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ON THE CANONICAL RING OF ALGEBRAIC VARIETIES

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1. Introduction

Let V be a compact algebraic manifold (over the complex numbers) of dimension d, with canonical divisor K_V. We define the *canonical ring* $R(V) = R(V, K_V) = \bigoplus_{n\geq 0} H^0(V, nK_V)$. One of the most vital unsolved problems in the classification theory of algebraic varieties is the following:

FUNDAMENTAL QUESTION: Is $R(V)$ finitely generated as an algebra over the complex numbers?

This question is of particular importance when V is of general type. In this case, if $R(V)$ is finitely generated, then we have a corresponding canonical model $\mathrm{Proj}(R(V))$. We may then attempt to classify these canonical models up to isomorphism.

In dimension 2, we know that the canonical ring is finitely generated (see [19]), and this enables one to produce a moduli space for surfaces of general type (see [12]). In dimension 3, work has been done on threefolds of general type assuming that the canonical ring is finitely generated (see [14] and [18]). For dimensions 3 and higher however, the main problem is still completely open.

In this paper, we start by broadening the question to the ring $R(V, D) = \bigoplus_{n\geq 0} H^0(V, nD)$, for an arbitrary divisor D, aiming to generalize some results of Zariski for surfaces [19]. In Section 1, we prove a useful geometric criterion for $R(V, D)$ to be finitely generated, valid in arbitrary dimensions. In Section 2, we extend a

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result of Zariski on arithmetically effective divisors to arbitrary dimensions.

In Section 3, we prove some elementary results on the canonical ring, which results will be useful in later sections. In particular, we investigate the fixed components of $|nK_V|$, when V is a threefold of general type.

In Section 4, we give two examples. Firstly a normal Gorenstein threefold V, with $\kappa(V, K_V) = 3$, but $R(V, K_V)$ not finitely generated. This example will also show that the arithmetic plurigenera (see [16] and [17]) of normal Gorenstein varieties are not invariant under algebraic deformations. Secondly, we give an example of a non-algebraic complex manifold of dimension 4 for which the canonical ring is not finitely generated. This explains why I have asked the Fundamental Question in the algebraic case only.

In both examples, the fixed locus of the n-canonical system $|nK_V|$ is rather simple; for n sufficiently large, $|nK_V|$ has no codimension > 1 fixed subvarieties, and the fixed components are normal (in fact smooth) and disjoint. In Section 5, we show however that if these conditions hold for a smooth threefold of general type, then the canonical ring must be finitely generated.

In the final section, various ways of progressing are discussed, considering in particular the important case when K_V is arithmetically effective. Here $R(V)$ is finitely generated if and only if $|nK_V|$ is without fixed points for some n. The ideas developed are then applied to the case of a non-normal fixed component on a smooth threefold of general type (cf. Section 5).

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1. The ring $R(V, D)$

Let V be a compact, complex algebraic manifold of dimension d. Let D be a divisor on V, and $\mathbb{N}(V, D) = \{n \geq 0 \text{ such that } h^0(V, nD) > 0\}$. Here of course, $h^0(V, nD)$ denotes the dimension of the vector space $H^0(V, \mathcal{O}_V(nD))$. In general h^i will denote the dimension of H^i, and in many places we shall use a divisor to denote also its corresponding invertible sheaf. For all $n \in \mathbb{N}(V, D)$, we can define a rational map $\phi_{nD}: V \to X_n \subset \mathbb{P}^N$, where $N = h^0(nD) - 1$ (see [7]). This map is a morphism outside the indeterminate locus of $|nD|$. Throughout this paper, I shall use the term *indeterminate locus* to mean the codimension > 1 fixed subvarieties of $|nD|$. The term *fixed locus* will mean all the fixed points of $|nD|$, i.e. indeterminate locus plus fixed components. The maps ϕ_{nD} enable us to define the number $\kappa(V, D)$, the *D-dimension* of V (see [7]).

Following Zariski [19], we define a ring $R(V, D) = \bigoplus_{n \geq 0} H^0(V, nD)$. If $R(V, D)$ is finitely generated, we can define a variety $X = \text{Proj}\ R(V, D)$ which has the property that for some $m \geq 1$, the varieties X_{nm} are isomorphic to X for $n \geq 1$. This ring is of particular interest when $\kappa(V, D) = d$, in which case the X_n are birationally equivalent to X for all n sufficiently large.

As however is well known, the ring $R(V, D)$ is not always finitely generated, and in the case of surfaces the question was investigated by Zariski in [19]. In this section we aim, for arbitrary dimension d, to translate the condition that $R(V, D)$ is finitely generated into a geometric condition concerning the fixed locus of $|nD|$ for n large. This is done in (1.2); first however we need the following result.

PROPOSITION 1.1: *Suppose that $|A|$ is a linear system on a compact algebraic manifold V, with ϕ_A birational. If F is an effective divisor on V which does not meet the fixed locus of $|A|$, and such that the map ϕ_A contracts down (to lower dimensions) every component of F, then for all $m \geq 1$, $|mA + F| = |mA| + F$.*

PROOF: We may clearly assume that $|A|$ has no indeterminate locus, and therefore that $\phi = \phi_A$ is a morphism. By Chow's theorem, we may also assume that V is projective. Suppose then that $V \subset \mathbb{P}^M$ say, and let $N = h^0(A) - 1$, so that $\phi(V) = X \subset \mathbb{P}^N$.

Let $|mA + F| = |A_m| + F'$, where the fixed components of $|A_m|$ do not meet F. Let F_s denote that part of F consisting of components E whose image under ϕ has dimension $\geq s$. Similarly we define F'_s; clearly $F'_s \leq F_s$. We show by induction that in fact $F'_s = F_s$ for all $s \leq d - 1$. For $s = 0$, this then gives the required result.

By assumption, the claim is trivially true for $s = d - 1$. Suppose then that $0 \leq s \leq d - 2$, and that the claim is true for all $s' > s$; we prove then that it is also true for s.

Let $\{E_i\}$ denote those components of F whose images $\bar{E}_i$ under ϕ have dimension s; i.e. the components of $F_s - F_{s+1}$. Take a section (say W) of X by a general linear space in $\mathbb{P}^N$ of codimension s. Thus we may assume:

(a) For any divisor E on V which is contracted under ϕ, with

$\bar{E} = \phi(E)$, the intersection $W \cap \bar{E}$ has the 'correct' dimension. In particular, W intersects each $\bar{E}_i$ in a finite number of points, and if a divisor E on V is contracted to a subvariety of dimension $<s$, then $W \cap \bar{E}$ is empty.

(b) The inverse image $U = \phi^{-1}(W)$ is smooth, and its intersection with the indeterminate locus of $|A_m|$ has at least codimension 2 in U. Note that U is merely the intersection of s general elements of $|A|$.

Thus U is a smooth variety of dimension $(d-s)$, which contains a collection $\{Y_i\}$ of divisors that are blown to points under the morphism $\phi \mid U : U \to W$. The $\{Y_i\}$ here are of course the intersections of the $\{E_i\}$ with U.

Now take a section of V by a general linear space H of codimension $(d-2-s)$ in $\mathbb{P}^M$, and let $S = H \cap U$. We then have the following: S is a smooth surface, on which there is a collection $\{C_i\}$ of curves that are blown down to points on $\bar{S} = \phi(S)$ under the morphism $\phi \mid S$. The curves $\{C_i\}$ are the intersections of $\{E_i\}$ with S. Also, for all components E of $F - F_s$, we have that $S \cap E$ is empty. Finally we may assume that the intersection of S with the indeterminate locus of $|A_m|$ has at least codimension 2 on S.

On V we have $F - F' \sim A_m - mA$ (linearly equivalent); restricting to S, we obtain an effective divisor $B = (F - F') \mid S$ which is supported on the curves $\{C_i\}$ (using the induction hypothesis that $F_{s+1} = F'_{s+1}$). We wish to show that $B = 0$.

If this were not so, then by a standard result (see [11]) we deduce that $B^2 < 0$. However, the general element of $|A|$ does not meet any of the curves $\{C_i\}$, (by assumption the fixed components are disjoint from $\{C_i\}$, and the mobile part does not meet $\{C_i\}$ by construction). By construction however, the complete linear system on S containing $A_m \mid S$ only has fixed components disjoint from $\{C_i\}$, and therefore $B^2 = B \cdot (A_m - mA) = B \cdot A_m \geq 0$. This then gives the required contradiction. □

THEOREM 1.2: *Let D be a divisor on a compact algebraic manifold V, where $\kappa(V, D) = d = \dim(V)$. The ring $R(V, D)$ is finitely generated if and only if there exists a positive integer n and a smooth modification $f : \tilde{V} \to V$ such that the map ϕ_{nf^*D} on $\tilde{V}$ is a birational morphism which contracts every fixed component of $|nf^*D|$.*

PROOF: For the 'only if' part, we essentially use an argument from [14]. If $R(V, D)$ is finitely generated, then for some $n \geq 1$ the ring $R(V, D)^{(n)} = \bigoplus_{m \geq 0} H^0(V, mnD)$ is generated by its degree one terms.

Now resolve the indeterminate locus of $|nD|$, obtaining $f: \tilde{V} \to V$ and $\tilde{D} = f^*D$. Thus $R(V, D) = R(\tilde{V}, \tilde{D})$ and $\phi = \phi_{n\tilde{D}}: \tilde{V} \to X$ is a morphism.

Put $|m\tilde{D}| = |A_m| + F_m$ for $m \geq 1$, and let E be any component of F_n. Denote by $r(m, E)$ the multiplicity of E in F_m; by construction $r(mn, E) = m.r(n, E)$ for all $m \geq 1$. Hence $h^0(\tilde{V}, mA_n) = h^0(\tilde{V}, mA_n + E)$ for all $m \geq 1$.

Now $A_n = \phi^*H$ for some hyperplane section H of X, and so for m sufficiently large, we have $h^1(X, \phi_* mA_n) = 0$. But $R^1\phi_*\mathcal{O}_{\tilde{V}}(mA_n) = R^1\phi_*\mathcal{O}_{\tilde{V}} \otimes \mathcal{O}_X(mH)$, and so using the Leray spectral sequence we deduce that $h^1(mA_n)$ behaves at worst like m^{d-2}. From the exact sequence

$$0 \to H^0(\tilde{V}, mA_n) \to H^0(\tilde{V}, mA_n + E) \to H^0(E, \mathcal{O}_E(mA_n + E)) \to H^1(\tilde{V}, mA_n)$$

we deduce that $h^0(E, \mathcal{O}_E(mA_n + E))$ behaves at worst like m^{d-2}, and thus E is contracted by ϕ.

We now prove the converse; let $|nf^*D| = |A_n| + F_n$, where $|A_n|$ is now without fixed points. By assumption ϕ_{nf^*D} contracts every component of F_n, and so we deduce from (1.1) that for all $m \geq 1$, $|mnf^*D| = |mA_n| + mF_n$. Since $|A_n|$ is without fixed points, we deduce (using [19], Theorem 6.5) that $R(\tilde{V}, f^*D)^{(n)}$ is finitely generated, and hence so also is $R(\tilde{V}, f^*D)$. □

2. Arithmetically effective divisors

Let V be a compact algebraic variety of dimension d, and D a Cartier divisor on V. Recall that D is called *arithmetically effective* (a.e.) if $D \cdot C \geq 0$ for all integral curves C on V. Arithmetically effective divisors on surfaces play a central role in Zariski's paper [19]. In this section we extend some of these ideas to higher dimensions.

DEFINITION 2.1: For a divisor D on V, we say that the fixed locus of $|nD|$ is *numerically bounded* if for every birational morphism $f: \tilde{V} \to V$, the fixed components of $|nf^*D|$ have bounded multiplicities as $n \to \infty$.

Thus the fixed locus is bounded if and only if it is numerically bounded and we can simultaneously resolve the indeterminate loci of $|nD|$ for all n sufficiently large. If the fixed locus of $|nD|$ is numerically bounded, then clearly $|nD|$ is *almost base point free* in the sense of Goodman [3].

The following result now gives a suitable generalization to higher dimensions of Theorem 10.1 of [19].

THEOREM 2.2: *If the fixed locus of $|nD|$ is numerically bounded, then D is* a.e. *Furthermore, if $\kappa(V, D) = \dim(V)$, then the converse holds.*

PROOF: Note first that if $f: \tilde{V} \to V$ is a birational morphism, then f^*D is a.e. if and only if D is a.e. (see [9], page 303). Furthermore, the multiplicity of a fixed component of $|nD|$ will be the same as the multiplicity of the corresponding fixed component in $|nf^*D|$. Thus by Chow's theorem and the results of Hironaka [6], we may assume that V is smooth and projective. The first part is now straightforward. In fact the stronger result holds, that if $|nD|$ is almost base point free, then D is a.e. (see [3], page 178).

Let us now prove the converse under the assumption that $\kappa(V, D) = \dim(V)$. From the above remarks and the results of Hironaka [6], it is sufficient to prove that if D is a.e. on a smooth projective variety V, and the fixed components of $|nD|$ are smooth, then these components have bounded multiplicities as $n \to \infty$.

Choose a very ample divisor H on V such that the divisors $H - K_V - E$ are ample for all fixed components E. Since D is a.e., it is also pseudo-ample ([5], Chapter I, Section 6). Thus for all $r \geq 1$, $m \geq 0$, $rH - K_V - E + mD$ is ample (using the Nakai criterion).

Now for a given E, let $V_i = E \cap H_1 \cap \cdots \cap H_{d-i-1}$ for $i = 1, 2, \ldots, d-1$, where the H_j are general elements of $|H|$. By Bertini's theorem the varieties V_i are smooth, and clearly $K_{V_i} = (K_V + E + (d-i-1)H) \mid V_i$. We prove by induction on i that $h^0(V_i, \mathcal{O}_{V_i}(mD + dH)) > 0$ for all $m \geq 0$. This is clear for $i = 1$; we assume therefore that $1 < i \leq d-1$ and that the claim is true for $(i-1)$. Since $iH - K_V - E + mD$ is ample, we deduce that $h^1(V_i, \mathcal{O}_{V_i}(mD + (d-1)H)) = 0$. We therefore have an exact sequence

$$0 \to H^0(\mathcal{O}_{V_i}(mD + (d-1)H)) \to H^0(\mathcal{O}_{V_i}(mD + dH)) \\ \to H^0(\mathcal{O}_{V_{i-1}}(mD + dH)) \to 0.$$

Hence the claim is true for all $i \leq d-1$; in particular we have that $h^0(\mathcal{O}_E(mD + dH)) > 0$ for all $m \geq 0$.

Now from the exact sequence

$$0 \to H^0(mD + dH - E) \to H^0(mD + dH) \to H^0(\mathcal{O}_E(mD + dH)) \to 0$$

we deduce that $|mD + dH|$ does not have E as a fixed component for all $m \gg 0$. As this is true for all fixed components E, we deduce that the linear system $|mD + dH|$ has no fixed components for $m \gg 0$.

Since $\kappa(V, D) = \dim(V)$, a standard argument now shows that there exists an integer a such that the divisor aD-dH is effective. Thus the fixed components of $|(m + a)D|$ have bounded multiplicities as $m \to \infty$, which is the result that we wanted to prove. □

REMARKS 2.3:

1. If $\kappa(V, D) = \dim(V)$, then the fixed locus of $|nD|$ is numerically bounded if and only if $|nD|$ is almost base point free, which is if and only if D is a.e. (by (2.2) and [3], page 178).
2. For D a.e. with $\kappa(V, D) = \dim(V)$, we have that $R(V, D)$ is finitely generated if and only if $|nD|$ has no fixed points for some positive integer n.

3. The canonical ring

Now let V be a compact algebraic manifold of dimension d, with canonical divisor K_V. We wish to know whether the canonical ring $R(V) = R(V, K_V)$ is finitely generated. The question will be of particular importance when V is of general type, i.e. $\kappa(V) = \dim(V)$. In this case, the following is an easy corollary of (1.2).

PROPOSITION 3.1: *The canonical ring $R(V)$ is finitely generated if and only if there is a positive integer n and a birational (smooth) model $\tilde{V}$ on which $\phi_{nK_{\tilde{V}}}$ is a birational morphism contracting every fixed component of $|nK_{\tilde{V}}|$.* □

Suppose now that E is a fixed component of $|nK_V|$ for all n sufficiently large, and let $r(n, E)$ denote the multiplicity of E in $|nK_V|$ (essentially independent of the birational model V). Clearly E is Gorenstein, and we let K_E denote a dualizing divisor. We put $|nK_V| = |A_n| + F_n$, where $|A_n|$ denotes the mobile part.

LEMMA 3.2: *If ϕ_{nK} is a birational morphism and $r(n, E) \leq r(n + 1, E)$, then $h^0(E, \mathcal{O}_E(K_E + A_n)) = 0$. In particular $h^0(\mathcal{O}_E(K_E)) = 0$.*

PROOF: By the Grauert-Riemenschneider form of Kodaira vanishing [4], we know that $h^1(K_V + A_n) = 0$. Thus we have an exact sequence:

$$0 \to H^0(K_V + A_n) \to H^0(K_V + A_n + E) \to H^0(E, \mathcal{O}_E(K_E + A_n)) \to 0$$

Since $r(n, E) \leq r(n+1, E)$, the first map is an isomorphism, and so the result follows. $\square$

COROLLARY 3.3: *If V is of general type, E a fixed component of $|nK_V|$ for all n sufficiently large, then $p_g(E) = 0$.*

PROOF: By resolving the singularities of E in V (see [6]), we may assume that E is smooth. Choose n such ϕ_{nK_V} is birational, and that $r(n, E) \leq r(n+1, E)$. Resolve the base locus of $|nK_V|$ and use (3.2). $\square$

Conjecture (Ueno). If E is such a fixed component, then $\kappa(E) = -\infty$.

We now look at the case when V is a threefold of general type. Let E be a fixed component of $|mK_V|$ for all m sufficiently large. Motivated by (3.1), we consider the case when ϕ_{nK_V} is a birational morphism which does not contract E. As in (3.3) we could assume that E is smooth - here however we shall only assume that it is normal (this will be needed in Section 5).

LEMMA 3.4: *If $r(n, E) \leq r(n+1, E)$, then E contains only rational double point singularities, and $\kappa(E) = -\infty$. Putting $B = A_n \mid E$ and $\chi(\mathcal{O}_E) = 1 - q$, we have that the genus of B, $p(B) = q$. Except in the case when E is rational and its image under ϕ_{A_n} is $\mathbb{P}^2$, we have $h^0(\mathcal{O}_E(K_E + 2B)) > 0$, and so the linear system $|K_V + 2A_n + E|$ on V does not have E as a fixed component. Thus $r(2n+1, E) < 2.r(n, E)$. In the exceptional case, $h^0(\mathcal{O}_E(K_E + 3B)) > 0$, and so the linear system $|K_V + 3A_n + E|$ does not have E as a fixed component.*

PROOF: Using an argument involving Kodaira vanishing and the Leray spectral sequence, it is easily verified that $h^1(\mathcal{O}_E(K_E + B)) = 0$. Since E is Gorenstein, the Riemann–Roch theorem holds for Cartier divisors on E (proved in a purely formal manner). Thus we deduce (using (3.2)) that

$$0 = \chi(\mathcal{O}_E) + 1/2(K_E + B) \cdot B \tag{$*$}$$

Letting $\chi(\mathcal{O}_E) = 1 - q(E)$ by (3.2), we deduce that the genus of B, $p(B) = q$. Since E is normal, the general element of $|B|$ is a non-singular curve.

As commented above however, we may resolve the singularities of E in V, and apply the same argument to the corresponding smooth component $\tilde{E}$. With the obvious notation, it is clear that $p(B)=p(\tilde{B})$; hence the above argument yields $\chi(\mathcal{O}_E)=\chi(\mathcal{O}_{\tilde{E}})$, in other words that the singularities of E are rational. Since E is also Gorenstein, these singularities must be rational double points.

If $\kappa(E)\geq 0$, then $\chi(\mathcal{O}_E)=\chi(\mathcal{O}_{\tilde{E}})\geq 0$. Also, for some positive n we have that $h^0(\mathcal{O}_E(nK_E))=h^0(\mathcal{O}_{\tilde{E}}(nK_{\tilde{E}}))>0$. Thus we have $K_E\cdot B\geq 0$. Since $B^2>0$, this gives an immediate contradiction from (*).

For the next part, we note by Riemann–Roch that $h^0(\mathcal{O}_E(K_E+2B))=\chi(\mathcal{O}_E)+(K_E+2B)\cdot B=q-1+B^2$. Hence if E is not rational (i.e. $q>0$), or if the image of E under ϕ_{A_n} is not $\mathbb{P}^2$ (i.e. $B^2>1$), we have that $h^0(\mathcal{O}_E(K_E+2B))>0$. The next statement follows from the exact sequence

$$0\to H^0(K_V+2A_n)\to H^0(K_V+2A_n+E)\to H^0(E,\mathcal{O}_E(K_E+2B))\to 0.$$

The final sentence of the Lemma follows similarly. □

For a surface of general type, it is well known that the canonical ring is finitely generated. This may be shown in a number of ways. Originally it was proved in the Mumford Appendix to [19], which proof relies heavily on the results of Artin concerning the contractability of curves on a surface. For a more cohomological approach, the result follows as an immediate corollary to the work of Bombieri [2], which paper uses crucially the idea of connectedness of divisors on surfaces. As a third alternative, one can produce a mechanical (but rather long) cohomological proof using the methods of this section. This last approach is close in spirit to that of Artin (in that one considers the possible configurations of curves in the fixed locus), but seems to rely less on special facts concerning curves on surfaces.

In all three approaches however, the first step is to note that there exists a smooth model on which the canonical divisor is a.e. Such a model does not of course exist in higher dimensions (see [14] and [15]). We shall return to this point in Section 6.

4. Two examples

At this stage it seems appropriate to give a couple of examples. Both examples are motivated by the Zariski examples in [19] of divisors D on a surface S for which $R(S,D)$ is not finitely generated.

EXAMPLE 4.1: Let V be a normal Gorenstein variety with canonical divisor K_V. Is $R(V, K_V)$ finitely generated in general? We give an example here of a normal Gorenstein threefold V with $\kappa(V, K_V) = 3$ for which the above ring is not finitely generated.

Let $S \subset \mathbb{P}^3$ be a cone on a smooth elliptic curve C, vertex P say, and let H denote a hyperplane in $\mathbb{P}^3$. Thus $K_{\mathbb{P}^3} \sim -S - H$ (linearly equivalent).

If we now blow up P, say $f : \tilde{\mathbb{P}} \to \mathbb{P}^3$, we obtain a smooth surface $\tilde{S}$, the proper transform of S. Letting E denote the exceptional divisor in $\tilde{\mathbb{P}}$, and H denote f^*H, we have $K_{\tilde{\mathbb{P}}} \sim -\tilde{S} - H - E$.

Now $\tilde{S}$ is a ruled surface over the curve C, with structure map $\pi : \tilde{S} \to C$ say. The zero section C_0 has self-intersection -3. If we let $\tilde{H}$ represent the proper transform of a general hyperplane through P, i.e. $\tilde{H} + E \sim H$, we see that $\tilde{H} \sim \pi^* h$ for some divisor h of degree 3 on C. Choose points $P_1, P_2, \ldots, P_{12}$ on C such that $(4h - \Sigma_i P_i)$ is not a torsion element of $\mathrm{Pic}^\circ(C) \simeq C$. Letting $L_i = \pi^* P_i$, we now blow $\tilde{\mathbb{P}}$ up in $L_1, \ldots, L_{12}$, say $g : W \to \tilde{\mathbb{P}}$. We let S', H' and E' denote the proper transforms of $\tilde{S}, \tilde{H}$ and E respectively, and H denote the pullback of f^*H by g (i.e. $H \sim H' + E'$). Therefore $K_W \sim -S' - H - E' \sim -S' - H' - 2E'$.

As S' is isomorphic to $\tilde{S}$, we shall use π to denote also the structure map of S'. We denote by E_i the exceptional divisor on W corresponding to L_i on $\tilde{\mathbb{P}}$. We then have

$$3H' + 3E' \sim g^* f^* S \sim S' + 3E' + \sum_i E_i.$$

Thus $(S' + H') \mid S' \sim \pi^*(4h - \Sigma_i P_i)$. Therefore, it is clear that S' is a fixed component of the linear systems $|n(S' + H')|$ for $n \geq 1$. I claim however that for all $n \geq 1$, the linear system $|(n-1)S' + nH'|$ has no fixed points in W. This is trivial for $n = 1$, so we assume that $n \geq 2$.

Clearly, the linear system on S' containing $((n-1)S' + nH') \mid S'$ has no fixed points. From the exact sequence

$$0 \to \mathcal{O}_W((n-2)S' + nH') \to \mathcal{O}_W((n-1)S' + nH') \to \mathcal{O}_{S'}((n-1)S' + nH') \to 0$$

it is sufficient to prove that $h^1(\mathcal{O}_W((n-2)S' + nH')) = 0$.

We prove by induction that $h^1(\mathcal{O}_W((r-2)S' + nH')) = 0$ for all $1 \leq r \leq n$. For $r = 1$, it is clear that $h^1(\mathcal{O}_W(-S' + nH')) = h^2(\mathcal{O}_W(-2H - (n-1)H')) = 0$ by the Grauert–Riemenschneider vanishing theorem [4]. Suppose then that $h^1(\mathcal{O}_W((r-2)S' + nH')) = 0$ for some r, $1 \leq r <$

n. We have an exact sequence

$$0 \to \mathcal{O}_W((r-2)S' + nH') \to \mathcal{O}_W((r-1)S' + nH') \to \mathcal{O}_{S'}((r-1)S' + nH') \to 0.$$

In order to complete the induction step, it is sufficient therefore to prove that $h^1(\mathcal{O}_{S'}((r-1)S' + nH')) = 0$.

However, $([r-1]S' + nH') \mid S' \sim \pi^* D$, where D is a divisor on C of degree $3(n+1-r) \geq 6$. An elementary agrument using the Leray spectral sequence now shows that $h^1(\mathcal{O}_{S'}(\pi^* D)) = 0$. Hence the induction is complete, and $|(n-1)S' + nH'|$ has no fixed points.

Since $-K_W \sim S' + H' + 2E'$, we see that W is an example of a smooth variety for which $R(W, -K_W)$ is not finitely generated. This is because S' will be a fixed component of multiplicity one in $|-nK_W|$ for all $n \geq 1$ (see [19], page 562).

Now consider the linear systems $|n(S'+H)|$ on W. As in the above case S' will be a fixed component for all $n \geq 1$. It is then clear that the only fixed points of $|(n-1)S' + nH|$ lie on the curve $E' \cdot S' = C_0$. Therefore, if we take the double cover of W ramified over the general element of $|4(S'+H)|$, say $\alpha : V \to W$, V will have non-canonical singularities (see [14]) only over the curve C_0. Since V is clearly Gorenstein, we deduce from Serre's criterion that it must be normal.

On V, $K_V \sim \alpha^*(K_W + 2S' + 2H) \sim \alpha^*(S' + H')$. Now for all $n \geq 1$, we have

$$\begin{aligned}\alpha_* \mathcal{O}_V(nK_V) &= \alpha_* \mathcal{O}_V \otimes \mathcal{O}_W(nS' + nH') \\ &= \mathcal{O}_W(nS' + nH') \oplus \mathcal{O}_W((n-2)S' \\ &\quad + (n-2)H' - 2E').\end{aligned}$$

Since $H^0(W, \mathcal{O}_W(mS' + mH' - 2E')) = 0$ for all m, we deduce that $H^0(V, nK_V) \simeq H^0(W, \mathcal{O}_W(nS' + nH'))$. Thus $R(V, K_V) \simeq R(W, S' + H')$ is not finitely generated, and the divisor $\alpha^* S'$ is fixed in $|nK_V|$ for all $n \geq 1$.

REMARK 4.2: By varying the points P_i, we may obtain a family of normal Gorenstein threefolds, $\theta : \mathcal{V} \to C$ say, for which $R(\mathcal{V}_t, K_{\mathcal{V}_t})$ is finitely generated on a countable dense subset of C (corresponding to the torsion points), and is not finitely generated elsewhere. In particular therefore, we see that the arithmetic plurigenera $\bar{P}_n(\mathcal{V}_t) = h^0(\mathcal{V}_t, nK_{\mathcal{V}_t})$ cannot be constant in the family. This contrasts with known results for normal Gorenstein surfaces (see [16] and [17], and also cf. [8]).

EXAMPLE 4.3: We may of course extend the Fundamental Question in another direction, and ask whether for an abitrary compact complex manifold M, the canonical ring $R(M)$ is finitely generated. When the algebraic dimension $a(M) = \dim(M)$, this reduces to the algebraic case; when $a(M) < \dim(M)$, the answer in general is No, as is seen from the following example.

Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and H a line. Blow up 12 general points $P_1, \dots, P_{12}$ on C, and one point P not on C. Let $f: \tilde{\mathbb{P}} \to \mathbb{P}^2$ denote this modification, with exceptional curves $E_1, \dots, E_{12}$ and E. Let C' be the proper transform of $C, H' = f^*H - E$ and take $D = C' + H'$.

It is clear that C' must be a fixed component of $|nD|$ for all $n \geq 1$. The following claim is left as an exercise for the reader (cf. (4.1)).

CLAIM: For all $n \geq 1$, $nH' + (n-1)C'$ is very ample.
Thus $R(\tilde{\mathbb{P}}, D)$ is not finitely generated (see [19], page 562).

Now let S be the double cover of $\tilde{\mathbb{P}}$ ramified over the general element of $|6C' + 6H'|$. From the above claim, the general element of this linear system only has ordinary double point singularities, and so S only has rational double point singularities. Let $\tilde{S}$ denote the minimal desingularization of S; we therefore have a morphism $\alpha: \tilde{S} \to \tilde{\mathbb{P}}$. An elementary calculation confirms that $K_{\tilde{S}} \sim \alpha^*(2C' + 3H' + E)$.

We now use a construction due to Atiyah [1], obtaining an analytic fibre space of complex tori over $\tilde{\mathbb{P}}$. Let L denote the line bundle over $\tilde{\mathbb{P}}$ corresponding to the divisor H'. Take sections $s_1, s_2 \in H^0(\tilde{\mathbb{P}}, L)$ such that s_1 and s_2 are never simultaneously zero.

Now consider the quaternion matrices in $M_2(\mathbb{C})$:

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \text{ and } I_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Note that for $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, $\det(\Sigma a_i I_i) = 0$ if and only if $(a_1, a_2, a_3, a_4) = (0, 0, 0, 0)$.

We can therefore consider the analytic family of tori

$$\begin{array}{c} W = L \oplus L/\Lambda \\ \big\downarrow{\scriptstyle \pi} \\ \tilde{\mathbb{P}} \end{array}$$

where

$$\Lambda = \left(I_1\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, I_2\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, I_3\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, I_4\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\right).$$

In other words for $z \in \tilde{\mathbb{P}}$, the fibre $W_z = \mathbb{C} \oplus \mathbb{C}/\Lambda(z)$, where $\Lambda(z)$ is the lattice $\left\{I_1\begin{pmatrix} s_1(z) \\ s_2(z)\end{pmatrix}, I_2\begin{pmatrix} s_1(z) \\ s_2(z)\end{pmatrix}, I_3\begin{pmatrix} s_1(z) \\ s_2(z)\end{pmatrix}, I_4\begin{pmatrix} s_1(z) \\ s_2(z)\end{pmatrix}\right\}$ in $\mathbb{C}^2$. It is easily checked that the relative canonical sheaf $\omega_{W/\tilde{\mathbb{P}}} = \pi^*\mathcal{O}_{\tilde{\mathbb{P}}}(-2H')$.

Now take the fibre product of $\alpha : \tilde{S} \to \tilde{\mathbb{P}}$ and $\pi : W \to \tilde{\mathbb{P}}$, obtaining $g : V \to \tilde{\mathbb{P}}$ say. In other words, V is obtained by taking the Atiyah fibration over $\tilde{S}$ corresponding to the divisor α^*H'.

V is then a complex (non-Kähler) 4-dimensional analytic manifold with $a(V) = 2 = \kappa(V)$, and such that

$$\omega_V = g^*\mathcal{O}_{\tilde{\mathbb{P}}}(2C' + 3H' + E - 2H') = g^*\mathcal{O}_{\tilde{\mathbb{P}}}(2C' + H' + E).$$

It is now easily checked that $R(V, K_V)$ is not finitely generated.

By varying the points P_i again, we note that the canonical ring behaves badly under smooth deformations, this being for a similar reason as exhibited in (4.2).

5. Threefolds of general type

In both the examples of Section 4, we have that $|nK_V|$ is without indeterminate locus for n sufficiently large (in fact for $n \geq 1$), and that the fixed components are normal (in fact smooth) and disjoint. In this section we show that if these conditions hold for a threefold of general type, then the canonical ring is in fact finitely generated.

So let V denote a compact smooth algebraic threefold of general type. Suppose that for n sufficiently large, E is a fixed component of $|nK_V|$ which meets neither the indeterminate locus nor any other fixed component, i.e. E is isolated in the fixed locus.

PROPOSITION 5.1: *If E is normal, then there exists N with ϕ_{NK_V} a birational map which contracts E.*

PROOF: Choose n sufficiently large such that ϕ_{mK} is birational for $m \geq n$, and such that $r(n, E) \leq r(n+1, E)$. Resolve the indeterminate locus of $|nK_V|$; we may therefore assume that $|nK_V| = |A_n| + F_n$, where $|A_n|$ is without fixed points and E is an isolated component of F_n. If E is contracted by ϕ_{nK}, there is nothing to prove; we assume therefore that it is not contracted.

(a) Letting $B = A_n \mid E$, we consider first the case when $B^2 = 1$. Thus the image of E under ϕ_{nK} is $\mathbb{P}^2$, and B corresponds to a line H on $\mathbb{P}^2$.

Using (3.4), we deduce that the map on E corresponding to the linear system $|K_E + 3B|$ is a morphism which contracts E to a point. Noting that $nK_E \sim B + (n+r)E \mid E$, where $r = r(n, E)$, it is clear that the divisor $E \mid E$ on E corresponds on $\mathbb{P}^2$ to $-aH$ for some $a > 0$. Recall from (3.4) that the linear system $|K_V + 3A_n + E|$ does not contain E as a fixed component. Letting $|(3n+1)K_V| = |A_{3n+1}| + F_{3n+1}$ (where $|A_{3n+1}|$ is the mobile part), we deduce that $A_{3n+1} = K_V + 3A_n + E$ (modulo components E' of F_n disjoint from E). It therefore follows that E is contracted to a point by ϕ_{NK_V}, where $N = 3n+1$.

(b) By (3.4), we may therefore assume that $h^0(\mathcal{O}_E(K_E + 2B)) > 0$. Let $|(2n+1)K_V| = |A_{2n+1}| + F_{2n+1}$, where by assumption $|A_{2n+1}|$ has no fixed points on E. If E is not isomorphic to $\mathbb{P}^2$, we shall see that E is contracted by ϕ_{NK_V}, where $N = 2n+1$.

By (3.4), $|K_V + 2A_n + E|$ does not have E as a fixed component; thus we deduce that for some non-negative integer s, $A_{2n+1} = K_V + 2A_n + (s+1)E$ (modulo components E' of F_n disjoint from E). It therefore suffices to prove that the morphism on E corresponding to the divisor $A_{2n+1}|E = (K_E + 2B + sE)|E$ must then contract E.

Note that $n(K_E + 2B) = (2n+1)B + (n+r)E \mid E$, where $r = r(n, E)$, and that $nA_{2n+1}|E = (2n+1)B + (n+r+sn)E|E$. Hence

$$(r+n+sn)n(K_E + 2B) = (r+n)A_{2n+1} \mid E + ns(2n+1)B,$$

and so the corresponding linear system is without fixed points. If E is contracted by the morphism corresponding to $(r+n+sn)n(K_E + 2B)$, then it is also contracted by the morphism corresponding to $(r+n)A_{2n+1} \mid E$. Since $|A_{2n+1}|$ has no fixed points on E, this implies that E is contracted by the morphism corresponding to $A_{2n+1} \mid E$, which is the result that we are trying to prove.

Since $|(r+n+ns)n(K_E + 2B)|$ is without fixed points, it suffices to prove that $(K_E + 2B)^2 \leq 0$. However, an elementary calculation shows that $(K_E + 2B)^2 = 4(K_E + B) \cdot B + K_E^2 = -8\chi(\mathcal{O}_E) + K_E^2$ (see (3.4)). By (3.4), we know that E only has rational double point singularities; thus, if E is not isomorphic to $\mathbb{P}^2$, then the classification of ruled surfaces yields the required result.

(c) Finally therefore, we consider the case when $h^0(\mathcal{O}_E(K_E + 2B)) > 0$ and E is isomorphic to $\mathbb{P}^2$. By (3.2), we deduce that $B \sim 2H$, where H denotes a line on $\mathbb{P}^2$. We also have $E \mid E \sim -aH$ for some $a > 0$, and that the linear system $|K_V + 2A_n + E|$ does not contain E as a fixed component. Noting that $K_E + 2B \sim H$ on E, we therefore deduce that either E is contracted to a point by the morphism corresponding to $|A_{2n+1}|$, or $A_{2n+1} = K_V + 2A_n + E$ (modulo components E' of F_n disjoint from E).

In the first case, E is contracted by ϕ_{NK_V}, where $N = 2n + 1$. In the second case, we have that $A_{2n+1} \mid E \sim H$, and we may therefore apply the argument of part (a) to deduce that E is contracted by ϕ_{NK_V}, where $N = 3(2n+1)+1$. $\square$

THEOREM 5.2: *Let V be a threefold of general type, and suppose that for n sufficiently large, $|nK_V|$ has no indeterminate locus and the fixed components are normal and disjoint. Then the canonical ring $R(V)$ is finitely generated.*

PROOF: Let the fixed components be $E_1, \ldots, E_s$, and put $|nK_V| = |A_n| + F_n$, where $|A_n|$ is without fixed points for all n sufficiently large. By (5.1), for each $1 \leq i \leq s$, there exists an integer N_i such that the morphism corresponding to $|A_{N_i}|$ is birational and contracts E_i. Therefore, by (1.1), we deduce that $r(mN_i, E_i) = m.r(N_i, E_i)$ for all $m \geq 1$.

Let N_0 be an integer for which the fixed components of $|N_0K_V|$ are precisely $E_1, \ldots, E_s$. Now put $N = N_0N_1N_2 \ldots N_s$; we therefore have that $F_{mN} = mF_N$ for all $m \geq 1$. Since $|A_N|$ is without fixed points, we deduce (using [19], Theorem 6.5) that the canonical ring is finitely generated. $\square$

REMARK 5.3: Let V be a normal Gorenstein threefold with $\kappa(V, K_V) = 3$. Suppose that for n sufficiently large, $|nK_V|$ has no indeterminate locus and the fixed components are disjoint and normal. If the fixed components are also Cartier divisors on V, then the above proof works essentially without modification to show that $R(V, K_V)$ is finitely generated. In our example (4.1) however, the fixed component T on V (corresponding to S' on W) is not a Cartier divisor, although $2T = \alpha^*(S')$ clearly is.

I should add on the practical side, that the main use of (5.2) is to rule out immediately many of the possible counter-examples that one would otherwise consider.

6. Arithmetically effective canonical divisors

Turning to the proof that the canonical ring is finitely generated in dimension 2, we see that it is vital that we can demonstrate the existence of a smooth model on which the canonical divisor is a.e. Unfortunately, in dimensions 3 and higher, such a smooth model does not always exist (see [14] and [15] for examples). However, it is reasonable to ask about what may be said if such a model does exist. This is all the more reasonable in dimension 3, in view of recent results of Mori concerning the case when the canonical divisor is not a.e. (see [10]).

THEOREM 6.1: (*Mori*). *Suppose that V is a smooth projective threefold of general type on which K_V is not* a.e. *Then there exists an irreducible divisor D on V such that V is the blow-up of some projective variety W, $\psi: V \to W$ say, with centre the subvariety $\psi(D)$. Furthermore, one of the following five cases holds*:

1. *$\psi(D)$ is a non-singular curve and W is non-singular.*
2. *$Q = \psi(D)$ is a non-singular point of W.*
3. *$Q = \psi(D)$ is an ordinary double point of W; $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_D(D) \simeq p_1^*\mathcal{O}_{\mathbb{P}}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}}(-1)$, the p_i denoting the projections.*
4. *$Q = \psi(D)$ is a double point of W; D isomorphic to an irreducible reduced singular quadric surfaces in $\mathbb{P}^3$, $\mathcal{O}_D(D) \simeq \mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}}(-1)$.*
5. *$Q = \psi(D)$ is a quadruple point of W, locally isomorphic to a cone on the Veronese surface* (cf. [14] *and* [15]). *Here D is isomorphic to $\mathbb{P}^2$, and $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbb{P}}(-2)$.*

Recall that for a surface S of general type, we have the following equivalent conditions:

i. K_S is a.e.
ii. $|nK_S|$ has no fixed points for n sufficiently large.
iii. S is minimal.
iv. $h^1(nK_S) = 0$ for all $n > 1$.

Let us extend this to threefolds.

THEOREM 6.2: *Let V be a smooth projective threefold of general type. The following conditions are then equivalent*:

a. K_V is a.e.
b. The fixed locus of $|nK_V|$ is numerically bounded.
c. V is relatively minimal, and $h^2(nK_V) = 0$ for all $n > 1$.
If the canonical ring is finitely generated, then we have a further equivalent condition:
d. $|nK_V|$ has no fixed points for n sufficiently large.

PROOF: By (2.2), (a) and (b) are equivalent. If K_V is a.e., then we deduce from the Ramanujam form of Kodaira vanishing [13], that $h^2(nK_V) = h^1(-(n-1)K_V) = 0$ for all $n > 1$.

Suppose now that V is not relatively minimal. We therefore have a birational morphism $f: V \to \bar{V}$ onto a smooth threefold $\bar{V}$, with $K_V = f^*K_{\bar{V}} + D$ for some positive divisor D on V. Since $P_n(V) = P_n(\bar{V})$ for all $n \geq 1$, we deduce that nD is fixed in $|nK_V|$ for all $n \geq 1$. Therefore, condition (b) implies that V is relatively minimal, and hence also condition (c).

Suppose now that condition (c) holds. If K_V is not a.e., then we may use (6.1). Since V is relatively minimal by assumption, we deduce that there exists an irreducible divisor D on V of type (3), (4) or (5). An easy argument using the Leray spectral sequence, shows that in all these cases, $h^2(nK_V) > 0$ for n sufficiently large.

Finally, if $R(V)$ is finitely generated, then a non-empty numerically bounded fixed locus cannot occur. The equivalence of (b) and (d) is then clear. □

REMARK (UENO): Given any variety V of dimension d and a very ample divisor H on V such that $K_V + H$ is ample, we may blow up the base locus of a Lefschetz pencil containing H, say $f: \tilde{V} \to V$. Using the Leray spectral sequence, it is then easy to see that $h^i(nK_{\tilde{V}}) = 0$ for all $i > 1$ and $n > 1$. Thus in condition (c) of (6.2), the stipulation that V is relatively minimal is essential.

COMMENT: Note that if K_V is numerically positive on a threefold V with $\kappa(V) > 0$ (i.e. $K_V \cdot C > 0$ for every integral curve C on V), then K_V is in fact ample (see Proposition 2.3 of [18]), and thus $R(V)$ is obviously finitely generated.

As to the next step, it seems that the following question should be asked.

QUESTION: Let V be a smooth threefold of general type on which K_V is a.e., and such that $|nK_V|$ has no indeterminate locus for n sufficiently large. Then is the canonical ring finitely generated?

Since the fixed components would have bounded multiplicities if they existed, there are some rather strong numerical conditions implied in this case. As in (5.2) however, one of the major problems is that of non-normal fixed components.

Finally, let us apply the above ideas to the case of non-normal isolated fixed components, which case was excluded from consideration in (5.1) and (5.6). Let V denote a smooth threefold of general type, and suppose that E is a non-normal fixed component of $|nK_V|$, which for sufficiently large n meets neither the indeterminate locus nor any other fixed component. We shall also assume here that $K_V \cdot C \geq 0$ for all curves C on E. This last condition is almost certainly implied by the others, using methods similar to those say of [10]. In particular, the condition follows as an immediate corollary to (6.1) in the case when E is the only fixed component. Note that if $K_V \cdot C < 0$ for some curve C on E, then C can only move in an algebraic family within E, since otherwise we would have another fixed component E' meeting E.

THEOREM 6.3: *With the notation as above, E has bounded multiplicity in $|nK_V|$, and $K_V \mid E$ is numerically equivalent to zero. Furthermore, the desingularization of E must be a rational surface.*

PROOF: Since $K_V \cdot C \geq 0$ for all curves C on E, it is an easy exercise from (2.2) to show that E has bounded multiplicity in $|nK_V|$. From (1.1), we deduce that for all n sufficiently large, ϕ_{nK} does not contract E.

By (3.2) we know that $h^0(\mathcal{O}_E(K_E + A_n)) = 0$ for infinitely many n. For such n, we deduce formally from Riemann–Roch that

$$-h^1(\mathcal{O}_E(-A_n)) = \chi(\mathcal{O}_E) + A_n \cdot (K_E + A_n \mid E)/2.$$

Letting $r = r(n, E)$, we have that $A_n|E = (nK_V - rE)|E$. Thus

$$\begin{aligned} -h^1(\mathcal{O}_E(-A_n)) = \chi(\mathcal{O}_E) + n(n+1)K_V^2 \cdot E/2 \\ -(2nr - n + r)K_V \cdot E^2/2 + r(r-1)E^3/2. \end{aligned}$$

Since $K_V \mid E$ is a.e., we know that $K_V^2 \cdot E \geq 0$. Since the above equation holds for infinitely many n, and $r(n, E)$ is bounded, we see that $K_V^2 \cdot E = 0$ and $K_V \cdot E^2 \geq 0$.

However, we know that $(nK_V - rE)^2 \cdot E > 0$ for infinitely many n. Thus we have $K_V \cdot E^2 \leq 0$, and hence that $K_V \cdot E^2 = 0$.

Putting $B = A_n|E = (nK_V - rE)|E$, we see that $B \cdot (K \mid E) = 0$. Also we have $B^2 > 0$ and $(K \mid E)^2 = 0$. Hence, using the Hodge index theorem (on a resolution of E), we deduce that $K \mid E$ is numerically equivalent to zero.

For n sufficiently large, we know that E is isolated in the fixed locus of $|nK_V|$, and that ϕ_{nK_V} is a birational map which does not contract E. Choose such an n with $r(n, E) \leq r(n+1, E)$.

Now resolve the singularities of E in V by the standard procedure, say $f : \tilde{V} \to V$, with $\tilde{E}$ the smooth surface on $\tilde{V}$ corresponding to E on V. Let $g = f \mid \tilde{E}$, and $|\tilde{A}_n|$ denote the mobile part of $|nK_{\tilde{V}}|$.

From (3.4) we know that $\tilde{E}$ is (birationally) a ruled surface of genus q, and that putting $\tilde{B} = \tilde{A}_n \mid E$, we have $p(\tilde{B}) = q$. We need to show that $q = 0$. Since $K_V \mid E \equiv 0$ (numerical equivalence) on E, we have $K_E \equiv E \mid E \equiv -(1/r)B$, and so on $\tilde{E}$, $g^*K_E \equiv -(1/r)\tilde{B}$.

For the case $q > 1$, we obtain an immediate contradiction. By Hurwitz's theorem, $\tilde{B}$ is a single section of the birational ruling, i.e. $\tilde{B} \cdot L = 1$ for a general fibre L. Thus from the above equation for g^*K_E, we deduce that $r = 1$. Therefore $p(B) = 1 + 1/2(K_E + B) \cdot B = 1$, and hence $p(\tilde{B}) \leq 1$. Thus $q(\tilde{E}) \leq 1$, which is the required contradiction.

Let us now suppose therefore that $q(\tilde{E}) = 1$, and thus $p(\tilde{B}) = 1$. The standard desingularization procedure yields that $K_{\tilde{E}} = g^*K_E - \Delta$ for some effective divisor Δ on $\tilde{E}$. Since $K_{\tilde{E}} \cdot L = -2$ for the general fibre L of the birational ruling, and $g^*K_E \equiv -(1/r)\tilde{B}$, we deduce that $\Delta \cdot L = 0$ or 1.

(a) The easy case is when $\Delta \cdot L = 0$, i.e. Δ is concentrated in fibres. Consider $h: \tilde{E} \to \bar{E}$, where $\bar{E}$ is any minimal model of $\tilde{E}$. Since $rK_{\tilde{E}} \equiv -\tilde{B} - r\Delta$, we deduce that $rK_{\bar{E}} \equiv -\bar{B} - r\bar{\Delta}$, where $\bar{B}$ (respectively $\bar{\Delta}$) is the image under h of $\tilde{B}$ (respectively Δ). Thus $r^2K_{\bar{E}}^2 = (\bar{B} + r\bar{\Delta})^2 > 0$, which is impossible unless $q(\tilde{E}) = 0$.

(b) We are therefore left with the case when $\Delta \cdot L = 1$ for the general fibre L. Thus Δ contains a (multiplicity one) curve C_0 which is a section of the ruling. Hence $p(C_0) = 1$. Therefore $0 = (K_{\tilde{E}} + C_0) \cdot C_0 = -(1/r)\tilde{B} \cdot C_0 - (\Delta - C_0) \cdot C_0$. Thus $\tilde{B} \cdot C_0 = 0$ and $(\Delta - C_0) \cdot C_0 = 0$, i.e. C_0 is isolated in Δ, and does not meet $\tilde{B}$.

Now suppose C_1 is a component of a fibre (not a whole fibre) meeting C_0. Therefore C_1 is rational, $C_1^2 < 0$ and C_1 does not appear in Δ. Thus

$$-2 = (K_{\tilde{E}} + C_1) \cdot C_1 = -(1/r)\tilde{B} \cdot C_1 + C_1^2 - 1 - (\Delta - C_0) \cdot C_1.$$

Hence $\tilde{B} \cdot C_1 = 0 = (\Delta - C_0) \cdot C_1$, and $C_1^2 = -1$. In particular, C_1 is an exceptional curve of the first kind which meets neither $\tilde{B}$ nor $(\Delta - C_0)$. We may therefore contract C_1, and repeat the above argument. In this way we deduce that $\Delta = C_0$.

Since $p(\tilde{B}) = 1$, we now have that $0 = (K_{\tilde{E}} + \tilde{B}) \cdot \tilde{B} = (1 - 1/r)\tilde{B}^2$. Hence $r = 1$, and $K_{\tilde{E}} \equiv -\tilde{B} - C_0$. Since $\tilde{B}$ does not meet C_0, it is also easy to see that the general element of the linear system $|B|$ on E does not meet the singular locus of E.

As $q(\tilde{E})>0$, we deduce from (3.4) that $r(2n+1, E)=1$. In particular we may apply the above results in this case also. Let $|A_{2n+1}|$ (respectively $|\tilde{A}_{2n+1}|$) denote the mobile part of $|(2n+1)K_V|$ (respectively $|(2n+1)K_{\tilde{V}}|$). Clearly $A_{2n+1} \mid E \sim K_E + 2B$ (linear equivalence).

We now calculate the quantities $\chi(\mathcal{O}_E(A_{2n+1}))$ and $\chi(\mathcal{O}_{\tilde{E}}(\tilde{A}_{2n+1}))$. Letting $g = h^1(\mathcal{O}_E)$ and applying Rieman–Roch formally on E, we obtain the equation

$$h^0(\mathcal{O}_E(A_{2n+1})) - h^1(\mathcal{O}_E(-2B)) = 1 - g + A_{2n+1} \cdot B.$$

Similarly on $\tilde{E}$, we see that

$$h^0(\mathcal{O}_{\tilde{E}}(\tilde{A}_{2n+1})) - h^1(\mathcal{O}_{\tilde{E}}(\tilde{A}_{2n+1})) = \tilde{A}_{2n+1} \cdot \tilde{B} = A_{2n+1} \cdot B$$

(using the fact here that $\tilde{A}_{2n+1}$ does not meet C_0). Now $\tilde{A}_{2n+1} \mid \tilde{E} \sim g^*K_E + 2\tilde{B}$, and hence $h^1(\mathcal{O}_{\tilde{E}}(\tilde{A}_{2n+1})) = h^1(\mathcal{O}_{\tilde{E}}(-C_0 - 2\tilde{B}))$. From the exact sequence

$$\mathcal{O}_{\tilde{E}}(-C_0 - 2\tilde{B}) \to \mathcal{O}_{\tilde{E}}(-2\tilde{B}) \to \mathcal{O}_{C_0}$$

we deduce that $h^1(\mathcal{O}_{\tilde{E}}(-C_0 - 2\tilde{B})) = h^0(\mathcal{O}_{C_0}) = 1$. Finally, using the fact that A_{2n+1} does not meet the singular locus of E, we see that $h^0(\mathcal{O}_{\tilde{E}}(\tilde{A}_{2n+1})) = h^0(\mathcal{O}_E(A_{2n+1}))$. The above equations therefore yield $h^1(\mathcal{O}_E(-2B)) = g$.

We now use the fact that $h^0(\mathcal{O}_E(K_E + B)) = 0$ (see (3.2)). Combined with a formal use of Riemann–Roch, this gives $h^1(\mathcal{O}_E(-B)) = g - 1$. However, if we consider the cohomology exact sequence derived from the sequence of sheaves

$$\mathcal{O}_E(-2B) \to \mathcal{O}_E(-B) \to \mathcal{O}_B(-B),$$

we see that $h^1(\mathcal{O}_E(-2B)) \leq h^1(\mathcal{O}_E(-B))$. This then is the required contradiction. □

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