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On the Cancellativity of AG-Groupoids

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Abstract. An AG-groupoid is a non-associative groupoid in general in which the identity $(ab)c = (cb)a$ holds. In this paper we study some structural properties of AG-groupoids with respect to the cancellativity. We prove that cancellative and non-cancellative elements of an AG-groupoid S partition S and the two classes are AG-subgroupoids of S if S has left identity e. Cancellativity and invertibility coincide in a finite AG-groupoid S with left identity e. For a finite AG-groupoid S with left identity e having at least one non-cancellative element, the set of non-cancellative elements form a maximal ideal. We also prove that for an AG-groupoid S, the conditions (i) S is left cancellative (ii) S is right cancellative (iii) S is cancellative, are equivalent.

Keywords: Cancellative AG-groupoid, Non-cancellative AG-groupoid, AG-group

Introduction: An AG-groupoid is a groupoid satisfying the left invertive law: $(ab)c = (cb)a$. In literature this structure has also been called by different names by different authors like left almost semigroup or shortly LA-semigroup in [3], left invertive groupoid in [2], while right modular groupoid in [1, line 35]. Cancellativity plays an important role in groups and loops and many results therein occur due to cancellativity. AG-groupoids are not necessarily cancellative but all or some of the elements of an AG-groupoids can be cancellative and hence can enjoy some special properties that a general AG-groupoid cannot possess. In the present paper we see this aspect of AG-groupoids. We remove in Theorem 1 a wrong impression existing in the literature that a right cancellative AG-groupoid is not cancellative in general. We prove that cancellative and non-cancellative elements of an AG-groupoid S partition S and the two classes are AG-subgroupoids of S if S has left identity e (Theorem 6). Cancellativity and invertibility coincide in a finite AG-groupoid S with left identity e (Theorem 7). For a finite AG-groupoid S with left identity e having at least one non-cancellative element, the set of non-cancellative elements form a maximal ideal (Corollary 6). The direct product $S_1 \times S_2$ of two cancellative AG-groupoids S_1 and S_2 is cancellative (Theorem 8).

Preliminaries: An AG-groupoid $(S, \cdot)$ always satisfies the medial law: $(ab)(cd) = (ac)(bd)$ [1, Lemma 1.1 (i)] while an AG-groupoid $(S, \cdot)$ with left identity e satisfies paramedial law: $(ab)(cd) = (db)(ca)$ [1, Lemma 1.2 (ii)]. An AG-groupoid $(G, \cdot)$ is called an AG-group or a left almost group (LA-group), if there exists left identity $e \in G$ (that is, $ea = a$ for all $a \in G$), for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$ [5]. A non-empty subset H of an AG-groupoid S is called an AG-subgroupoid if $ab \in H$ for all $a, b \in H$. A subset I of an AG-groupoid S is called left ideal (right ideal) if $SI \subseteq I$ ($IS \subseteq I$). A subset I of an AG-groupoid S is called ideal if it is both left and right ideal. An element a of an AG-groupoid S is called left cancellative if $ax = ay \Rightarrow x = y$ for all $x, y \in S$. Similarly an element a of an AG-groupoid S is called right cancellative if $xa = ya \Rightarrow x = y$ for all $x, y \in S$. An element a of an AG-groupoid S is called cancellative if it is both left and right cancellative. An AG-groupoid S is called left cancellative (right cancellative, cancellative) if every element of S is left cancellative (right cancellative, cancellative).

1. Cancellativity of AG-groupoids

In [6, Theorem 2.6], this has been proved that every left cancellative AG-groupoid S is cancellative while in [7] this has been said through a reference to [6] and without giving a counterexample that the converse is not true in general but true only if S has left identity. We prove that this is incorrect. The converse is also true in general and does not require the existence of left identity. That is, every right cancellative AG-groupoid S is also left cancellative. So we begin by the following theorem.

Theorem 1. *The following conditions are equivalent for an AG-groupoid S*
(i) S is left cancellative
(ii) S is right cancellative
(iii) S is cancellative.

Proof. $(i) \Rightarrow (ii)$ Let S be left cancellative. Let a be an arbitrary element of S and let $xa = ya$ for all $x, y \in S$. Suppose k is any element of S. Then $(ka)x = (xa)k = (ya)k = (ka)y$ which by left cancellativity implies that $x = y$. Thus S is right cancellative $(ii) \Rightarrow (iii)$ Let S be right cancellative and let $ax = ay$ for all $x, y \in S$. Suppose k is any element of S. Then $((xk)a)a = (aa)(xk) = (ax)(ak) = (ay)(ak) = (aa)(yk) = ((yk)a)a$ which by repeated use of right cancellativity implies that $x = y$. Thus S is left cancellative. $(iii) \Rightarrow (i)$ Obvious. □

Corollary 1. *The following conditions are equivalent for an AG-groupoid S.*
(i) S is left quasigroup
(ii) S is right quasigroup
(iii) S is quasigroup.

The previous discussion was about the whole left cancellativity or right cancellativity of the AG-groupoid. In what follows we focus on the cancellativity of an individual element of an AG-groupoid when the whole AG-groupoid is not necessarily left cancellative or right cancellative. But first observe that an AG-groupoid can have all, some or none of its elements as cancellative. For example all the elements of the following AG-groupoid are cancellative.

Example 1. *A cancellative AG-groupoid with left identity* 0*:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

The following AG-groupoid has two cancellative elements which are the left identity 0 and 3.

Example 2. *An AG-groupoid with left identity* 0*:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	4	2	2	4	4
2	2	2	2	2	2
3	3	1	2	0	4
4	1	1	2	1	2

The following AG-groupoid has four cancellative elements and one non-cancellative.

Example 3. *An AG-groupoid with* $\{0, 1, 2, 3\}$ *as cancellative elements and only* $\{4\}$ *as non-cancellative.*

$\cdot$	0	1	2	3	4
0	0	2	3	1	4
1	3	1	0	2	4
2	1	3	2	0	4
3	2	0	1	3	4
4	4	4	4	4	4

The following AG-groupoid has no cancellative element.

Example 4. *An AG-groupoid without left identity and without any cancellative element:*

$\cdot$	0	1	2	3	4
0	2	2	2	2	2
1	2	0	2	2	4
2	2	2	2	2	2
3	0	0	2	4	4
4	2	2	2	2	2

Theorem 2. *Every right cancellative element of an AG-groupoid S is (left) cancellative.*

Proof. Let S be an AG-groupoid. Let a be an arbitrary right cancellative element of S. Suppose $ax = ay$ for all $x, y \in S$. Then $((xa)a)a = (aa)(xa) = (ax)(aa) = (ay)(aa) = (aa)(ya) = ((ya)a)a$ which by repeated use of right cancellativity implies $x = y$. Thus a is left cancellative. Hence every right cancellative element of S is left cancellative. □

Next we need the following theorem from [6].

Theorem 3. *In an AG-groupoid S with left identity e, $ab = cd \Rightarrow ba = dc$ for all $a, b, c, d \in S$.*

Theorem 4. *Let S be an AG-groupoid with left identity e. Then every left cancellative element is also right cancellative.*

Proof. Let a be an arbitrary left cancellative element of S. Suppose $xa = ya$ for all $x, y \in S$. Then by Theorem 3, we have $ax = ay$. Which by left cancellativity implies $x = y$. Thus a is right cancellative. Hence every left cancellative element of S is right cancellative. □

Remark 1. From Theorem 3, this is clear that if the AG-groupoid S has left identity e then e will always be cancellative because e by its definition is left cancellative.

Next we prove that the set of cancellative elements and the set of non-cancellative elements of an AG-groupoid S form a partition of S.

Theorem 5. *Let S be an AG-groupoid and let $a, b, c \in S$. Define on S the relation $\sim$ as*

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are both cancellative or non-cancellative.}$$

Then $\sim$ is an equivalence relation.

Proof. Since a, a are both cancellative or non-cancellative. Therefore $a \sim a$. Thus $\sim$ is reflexive. Suppose now $a \sim b$ then a and b are both cancellative or non-cancellative. Which implies that b and a are both cancellative or non-cancellative. Which implies that $b \sim a$. Thus $\sim$ is symmetric. Next suppose that $a \sim b$ and $b \sim c$ then a and b are both cancellative or non-cancellative and b and c are both cancellative or non-cancellative. Which implies that a and c are both cancellative or non-cancellative and so $a \sim c$. Thus $\sim$ is transitive. Hence $\sim$ is an equivalence relation. □

Corollary 2. *Cancellative and non-cancellative elements of an AG-groupoid S partition S.*

Next we prove that the two classes will be AG-subgroupoids of S if S has left identity.

Lemma 1. *The set of cancellative elements of an AG-groupoid S with left identity e is an AG-subgroupoid of S.*

Proof. Let $H = \{a \in S\colon\ a \text{ is cancellative}\}$. Clearly H is non-empty as $e \in H$ by Remark 1. Now let $a_1, a_2 \in H$ and let $a = a_1a_2$. We show that a is cancellative. Suppose $ax = ay$ for all $x, y \in S$ then $(xa_2)a_1 = (a_1a_2)x = ax = ay = (a_1a_2)y = (ya_2)a_1$ which by cancellativity of a_1 and a_2 implies $x = y$. Thus a is left cancellative and hence cancellative by Theorem 4. This implies $a \in H$. Hence H is an AG-subgroupoid of S. □

In Example 1, $H = S$, in Example 2, $H = \{0, 3\}$, in Example 3, $H = \{0, 1, 2, 3\}$ that can be easily seen as an AG-subgroupoid of S.

Remark 2. Computer search shows that the smallest non-associative AG-groupoid is of order 3. But how many non-isomorphic AG-groupoids of order 3 or higher order exist no one has counted yet, neither computationally nor algebraically. So we suggest this as a future problem.

Example 5. *A non-associative AG-groupoid of order* 3 :

$\cdot$	0	1	2
0	2	2	2
1	0	2	2
2	2	2	2

Lemma 2. *Every cancellative element of an AG-subgroupoid S with left identity e is the product of two cancellative elements of S.*

Proof. By Remark 1, e is cancellative and $e = ee$. Let a be an arbitrary non-trivial cancellative element of S. Let $a = a_1a_2$. We show that a_1, a_2 are both cancellative. Suppose $xa_2 = ya_2$ for all $x, y \in S$. Then $ax = (a_1a_2)x = (xa_2)a_1 = (ya_2)a_1 = (a_1a_2)y = ay$ which implies that $x = y$ since a is cancellative. Thus a_2 is right cancellative and hence cancellative by Theorem 2. Now suppose $a_1x = a_1y$. Then

$$\begin{aligned} a(xa_2) &= (a_1a_2)(xa_2) \\ &= (a_1x)(a_2a_2) = (a_1y)(a_2a_2) \\ &= (a_1a_2)(ya_2) = a(ya_2), \end{aligned}$$

which by cancellativity of a and a_2 implies that $x = y$. Thus a_1 is left cancellative and hence cancellative by Theorem 4. □

Note that Lemma 2 does not hold for a non-cancellative element. A non-cancellative element can be expressed as the product of a cancellative element and a non-cancellative element or as the product of two non-cancellative elements as in Example 2, $2 = 0 \cdot 2$ or $2 = 2 \cdot 2$ where 0 is cancellative and 2 is non-cancellative elements of S.

Corollary 3. *In an AG-subgroupoid S with left identity e the product of two non-cancellative elements or one cancellative and one non-cancellative is always non-cancellative, that is, if a or b is non-cancellative then ab is non-cancellative.*

Lemma 3. *The set of all non-cancellative elements of an AG-groupoid S with left identity e is either empty or an AG-subgroupoid of S.*

Proof. Let $K = \{a \in S\colon\ a$ is non-cancellative$\}$. Clearly K is empty if S is cancellative. Suppose S is not cancellative. Then $e \notin K$ since e is always cancellative. Now let $a_1, a_2 \in K$ and let $a = a_1a_2$. We show that a is non-cancellative,that is, $a \in K$. Suppose $a \notin K$ then a is cancellative and consequently a_1, a_2 are cancellative by Lemma 2 and thus $a_1, a_2 \notin K$, which is a contradiction. Therefore $a \in K$. Thus K is an AG-subgroupoid of S. □

In Example 1, $K = \phi$, in Example 2, $K = \{1, 2, 4\}$, in Example 3, $K = \{4\}$ and in Example 4, $K = S$ that can easily be seen as an AG-subgroupoid of S in the non-empty case.

Thus from Corollary 2, Lemma 1 and Lemma 3, it follows that:

Theorem 6. *Cancellative and non-cancellative elements of an AG-groupoid S with left identity e partition S into two AG-subgroupoids of S.*

As an application of our theory to the ideal theory of AG-groupoids, we have the following:

Corollary 4. *A proper (left,right) ideal of an AG-groupoid S with left identity e cannot be a subset of H.*

Proof. Since the product of the non-cancellative elements of S with the elements of H cannot be contained in H by Lemma 2. So a proper (left,right) ideal of S cannot be a subset of H. □

Next we show that none of the elements of the proper (left,right) ideal can lie in H at least in finite case.

Corollary 5. *A proper (left,right) ideal of a finite AG-groupoid S with left identity e is a subset of K.*

Proof. Let $S = \{s_1, s_2, ..s_n\}$ and let I be a proper left ideal of S. Let $a \in I$ be such that $a \in H$. Then since a is cancellative therefore $s_1a, s_2a, ..s_na \in I$ are all distinct. This implies that I and S have the same number of elements which is a contradiction. So $a \notin H$. Therefore $I \subseteq K$. Other cases are similar. □

Corollary 6. *For an AG-groupoid S with left identity e having at least one non-cancellative element, K is always a maximal ideal.*

Proof. It follows from Corollary 3 and 5. □

Next we prove that cancellativity and invertibility coincide in a finite AG-groupoid S with left identity e.

Lemma 4. *Every invertible element of an AG-groupoid with left identity e is cancellative.*

Proof. Suppose a is an invertible element then there exists $a^{-1} \in S$ such that $aa^{-1} = a^{-1}a = e$. Suppose $xa = ya$ then $x = ex = (a^{-1}a)x = (xa)a^{-1} = (ya)a^{-1} = (a^{-1}a)y = ey = y$. Thus a is right cancellative and hence cancellative. □

Corollary 7. *An AG-group G is cancellative* [5].

Lemma 5. *Every cancellative element of a finite AG-groupoid S with left identity e is invertible.*

Proof. Let $S = \{s_1, s_2, ..s_n\}$ and let a be an arbitrary cancellative element of S. Then clearly $as_1, as_2, ..as_n$ are all distinct. Since S is finite therefore there must exists a positive integer $i \in \{1, 2, ..n\}$ such that $as_i = e$ but then $s_ia = e$ follows by Theorem 3. Hence a is invertible. □

Now the following theorem follows.

Theorem 7. *Let S be a finite AG-groupoid with left identity e then a is invertible $\Leftrightarrow$ a is cancellative.*

In Example 1, all elements are cancellative as well as invertible, in Example 2, 0 and 3 are cancellative as well as invertible elements, in Example 3, the elements $0, 1, 2, 3$ are both cancellative and invertible and in Example 4, there is no cancellative and no invertible element.

Remark 3. If the AG-groupoid S does not have left identity, then Theorem 7 does not hold as the following example shows:

Example 6. *A cancellative AG-groupoid without left identity:*

·	0	1	2	3	4
0	2	1	0	4	3
1	0	4	3	2	1
2	3	2	1	0	4
3	1	0	4	3	2
4	4	3	2	1	0

Corollary 8. *A finite cancellative AG-groupoid with left identity e is an AG-group.*

The AG-groupoid in Example 1 is an AG-group.

Theorem 8. *The direct product $S_1 \times S_2$ of two cancellative AG-groupoids S_1 and S_2 is cancellative.*

Proof. Suppose the AG-groupoids S_1 and S_2 are cancellative. Then $S_1 \times S_2$ is also an AG-groupoid by [4, page 462, line 9]. Now let $a, x_1, y_1 \in S_1$ and $b, x_2, y_2 \in S_2$ then consider $(a, b)(x_1, y_1) = (a, b)(x_2, y_2)$, which implies

$(ax_1, by_1) = (ax_2, by_2)$, from this we get that $ax_1 = ax_2, by_1 = by_2$, which by cancellativity of S_1 and S_2 implies that $x_1 = x_2$ and $y_1 = y_2$. Thus $S_1 \times S_2$ is cancellative. □

Finally let us apply the concept of cancellativity in the proof of Theorem 3 which has been proved in [6] without this. The proof becomes a bit easier.

Proof. $(ba)e = (ea)b = ab = cd = (ec)d = (dc)e \Rightarrow ba = dc$, since e is cancellative. □

Conclusion: In this paper we have proved that a right cancellative element of an AG-groupoid S (not necessarily having left identity) is left cancellative. This has also been shown that a left cancellative element of an AG-groupoid S is right cancellative if either S is cancellative or if S has left identity. But if the whole S is not cancellative or S does not have a left identity then we are unable to prove that a left cancellative element is also right cancellative. Thus we had to take an AG-groupoid S with left identity e. This requires further investigation to remove this condition. If this could be proved then most of our results will hold in general. So we suggest it as an open problem:

Problem 1. *Prove or disprove that every left cancellative element is also right cancellative of an AG-groupoid S without left identity.*

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