{\alpha^2} - j\omega(\gamma - 1)\hat{p}_{\alpha^2} - \frac{1}{a^2 d^2\,\mathrm{Pr}} \frac{\partial^2 \hat{\vartheta}_1}{\partial z^2} - \frac{1}{d^2\,\mathrm{Pr}}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \hat{\vartheta}_1}{\partial r}\right) = 0. \tag{120}$$

Substitution of Eq. (111) into Eq. (119) yields

$$\hat{\rho}_1 = \hat{p}_1, \quad \hat{\rho}_\alpha = \hat{p}_\alpha \tag{121}$$

Inspection of Eqs. (101), (104), (109), and (114) shows that the leading order pressure perturbations in the core region $\hat{p}_1$ satisfy the Helmholtz equation

$$\omega^2 \hat{p}_1 + \frac{\partial^2 \hat{p}_1}{\partial z^2} + a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \hat{p}_1}{\partial r}\right) = 0, \tag{122}$$

together with the boundary condition

$$r = 1: \quad \frac{\partial \hat{p}_1}{\partial r} = 0. \tag{123}$$

The solution is given by the modal decomposition

$$\hat{p}_1 = \sum_{n=0}^{\infty}\left(\hat{C}_{1n1}(z_1, z_2)\frac{\mathrm{J}_0(\gamma_n r)}{\mathrm{J}_0(\gamma_n)} e^{-j\hat{k}_n z} + \hat{C}_{2n1}(z_1, z_2)\frac{\mathrm{J}_0(\gamma_n r)}{\mathrm{J}_0(\gamma_n)} e^{j\hat{k}_n z}\right), \tag{124}$$

where the parameters γ_n, $n=0,1,2,...$ are the zeros of the first-order Bessel function $J_1(\hat{\xi})$, with $\gamma_0=0$ and the quantities

$$\hat{k}_n=\begin{cases}\sqrt{\omega^2-a^2\gamma_n^2}, & \omega>a\gamma_n\\ -j\sqrt{a^2\gamma_n^2-\omega^2}, & \omega<a\gamma_n\end{cases} \tag{125}$$

are, respectively, the axial wave numbers for the propagating and evanescent modes.

Combining Eqs. (105), (109), (114), and (115) yields the inhomogeneous Helmholtz equation for the second-order term $\hat{p}_\alpha$,

$$\omega^2\hat{p}_\alpha+\frac{\partial^2\hat{p}_\alpha}{\partial z^2}+a^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{p}_\alpha}{\partial r}\right)=-2\frac{\partial^2\hat{p}_1}{\partial z\partial z_1}. \tag{126}$$

With Van Dyke's matching rules applied to the expressions for $\hat{V}_{z1}$, $\hat{v}_{z1}$, $\hat{\Theta}_1$, and $\hat{\vartheta}_1$ resulting from Eqs. (112), (114), (117), (119), and (23), the solutions

$$\begin{aligned}\hat{V}_{z1}&=\hat{v}_{z1}|_{r=1}(1-e^{-(1+j)ds\sqrt{(\omega/2)}}),\\ \hat{\Theta}_1&=\hat{p}_1|_{r=1}(\gamma-1)(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})\end{aligned} \tag{127}$$

are derived. It then follows from Eqs. (102) and (111) that

$$\begin{aligned}a\hat{V}_{r\alpha}=\hat{p}_1|_{r=1}&\left[j\omega s+\frac{1+j}{d}\sqrt{\frac{\omega}{2}}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right.\\ &\left.\times(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}})\right]+\left.\frac{\partial^2\hat{p}_1}{\partial z^2}\right|_{r=1}\\ &\times\frac{1}{\omega^2}\left[j\omega s-\frac{1+j}{d}\sqrt{\frac{\omega}{2}}(1-e^{-(1+j)ds\sqrt{(\omega/2)}})\right].\end{aligned} \tag{128}$$

Consequently, the matching principle gives the boundary condition for the outer expansion term $\hat{p}_\alpha$ in the form

$$r=1:\quad \frac{\partial\hat{p}_\alpha}{\partial r}=\frac{1-j}{a^2d}\omega\sqrt{\frac{\omega}{2}}\left(\hat{p}_1\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}-\frac{\partial^2\hat{p}_1}{\partial z^2}\frac{1}{\omega^2}\right). \tag{129}$$

The solution $\hat{p}'_\alpha$ to the homogeneous part of Eq. (126) subject to the homogeneous boundary conditions at $r=1$ assumes the same form as the expression for $\hat{p}_1$ from Eq. (124). However, the functions $\hat{C}_{1n1}$ and $\hat{C}_{2n1}$ have to be replaced with $\hat{C}_{1n\alpha}$ and $\hat{C}_{2n\alpha}$, respectively. In order to determine a particular solution $\hat{p}''_\alpha$ such that $\hat{p}_\alpha=\hat{p}'_\alpha+\hat{p}''_\alpha$ the ansatz

$$\begin{aligned}\hat{p}''_\alpha=\sum_{n=0}^{\infty}&\left[\left(\hat{f}_{1n\alpha}(z,z_1,z_2)\frac{J_0(\gamma_n r)}{J_0(\gamma_n)}\right.\right.\\ &\left.+\hat{g}_{n\alpha}(r)\hat{C}_{1n1}(z_1,z_2)\right)e^{-j\hat{k}_n z}\\ &+\left(\hat{f}_{2n\alpha}(z,z_1,z_2)\frac{J_0(\gamma_n r)}{J_0(\gamma_n)}\right.\\ &\left.\left.+\hat{g}_{n\alpha}(r)\hat{C}_{2n1}(z_1,z_2)\right)e^{j\hat{k}_n z}\right]\end{aligned} \tag{130}$$

is made, which is required to fulfill the conditions (24) and (129). Furthermore, the functional dependence on the coordinate r of the terms that are generated by the left-hand side of Eq. (126) for each mode $n=0,1,2,...$ on substituting Eq. (130) must be given by the radial eigenfunctions $J_0(\gamma_n r)$. Together, these requirements imply that

$$\hat{g}_{0\alpha}=\frac{\omega^2\hat{F}_0}{2a^2}r^2,\quad \hat{g}_{n\alpha}=\frac{\hat{k}_n^2\hat{F}_n}{a^2}\frac{rJ_1(\gamma_n r)}{\gamma_n J_0(\gamma_n)},\quad n=1,2,..., \tag{131}$$

with the parameters $\hat{F}_n$ defined as

$$\hat{F}_n=\frac{1-j}{d\sqrt{2\omega}}\left(1+\frac{\omega^2}{\hat{k}_n^2}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right),\quad n=0,1,2,.... \tag{132}$$

Substitution of expression (130) into Eq. (126) then results in

$$\begin{aligned}&\frac{\partial^2\hat{f}_{1n\alpha}}{\partial z^2}-2j\hat{k}_n\frac{\partial\hat{f}_{1n\alpha}}{\partial z}=-2\hat{F}_n\hat{k}_n^2\hat{C}_{1n1}+2j\hat{k}_n\frac{\partial\hat{C}_{1n1}}{\partial z_1},\\ &\frac{\partial^2\hat{f}_{2n\alpha}}{\partial z^2}+2j\hat{k}_n\frac{\partial\hat{f}_{2n\alpha}}{\partial z}=-2\hat{F}_n\hat{k}_n^2\hat{C}_{2n1}-2j\hat{k}_n\frac{\partial\hat{C}_{2n1}}{\partial z_1}.\end{aligned} \tag{133}$$

The forcing terms on the right-hand sides are resonant and would produce secular terms in in the functions $\hat{f}_{1n\alpha}$ and $\hat{f}_{2n\alpha}$. Therefore, they must be annihilated, yielding

$$\begin{aligned}&\hat{C}_{1n1}(z_1,z_2)\rightarrow\hat{C}_{1n1}(z_2)e^{-j\hat{k}_n\hat{F}_n z_1},\\ &\hat{C}_{2n1}(z_1,z_2)\rightarrow\hat{C}_{2n1}(z_2)e^{j\hat{k}_n\hat{F}_n z_1},\end{aligned} \tag{134}$$

whereupon the functions $\hat{f}_{1n\alpha}$ and $\hat{f}_{2n\alpha}$ can simply be set to zero, since any other solution would lead to expressions for $\hat{p}''_\alpha$ that could be incorporated into the homogeneous solution $\hat{p}'_\alpha$. Please note that Eq. (134) could also have been derived by using a different concept: As a result of the homogeneous problem

$$(\omega^2-\hat{k}_n^2)\hat{\psi}+a^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\hat{\psi}}{\partial r}\right)=0,\quad r=1:\quad \frac{\partial\hat{\psi}}{\partial r}=0 \tag{135}$$

having a nontrivial solution and the operator being self-adjoint, the inhomogeneous problem has a solution only if the forcing terms are orthogonal to the homogeneous solution, see, e.g., Ref. 11. This is known as the *Fredholm alternative* and introduces two solvability conditions for Eq. (126), which are identical to the right-hand sides of Eq. (133).

The quantities $\Im(-\omega\hat{F}_0)=\Im(-\omega\hat{F})$ and $\Im(-\hat{k}_n\hat{F}_n)$, $n=1,2,...$, turn out to be, respectively, the leading order decay rate of the fundamental mode and the leading order decay rates of the higher order modes due to boundary layer attenuation, which is in accordance with Beatty's[23] results for the axisymmetric case obtained by using the concept of boundary layer admittance.

In order to calculate the corresponding expressions for the third-order terms, a procedure very similar to that used before is performed: Combination of Eqs. (106), (110), (114)–(116), (119), and (120) and multiple application of Eq. (122) shows that

$$\hat{v}_{z\alpha^2} = -\frac{1}{a^2 d^2}\frac{\partial \hat{p}_1}{\partial z}\left(\frac{4}{3}+\eta\right) + \frac{j}{\omega}\left(\frac{\partial \hat{p}_1}{\partial z_2} + \frac{\partial \hat{p}_\alpha}{\partial z_1} + \frac{\partial \hat{p}_{\alpha^2}}{\partial z}\right) \tag{136}$$

and that $\hat{p}_{\alpha^2}$ has to satify the inhomogeneous equation

$$\omega^2 \hat{p}_{\alpha^2} + \frac{\partial^2 \hat{p}_{\alpha^2}}{\partial z^2} + a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{p}_{\alpha^2}}{\partial r}\right) = -2\frac{\partial^2 \hat{p}_1}{\partial z \partial z_2} - \frac{\partial^2 \hat{p}_1}{\partial z_1^2} + j\omega^3 \frac{1}{a^2 d^2}\hat{p}_1\left(\frac{4}{3}+\eta+\frac{\gamma-1}{\mathrm{Pr}}\right) - 2\frac{\partial^2 \hat{p}_\alpha}{\partial z \partial z_1}, \tag{137}$$

where the third term on the right-hand side obviously incorporates the effects of heat conduction and viscosity in the core region associated with the propagation of the leading order pressure fluctuations $\hat{p}_1$. This is in contrast to the analysis in Sec. IV where, due to the long wavelength assumption $l \ll 1$, these effects do not affect the wave equations [see Eq. (70)]. Moreover, Eqs. (113) and (118) are solved for the axial velocity and the temperature in the boundary layer, leading to

$$\begin{aligned}\hat{V}_{z\alpha} &= \hat{v}_{z\alpha}|_{r=1}\left(1-e^{-(1+j)ds\sqrt{(\omega/2)}}\right) - \hat{v}_{z1}|_{r=1}\frac{s}{2}e^{-(1+j)ds\sqrt{(\omega/2)}},\\ \hat{\Theta}_\alpha &= \hat{p}_\alpha|_{r=1}(\gamma-1)\left(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}\right) - \hat{p}_1|_{r=1}(\gamma-1)\frac{s}{2}e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}.\end{aligned} \tag{138}$$

After substitution for $\hat{\Psi}_\alpha$, $\hat{V}_{z1}$, $\hat{V}_{r\alpha}$, and $\hat{V}_{z\alpha}$ from Eqs. (111), (127), (128), and (138), respectively, integration of Eq. (103) with respect to the inner coordinate s gives

$$\begin{aligned} a\hat{V}_{r\alpha^2} &= \hat{p}_1|_{r=1}\left[\frac{j\omega}{2}s^2 - \frac{\gamma-1}{2d^2\mathrm{Pr}}\left(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}\right) + \frac{1+j}{2d}\sqrt{\frac{\omega}{2}}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}s\left(2-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}\right)\right]\\ &+ \left.\frac{\partial^2 \hat{p}_1}{\partial z^2}\right|_{r=1}\frac{1}{\omega^2}\left[\frac{j\omega}{2}s^2 + \frac{1}{2d^2}\left(1-e^{-(1+j)ds\sqrt{(\omega/2)}}\right) - \frac{1+j}{2d}\sqrt{\frac{\omega}{2}}s\left(2-e^{-(1+j)ds\sqrt{(\omega/2)}}\right)\right]\\ &+ \hat{p}_\alpha|_{r=1}\left[j\omega s + \frac{1+j}{d}\sqrt{\frac{\omega}{2}}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-e^{-(1+j)ds\sqrt{(\omega\,\mathrm{Pr}/2)}}\right)\right]\\ &+ \left(\left.\frac{\partial^2 \hat{p}_\alpha}{\partial z^2}\right|_{r=1} + 2\left.\frac{\partial^2 \hat{p}_1}{\partial z \partial z_1}\right|_{r=1}\right)\frac{1}{\omega^2}\left[j\omega s - \frac{1+j}{d}\sqrt{\frac{\omega}{2}}\left(1-e^{-(1+j)ds\sqrt{(\omega/2)}}\right)\right]. \end{aligned} \tag{139}$$

Furthermore, upon applying the matching rules to the inner and outer expansions of the radial velocity, the boundary condition

$$r=1:\quad \frac{\partial \hat{p}_{\alpha^2}}{\partial r} = \frac{j\omega}{2a^2 d^2}\left(\hat{p}_1\frac{\gamma-1}{\mathrm{Pr}} - \frac{\partial^2 \hat{p}_1}{\partial z^2}\frac{1}{\omega^2}\right) + \frac{1-j}{a^2 d}\omega\sqrt{\frac{\omega}{2}}\left[\hat{p}_\alpha\frac{\gamma-1}{\sqrt{\mathrm{Pr}}} - \left(\frac{\partial^2 \hat{p}_\alpha}{\partial z^2} + 2\frac{\partial^2 \hat{p}_1}{\partial z \partial z_1}\right)\frac{1}{\omega^2}\right] \tag{140}$$

is obtained. It should be mentioned that the particular solution $\hat{p}''_\alpha$ vanishes at the boundary for all higher order modes $n=1,2,\ldots$.

A particular solution $\hat{p}''_{\alpha^2}$ to Eq. (137) can be found by employing the ansatz

$$\begin{aligned}\hat{p}''_{\alpha^2} = \sum_{n=0}^{\infty}\Bigg[&\Bigg(\hat{f}_{1n\alpha^2}(z,z_1,z_2)\frac{\mathrm{J}_0(\gamma_n r)}{\mathrm{J}_0(\gamma_n)} + \hat{g}_{n\alpha^2}(r)\hat{C}_{1n1}(z_2)e^{-j\hat{F}_n\hat{k}_n z_1} + \hat{h}_{n\alpha^2}(r)\hat{C}_{1n\alpha}(z_1,z_2)\Bigg)e^{-j\hat{k}_n z}\\ &+\Bigg(\hat{f}_{2n\alpha^2}(z,z_1,z_2)\frac{\mathrm{J}_0(\gamma_n r)}{\mathrm{J}_0(\gamma_n)} + \hat{g}_{n\alpha^2}(r)\hat{C}_{2n1}(z_2)e^{j\hat{F}_n\hat{k}_n z_1} + \hat{h}_{n\alpha^2}(r)\hat{C}_{2n\alpha}(z_1,z_2)\Bigg)e^{j\hat{k}_n z}\Bigg],\end{aligned} \tag{141}$$

such that conditions (24) and (140) are satisfied and the operators on the left-hand side of Eq. (137) produce terms that have the same functional dependence on r as the terms on the right-hand side. As a consequence, one obtains

$$\hat{g}_{0\alpha^2}=\frac{\omega^2}{2a^2}\left(\hat{G}_0+\frac{\omega^2\hat{F}_0^2}{4a^2}\right)r^2+\frac{\omega^4\hat{F}_0^2}{16a^4}r^4,$$

$$\hat{g}_{n\alpha^2}=\frac{\hat{k}_n^2}{a^2}\left(\hat{G}_n+\frac{\hat{k}_n^2\hat{F}_n^2}{a^2\gamma_n^2}\right)\frac{r\mathrm{J}_1(\gamma_n r)}{\gamma_n\mathrm{J}_0(\gamma_n)}$$

$$-\frac{\hat{k}_n^4\hat{F}_n^2}{2a^4}\frac{r^2\mathrm{J}_0(\gamma_n r)}{\gamma_n^2\mathrm{J}_0(\gamma_n)},\quad n=1,2,..., \tag{142}$$

$$\hat{h}_{0\alpha^2}=\frac{\omega^2\hat{F}_0}{2a^2}r^2,\quad \hat{h}_{n\alpha^2}=\frac{\hat{k}_n^2\hat{F}_n}{a^2}\frac{r\mathrm{J}_1(\gamma_n r)}{\gamma_n\mathrm{J}_0(\gamma_n)},\quad n=1,2,...,$$

where

$$\hat{G}_n=-\frac{j}{2d^2\omega}\left[3-\frac{\omega^2}{\hat{k}_n^2}(\gamma-1)\left(\frac{1}{\mathrm{Pr}}-\frac{4}{\sqrt{\mathrm{Pr}}}\right)\right],$$

$$n=0,1,2,\dots . \tag{143}$$

Substitution of Eq. (130) into Eq. (137) then yields

$$\frac{\partial^2\hat{f}_{1n\alpha^2}}{\partial z^2}-2j\hat{k}_n\frac{\partial\hat{f}_{1n\alpha^2}}{\partial z}=\left(-2\hat{H}_n\hat{k}_n^2\hat{C}_{1n1}+2j\hat{k}_n\frac{\partial\hat{C}_{1n1}}{\partial z_2}\right)$$
$$\times e^{-j\hat{F}_n\hat{F}_n z_1}-2\hat{F}_n\hat{k}_n^2\hat{C}_{1n\alpha}$$
$$+2j\hat{k}_n\frac{\partial\hat{C}_{1n\alpha}}{\partial z_1}, \tag{144}$$

$$\frac{\partial^2\hat{f}_{2n\alpha^2}}{\partial z^2}+2j\hat{k}_n\frac{\partial\hat{f}_{2n\alpha^2}}{\partial z}=\left(-2\hat{H}_n\hat{k}_n^2\hat{C}_{1n1}-2j\hat{k}_n\frac{\partial\hat{C}_{2n1}}{\partial z_2}\right)$$
$$\times e^{-j\hat{F}_n\hat{K}_n z_1}-2\hat{F}_n\hat{k}_n^2\hat{C}_{2n\alpha}$$
$$-2j\hat{k}_n\frac{\partial\hat{C}_{2n\alpha}}{\partial z_1},$$

where

$$\hat{H}_0=\hat{G}_0-\frac{\hat{F}_0^2}{2}+\frac{\omega^2\hat{F}_0^2}{4a^2}-\frac{j\omega}{2a^2d^2}\left(\frac{4}{3}+\eta+\frac{\gamma-1}{\mathrm{Pr}}\right)$$
$$=-\frac{j}{d^2\omega}\left[1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-\frac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right]$$
$$-\frac{j\omega}{2a^2d^2}\left[\frac{11}{6}+\eta+\frac{\gamma-1}{2\sqrt{\mathrm{Pr}}}\left(2+\frac{1+\gamma}{\sqrt{\mathrm{Pr}}}\right)\right],$$
$$\hat{H}_n=\hat{G}_n-\frac{\hat{F}_n^2}{2}-\frac{j\omega^3}{2a^2d^2\hat{k}_n^2}\left(\frac{4}{3}+\eta+\frac{\gamma-1}{\mathrm{Pr}}\right) \tag{145}$$
$$=-\frac{j}{d^2\omega}\left[1+\frac{\omega^2}{\hat{k}_n^2}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left[1-\frac{1}{2\sqrt{\mathrm{Pr}}}\left(1+\frac{\omega^2}{\hat{k}_n^2}(\gamma-1)\right)\right]\right]$$
$$-\frac{j\omega^3}{2a^2d^2\hat{k}_n^2}\left(\frac{4}{3}+\eta+\frac{\gamma-1}{\mathrm{Pr}}\right),\quad n=1,2,\dots .$$

As before, the right-hand sides of Eq. (144) have to be annihilated in order to rule out secular terms in the functions $\hat{f}_{1n\alpha^2}$ and $\hat{f}_{2n\alpha^2}$. The resulting solvability conditions thus read

$$2\hat{F}_n\hat{k}_n^2\hat{C}_{1n\alpha}-2j\hat{k}_n\frac{\partial\hat{C}_{1n\alpha}}{\partial z_1}=\left(-2\hat{H}_n\hat{k}_n^2\hat{C}_{1n1}+2j\hat{k}_n\frac{\partial\hat{C}_{1n1}}{\partial z_2}\right)$$
$$\times e^{-j\hat{F}_n\hat{k}_n z_1},$$
$$2\hat{F}_n\hat{k}_n^2\hat{C}_{2n\alpha}+2j\hat{k}_n\frac{\partial\hat{C}_{2n\alpha}}{\partial z_1}=\left(-2\hat{H}_n\hat{k}_n^2\hat{C}_{1n1}-2j\hat{k}_n\frac{\partial\hat{C}_{2n1}}{\partial z_2}\right) \tag{146}$$
$$\times e^{-j\hat{F}_n\hat{k}_n z_1},$$

,

and, furthermore, $\hat{f}_{1n\alpha^2}$ and $\hat{f}_{2n\alpha^2}$ can be set to zero. The right-hand sides of Eq. (146) are again identified as resonant forcing terms, finally leading to

$$\hat{C}_{1n1}(z_2)e^{-j\hat{F}_n\hat{k}_n z_1}\to\hat{C}_{1n1}e^{-j\hat{k}_n(\hat{F}_n z_1+\hat{H}_n z_2)},$$
$$\hat{C}_{2n1}(z_2)e^{j\hat{F}_n\hat{k}_n z_1}\to\hat{C}_{2n1}e^{j\hat{k}_n(\hat{F}_n z_1+\hat{H}_n z_2)} \tag{147}$$

and, additionally,

$$\hat{C}_{1n\alpha}(z_1,z_2)\to\hat{C}_{1n\alpha}(z_2)e^{-j\hat{k}_n\hat{F}_n z_1},$$
$$\hat{C}_{2n\alpha}(z_1,z_2)\to\hat{C}_{2n\alpha}(z_2)e^{j\hat{k}_n\hat{F}_n z_1}. \tag{148}$$

Results. One interesting property of the parameter $\hat{H}_0$ for the fundamental mode is that it cannot be derived from the quantities $\hat{H}_n$, $n=1,2,...$, for the higher order modes simply by setting $\hat{k}_n=\omega$. However, it reduces to the parameter $\hat{H}$ defined in Sec. IV, when the limit as $a\to\infty$ is taken and thus, formally, the long wavelength assumption $l\ll 1$ is reintroduced. In accordance with this, the axial wave numbers $\hat{k}_n$, $n=1,2,...$, then approach $-j\infty$ and all higher order modes die away immediately.

The definition of the propagation parameter from Eq. (3) can easily be extended, such that each mode is treated separately: To this end, the quantities

$$\hat{\Gamma}_n=\sqrt{\frac{1}{\hat{p}_{ns}\hat{v}_{nz}}\frac{\partial\hat{p}_{ns}}{\partial z}\frac{\partial\hat{v}_{nz}}{\partial z}}=\sqrt{\frac{1}{\hat{p}_{ns}}\frac{\partial^2\hat{p}_{ns}}{\partial z^2}}, \tag{149}$$

where $\hat{p}_{ns}$ and $\hat{v}_{nz}$ denote the sound pressures and the axial velocities associated with the modes $n=0,1,2,...$, are introduced. Upon replacing the partial derivatives with respect to z with derivatives with respect to the three length scales used here according to Eq. (28), application of Eqs. (134), (147), and (148) implicates that the parameters $\hat{\Gamma}_n$ are given by the relationship

$$\hat{\Gamma}_n=\sqrt{-\hat{k}_n^2[1+2\alpha\hat{F}_n+\alpha^2(2\hat{H}_n+\hat{F}_n^2)]+\cdots},$$
$$n=0,1,2,..., \tag{150}$$

where $\hat{F}_n$ and $\hat{H}_n$ are the quantities already introduced in Eqs. (132) and (145), respectively. If $|\hat{k}_n|=O(1)$ this result simplifies to

$$\hat{\Gamma}_n=j\hat{k}_n+\alpha j\hat{k}_n\hat{F}_n+\alpha^2 j\hat{k}_n\hat{H}_n+\dots,\quad n=0,1,2,\dots . \tag{151}$$

Inspection of the definitions (132) and (145) shows that in the limiting case of a mode $n=m$ having a cut-off frequency so close to the driving frequency that $|\hat{k}_m| \ll 1$ the orders of magnitude of the different terms appearing in Eq. (150) may change completely. Before turning to a detailed analysis of that problem, please note that one aspect of the first- and second-order solutions derived by Anderson and Vaidya[13] in their study of linear sound propagation in slit-shaped waveguides carries over unchanged to the case of circular cross sections considered here: The solvability conditions resulting from Eqs. (133) and (144) are singular at the cut-off frequencies $\omega = a\gamma_n$, $n=1,2,...$, predicting that in the limit as $\omega \to a\gamma_m$, $\hat{k}_m \to 0$, the attenuation of the mode $n=m$ takes place over a much shorter spatial range than that defined by $z_1 = O(1)$. A closer examination of the solvability conditions then indicates that in order to investigate the modulation of a mode m featuring a cut-off frequency in proximity to the driving frequency ω such that

$$\omega = \omega_m + \alpha\omega_\alpha, \quad \omega_m = a\gamma_m, \quad m \geqslant 1 \tag{152}$$

holds, for this mth mode, the length scales z, z_1, and z_2 have to be replaced with the length scales

$$\overline{z_1} = \alpha^{1/2} z, \quad \overline{z_2} = \alpha^{3/2} z. \tag{153}$$

Moreover, this necessitates expressing the pressure and, similarly, the velocity components and the other thermodynamic quantities in the form:

$$p = \frac{1}{\gamma} + M(p_1 + \alpha^{1/2} p_{\alpha^{1/2}} + \alpha p_\alpha + \alpha^{3/2} p_{\alpha^{3/2}} + \alpha^2 p_{\alpha^2} + \cdots) + M^2(\cdots) + \cdots. \tag{154}$$

Application of the MMS then leads to the following results: (a) The leading order solutions given by Eqs. (109), (114), and (124) remain unchanged. However, the functions $\hat{C}_{1m1}(z_1,z_2)$ and $\hat{C}_{2m1}(z_1,z_2)$ have to be replaced with $\hat{C}_{1m1}(\overline{z_1},\overline{z_2})$ and $\hat{C}_{2m1}(\overline{z_1},\overline{z_2})$, respectively, and, additionally, ω must be replaced with ω_m, resulting in $\hat{k}_m = 0$. (b) The perturbations $\hat{v}_{r\alpha^{1/2}}$, $\hat{v}_{z\alpha^{1/2}}$, and $\hat{p}_{\alpha^{1/2}}$ satisfy Eq. (109),

$$j\omega_m \hat{v}_{z\alpha^{1/2}} + \frac{\partial \hat{p}_{\alpha^{1/2}}}{\partial z} + \frac{\partial \hat{p}_1}{\partial \overline{z_1}} = 0, \tag{155}$$

and the homogeneous Helmholz equation (122) subject to the boundary condition (123). (c) The sound propagation at the order $O(\alpha)$ is now governed by the set

$$\begin{gathered} j\omega_m \hat{v}_{r\alpha} + j\omega_\alpha \hat{v}_{r1} + a\frac{\partial \hat{p}_\alpha}{\partial r} = 0, \\ j\omega_m \hat{v}_{z\alpha} + j\omega_\alpha \hat{v}_{z1} + \frac{\partial \hat{p}_\alpha}{\partial z} + \frac{\partial \hat{p}_{\alpha^{1/2}}}{\partial \overline{z_1}} + \frac{\partial \hat{p}_1}{\partial z_1} = 0, \\ \omega_m^2 \hat{p}_\alpha + \frac{\partial^2 \hat{p}_\alpha}{\partial z^2} + a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{p}_\alpha}{\partial r}\right) = -2\omega_m\omega_\alpha \hat{p}_1 - 2\frac{\partial^2 \hat{p}_1}{\partial z \partial z_1} - \frac{\partial^2 \hat{p}_1}{\partial \overline{z}_1^2}, \end{gathered} \tag{156}$$

and the boundary condition (129). By introducing the ansatz (130), with $\hat{g}_{m\alpha}$ redefined as

$$\hat{g}_{m\alpha} = \frac{\hat{\hat{F}}_m}{a^2}\frac{r J_1(\gamma_m r)}{\gamma_m J_0(\gamma_m)}, \quad \hat{\hat{F}}_m = \omega_m^{3/2}\frac{1-j}{d\sqrt{2}}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}, \tag{157}$$

the following modified solvability conditions are obtained:

$$\begin{gathered} -\hat{C}_{1n1}(\hat{F}_n \hat{k}_n^2 + \omega_m\omega_\alpha) + j\hat{k}_n \frac{\partial \hat{C}_{1n1}}{\partial z_1} + j\hat{k}_n \frac{\partial \hat{C}_{1n\alpha^{1/2}}}{\partial \overline{z_1}} = 0, \\ \hat{C}_{2n1}(\hat{F}_n \hat{k}_n^2 + \omega_m\omega_\alpha) + j\hat{k}_n \frac{\partial \hat{C}_{2n1}}{\partial z_1} + j\hat{k}_n \frac{\partial \hat{C}_{2n\alpha^{1/2}}}{\partial \overline{z_1}} = 0, \\ n \neq m, \end{gathered} \tag{158}$$

where ω in $\hat{k}_n$ and $\hat{F}_n$ has to be replaced with ω_m, and

$$2(\hat{C}_{1m1} + \hat{C}_{2m1})(\hat{\hat{F}}_m + \omega_m\omega_\alpha) + \frac{\partial^2}{\partial \overline{z}_1^2}(\hat{C}_{1m1} + \hat{C}_{2m1}) = 0 \tag{159}$$

for the mth mode. Equations (158) and (159) thus yield

$$\begin{gathered} \hat{C}_{1n1}(z_1,z_2) \to \hat{C}_{1n1}(z_2) e^{-j\hat{k}_n(\hat{F}_n + \omega_m\omega_\alpha/\hat{k}_n^2)z_1}, \\ \hat{C}_{2n1}(z_1,z_2) \to \hat{C}_{2n1}(z_2) e^{j\hat{k}_n(\hat{F}_n + \omega_m\omega_\alpha/\hat{k}_n^2)z_1}, \quad n \neq m, \end{gathered} \tag{160}$$

$$\begin{gathered} \hat{C}_{1m1}(\overline{z_1},\overline{z_2}) \to \hat{C}_{1m1}(\overline{z_2}) e^{-j(2\hat{\hat{F}}_m + 2\omega_m\omega_\alpha)^{1/2}\overline{z_1}}, \\ \hat{C}_{2m1}(\overline{z_1},\overline{z_2}) \to \hat{C}_{2m1}(\overline{z_2}) e^{j(2\hat{\hat{F}}_m + 2\omega_m\omega_\alpha)^{1/2}\overline{z_1}}. \end{gathered} \tag{161}$$

(d) Furthermore, the solvability conditions for $\hat{p}_{\alpha^{3/2}}$ imply that

$$\begin{gathered} \hat{C}_{1m\alpha^{1/2}}(\overline{z_1},\overline{z_2}) \to \hat{C}_{1m\alpha^{1/2}}(\overline{z_2}) e^{-j(2\hat{\hat{F}}_m + 2\omega_m\omega_\alpha)^{1/2}\overline{z_1}}, \\ \hat{C}_{2m\alpha^{1/2}}(\overline{z_1},\overline{z_2}) \to \hat{C}_{2m\alpha^{1/2}}(\overline{z_2}) e^{j(2\hat{\hat{F}}_m + 2\omega_m\omega_\alpha)^{1/2}\overline{z_1}}. \end{gathered} \tag{162}$$

Substitution of ω from Eq. (152) into Eq. (134) multiplied by $\exp(\mp j\hat{k}_n z)$, series expansion with respect to α, and comparison with Eq. (160) immediately shows the equivalence of both formulations provided that $n \neq m$. As expected, relationships (150) and (151) for the propagation parameter are left unchanged for all modes having cut-off frequencies not close to the driving frequency. However, if there is a mode $m \geqslant 1$ such that Eq. (152) is fulfilled, Eqs. (161) and (162) lead to the interesting result that the propagation parameter $\hat{\Gamma}_m$ is then given by

$$\hat{\Gamma}_m = \alpha^{1/2} j(2\hat{\hat{F}}_m + 2\omega_m\omega_\alpha)^{1/2} + \alpha^{3/2}\hat{\Gamma}_{m\alpha^{3/2}} + \cdots, \tag{163}$$

which is valid for all frequencies ω arbitrarily near the cut-off frequency $a\gamma_m$, i.e., even for $\omega_\alpha = 0$. This solution agrees perfectly with the approximation derived by Hudde,[24] using the concept of boundary layer admittance; the experiments reported in this paper are also in good accordance with the theoretically predicted attenuation rate. In principle, the higher order correction term $\hat{\Gamma}_{m\alpha^{3/2}}$ could be calculated by evaluating the resonant forcing terms appearing in the equations governing the pressure perturbation $\hat{p}_{\alpha^2}$. Here, however, a different (more intuitive) approach shall be used: In contrast to the limits of the original solvabilty conditions (133) and (144) as $\hat{k}_n \to 0$, $n=1,2,...$, the corresponding limits of the propagation parameters given by Eq. (150) can be calculated without any difficulties. As it turns out, the leading order term of $\hat{\Gamma}_m$, which is of the order $O(\alpha^{1/2})$, can also be derived by substituting for ω from Eq. (152) in Eq. (150), setting $n=m$, and performing a series expansion with respect to the perturbation parameter α. Consequently, it is very reasonable to assume that for each mode, the expression for $\hat{\Gamma}_n$ from Eq. (150) *including* the $O(\alpha^2)$ terms retains its validity even for those driving frequencies that are very close to the cut-off frequency of this mode, which enables the computation of the higher order correction term of $\hat{\Gamma}_m$. Hence, one obtains

$$\hat{\Gamma}_{m\alpha^{3/2}}=\frac{j}{2}(2\hat{\hat{F}}_m+2\omega_m\omega_\alpha)^{1/2}\times\left(\frac{\omega_\alpha}{2\omega_m}+\frac{\frac{\omega_\alpha}{\omega_m}\hat{\hat{F}}_m+\sqrt{2\omega_m}\omega_\alpha\frac{1-j}{d}+\hat{\hat{K}}_m}{\hat{\hat{F}}_m+\omega_m\omega_\alpha}\right), \tag{164}$$

where

$$\hat{\hat{K}}_m=\lim_{\omega\to\omega_m}k_m^2\left(\hat{H}_m+\frac{\hat{F}_m^2}{2}\right)=j\omega_m\frac{\gamma-1}{2d^2}\left(\frac{1}{\text{Pr}}-\frac{4}{\sqrt{\text{Pr}}}\right)-\frac{j\omega_m^3}{2a^2d^2}\left(\frac{4}{3}+\eta+\frac{\gamma-1}{\text{Pr}}\right). \tag{165}$$

The corresponding result obtained when the general expression for the propagation parameters from Eq. (150) is rewritten in terms of dimensional quantities can be found in Appendix C. As far as the fundamental mode is concerned it agrees perfectly with the solution given by Kergomard.[25] This applies even for the terms appearing in $\hat{H}_0$ that result from heat conduction and viscosity in the core region. Unfortunately, in that paper, no derivations were presented. Later Bruneau *et al.*[26] calculated the propagation parameters for the higher order modes, starting from a generalized dispersion equation. However, in the intermediate steps that then followed only boundary layer effects were taken into account (the mentioned dispersion relation was corrected in a subsequent paper, see Ref. 27; if, though, only the axisymmetric modes are considered, as is the case here, it remains unchanged). The expressions for the propagation parameters $\hat{\Gamma}_n$ presented in Eqs. (150) and (C4) are in complete accordance with the results in Ref. 26, if the terms proportional to $1/a^2$ appearing in the definitions (145) are formally omitted, so that the quantities $\hat{H}_n$ reduce to $\hat{H}_n=\hat{G}_n-\hat{F}_n^2/2$, $n=0,1,2,\ldots$. However, it should be kept in mind that within the framework of a correct asymptotic analysis such *a posteriori* simplifications are not appropriate, since in the case of $l=O(1)$ the terms in $\hat{H}_n$ resulting from boundary layer attenuation and those due to the viscous and thermal effects in the core region are of the same order of magnitude.

As can be seen from Eqs. (C4) and (C5), the quantity $\hat{\Gamma}_n\omega/c_0$ can be conveniently expressed in terms of the parameters St, He, Pr, γ, and η_0/μ_0. The real part of this function evaluated for the fundamental mode is displayed in Fig. 4, which clearly shows that the attenuation rate is significantly affected by the viscothermal damping mechanisms in the core region as soon as the Helmholtz number becomes an $O(1)$ quantity. In addition, Fig. 5 displays the graphs of $\Re(\hat{\Gamma}_n\omega/c_0)$ for $n=0$, 1, 2, 3 and $R=0.001$ m as functions of the Helmholtz number $\text{He}=O(1)$. Here, the (large) Stokes number has been eliminated using the relationship $\text{St}=\sqrt{\text{He}\,\text{Re}_R}$, where $\text{Re}_R=c_0R\rho_0/\mu_0\sim\text{Re}\gg1$ denotes the radial Reynolds number. The results plotted in this figure well illustrate the behavior of the modal damping rates when the frequency is increased such that a new mode becomes propagational: In conformity with Eqs. (151) and (163), $\Re(\hat{\Gamma}_n\omega/c_0)=O(1)$ if $\text{He}<\gamma_n$ and $\gamma_n-\text{He}=O(1)$, $\Re(\hat{\Gamma}_n\omega/c_0)=O(\text{St}^{-1/2})$ if $|\text{He}-\gamma_n|\ll1$, and $\Re(\hat{\Gamma}_n\omega/c_0)=O(\text{St}^{-1})$ if $\text{He}>\gamma_n$ and $\text{He}-\gamma_n=O(1)$.

A further point of interest is the calculation of the transfer functions that relate the pressure fluctuations at both ends

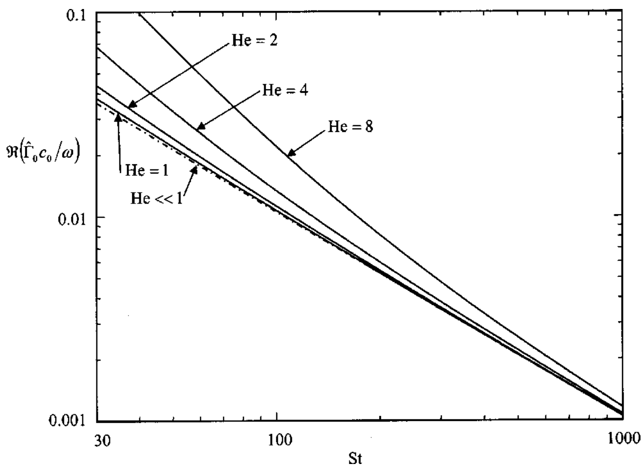

FIG. 4. Graphs of $\Re(\hat{\Gamma}_0c_0/\omega)$ as functions of St with He as parameter in double logarithmic scale; Pr=0.707, γ=1.402, $\eta_0/\mu_0=0.6$ (air at 300 K); the curve for $\text{He}\sim l\ll1$ corresponds to the long wavelength solution from Sec. IV; $\text{St}\gg1$: $\Re(\hat{\Gamma}_0c_0/\omega)=O(\text{St}^{-1})$.

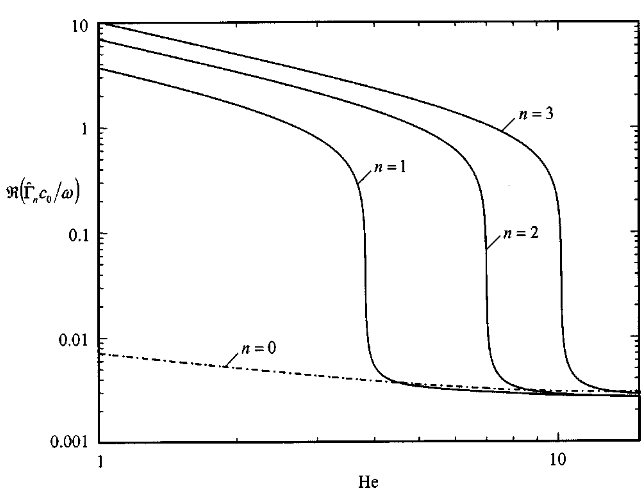

FIG. 5. Graphs of $\Re(\hat{\Gamma}_nc_0/\omega)$ as functions of He in double logarithmic scale, $R=0.001$ m, i.e., $\text{St}=\sqrt{\text{He}\,\text{Re}_R}$ with $\text{Re}_R=c_0R\rho_0/\mu_0$; Pr=0.707, γ=1.402, $\eta_0/\mu_0=0.6$, $c_0\rho_0/\mu_0=221.3732\times10^5\ \text{m}^{-1}$ (air at 300 K); $\gamma_1=3.8317$, $\gamma_2=7.0156$, $\gamma_3=10.1735$.

of a tube to the axial velocities (in the case of $l \ll 1$ these relations can easily be derived by solving the system of equations given by the transfer matrix $\hat{\mathbf{A}}$ for the sound pressures $\hat{p}_{s-}$ and $\hat{p}_{s+}$). Again, the tube is regarded as a transmission line whose length is large compared to the characteristic wavelength such that $L = L_1/\alpha$ with $L_1 = O(1)$. Due to the viscothermal processes taking place in the boundary layer, each transfer function associated with a distinct mode will turn out to be affected by all the other modes as well.

For convenience, the axial velocities in the core region at both ends of the duct are assumed to involve only terms of order $O(1)$. Decomposition through the radial eigenfunctions $\mathrm{J}_0(\gamma_n r)$ then results in

$$\frac{\hat{v}_{z-}}{M} = \hat{v}_{z1-} = \sum_{n=0}^{\infty} \hat{C}_{nv_z-} \frac{J_0(\gamma_n r)}{J_0(\gamma_n)},$$
$$\frac{\hat{v}_{z+}}{M} = \hat{v}_{z1+} = \sum_{n=0}^{\infty} \hat{C}_{nv_z+} \frac{J_0(\gamma_n r)}{J_0(\gamma_n)}. \quad (166)$$

Setting the origin of the axial coordinate z to the left end of the tube, evaluation of Eqs. (114) and (124) in combination with the solutions (134), (147), and (148) of the solvability conditions leads to

$$\hat{C}_{1n1} - \hat{C}_{2n1} = \hat{C}_{nv_z-} \frac{\omega}{\hat{k}_n},$$
$$\hat{C}_{1n1} + \hat{C}_{2n1} = \left(\hat{C}_{nv_z-} \coth(\hat{\Gamma}_n L) - \frac{\hat{C}_{nv_z+}}{\sinh(\hat{\Gamma}_n L)} \right) \frac{\omega}{\hat{k}_n}, \quad (167)$$

where, for an evanescent mode or a mode having a cut-off frequency close to the driving frequency, the terms $\coth(\hat{\Gamma}_n L)$ and $1/\sinh(\hat{\Gamma}_n L)$ can simply be replaced with 1 and 0, respectively, since in such cases the resulting corrections become exponentially small. Furthermore, annihilating the second- order solutions $\hat{v}_{z\alpha-}$ and $\hat{v}_{z\alpha+}$ given by Eqs. (115) and (130) yields

$$\hat{C}_{10\alpha} - \hat{C}_{20\alpha} = -\hat{C}_{0v_z-} \left(\hat{F}_0 + \frac{\omega^2 \hat{F}_0}{4a^2} \right) + \sum_{m=1}^{\infty} \hat{C}_{mv_z-} \frac{2\hat{k}_m^2 \hat{F}_m}{a^2 \gamma_m^2},$$
$$\hat{C}_{10\alpha} + \hat{C}_{20\alpha} = (\hat{C}_{10\alpha} - \hat{C}_{20\alpha}) \coth(\hat{\Gamma}_0 L) + \left[\hat{C}_{0v_z+} \left(\hat{F}_0 + \frac{\omega^2 \hat{F}_0}{4a^2} \right) - \sum_{m=1}^{\infty} \hat{C}_{mv_z+} \frac{2\hat{k}_m^2 \hat{F}_m}{a^2 \gamma_m^2} \right] \frac{1}{\sinh(\hat{\Gamma}_0 L)},$$
$$\hat{C}_{1n\alpha} - \hat{C}_{2n\alpha} = -\hat{C}_{nv_z-} \frac{\omega \hat{F}_n}{\hat{k}_n} - \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \hat{C}_{mv_z-} \frac{2\omega \hat{k}_m^2 \hat{F}_m}{a^2 \hat{k}_n (\gamma_n^2 - \gamma_m^2)}, \quad (168)$$
$$\hat{C}_{1n\alpha} + \hat{C}_{2n\alpha} = (\hat{C}_{1n\alpha} - \hat{C}_{2n\alpha}) \coth(\hat{\Gamma}_n L) + \left(\hat{C}_{nv_z+} \frac{\omega \hat{F}_n}{\hat{k}_n} + \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \hat{C}_{mv_z+} \frac{2\omega \hat{k}_m^2 \hat{F}_m}{a^2 \hat{k}_n (\gamma_n^2 - \gamma_m^2)} \right) \frac{1}{\sinh(\hat{\Gamma}_n L)},$$
$$n = 1,2,...,$$

where the relationship

$$\int_0^1 2r \frac{\mathrm{J}_0(\gamma_n r)}{\mathrm{J}_0(\gamma_n)} \frac{r \mathrm{J}_1(\gamma_m r)}{\gamma_m \mathrm{J}_0(\gamma_m)} dr = \begin{cases} \dfrac{2}{\gamma_n^2 - \gamma_m^2}, & n = 0,1,2..., \quad m = 1,2..., \quad m \neq n \\ 0, & m = n = 1,2... \end{cases} \quad (169)$$

has been used.

Equations (124), (130), (167), and (168) together with Eqs. (134), (147), and (148) are sufficient to determine the solutions

$$\frac{\hat{p}_{s-}}{M} = \hat{p}_{1-} + \alpha \hat{p}_{\alpha-} + \cdots = \sum_{n=0}^{\infty} \hat{C}_{np_s-} \frac{J_0(\gamma_n r)}{J_0(\gamma_n)},$$
$$\frac{\hat{p}_{s+}}{M} = \hat{p}_{1+} + \alpha \hat{p}_{\alpha+} + \cdots = \sum_{n=0}^{\infty} \hat{C}_{np_s+} \frac{J_0(\gamma_n r)}{J_0(\gamma_n)} \quad (170)$$

at $z=0$ and $z=L$, respectively, in terms of the radial eigenfunctions. The thus obtained expressions for the coefficients of the eigenfunction expansions can be written in the form

$$\hat{C}_{np_s\mp} = \left(\pm \hat{C}_{nv_z\mp} \coth(\hat{\Gamma}_n L) \mp \frac{\hat{C}_{nv_z\pm}}{\sinh(\hat{\Gamma}_n L)} \right) \times (1 - \alpha \hat{F}_n) \frac{\omega}{\hat{k}_n} \mp \alpha \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \hat{C}_{mv_z\mp} \frac{2\hat{k}_m^2 \hat{F}_m}{a^2(\gamma_n^2 - \gamma_m^2)} \times \left(\coth(\hat{\Gamma}_n L) \frac{\omega}{\hat{k}_n} - \coth(\hat{\Gamma}_m L) \frac{\omega}{\hat{k}_m} \right) \pm \alpha \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \hat{C}_{mv_z\pm} \frac{2\hat{k}_m^2 \hat{F}_m}{a^2(\gamma_n^2 - \gamma_m^2)} \times \left(\frac{1}{\sinh(\hat{\Gamma}_n L)} \frac{\omega}{\hat{k}_n} - \frac{1}{\sinh(\hat{\Gamma}_m L)} \frac{\omega}{\hat{k}_m} \right) + \cdots,$$
$$n = 0,1,2,... . \quad (171)$$

As pointed out earlier, the terms $\coth(\cdot)$ and $1/\sinh(\cdot)$ appearing in Eq. (171) can simply be replaced with 1 and 0, respectively, if n or m correspond to an evanescent mode or a mode having a cut-off frequency close to the driving frequency, which reveals the well-known fact that the pressure perturbations at one end can only be affected by velocity fluctuations at the other end that are associated with the propagating modes $n, m = 0,1,2,...,q$, where q denotes the highest mode such that $\hat{k}_q$ is a positive real of order $O(1)$. Therefore, as far as the propagating modes are concerned, the pressure fluctuations at one end can also be expressed in terms of the components of the axial velocity and the sound pressure at the other side by rearranging the systems (171), substituting the resulting equations mutually, and truncating after the second-order terms. It then follows that

$$\hat{C}_{np_s\mp}=\hat{C}_{np_s\pm}\cosh(\hat{\Gamma}_nL)\pm\hat{C}_{nv_z\pm}\sinh(\hat{\Gamma}_nL)(1-\alpha\hat{F}_n)\frac{\omega}{\hat{k}_n}+\alpha\sum_{\substack{m=0\\m\neq n}}^{q}\hat{C}_{mp_s\pm}\frac{2\hat{k}_m^2\hat{F}_m}{a^2(\gamma_n^2-\gamma_m^2)}(\cosh(\hat{\Gamma}_nL)-\cosh(\hat{\Gamma}_mL))\mp\alpha\sum_{\substack{m=0\\m\neq n}}^{q}\hat{C}_{mv_z\pm}\frac{2\hat{k}_m^2\hat{F}_m}{a^2(\gamma_n^2-\gamma_m^2)}\times\left(\sinh(\hat{\Gamma}_nL)\frac{\omega}{\hat{k}_n}-\sinh(\hat{\Gamma}_mL)\frac{\omega}{\hat{k}_m}\right)+\cdots,$$

$$n=0,1,2,...,q. \tag{172}$$

The expressions from Eqs. (171) and (172) written in dimensional form are presented in Appendix C.

In order to derive the asymptotically correct expressions for the volume flows at both ends, the axial Navier–Stokes equation (17) is averaged over the cross section and expanded with respect to the perturbation parameter α, yielding

$$j\omega\hat{\bar{v}}_{z1}+\frac{\partial\hat{\bar{p}}_1}{\partial z}=0,\quad j\omega\hat{\bar{v}}_{z\alpha}+\frac{2}{d^2}\frac{\partial\hat{V}_{z1}}{\partial s}\bigg|_{s=0}+\frac{\partial\hat{\bar{p}}_\alpha}{\partial z}+\frac{\partial\hat{\bar{p}}_1}{\partial z_1}=0. \tag{173}$$

Since the expansion terms of the pressure fluctuations in the boundary layer $\hat{P}_1$ and $\hat{P}_\alpha$ are independent of the inner coordinate s, the quantities $\hat{\bar{p}}_1$ and $\hat{\bar{p}}_\alpha$ can be conveniently calculated by integrating the inner solutions from Eqs. (124) and (130) over the core region. Hence, using Eqs. (127), (167), and (168) results in

$$\frac{\hat{u}_\mp}{M}=\frac{\hat{\bar{v}}_{z\mp}}{M}=\hat{\bar{v}}_{z1\mp}+\alpha\hat{\bar{v}}_{z\alpha\mp}+\cdots=\hat{C}_{0v_z\mp}-\alpha\frac{1-j}{d}\sqrt{\frac{2}{\omega}}\sum_{n=0}^{\infty}\hat{C}_{nv_z\mp}+\cdots. \tag{174}$$

In the limit as $a\rightarrow\infty$, i.e., $l\ll 1$, the solutions for the fundamental mode from Eq. (171) then reduce to

$$\hat{\bar{p}}_{s\mp}=M\hat{C}_{0p_s\mp}=\left(\pm\hat{u}_\mp\coth(\hat{\Gamma}_0L)\mp\frac{\hat{u}_\pm}{\sinh(\hat{\Gamma}_0L)}\right)\times\left[1+\alpha\frac{1-j}{d\sqrt{2\omega}}\left(1-\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)\right]+\cdots \tag{175}$$

and the matrix system (4), with the parameters $\hat{Z}_c$ and $\hat{\Gamma}$ given by Eq. (77), is recovered.

VI. CONCLUSIONS

In the work presented here the method of multiple scales in combination with a matched asymptotic analysis has been carried out to provide insight into the linear evolution of sound pressure waves in long hard-walled ducts. The calculations primarily proceed from the assumptions that the fluid is a perfect gas and, additionally, that the acoustic boundary layer is thin compared to the characteristic dimension of the cross-sectional area. Furthermore, two different assumptions concerning the diameter to wavelength ratio have been adopted in order to derive the transfer characteristics of a long tube up to the second-order terms.

In the first case, where the wavelengths are assumed to be large compared to the diameter, it is found that the small perturbation parameter ϵ introduced by that ratio plays only a passive role, that is to say, its smallness prevents higher order modes from being excited. As a consequence, the series expansion of the characteristic impedance up to second order and the series expansion of the propagation parameter, which has to be calculated up to the third order, depend only on the (small) scaling parameter α determining the thickness of the boundary layer in terms of the diameter. However, as shown in Secs. IV B and IV C, the parameter ϵ will become important if the effects resulting from radiation at the tube end are to be incorporated into the analysis.

In Sec. V the reduced frequency (or, equivalently, the Helmholtz number) is presupposed to be of order $O(1)$, i.e., $\epsilon=1$. As a consequence, the analysis has to account for the excitation of higher order modes and the interaction of the different modes in the acoustic boundary layer. To demonstrate this, the transfer functions relating the sound pressures at both ends of the duct to the axial velocities have been calculated. The series expansions derived for the propagation parameters extend the results given in the literature with additional terms resulting from shear and bulk viscosity and heat conduction in the core region. In addition, special emphasis has been placed on the asymptotically correct treatment of modes having cut-off frequencies close to the driving frequency.

Obviously, some of the simplifying assumptions made here can be relaxed in order to account for the physical mechanisms neglected so far. Examples include nonlinear effects and the consideration of the asymmetric modes.

APPENDIX A: DUCTS WITH RECTANGULAR CROSS SECTIONS

The derivations of $\hat{Z}$ and $\hat{Y}$ for the case of a rectangular tube, which were elaborated by Stinson[8] (see also Roh *et al.*[28]), result from his general procedure developed for ducts having arbitrary cross-sectional shape. As explicated in Ref. 8, this theory is primarily based on the hypotheses that both the characteristic wavelength λ as well as the inverse of the propagation parameter $\hat{\Gamma}=\sqrt{\hat{Z}\hat{Y}}$ are very large compared to the boundary layer thickness δ, the density perturbations and the sound pressure are of comparable magnitude when scaled by their equilibrium values, and, furthermore, the sound pressure does not vary significantly through the cross section. This set of assumptions then enables the simplification of the basic equations such that, in addition to the case of circular tubes, even in the case of tubes with rectangular cross section, the expressions for the shunt admittance and the series impedance can be given in closed form:

Let $2h$ and $2b$ be the height and the width of the tube with $D=2h$ as the characteristic dimension of the cross section and $b/h=O(1)$. Additionally, the following parameters

$$a_k=\frac{(k+1/2)\pi}{h},\qquad b_n=\frac{(n+1/2)\pi}{b} \tag{A1}$$

are introduced. Then the quantities $\hat{Z}$ and $\hat{Y}$ are constructed from

$$\hat{Z}=\frac{\mu_0 b^2 h^2}{4S\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{a_k^2 b_n^2\left(a_k^2+b_n^2+\frac{j\omega\rho_0}{\mu_0}\right)}},$$
$$\hat{Y}=\frac{j\omega S}{p_0}\left[1-\frac{4(\gamma-1)j\omega\rho_0 C_p}{\gamma\kappa_0 b^2 h^2}\times\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{a_k^2 b_n^2\left(a_k^2+b_n^2+\frac{j\omega\rho_0 C_p}{\kappa_0}\right)}\right], \tag{A2}$$

where $S=4bh$.

Since for every arbitrary $\hat{\xi}$ the relationship

$$\sum_{k=0}^{\infty}\frac{1}{h^2 a_k^2(h^2 a_k^2+\hat{\xi})}=\frac{1}{2\hat{\xi}}\left(1-\frac{\tanh(\sqrt{\hat{\xi}})}{\sqrt{\hat{\xi}}}\right) \tag{A3}$$

holds, expression (A2) can be recast into

$$\hat{Z}=\frac{\mu_0 b^2}{2Sh^2\sum_{n=0}^{\infty}\frac{1}{b_n^2\hat{\xi}_1}\left(1-\frac{\tanh(\sqrt{\hat{\xi}_1})}{\sqrt{\hat{\xi}_1}}\right)},$$
$$\hat{Y}=\frac{j\omega S}{p_0}\left[1-\frac{2(\gamma-1)j\omega\rho_0 C_p h^2}{\gamma\kappa_0 b^2}\times\sum_{n=0}^{\infty}\frac{1}{b_n^2\hat{\xi}_2}\left(1-\frac{\tanh(\sqrt{\hat{\xi}_2})}{\sqrt{\hat{\xi}_2}}\right)\right]. \tag{A4}$$

Here, $\hat{\xi}_1$ and $\hat{\xi}_2$ are abbreviations for

$$\hat{\xi}_1=h^2\left(b_n^2+\frac{j\omega\rho_0}{\mu_0}\right),\qquad \hat{\xi}_2=h^2\left(b_n^2+\frac{j\omega\rho_0 C_p}{\kappa_0}\right). \tag{A5}$$

Due to the infinite sums in Eqs. (A2) and (A4) the expansions in the limits as St→0 (LFL) or St→∞ (HFL), with the Stokes number defined as

$$\mathrm{St}=\sqrt{\frac{\omega\rho_0 h^2}{\mu_0}}, \tag{A6}$$

turn out to be very tedious. However, one could try to derive at least the asymptotically correct expressions for Eq. (A4) in the HFL by means of a multiple scales analysis based on the *low reduced frequency assumptions* stated in Eq. (5) in combination with a matched asymptotic expansion, as outlined in Sec. IV. Such an approach involving separate calculations for the acoustic flow in the core region, the main boundary regions, and, in principle, also in the corner regions of the boundary layer has the advantage that the series expansions of infinite sums can then be avoided. In contrast to the study in Sec. IV, the following analysis confines itself to investigating the *leading* order terms of the volume flow $\hat{u}$ and the sound pressure $\hat{p}_s$ generated by waves propagating over a distance $L\sim\mathrm{St}\,\lambda$ or, equivalently, the *leading* order terms and the *second*-order terms of the quantities $\hat{u}$ and $\hat{p}_s$ generated by waves propagating over a spatial range $L\sim\lambda$ only.

To this end, the nondimensional coordinates

$$x^*=\frac{x}{b},\quad y^*=\frac{y}{h} \tag{A7}$$

for the vertical and the horizontal direction, respectively, and the small scaling parameters

$$\epsilon=a\frac{h}{\lambda}\sim l,\quad \alpha=d\frac{\lambda}{h\sqrt{\mathrm{Re}}}\sim\mathrm{St}^{-1} \tag{A8}$$

are introduced. Omitting the superscripts $*$ denoting nondimensional quantities, the boundary layer coordinates are given by

$$s_{\pm y}=\frac{1\mp y}{\alpha},\quad s_{\pm x}=\frac{1\mp x}{\alpha}. \tag{A9}$$

As mentioned earlier, in this simplified analysis either the changes of the first- and second-order terms of the quantities $\hat{p}_s$ and $\hat{u}$ are to be resolved over a spatial range of the order $O(1)$ or just the leading order effects emerging over distances of the order $O(\alpha^{-1})$ are to be investigated. In both cases two length scales are sufficient, resulting in

$$\frac{\partial}{\partial z}\rightarrow\frac{\partial}{\partial z}+\alpha\frac{\partial}{\partial z_1}. \tag{A10}$$

In order to guarantee the linearity of the problem, the Mach number now has to be assumed to satisfy

$$M\ll\epsilon,\quad M\ll\alpha. \tag{A11}$$

Furthermore, it should be noted that the effects resulting from the viscothermal processes taking place in the very small corner regions of the boundary layer where both $s_{\pm x}$ as well as $s_{\pm y}$ are of order $O(1)$ will not enter the correction terms at second order in α, since their contribution to the volume flow is an order smaller than that of the main part of the boundary layer. Hence, deriving an exact solution of the boundary layer equations for the corner regions is not necessary.

A perturbation analysis for the HFL very similar to that carried out in Sec. IV then shows that the series impedance and the shunt admittance in dimensional form assume the limiting values

$$\hat{Z}=\frac{\omega\rho_0}{S}\left[j+(1+j)\left(1+\frac{h}{b}\right)\sqrt{\frac{\mu_0}{2\omega\rho_0 h^2}}+O(\mathrm{St}^{-2})\right], \tag{A12}$$

$$\hat{Y}=\frac{\omega S}{\gamma p_0}\left[j+(1+j)\left(1+\frac{h}{b}\right)(\gamma-1)\sqrt{\frac{\mu_0}{2\omega\rho_0 h^2}\frac{\kappa_0}{\mu_0 C_p}}+O(\mathrm{St}^{-2})\right].$$

The low reduced frequency assumptions from Eq. (5) together with $\mathrm{St}\sim h/\delta\gg1$ yield the ordering relationship $\lambda\gg h\gg\delta$ and Eq. (A12) implies that in the HFL, the inverse of the dimensional propagation parameter is of the order $O(\lambda)$. Furthermore, as in the case of tubes with circular cross-sectional shape, the sound pressure does not change significantly over the cross section. As a consequence, the basic assumptions adopted by Stinson (see above) are satisfied, which leads to the conclusion that Eq. (A12) represents the asymptotically correct approximations to Eq. (A2) in the limit of large Stokes numbers.

APPENDIX B: DUCTS WITH SLIT-SHAPED AND ARBITRARY CROSS SECTIONS

In the case of a rectangular slit-shaped tube with $D=2h$ as the characteristic dimension such that $h/b\ll1$, the expressions for $\hat{Z}$ and $\hat{Y}$ derived for $b/h=O(1)$ reduce to[8]

$$\hat{Z}=\frac{j\omega\rho_0}{S\left(1-\sqrt{\frac{\mu_0}{j\omega\rho_0}}\frac{\tanh\left(h\sqrt{\frac{j\omega\rho_0}{\mu_0}}\right)}{h}\right)}, \tag{B1}$$

$$\hat{Y}=\frac{j\omega S\left[1+(\gamma-1)\sqrt{\frac{\kappa_0}{j\omega\rho_0 C_p}}\frac{\tanh\left(h\sqrt{\frac{j\omega\rho_0 C_p}{\kappa_0}}\right)}{h}\right]}{\gamma p_0},$$

with $S=4bh$. The result for $\hat{Z}$ is in accordance with the solution already given in the 1975 paper by Backus;[29] however, please note that in the expression for the shunt admittance stated there, a factor of $\sqrt{j}$ is missing. Derivations of $\hat{Z}$ and $\hat{Y}$ can also be found in Ingard[30] (pp. 2/19–2/29).

The LFL can be calculated by means of a power series expansion with respect to the Stokes number from Eq. (A6) and reads

$$\text{LFL:}\quad \hat{Z}=\frac{3\mu_0}{h^2 S}\left(1+j\frac{2\omega\rho_0 h^2}{5\mu_0}+O(\mathrm{St}^4)\right), \tag{B2}$$

$$\hat{Y}=\frac{\omega S}{p_0}\left(j+\frac{\gamma-1}{\gamma}\frac{\omega\rho_0 h^2}{3\mu_0}\frac{\mu_0 C_p}{\kappa_0}+O(\mathrm{St}^4)\right).$$

Furthermore, since for $\mathrm{St}\rightarrow\infty$

$$\tanh(\hat{\xi})\rightarrow1, \tag{B3}$$

expression (B1) can be evaluated to give the HFL in the form

$$\text{HFL:}\quad \hat{Z}=\frac{\omega\rho_0}{S}\left[j+(1+j)\sqrt{\frac{\mu_0}{2\omega\rho_0 h^2}}+O(\mathrm{St}^{-2})\right], \tag{B4}$$

$$\hat{Y}=\frac{\omega S}{\gamma p_0}\left[j+(1+j)(\gamma-1)\sqrt{\frac{\mu_0}{2\omega\rho_0 h^2}\frac{\kappa_0}{\mu_0 C_p}}+O(\mathrm{St}^{-2})\right].$$

As expected, the relationships (A12) as well as Eqs. (B4) and (14) confirm the assumption stated in Morse and Ingard[14] (p. 475) and elsewhere that for $\mathrm{St}\rightarrow\infty$ terms resulting from viscous dissipation ($\hat{Z}$) and heat conduction ($\hat{Y}$) in the boundary layer are always proportional to P/S, with P being the perimeter of the cross-sectional area (however, in contrast to the results for $\hat{Z}$ and $\hat{Y}$ presented here, in Ref. 14 the effects of heat conduction are contained within the resistive part of the series impedance). For an arbitrary cross section the HFL can thus be written as

$$\text{HFL:}\quad \hat{Z}=\frac{\omega\rho_0}{S}\left[j+(1+j)\frac{P}{S}\sqrt{\frac{\mu_0}{2\omega\rho_0}}+\cdots\right], \tag{B5}$$

$$\hat{Y}=\frac{\omega S}{\gamma p_0}\left[j+(1+j)\frac{P}{S}(\gamma-1)\sqrt{\frac{\mu_0}{2\omega\rho_0}\frac{\kappa_0}{\mu_0 C_p}}+\cdots\right].$$

Further (semianalytical) results for the LFL and HFL in the case of rectangular and other cross sections, which were derived by using variational methods, are presented in Cummings.[31]

APPENDIX C: RESULTS IN DIMENSIONAL FORM

Expressions used in the following:

$$S=R^2\pi,\quad \mathrm{St}(\omega)=\sqrt{\frac{\omega\rho_0 R^2}{\mu_0}}\gg 1,$$

$$\mathrm{Pr}=\frac{\mu_0 C_p}{\kappa_0}=O(1), \tag{C1}$$

$$\mathrm{He}(\omega)=\frac{\omega R}{c_0}.$$

Equation (76):

$$\hat{Z}=\frac{\omega\rho_0}{S}\left[j+(1+j)\frac{\sqrt{2}}{\mathrm{St}(\omega)}+\frac{3}{\mathrm{St}(\omega)^2}+O(\mathrm{St}^{-3})\right],$$

$$\hat{Y}=\frac{\omega S}{\gamma p_0}\left[j+(1+j)\frac{(\gamma-1)}{\mathrm{St}(\omega)}\sqrt{\frac{2}{\mathrm{Pr}}}-\frac{(\gamma-1)}{\mathrm{St}(\omega)^2\,\mathrm{Pr}}+O(\mathrm{St}^{-3})\right],$$

$$\hat{Z}_c=\frac{c_0\rho_0}{S}\left[1+\frac{1-j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1-\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)-\frac{j}{2\,\mathrm{St}(\omega)^2}\times\left(2-2\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}-\frac{5\gamma-3\gamma^2-2}{\mathrm{Pr}}\right)+O(\mathrm{St}^{-3})\right], \tag{C2}$$

$$\hat{\Gamma}=\frac{j\omega}{c_0}\left[1+\frac{1-j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)-\frac{j}{\mathrm{St}(\omega)^2}\times\left[1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-\frac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right]+O(\mathrm{St}^{-3})\right].$$

Equation (81):

$$\hat{Z}_t=\frac{c_0\rho_0}{S}\left[\tanh\left(\frac{\omega L}{c_0}\left[j+\frac{1+j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)\right]\right)+\frac{1-j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1-\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)\tanh\left(\frac{\omega L}{c_0}\left[j+\frac{1+j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)\right]\right)\right.$$
$$\left.+\frac{0.8217j\,\mathrm{He}(\omega)+\dfrac{\omega L}{c_0\,\mathrm{St}(\omega)^2}\left[1+\dfrac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-\dfrac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right]}{\cosh\left(\dfrac{\omega L}{c_0}\left[j+\dfrac{1+j}{\sqrt{2}\,\mathrm{St}(\omega)}\left(1+\dfrac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right)\right]\right)^2}+O(\mathrm{St}^{-2})\right], \tag{C3}$$

where $\omega L/c_0=O(\mathrm{St})$ and $\mathrm{He}=O(\mathrm{St}^{-1})$.

Equation (150):

$$\hat{\Gamma}_n=\sqrt{-\hat{k}_n^2\left(1+2\frac{\hat{F}_n}{\mathrm{St}(\omega)}+\frac{2\hat{H}_n+\hat{F}_n^2}{\mathrm{St}(\omega)^2}\right)+O(\mathrm{St}^{-3})},\quad n=0,1,2,..., \tag{C4}$$

where

$$\hat{k}_n=\begin{cases}1/R\sqrt{\mathrm{He}(\omega)^2-\gamma_n^2}, & \mathrm{He}(\omega)>\gamma_n\\ -j/R\sqrt{\gamma_n^2-\mathrm{He}(\omega)^2}, & \mathrm{He}(\omega)<\gamma_n\end{cases},$$

$$\hat{F}_n=\frac{1-j}{\sqrt{2}}\left(1+\frac{\mathrm{He}(\omega)^2}{\hat{k}_n^2R^2}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right),\quad n=0,1,2,..., \tag{C5}$$

$$\hat{H}_0=-j\left[1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left(1-\frac{\gamma}{2\sqrt{\mathrm{Pr}}}\right)\right]-\frac{j\,\mathrm{He}(\omega)^2}{2}\left[\frac{11}{6}+\frac{\eta_0}{\mu_0}+\frac{\gamma-1}{2\sqrt{\mathrm{Pr}}}\left(2+\frac{1+\gamma}{\sqrt{\mathrm{Pr}}}\right)\right],$$

$$\hat{H}_n=-j\left[1+\frac{\mathrm{He}(\omega)^2}{\hat{k}_n^2R^2}\frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\left[1-\frac{1}{2\sqrt{\mathrm{Pr}}}\left(1+\frac{\mathrm{He}(\omega)^2}{\hat{k}_n^2R^2}\times(\gamma-1)\right)\right]\right]-\frac{j\,\mathrm{He}(\omega)^4}{2\hat{k}_n^2R^2}\left(\frac{4}{3}+\frac{\eta_0}{\mu_0}+\frac{\gamma-1}{\mathrm{Pr}}\right),$$
$$n=1,2,...,$$

and, furthermore, $\mathrm{He}=O(1)$.

Equations (171) and (172):

$$
\begin{aligned}
\hat{C}_{np_s\mp} = {} & \left(\pm \hat{C}_{nv_z\mp} \coth(\hat{\Gamma}_n L) \mp \frac{\hat{C}_{nv_z\pm}}{\sinh(\hat{\Gamma}_n L)} \right) \\
& \times \left(1 - \frac{\hat{F}_n}{\mathrm{St}(\omega)} \right) \frac{\mathrm{He}(\omega)}{\hat{k}_n R} \mp \sum_{\substack{m=0 \\ m\neq n}}^{\infty} \hat{C}_{mv_z\mp} \frac{2\hat{k}_m^2 R^2 \hat{F}_m}{\mathrm{St}(\omega)(\gamma_n^2 - \gamma_m^2)} \\
& \times \left(\coth(\hat{\Gamma}_n L) \frac{\mathrm{He}(\omega)}{\hat{k}_n R} - \coth(\hat{\Gamma}_m L) \frac{\mathrm{He}(\omega)}{\hat{k}_m R} \right) \\
& \pm \sum_{\substack{m=0 \\ m\neq n}}^{\infty} \hat{C}_{mv_z\pm} \frac{2\hat{k}_m^2 R^2 \hat{F}_m}{\mathrm{St}(\omega)(\gamma_n^2 - \gamma_m^2)} \left(\frac{1}{\sinh(\hat{\Gamma}_n L)} \frac{\mathrm{He}(\omega)}{\hat{k}_n R} \right. \\
& \left. - \frac{1}{\sinh(\hat{\Gamma}_m L)} \frac{\mathrm{He}(\omega)}{\hat{k}_m R} \right) + \cdots, \\
& n = 0,1,2,..., \qquad \text{(C6)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_{np_s\mp} = {} & \hat{C}_{np_s\pm} \cosh(\hat{\Gamma}_n L) \pm \hat{C}_{nv_z\pm} \sinh(\hat{\Gamma}_n L) \\
& \times \left(1 - \frac{\hat{F}_n}{\mathrm{St}(\omega)} \right) \frac{\mathrm{He}(\omega)}{\hat{k}_n R} \\
& + \sum_{\substack{m=0 \\ m\neq n}}^{q} \hat{C}_{mp_s\pm} \frac{2\hat{k}_m^2 R^2 \hat{F}_m}{\mathrm{St}(\omega)(\gamma_n^2 - \gamma_m^2)} (\cosh(\hat{\Gamma}_n L) \\
& - \cosh(\hat{\Gamma}_m L)) \mp \sum_{\substack{m=0 \\ m\neq n}}^{q} \hat{C}_{mv_z\pm} \frac{2\hat{k}_m^2 R^2 \hat{F}_m}{\mathrm{St}(\omega)(\gamma_n^2 - \gamma_m^2)} \\
& \times \left(\sinh(\hat{\Gamma}_n L) \frac{\mathrm{He}(\omega)}{\hat{k}_n R} - \sinh(\hat{\Gamma}_m L) \frac{\mathrm{He}(\omega)}{\hat{k}_m R} \right) + \cdots, \\
& n = 0,1,2,...,q, \qquad \text{(C7)}
\end{aligned}
$$

where $\hat{\Gamma}_n$, $\hat{k}_n$, and $\hat{F}_n$ are defined as in Eqs. (C4) and (C5), respectively, $\mathrm{He} = O(1)$ and $\omega L/c_0 = O(\mathrm{St})$. Then the sound pressure, the axial velocity, the average sound pressure, and the volume flow at the ends of the tube are given by

$$
\begin{aligned}
\hat{p}_{s\mp} &= c_0 \rho_0 \sum_{n=0}^{\infty} \hat{C}_{np_s\mp} \frac{\mathrm{J}_0(\gamma_n r/R)}{\mathrm{J}_0(\gamma_n)}, \\
\hat{v}_{z\mp} &= \sum_{n=0}^{\infty} \hat{C}_{nv_z\mp} \frac{\mathrm{J}_0(\gamma_n r/R)}{\mathrm{J}_0(\gamma_n)}, \\
\hat{\bar{p}}_{s\mp} &= c_0 \rho_0 C_{0p_s\mp}, \qquad \text{(C8)} \\
\hat{u}_{\mp} &= \hat{C}_{0v_z\mp} - (1-j) \frac{\sqrt{2}}{\mathrm{St}(\omega)} \sum_{n=0}^{\infty} \hat{C}_{nv_z\mp} + \cdots .
\end{aligned}
$$

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