On the Cauchy problem for dispersive equations with nonlinear terms involving high derivatives and with arbitrarily large initial data

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Abstract

The aim of this article is to study the Cauchy problem for general nonlinear dispersive equations involving derivatives in the nonlinearity. The use of some decay properties of the linear part allows us to address the case of arbitrarily large initial data.

1 Hypotheses and statement of the result

1.1 The evolution equation

The goal of this work is to study the Cauchy problem for equations of the form :

$$iu_t + Lu = F(u), \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and L is a linear (pseudo -) differential operator of order m with *real* valued symbol denoted by $l(\xi)$, and F is a nonlinear, possibly nonlocal operator. We will only consider the case where the linear part of (1) is dispersive, *i.e.* $l(\xi) \neq C\xi$. Actually, we will address cases where the linear group e^{itL} satisfies some "decay" properties, see Section 1.2 below.
The nonlinear term in (1) will be of the following form :

$$F(u) = \sum_{j=1}^{p} \mathcal{L}_j(f_j(u)),$$

where $\mathcal{L}_j$ is a pseudo-differential operator with constant coefficients of order l_j and $f_j : \mathbb{C} \to \mathbb{C}$ is smooth in the following sense: $f_j(u_1 + iu_2) = g_j(u_1, u_2)$, with $g_j \in \mathcal{C}^r(\mathbb{R}^2, \mathbb{R}^2)$.

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1.2 Hypotheses

In what follows, we denote by H^s and $W^{s,p}$ the Sobolev spaces $H^s(I\!R^n)$ and $W^{s,p}(I\!R^n)$. We write now precisely our hypotheses on $\mathcal{L}_j, f_j$ and L :

- (H1) we suppose that there exists $m \geq 2$, such that $e^{iLt} \equiv W_0(t)$, which is a group on every Sobolev space $H^s(I\!R^n)$, satisfies : for every $0 \leq \theta < \frac{2}{n}$ (in the case where $n = 1$, we impose $0 \leq \theta \leq 1$), denoting by (q,p) the pair $(\frac{4}{n\theta}, \frac{2}{1-\theta})$ and by (q',p') the conjugate exponents, we have the following estimates, for every $T \in I\!R$,

$$(H1)-1 \qquad |W_0(t)u_0|_{L^q(0;T;W^{\frac{\theta}{4}n(m-2),p})} \leq M(|T|)|u_0|_{L^2},$$

$$(H1)-2 \quad |\int_0^t W_0(t-\tau)g(\cdot;\tau)d\tau|_{L^q(0;T;W^{\frac{\theta}{2}n(m-2),p})} \leq M(|T|)|g|_{L^{q'}(0;T;L^{p'})},$$

$$(H1)-3 \qquad |\int_0^t W_0(t-\tau)g(\cdot;\tau)d\tau|_{L^\infty(0;T;L^2)} \leq M(|T|)|g|_{L^{q'}(0;T;L^{p'})},$$

where M is a non decreasing function from $I\!R^+$ into $I\!R^+$.

- (H2) The second hypothesis is on the operators $\mathcal{L}_j$ (we denote by $\widehat{\mathcal{L}_j}$ their symbols) :

$$|D^\alpha \widehat{\mathcal{L}_j}(\xi)| \leq C(1+|\xi|)^{l_j-|\alpha|} \text{ with } l_j < \frac{m-2}{2} \text{ and } |\alpha| < [\frac{n}{2}]+1.$$

Remark 1 : Thanks to Hörmander's multiplier's Theorem (see Torchinsky [11]), it is clear that $\mathcal{L}_j(I-\Delta)^{-l_j/2}$ maps L^p into L^p continuously for all $1 < p < \infty$. We will use this fact without notice in the course of the proof of Theorem 1.
We now denote by $l = \max_{j=1,\ldots,p} l_j$.

- (H3) The third hypothesis is on the nonlinearity :
$\exists s \geq 0, \sigma_j > 0$ and $0 \leq \theta < \frac{2}{n}$ (in the case where $n = 1$, we impose $0 \leq \theta \leq 1$) such that

$$|f_j(u)|_{W^{s+l,\beta'}} \leq C|u|_{H^s}^{\sigma_j}|u|_{W^{s+k,\beta}},$$

$$|f_j(u)-f_j(v)|_{L^{\beta'}} \leq C(|u|_{H^s}^{\sigma_j}+|v|_{H^s}^{\sigma_j})|u-v|_{L^\beta},$$

with $k = \frac{\theta n(m-2)}{4}$, $\theta = 1-\frac{2}{\beta}$ and $\frac{1}{\beta}+\frac{1}{\beta'} = 1$.

With the previous hypotheses and notations, our main result reads as follows.

Theorem 1

- *Under (H1), (H2), (H3), if $u_0 \in H^s$ then (1) has an unique maximal solution on $[0; T(u_0)[$ in $\mathcal{C}([0; T(u_0)[; H^s)$. Moreover $u \in L^q(0; t; W^{k+s,\beta})$ for*

$$\frac{2}{q} = n(\frac{1}{2} - \frac{1}{\beta}), \quad \forall t < T(u_0).$$

- *$u(t)$ depends continuously on u_0 in the following sense : if $u_0^n \to u_0$ in H^s, then $\forall T < T(u_0)$, if n is sufficiently large $u^n(t)$ exist on a common interval $[0; T]$ and $u^n(t) \to u(t)$ in $\mathcal{C}([0; T]; H^s) \cap L^q(0; T; W^{k+s,\beta})$.*

Remark 2 : It is well known that for some equations $T(u_0) < +\infty$, (even for semilinear equations with $F(u) = -|u|^2 u$ and $L = \Delta$) see for example Glassey [3].

1.3 Comments on (H1), (H2), and (H3)

In [4] Kenig, Ponce and Vega prove that if the symbol of L is an elliptic polynomial, or if it is tensorial, then L satisfies (H1).
In section 2 below, we show that if the functions f_j are $\mathcal{C}^r$ with $r \geq [k]+1$ and if there exists $\sigma_j \geq \frac{8l}{n(m-2)}$ such that

$$\forall \alpha \text{ with } |\alpha| \leq r, \ |(D^\alpha f_j)(u)| \leq c_{\alpha,j} |u|^{\sigma_j + 1 - |\alpha|}, \tag{2}$$

then f_j satisfies (H3).
The estimate (2) means that f_j and its derivatives behave like a power of u, so that Theorem 1 applies.
For example, we have :

Theorem 2 *The problem*

$$\begin{cases} iu_t + (-\Delta)^a u &= \sum_{j=1}^{p} a_j D^{l_j}(|u|^{\sigma_j} u), \\ u(x; 0) &= u_0(x), \end{cases}$$

with $l = \max l_j < a - 1, \sigma_j \geq \frac{4l}{n(a-1)}$ is locally well-posed in H^s for $s \geq n\left[\frac{1}{2} - \frac{2l}{n(a-1)\sigma}\right]$ with $\sigma = \max \sigma_j$.

Some results on the Cauchy problem for dispersive equations are available in the litterature; in [5], Kenig, Ponce, Vega prove that $\frac{\partial u}{\partial t} + \partial_x(u^k) + \partial_{x^3} u = 0$ is locally well posed in H^s for s depending on k. In dimension $n \geq 2$, they show in [6] that $\partial_t u = i\Delta u + P(u; P_x u; \bar{u}; \nabla_x \bar{u})$ where P is a complex polynomial is well posed in some weighted Sobolev spaces for small initial data. There exists another result (Klainerman-Ponce [7]) for $iu_t - \Delta u = F(u; \nabla u)$ in $I\!R^n$ for small initial data in $H^s, s > \frac{n}{2} + 2$ under restrictive hypotheses on the form of F.

The difference with our work is that we do not impose to the initial data to be small, and we work in the spaces $H^s(I\!\!R^n)$ which are the natural spaces corresponding to the linear part. Actually, our results are more restrictive on the nonlinearity since we consider general situations, and therefore, we do not have a large variety of estimates on the linear group, as it is the case for the Airy equation which is the linear part of the KdV equation [5].

We can compare our results with the work of J.C. Saut [9]. He proves that $\frac{\partial u}{\partial t} + \sum_{i=1}^n \partial_{x_i}[f(t,u) + L(x;u)] = g(x;t)$ is well posed, where L is an elliptic operator and $f(t,u)$ a polynomial in u which maximal degree depends on L. The method used in our paper gives a slightly different result. Indeed, the degree of f is not limited, but we replace $\frac{\partial}{\partial x_i}$ by a pseudo-differential operator whose order is limited by that of L. Moreover our result holds for non elliptic operators as long as they satisfy (H1).

2 Examples of nonlinearities satisfying (H3)

Suppose that for $r \geq [k]+1$, f_j is $\mathcal{C}^r$ and that there exists $\sigma_j \geq Max(\frac{8l}{n(m-2)}, r-1)$ such that

$$\forall \alpha \text{ with } |\alpha| \leq r \text{ then } |D^\alpha f_j(u)| \leq C_{\alpha,j}|u|^{\sigma_j+1-|\alpha|}.$$

We have :

Proposition 1 *: If $u \in W^{s+l,\beta} \cap L^{z_j}$, then $f_j(u) \in W^{s+l,\beta'}$, with $\frac{1}{\beta'} = \frac{\sigma_j}{z_j} + \frac{1}{\beta}$ for $+\infty > \beta > 1$, $z_j > 1$ and*

$$|f_j(u)|_{W^{s+l,\beta'}} \leq c|u|^{\sigma}_{L^{z_j}}|u|_{W^{s+l,\beta}}.$$

This kind of inequality is essentially due to Y. Meyer [8] for the case $z_j = +\infty$. Christ and Weinstein in [2] prove a related result :

$$|D^\alpha F(u)|_{L^p} \leq C|F'(u)|_{L^q}|D^\alpha u|_{L^r}$$
$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \qquad 0 < \alpha < 1 \text{ and } u : I\!\!R \to I\!\!R.$$

We give the proof of Proposition 1 in the Appendix.

Now we prove that f_j satisfies (H3).
Let $\sigma = Max\sigma_j$, take θ given by $l = \frac{\theta n(m-2)}{4}$ (recall that $l = Maxl_j$) and z given by $\theta = \frac{\sigma}{z}$.
Choose z_j such that $\frac{\sigma_j}{z_j} = \frac{\sigma}{z}$ ie $z_j = \frac{z\sigma_j}{\sigma} \leq z$. And since we impose the condition $\sigma_j \geq \frac{8l}{n(m-2)}$, we have $z_j \geq 2$.
We now choose s such that $H^s \hookrightarrow L^z$ and since $2 \leq z_j \leq z$, $H^s \hookrightarrow L^{z_j}$ so that the proposition implies that (H3) is satisfied. The application of this fact and of Theorem 1 prove Theorem 2.

3 Proof of Theorem 1

First we transform (1) :

$$\begin{cases} iu_t + Lu & = \sum_{j=1}^{p} \mathcal{L}_j(f_j(u)), \\ u(x;0) & = u_0(x), \end{cases}$$

into the equivalent integral equation (we shall prove later on that they are equivalent, see Section 3.4 below).

$$(INT) \ \ \mathcal{T}(u) \equiv W_0(t)u_0 - i\int_0^t W_0(t-s)(\sum_{j=1}^{p} \mathcal{L}_j(f_j(u))(s))ds = u(t).$$

3.1 Some estimates on $\mathcal{T}(u)$:

Lemma 1 $\mathcal{T}(u)$ *satisfies*

a)
$$|\mathcal{T}(u)|_{L^\infty(0;T;H^s)\cap L^q(0;T;W^{k+s,\beta})}$$

$$\leq C_1(1+M(|T|))|u_0|_{H^s} + C_2 M(|T|)T^\delta(\sum_{j=1}^{p}|u|^{\sigma_j}_{L^\infty(0;T;H^s)})|u|_{L^q(0;T;W^{k+s,\beta})},$$

and

b)
$$|\mathcal{T}(u)-\mathcal{T}(v)|_{L^q(o;T;L^\beta)} \leq C_3 M(|T|)T^\delta(\sum_{j=1}^{p}|u|^{\sigma_j}_{L^\infty(0;T;H^s)}$$

$$+|v|^{\sigma_j}_{L^\infty(0;T;H^s)})|u-v|_{L^q(0;T;L^\beta)},$$

$$\textit{where } \delta = \frac{1}{q'} - \frac{1}{q} > 0.$$

Proof : a) In order to estimate $\mathcal{T}(u)$ in $L^\infty(0;T;H^s)\cap L^q(0;T;W^{k+s,\beta})$, we need to estimate two terms :

i) The linear term $W_0(t)u_0$:

$$|W_0(t)u_0|_{H^s} = |u_0|_{H^s} \tag{3}$$

since $W_0(t)$ is unitary in H^s.

$$|W_0(t)u_0|_{W^{k+s,\beta}} \leq C|W_0(t)(I-\Delta)^{s/2}u_0|_{W^{k,\beta}},$$

but since $k = \frac{\theta n(m-2)}{4}$ by (H1) -1, we have

$$|W_0(t)u_0|_{L^q(0;T;W^{k,+s,\beta})} \leq CM(|T|)|(I-\Delta)^{s/2}u_0|_{L^2} = C|u_0|_{H^s}. \tag{4}$$

Then (3) and (4) lead to

$$|W_0(t)u_0|_{L^\infty(0,T,H^s)\cap L^q(0,T,W^{k+s,\beta})} \leq C_1(1+M(|T|))|u_0|_{H^s}. \qquad (5)$$

ii) The nonlinear term $\int_0^t W_0(t-s)\sum_{j=1}^p \mathcal{L}_j(f_j(u))(s)ds$:
We estimate each term separately :

- $|\int_0^t W_0(t-s)\mathcal{L}_j f_j(u)(s)ds|_{L^\infty(0;T;H^s)}$

$$\leq C|\int_0^t W_0(t-s)(I-\Delta)^{s/2}\mathcal{L}_j(f_j(u))ds|_{L^\infty(0;T;L^2)},$$

$$\leq CM(|T|)|(I-\Delta)^{s/2}\mathcal{L}_j(f_j(u))|_{L^{q'}(0;T:L^{\beta'})}$$

by (H1)-3,

$$\leq CM(|T|)|f_j(u)|_{L^{q'}(0;T;W^{s+l_j,\beta_j})}$$

by (H2). So that

$$|\int_0^t W_0(t-s)\mathcal{L}_j f_j(u)(s)ds|_{L^\infty(0;T;H^s)} \leq CM(|T|)T^\delta|u|^{\sigma_j}_{L^\infty(0;T;H^s)}|u|_{L^q(0;T;W^{s+k,\beta})} \qquad (6)$$

by (H3) and Hölder's inequality with respect to time.

- On the other hand :

$$|\int_0^t W_0(t-s)\mathcal{L}_j(f_j(u))(s)ds|_{L^q(0;T;W^{k+s,\beta})}$$

$$\leq C|\int_0^t W_0(t-s)(I-\Delta)^{s/2}f_j(u)ds|_{L^q(0;T;W^{k+l_j,\beta_j})}$$

by (H2),

$$\leq CM(|T|)|f_j(u)|_{L^{q'}(0;T;W^{s,\beta'_j})}$$

by (H1)-2 since $k+l\leq\frac{\theta n(m-2)}{2}$,

$$|\int_0^t W_0(t-s)\mathcal{L}_j(f_j(u))(s)ds|_{L^q(0;T;W^{k+s,\beta})} \leq CM(|T|)T^\delta|u|^{\sigma_j}_{L^\infty(0;T;H^s)}|u|_{L^q(0;T;W^{s+k,\beta})} \qquad (7)$$

by (H3) and Hölder's inequality with respect to time; (5), (6) and (7) together give the estimate of the a) of Lemma 1.

b) $$|\mathcal{T}(u)-\mathcal{T}(v)|_{L^q(0;T;L^\beta)} = |\int_0^t W_0(t-s)\sum_{j=1}^p \mathcal{L}_j(f_j(u)-f_j(v))ds|_{L^q(0;T;L^\beta)}$$

$$\leq CM(|T|)\sum_{j=1}^p |f_j(u)-f_j(v)|_{L^{q'}(0;T;L^{\beta'})}$$

by(H1)-2,

$$\leq CM(|T|)T^{\delta}\sum_{j=1}^{p}(|u|^{\sigma_j}_{L^\infty(0;T;H^s)}+|v|^{\sigma_j}_{L^\infty(0;T;H^s)})|u-v|_{L^q(0;T;L^\beta)}$$

by (H3) and Hölder's inequality with respect to time.
The proof of Lemma 1 is complete. ∎

3.2 Existence and uniqueness

Now we fix $T_1 > 0$ and we consider $R' \equiv C_1|u_0|_{H^s}(1+M(|T_1|))$, ($C_1$ is the constant appearing in a) of Lemma 1).

Proposition 2 *Let $R > R'$, if T is sufficiently small, then $\mathcal{T}$ maps the ball of radius R in $L^\infty(0;T;H^s)\cap L^q(0;T;W^{k+s,\beta})$ in itself and it is a contraction in the norm of $L^q(0;T;L^\beta)$.*

Proof:
• Indeed by a) of Lemma 1, if $T \leq T_1$ then

$$|\mathcal{T}(u)|_{L^\infty(0;T;H^s)\cap L^q(0;T;W^{k+s;\beta})} \leq R' + C_2M(|T|)T^\delta(\sum_{j=1}^{p}R^{\sigma_j})R.$$

It follows that if T is sufficiently small,

$$|\mathcal{T}(u)|_{L^\infty(0;T;H^s)\cap L^q(0;T;W^{k+s,\beta})} \leq R.$$

• b) of Lemma 2 gives :

$$|\mathcal{T}(u)-\mathcal{T}(v)|_{L^q(0;T;L^\beta)} \leq C_3M(|T|)T^\delta 2\sum_{j=1}^{p}R^{\sigma_j}|u-v|_{L^q(0;T;L^\beta)}.$$

We take T such that $C_3M(|T|)T^\delta 2\sum_{j=1}^{p}R^{\sigma_j} \leq 1/2$, thereby proving the proposition. ∎

Now we remark that a ball in $L^\infty(0;T;H^s)\cap L^q(0;T;W^{s+k,\beta})$ is complete for the norm of $L^q(0;T;L^\beta)$. Hence the contraction principle gives a local solution to (INT). Since T depends only on R which in its turn depends only on $|u_0|_{H^s}$, the existence time of the maximal solution depends only on $|u|_{H^s}$ *i.e.* :
if $T(u_0)$ (= the existence time of the maximal solution) is finite then

$$\lim_{t\to T(u_0)}|u(t)|_{H^s} = +\infty.$$

Now if u, v are two solutions to (INT) in $\mathcal{C}([0;T_0];H^s)$ then

$$\forall T < T_0, |u-v|_{L^q(0;T;L^\beta)} = |\mathcal{T}(u)-\mathcal{T}(v)|_{L^q(0;T;L^\beta)},$$

$$\leq C_3 M(|T|)T^\delta(\sum_{j=1}^{p}|u|^{\sigma_j}_{L^\infty(0;T;H^s)}+|v|^{\sigma_j}_{L^\infty(0;T;H^s)})|u-v|_{L^q(0;T;L^\beta)}$$

by Lemma 1 b).
Therefore taking T sufficiently small :

$$|u-v|_{L^q(0;T;L^\beta)} \leq 1/2|u-v|_{L^q(0;T;L^\beta)}$$

and $u \equiv v$ on $[0;T]$, thereby proving local uniqueness. ■

We have proved :

Proposition 3 *For all $u_0 \in H^s$, there exists a unique maximal solution u to (INT) in $\mathcal{C}([0;T(u_0)[;H^s)$. Moreover $\forall t < T(u_0)$, $u \in L^q(0;t;W^{k+s,\beta})$.*

3.3 Continuous dependence with respect to the initial data

Let $u_0 \in H^s$ and $u_0^n \in H^s$ with $u_0^n \to u_0$ in H^s. We note

$$\mathcal{T}_n(v(t)) \equiv W_0(t)u_0^n - i\int_0^t W_0(t-s)\sum_{j=1}^{p}\mathcal{L}_j(f_j(v))ds).$$

We call $u^n(t)$ the solution to :

$$(INT)_n \qquad\qquad \mathcal{T}_n(v)=v.$$

Proposition 4 *Let $T < T(u_0)$, if n is sufficiently large, then the solutions to $(INT)_n$ exist on $[0;T]$ and $u_n \to u$ in $\mathcal{C}(0;T;H^s)\cap L^q(0;T;W^{k+s,\beta})$.*

Proof : By standard arguments, it is sufficient to prove a local version.
Now the calculations of section 3.2 show that there exists $\tilde{T}>0$ depending only on $|u_0|_{H^s}$ such that the $(\mathcal{T}_n)$ are contraction in the ball of radius R in $L^\infty(0;\tilde{T};H^s)\cap L^q(0;\tilde{T};W^{s+k,\beta})$ for the norm of $L^q(0;\tilde{T};L^\beta)$; the rate of contraction being 1/2. The continuous dependence of the fixed point follows by standard arguments. ■

3.4 Equivalence between the integral equation and the partial differential equation

In order to finish the proof of Theorem 1, we need to show :

Proposition 5 *Let $u \in \mathcal{C}([0;T];H^s)$,*
u satisfies (1) if and only if u satisfies (INT).

Proof : Let $u \in \mathcal{C}([0;T];H^s)$, then $f_j(u) \in L^{\beta'} \hookrightarrow H^{-s}$ by (H3), so that by (H2) $\mathcal{L}_j(f_j(u)) \in H^{-s-l}$.
Next we need :

Lemma 2 *i) if* $u_0 \in H^\alpha$ *then* $W_0(t)u_0 \in \mathcal{C}([0;T];H^\alpha) \cap \mathcal{C}^1([0;T];H^{\alpha-m})$ *and*

$$i\frac{\partial}{\partial t}(W_0(t)u_0) + L(W_0(t)u_0) = 0,$$

$$W_0(0)u_0 = u_0.$$

ii) If $f \in L^1(0;T;H^\alpha)$ *then*

$$\Lambda f \equiv -i\int_0^t W_0(t-s)f(s)ds \in \mathcal{C}([0;T],H^\alpha) \cap AC([0;T];H^{\alpha-m})$$

and

$$\frac{i\partial}{\partial t}(\Lambda f) + L(\Lambda f) = f,$$

$$\Lambda f(0) = 0.$$

iii) if $v, f \in L^1(0;T;H^\alpha)$ *satisfy*

$$iv_t + Lv = f,$$

then $v(0) \in H^{\alpha-m}$ *exists and*

$$v = W_0(t)v(0) - i\int_0^t W_0(t-s)f(s)ds.$$

These are classic tools for unitary operators and we shall omit the proof of these results. Now since $\mathcal{L}_j(f_j(u)) \in H^{-s-l}$ and $u \in \mathcal{C}([0;T];H^s)$, Proposition 5 follows from Lemma 2.
The proof of Theorem 1 is now complete. ■

Appendix

The aim of this Appendix is to prove the following theorem :

Theorem 3 *Let* $s > 0, F \in \mathcal{C}^r(I\!R^2)$ *with* $r \geq [s]+1$. *Suppose that* $\exists \sigma \geq r-1$ *such that* $\forall_j$ *with* $|j| \leq r$,

$$|D^j F(\xi)| \leq C|\xi|^{\sigma+1-|j|}, \ \ \forall \xi \in I\!R^2.$$

Then :

$$|F(u)|_{W^{s,\beta}} \leq C|u|^\sigma_{L^z}|u|_{W^{s,p}},$$

with

$$\frac{1}{\beta} = \frac{\sigma}{z} + \frac{1}{p}, \ 1 < \beta, p < \infty,$$

$$1 < z \leq +\infty.$$

In [8], Y. Meyer shows that if $F \in \mathcal{C}^\infty(I\!R^2)$ with $F(0) = 0$ and $u \in W^{s,p} \cap L^\infty$ then $F(u) \in W^{s,p}$. We shall extend his proof to the case $W^{s,p} \cap L^z$, and we derive the inequality of Theorem 3.

Proof : We take a radial nonincreasing function $\varphi \in \mathcal{D}(I\!R^n)$ such that $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varphi(\xi) = 0$ if $|\xi| \geq 1$. We denote by $S_k(f) = f_k$ the partial sum defined by :

$$\widehat{S_k(f)} = \varphi(\frac{\xi}{2^k})\hat{f},$$

and by $\Delta_k(f)$ the dyadic block : $\Delta_k(f) = S_{k+1}(f) - S_k(f)$, *i.e.*

$$\widehat{\Delta_k(f)} = \psi(\frac{\xi}{2^k})\hat{f} \text{ with } \psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi).$$

The spectrum of $\Delta_k(f)$ is inclued in $\Gamma_k = \{\xi \in I\!R^n / \frac{2^k}{2} \leq |\xi| \leq 2.2^k\}$ and $f = S_0(f) + \Delta_0(f) + \cdots + \Delta_k(f) + \cdots$

We recall the characterization of $W^{s,p} = \{f \in S'/(1-\Delta)^{s/2} f \in L^p\}$ for $1 < p < \infty$. Then $f \in W^{s,p}$ if and only if $S_0(f) \in L^p$ and $(\sum_{k=0}^{+\infty} 4^{ks}|\Delta_k f|^2)^{1/2} \in L^p$. Moreover $||S_0(f)||_{L^p} + ||(\sum_{k=0}^{\infty} 4^{ks}|\Delta_k(f)|^2)^{1/2}||_{L^p}$ and $||(I-\Delta)^{1/2} f||_{L^p}$ are two equivalent norms on $W^{s,p}$.
We first we have :

Lemma 3 *$\exists C$ independent of k and f such that :*

$$\forall \alpha \in N^n \, |\partial^\alpha f_k|_{L^p} \leq C 2^{k|\alpha|}|f|_{L^p},$$

$$\frac{1}{C} 2^{kq}|\Delta_k(f)|_{L^p} \leq \sum_{|\alpha|=q} |\partial^\alpha \Delta_k(f)|_{L^p} \leq C 2^{kq}|\Delta_k(f)|_{L^p},$$

$$1 < p < +\infty.$$

Proof : For $1 < p < \infty$, it is the application of Hörmander's Multiplier's Theorem. For the case $p = \infty$, see Alinhac - Gérard [1]. ■

Proof of the Theorem :
We write

$$f = S_0(f) + \sum_{k=0}^{+\infty} \Delta_k(f)$$

and

$$F(f) = F(f_0) + \sum_{k=0}^{+\infty} [F(f_{k+1}) - F(f_k)].$$

We estimate each term :

i) The term $F(f_0)$:

* $|F(f_0)| \leq C|f_0|^{\sigma+1}$, Lemma 3 implies $|f_0|_{L^p} \leq C|f|_{L^p}$ and $|f_0|_{L^z} \leq C|f|_{L^z}$, so that

$$|F(f_0)|_{L^\beta} \leq C|f_0|^\sigma_{L^z}|f_0|_{L^p}. \tag{8}$$

* We estimate now :

$$\partial^\alpha F(f_0) = \sum_{\gamma+\cdots+\gamma_q=\alpha} (\partial^{|\alpha|}F)(f_0)\partial^{\gamma_1} f_0 \cdots \partial^{\gamma_q} f_0 \text{ with } |\alpha| < [s+1].$$

Then by Lemma 3, $|\partial^{\gamma_1} f_0|_{L^p} \leq C|f_0|_{L^p}$ and for $i > 1, |\partial^{\gamma_i} f_0|_{L^z} \leq C|f_0|_{L^z}$, so that

$$|\partial^\alpha F(f_0)|_{L^\beta} \leq C|f|^\sigma_{L^z}|f|_{L^p}, \forall |\alpha| < [s+1]. \tag{9}$$

With (8) and (9) we obtain

$$|F(f_0)|_{W^{s,\beta}} \leq C|f|^\sigma_{L^z}|f|_{L^p}. \tag{10}$$

ii) The terms $F(f_{k+1}) - F(f_k)$ *:*

$F(f_{k+1}) - F(f_k) = \Delta_k(f) \int_0^1 F'(f_k + t\Delta_k(f))dt = \Delta_k(f)m_k.$

• We first suppose that the spectrum of $m_k(x)\Delta_k(f)(x)$ is include in $\{|\xi| \leq 100.2^k\}$ and we consider $h_k(x) = m_k(x)\Delta_k(f)(x)$.

Let $\sigma(x) = (\sum_j 4^{js}|\Delta_j(\sum_k h_k)|^2)^{1/2}$.

Lemma 4

$$|\sigma(x)|_{L^\beta} \leq C||(\sum_k m_k(x)\Delta_k(f)|^2 4^{ks})^{1/2}||_{L^\beta}$$

Proof : See Meyer [8].

To continue, we need

Lemma 5 *For any f,*

$$\forall |N| < \sigma + 1, |\partial^N m_k(x)| \leq C|M(f)(x)|^\sigma 2^{|N|k},$$

where $M(f)$ is the maximal function of f.

The proof of Lemma 5 depends on

Lemma 6 *Let $\varphi \in S$ and $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$, then*

$$\sup_{\varepsilon>0} |(f * \varphi_\varepsilon)(x)| \leq CM(f)(x),$$

for $f \in L^p$, $1 \leq p \leq +\infty$.

For the proof of Lemma 6, see Stein [10] p. 62-64.

We can now prove Lemma 5 :

$$|\partial^N m_k(x)| \leq C \sum_{\alpha_i+\cdots+\alpha_q=N} (D^{N+1}F)(f_k)|\partial^{\alpha_1} f_k| \cdots |\partial^{\alpha_q} f_k|$$

and by Lemma 6, $|f_k(x)| \leq CM(f)(x)$;

moreover

$$\partial^{\alpha_i} f_k(x) = \Im^{-1}(\xi^{\alpha_i}\varphi(\frac{\xi}{2^k})\hat{f}),$$

$$= 2^{|\alpha_i|k}\Im^{-1}((\frac{\xi}{2^k})^{\alpha_i}\varphi(\frac{\xi}{2^k})\hat{f}).$$

We apply Lemma 6 again with the function $\Im^{-1}((\frac{\xi}{2^k})^{\alpha_i}\varphi(\frac{\xi}{2^k}))$; we obtain $|\partial^{\alpha_i} f_k(x)| \leq C2^{|\alpha_i|k}CM(f)(x)$ and Lemma 5 follows. ∎

Now

$$|\sigma(x)|_{L^\beta} \le C|\ |M(f)(x)|^\sigma (\sum_k |\Delta_k(f)|^2 4^{ks})^{1/2}|_{L^\beta},$$

$$|\sigma(x)|_{L^\beta} \le C|M(f)(x)|^\sigma_{L^z} |f|_{W^{s,p}}. \tag{11}$$

• General case : we decompose $m_k(x)$ with the following partition of 1 :

$$1 = \varphi\left(\frac{\xi}{100.2^k}\right) + \sum_{m=0}^{+\infty} \psi(\frac{\xi}{100.2^{k+m}}),$$

i.e. $m_k(x) = q_k(x) + \sum_{m=0}^{\infty} P_{k,m}(x)$.

The spectrum of q_k is inclued in $\{|\xi| \le 100\, 2^k\}$, so that the proof of the preceding case applies to q_k and thanks to (11)

$$|\sum_k q_k(x)\Delta_k(f)|_{W^{s,\beta}} \le C|f|^\sigma_{L^z}|f|_{W^{s,p}}. \tag{12}$$

Now the spectrum of $P_{k,m}$ is inclued in the ring

$$\{\frac{100}{2} 2^{k+m} \le |\xi| \le 2.100.2^{k+m}\}.$$

We define $l_m = \sum_{k=0}^{+\infty} P_{k,m}(x)\Delta_k(f)(x)$. The spectrum of $P_{k,m}\Delta_k(f)$ is inclued in the ring $\{\frac{100}{3}2^{k+m} \le |\xi| \le 3.100.2^{k+m}\}$, these rings taken 5 by 5 are disjoint and we can apply the Littlewood - Paley theory on

$$S_r(x) = \sum_{k\in 5I\!N+r} P_{k,m}\Delta_k(f) :$$

We estimate

$$|S_r|_{W^{s,\beta}} = ||\left(\sum_k |P_{k,m}|^2 |\Delta_k(f)|^2 4^{(k+m)s}\right)^{1/2} ||_{L^\beta},$$

but

$$P_{k,m}(x) = \int e^{ix.\xi}\psi(\frac{\xi}{100.2^{k+m}})\widehat{m_k}(\xi)d\xi.$$

We introduce a partition of unity on the sphere S^{n-1}, $(\chi_p)_{p=1..n}$, such that on suppχ_p, $\xi_p \ne 0$. We extend χ_p into $R^n/\{0\}$ by $\chi_p(\xi) = \chi_p(\frac{\xi}{|\xi|})$.
Let $N = [s]+1$, we obtain

$$P_{k,m}(x) = \sum_{p=1}^n \int e^{ix.\xi}\psi(\frac{\xi}{100.2^{k+m}})\widehat{m_k}(\xi)\chi_p(\xi)d\xi,$$

$$= \sum_{p=1}^n \int e^{ix.\xi}\frac{1}{\xi_p^N}\psi(\frac{\xi}{100.2^{k+m}})\xi_p^N\widehat{m_k}(\xi)\chi_p(\xi)d\xi,$$

$$= \sum_{p=1}^n \frac{2^{-(k+m)N}}{100}\int e^{ix.\xi}100\frac{2^{(k+m)N}}{\xi_p^N}\psi(\frac{\xi}{100.2^{k+m}})\widehat{\frac{\partial^N m_k}{\partial x_p^N}}(\xi)\chi_p(\frac{\xi}{100.2^{k+m}})d\xi,$$

since χ_p is homogeneous of degree 0,

$$\sum_{p=1}^{n} \frac{2^{-(k+m)N}}{100} \mathcal{F}^{-1}((\frac{\psi}{\xi_p^N}\chi_p)(\frac{\xi}{100.2^{k+m}})) * \frac{\partial^N m_k}{\partial x_p^N}.$$

So that

$$|P_{k,m}(x)| \leq \sum_{p=1}^{n} C\frac{2^{-(k+m)N}}{100} M(\frac{\partial^N m_k}{\partial x_p^n})(x),$$

by lemma 6.
It follows that

$$|P_{k,m}(x)| \leq C\frac{2^{-(k+m)N}}{100} M(|M(f)(x)|^\sigma)2^{kN}$$

by lemma 5,

$$\leq C2^{-mN} M(|M(f)(x)|^\sigma).$$

So that

$$|S_r|_{W^{s,\beta}} \leq ||M(|M(f)(x)|^\sigma)(\sum_k |\Delta_k(f)|^2 4^{ks})^{1/2} 2^{-m(N-s)}||_{L^\beta},$$

$$\leq C|f|_{L^z}^\sigma |f|_{W^{s,p}} 2^{-m(N-s)};$$

We obtain $|l_m|_{W^{s,\beta}} \leq C2^{-m[N-s]}|f|_{L^z}^\sigma |f|_{W^{s,p}}$ and the serie $\sum_m l_m$ converges normaly in $W^{s,\beta}$, with

$$|\sum_m l_m|_{W^{s,\beta}} \leq C|f|_{L^z}^\sigma |f|_{W^{s,p}}. \tag{13}$$

(10), (12) and (13) together lead to the estimate of Theorem 3. ■

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