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# ON THE CAUCHY PROBLEM FOR PARABOLIC PSEUDO-DIFFERENTIAL EQUATIONS

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## 1. Introduction

In the recent paper [8] S. Kaplan has obtained an analogue of Gårding's inequality for parabolic differential operators and applied it to a Hilbert space treatment of the Cauchy problem. D. Ellis [3] has extended those results to higher order parabolic differential operators (see also [4]). On the other hand in [13] the author has studied a Hilbert space treatment of the Cauchy problem for parabolic pseudo-differential equations and generalized the results of S. Kaplan [8].

In the present paper we shall study the Cauchy problem for higher order parabolic pseudo-differential equations of the form

$$Lu = D_t^k u(t, x) + \sum_{j=1}^{k} p_j(t, X, D_x) D_t^{k-j} u(t, x) = f(t, x)$$

where $p_j(t, x, \xi)$ are symbols of the class $S_{0,\lambda}^{m_j}$ introduced in [11] and [12]. We need not assume that the basic weight function $\lambda(\xi)$ tends to infinity as $|\xi| \to \infty$. Therefore the theory can be applied to more general classes of operators (including difference operators) than the class of usual parabolic differential operators.

In section 2 we give definitions and lemmas for pseudo-differential operators. In section 3 the algebras and $L^2$-theory are stated. The $L^2$-continuity of pseudo-differential operators has been studied in many papers (see for example, Calderón and Vaillancourt [1], [2], Hörmander [7] and Kumano-go [10]. In the present paper the $L^2$-continuity theorem by Calderón and Vaillancourt in [1] plays an essential role. In section 4 we define the space $H_{r,s}(\Omega)$ which is needed to study the Cauchy problem. In section 5 we derive energy inequalities for the parabolic system which is associated with a higher order parabolic pseudo-differential operator. These energy inequalities are very similar to those of D. Ellis [3] and [4]. To obtain the energy inequalities the idea of double symbols of pseudo-differential operators is very important. In section 6, using the results in section 4 and 5, we discuss a Hilbert space treatment of Cauchy problem for parabolic systems. In section 7 finally we state the main results for the Cauchy problem for higher order parabolic pseudo-differential equations.

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## 2. Definitions and lemmas

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be a multi-integer of $\alpha_j \geqq 0$, $j=1, \cdots, n$. We put $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha!=\alpha_1!\cdots\alpha_n!$ and $\partial_\xi^\alpha=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}$.

DEFINITION 2.1. Let $\lambda(\xi)$ be a real valued $C^\infty$ function defined on the $n$-dimensional real space $R_\xi^n$. We say that $\lambda(\xi)$ is a basic weight function when $\lambda(\xi)$ satisfies that

$$\lambda(\xi)\geqq 1\,, \tag{2.1}$$

$$|\partial_\xi^\alpha\lambda(\xi)|\leqq C_\alpha\lambda(\xi)^{1-|\alpha|} \text{ for any } \alpha\,, \tag{2.2}$$

(see [9] and [13]).

We can see that the function $\langle\xi\rangle=(1+|\xi|^2)^{1/2}=(1+\xi_1^2+\cdots+\xi_n^2)^{1/2}$ is a basic weight function.

The following lemma was proved in [13].

**Lemma 2.2.** *Let $\lambda(\xi)$ be a basic weight function and $\delta$ and $m$ be real numbers satisfying $0\leqq\delta<1$. Then we have*

$$\lambda(\xi)\leqq C_1\langle\xi\rangle\,, \tag{2.3}$$

$$\lambda(\xi+\eta)\leqq\lambda(\xi)+C_2|\eta|\leqq C_2\lambda(\xi)\langle\eta\rangle\,, \tag{2.4}$$

$$C_\delta^{-1}\lambda(\xi)\leqq\lambda(\xi+\lambda(\xi)^\delta\sigma)\leqq C_\delta\lambda(\xi) \tag{2.5}$$

*for any $\sigma\in R^n$ satisfying $|\sigma|\leqq 1$,*

$$\lambda(\xi+\eta)^m\leqq C_m\lambda(\xi)^m\langle\eta\rangle^{|m|}\,, \tag{2.6}$$

*where $C_1$, $C_2$, $C_\delta$ and $C_m$ are positive constants which are independent of $\xi$, $\eta$ and $\sigma$.*

Throughout this paper the letter $C$ with or without indices will denote positive constants not necessarily the same at each occurence.

**Lemma 2.3.** *Let $\lambda_0(\xi)$ be a real valued $C^1$ function such that $\lambda_0(\xi)\geqq c_0$ for some positive constant $c_0$ and $\partial_{\xi_j}\lambda_0(\xi)$ $(j=1, \cdots, n)$ are bounded. Then there exists a basic weight function $\lambda(\xi)$ which satisfies that*

$$c_1\lambda_0(\xi)\leqq\lambda(\xi)\leqq c_2\lambda_0(\xi) \tag{2.7}$$

*for some positive constants $c_1$ and $c_2$.*

Proof. By assumptions for $\lambda_0(\xi)$ we have $|\lambda_0(\xi)-\lambda_0(\eta)|\leqq C|\xi-\eta|$, so taking $\varepsilon_0=\frac{1}{2C}$ it holds that $(1/2)\lambda_0(\xi)\leqq\lambda_0(\eta)\leqq 2\lambda_0(\xi)$ for $|\xi-\eta|\leqq\varepsilon_0\lambda_0(\eta)$.

Let $\varphi(\eta)\in C_0^\infty(R^n)$ satisfy that $\int_{R^n}\varphi(\eta)d\eta=1$, $0\leqq\varphi(\eta)\leqq C_1$, supp $\varphi\subset\{\eta; |\eta|\leqq\varepsilon_0\}$ and $\varphi(\eta)\geqq C_1^0>0$ for $|\eta|\leqq\varepsilon_0/2$. Then the function $\lambda(\xi)=\int_{R^n}\varphi((\xi-\eta)/\lambda_0(\eta))\lambda_0(\eta)^{-n+1}d\eta$ is a basic weight function and satisfies the inequality (2.7). In fact,

$$\partial_\xi^\alpha\lambda(\xi)=\int_{R^n}\varphi^{(\alpha)}((\xi-\eta)/\lambda_0(\eta))\lambda_0(\eta)^{-n+1-|\alpha|}d\eta$$

where $\varphi^{(\alpha)}(\eta)=\partial_\eta^\alpha\varphi(\eta)$, so

$$\begin{aligned}|\partial_\xi^\alpha\lambda(\xi)| &\leqq C_\alpha\int_{|\xi-\zeta|\leqq\varepsilon_0\lambda_0(\zeta)}\lambda_0(\zeta)^{-n+1-|\alpha|}d\zeta\\ &\leqq C_\alpha\int_{|\xi-\zeta|\leqq 2\varepsilon_0\lambda_0(\xi)}\lambda_0(\xi)^{-n+1-|\alpha|}d\zeta\leqq C_\alpha\lambda_0(\xi)^{1-|\alpha|},\end{aligned}$$

$$\begin{aligned}\lambda_0(\xi) &= c_n\int_{|\xi-\zeta|\leqq\varepsilon_0\lambda_0(\xi)/4}\lambda_0(\xi)^{-n+1}d\zeta\\ &\leqq\left(\frac{c_n}{C_1^0}\right)C_1^0\int_{|\xi-\zeta|\leqq\varepsilon_0\lambda_0(\zeta)/2}\lambda_0(\xi)^{-n+1}d\zeta\\ &\leqq C\int_{R^n}\varphi((\xi-\zeta)/\lambda_0(\zeta))\lambda_0(\zeta)^{-n+1}d\zeta=C\lambda(\xi)\\ &\leqq C'\int_{|\xi-\zeta|\leqq\varepsilon_0\lambda_0(\zeta)}\lambda_0(\zeta)^{-n+1}d\zeta\\ &\leqq C'\int_{|\xi-\zeta|\leqq 2\varepsilon_0\lambda_0(\xi)}\lambda_0(\xi)^{-n+1}d\zeta=C'\lambda_0(\xi)\,.\end{aligned}$$

By these inequalities we obtain Lemma 2.3. Q.E.D.

Let $B(R^n)=\{f(x)\in C^\infty(R^n);\ |\partial_x^\alpha f(x)|\leqq C_\alpha$ for any $\alpha\}$, $S=S(R^n)=\{f(x)\in C^\infty(R^n);\ \lim_{|x|\to\infty}|x|^m|\partial_x^\alpha f(x)|=0$ for any $\alpha$ and real number $m\}$ and let $S'$ denote the dual space of $S$.

DEFINITION 2.4. Let $\lambda(\xi)$ be a basic weight function.

(i) We say that $p(x,\xi)$ belongs to $S_{0,\lambda}^m$ when $p(x,\xi)\lambda(\xi)^{-m}\in B(R^{2n})$.

(ii) We say that $p(x,\xi,x')$ belongs to $S_{0,\lambda}^m$ when $p(x,\xi,x')\lambda(\xi)^{-m}\in B(R^{3n})$.

(iii) We say that $p(x,\xi,x',\xi')$ belongs to $S_{0,\lambda}^{m,m'}$ when $p(x,\xi,x',\xi')\lambda(\xi)^{-m}\lambda(\xi')^{-m'}\in B(R^{4n})$.

(iv) We set $S_{0,\lambda}^\infty=\bigcup_{-\infty<m<\infty}S_{0,\lambda}^m$ and $S_{0,\lambda}^{-\infty}=\bigcap_{-\infty<m<\infty}S_{0,\lambda}^m$.

(v) Let $\lambda(\xi)$ and $\lambda'(\xi)$ be basic weight functions. Then we say that $p(x,\xi,x',\xi')$ belongs to $S_{0,\lambda,\lambda'}^{m,m'}$ when $p(x,\xi,x',\xi')\lambda(\xi)^{-m}\lambda'(\xi')^{-m'}\in B(R^{4n})$.

We use the notation: $D_x^\alpha=(-i)^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ for any $\alpha$. Then we set $p_{(\beta)}^{(\alpha)}(x,\xi)=D_x^\beta\partial_\xi^\alpha p(x,\xi)$, $p_{(\beta,\beta')}^{(\alpha)}(x,\xi,x')=D_x^\beta D_{x'}^{\beta'}\partial_\xi^\alpha p(x,\xi,x')$ and $p_{(\beta,\beta')}^{(\alpha;\alpha')}(x,\xi,x',\xi')=D_x^\beta D_{x'}^{\beta'}\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}p(x,\xi,x',\xi')$ for any $\alpha$, $\alpha'$, $\beta$ and $\beta'$.

We can see that

(i) $p(x, \xi)\in S_{0,\lambda}^{m}$ if and only if $|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{m}$ for any $\alpha$ and $\beta$,

(ii) $p(x, \xi, x')\in S_{0,\lambda}^{m}$ if and only if $|p_{(\beta,\beta')}^{(\alpha)}(x, \xi, x')|\leqq C_{\alpha,\beta,\beta'}\lambda(\xi)^{m}$ for any $\alpha$, $\beta$ and $\beta'$,

(iii) $p(x, \xi, x', \xi')\in S_{0,\lambda,\lambda'}^{m,m'}$ if and only if $|p_{(\beta,\beta')}^{(\alpha,\alpha')}(x, \xi, x', \xi')|\leqq C_{\alpha,\alpha',\beta,\beta'}\lambda(\xi)^{m}\lambda'(\xi')^{m'}$ for any $\alpha$, $\alpha'$, $\beta$ and $\beta'$,

(iv) when $m_1\geqq m_2$, it holds that $S_{0,\lambda}^{m_1}\supset S_{0,\lambda}^{m_2}$.

In this paper we write $\int f(x)dx$ for $\int_{R^n} f(x)dx$ and $d\xi$ for $(2\pi)^{-n}\, d\xi$.

DEFINITION 2.5. (i) For $p(x, \xi)\in S_{0,\lambda}^{\infty}$, we define the pseudo-differential operator $p(X, D_x)$ by

(2.8) $p(X, D_x)u(x) = \int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi$ for $u\in S$, where $\hat{u}(\xi)$ denote the Fourier transform $\int e^{-ix\cdot\xi}u(x)dx$ of $u(x)$ and $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$.

(ii) For $p(x, \xi, x')\in S_{0,\lambda}^{\infty}$, we define the operator $p(X, D_x, X')$ by

(2.9) $p(X, D_x, X')u(x) = \iint e^{i(x-x')\cdot\xi}p(x, \xi, x')u(x')dx'\cdot d\xi$ for $u\in S$,

where $dx'\cdot d\xi$ means the integration in $\xi$ follows the integration in $x'$.

(iii) For $p(x, \xi, x', \xi')\in S_{0,\lambda}^{m,m'}$ or $S_{0,\lambda,\lambda'}^{m,m}$, we define the operator $p(X, D_x, X', D_{x'})$ by

(2.10) $p(X, D_x, X', D_{x'})u(x) = \iiint e^{i(x-x')\cdot\xi+ix'\cdot\xi'}p(x, \xi, x', \xi')\hat{u}(\xi')d\xi'\cdot dx'\cdot d\xi$ for $u\in S$.

We can see that the above operators $p(X, D_x)$ and $p(X, D_x, X')$ are continuous linear operators from $S(R^n)$ to $S(R^n)$. We say that the functions $p(x, \xi)$, $p(x, \xi, x')$ and $p(x, \xi, x', \xi')$ are symbols of the pseudo-differential operators $p(X, D_x)$, $p(X, D_x, X')$ and $p(X, D_x, X', D_{x'})$ respectively and in particular $p(x, \xi, x', \xi')$ is often called a double symbol.

DEFINITION 2.6. Let $\lambda(\xi)$ be a basic weight function and $s$ be a real number. We define a Sobolev space $H_s$ by

$$H_s = H_{s,\lambda} = \{u\in S';\ \hat{u}(\xi)\in L_{loc}^1(R^n),\ \lambda(\xi)^s\hat{u}(\xi)\in L^2(R^n)\}\,.$$

We can see that $H_{s,\lambda}$ is a Hilbert space with inner product

$$(2.11) \qquad (u, v)_s = (u, v)_{s,\lambda} = \int \lambda(\xi)^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$$

and the set $S=S(R^n)$ is a dense subset of $H_{s,\lambda}$

For $s=0$, $H_{0,\lambda}=L^2(R^n)$. When $s_1\leqq s_2\lneqq s_3$, for any $\varepsilon>0$ there exists a constant $C=C_{s_1,s_2,s_3,\varepsilon}$ such that

$$(2.12) \qquad ||u||_{s_2}^2\leqq\varepsilon||u||_{s_3}^2+C||u||_{s_1}^2 \text{ for any } u\in S,$$

where $||u||_s=\sqrt{(u, u)_{s,\lambda}}$ (see [13]).

When $\boldsymbol{P}(x, \xi)=(p_{i,j}(x, \xi))$ is a $k\times k$ matrix function, we say that $\boldsymbol{P}(x, \xi)$ belongs to $S^m_{0,\lambda}$ if all the elements $p_{i,j}(x, \xi)$ belong to $S^m_{0,\lambda}$ in the sense of Definition 2.4 (i). By the same way we define $\boldsymbol{P}(x, \xi, x')\in S^m_{0,\lambda}$ and $\boldsymbol{P}(x, \xi, x', \xi')\in S^{m,m'}_{0,\lambda}$ or $S^{m,m'}_{0,\lambda,\lambda'}$. For $\boldsymbol{P}(x, \xi)=(p_{i,j}(x, \xi))\in S^\infty_{0,\lambda}$, we define the pseudo-differential operator $\boldsymbol{P}(X, D_x)$ by $\boldsymbol{P}(X, D_x)\boldsymbol{U}(x)=\int e^{ix\cdot\xi}\boldsymbol{P}(x, \xi)\hat{\boldsymbol{U}}(\xi)d\xi$, where $\boldsymbol{U}(x)={}^t(u_1(x), \cdots, u_k(x))\in\{S\}^k$ and $\boldsymbol{P}(x, \xi)\hat{\boldsymbol{U}}(\xi)=\begin{pmatrix}\sum_{j=1}^k p_{1,j}(x, \xi)\hat{u}_j(\xi)\\ \vdots\\ \sum_{j=1}^k p_{k,j}(x, \xi)\hat{u}_j(\xi)\end{pmatrix}$.

By the same way we can define the operators $\boldsymbol{P}(X, D_x, X')$ and $\boldsymbol{P}(X, D_x, X', D_{x'})$.

REMARK 2.7. With the aid of Lemma 2.3, we can see that

(i) for any basic weight functions $\lambda_1(\xi)$ and $\lambda_2(\xi)$, there exists a basic weight function $\lambda(\xi)$ such that $c_1\lambda(\xi)\leqq\lambda_1(\xi)+\lambda_2(\xi)\leqq c_2\lambda(\xi)$,

(ii) for any basic weiht function $\lambda(\xi)$ in $R^n$ and real number $m\geqq 1$, there exists a basic weight function $\lambda_1(\tau, \xi)$ in $R^{n+1}$ such that $c_1\lambda_1(\tau, \xi)\leqq(\tau^2+\lambda(\xi)^{2m})^{1/2m}\leqq c_2\lambda_1(\tau, \xi)$ (see [12] and [13]).

The fact of Remark 2.7 (ii) is important to define the spaces which are necessary to study the Cauchy problem for parabolic pseudo-differential equations.

REMARK 2.8. From the definition of basic weight functions, if $\lambda(\xi)$ is a basic weight function in $R^n$, $\lambda(\xi)$ is also a basic weight function in $R^{n+1}_{\tau,\xi}$.

## 3. Properties of pseudo-differential operators

All the theorems and corollaries of this section are stated in [12] and [13], so we omit the proofs.

**Theorem 3.1.** *Let $\lambda(\xi)$ and $\lambda'(\xi)$ be basic weight functions and let $p(x, \xi, x', \xi')\in S^{m,m'}_{0,\lambda,\lambda'}$. Then there exists a function $p_L(x, \xi)$ such that*

$$p_L(x, \xi)\lambda(\xi)^{-m}\lambda'(\xi)^{-m'}\in B(R^{2n}) \tag{3.1}$$

*and*

$$p_L(X, D_x)u=p(X, D_x, X', D_{x'})u \quad \text{for any } u\in S. \tag{3.2}$$

**Corollary 3.2.** (i) *Let $p_1(x, \xi)\in S^m_{0,\lambda}$ and $p_2(x, \xi)\in S^{m'}_{0,\lambda'}$. Then there exists a function $p_L(x, \xi)$ such that*

$$p_L(x, \xi)\lambda(\xi)^{-m}\lambda'(\xi)^{-m'}\in B(R^{2n}) \tag{3.3}$$

*and*

$$p_L(X, D_x)u=p_1(X, D_x)\cdot p_2(X, D_x)u \quad \text{for any } u\in S. \tag{3.4}$$

(ii) *For $p(x, \xi)\in S^m_{0,\lambda}$, there exists a symbol $p^*(x, \xi)\in S^m_{0,\lambda}$ such that*

$(p(X, D_x)u, v)_0=(u, p^*(X, D_x)v)_0$ *for any* $u, v\in S$.

When $\lambda(\xi)=\lambda'(\xi)$, the assertions in Corollary 3.2 mean that the class of pseudo-differential operators defined by the symbols in $S^{\infty}_{0,\lambda}$ forms an algebra.

**Theorem 3.3.** *Let* $0<\delta\leqq 1$ *and* $p(x, \xi, x', \xi')\in S^{m,m'}_{0,\lambda,\lambda'}$. *We assume that* $\partial_{\xi_j}p(x, \xi, x', \xi')\in S^{m-\delta,m'}_{0,\lambda,\lambda'}$. *Then for* $p_L(x, \xi)$ *in Theorem* 3.1 *and* $p_0(x, \xi)=p(x, \xi, x, \xi)$, *it holds that*

$$\{p_L(x, \xi)-p_0(x, \xi)\}\lambda(\xi)^{-m+\delta}\lambda'(\xi)^{-m'}\in B(R^{2n}). \tag{3.5}$$

**Corollary 3.4.** (i) *Let* $p_1(x, \xi)\in S^m_{0,\lambda}$ *and* $p_2(x, \xi)\in S^{m'}_{0,\lambda'}$. *Assume that* $\partial_{\xi_j}p_1(x, \xi)\in S^{m-\delta}_{0,\lambda}(j=1, \cdots, n)$ *for some* $\delta\in(0,1]$. *Then*

$$\{p_L(x, \xi)-p_1(x, \xi)p_2(x, \xi)\}\lambda(\xi)^{-m+\delta}\lambda'(\xi)^{-m'}\in B(R^{2n}), \tag{3.6}$$

*where* $p_L(x, \xi)$ *is the function defined in Corollary* 3.2.

(ii) *Assume that* $p(x, \xi)\in S^m_{0,\lambda}$ *and* $\partial_{\xi_j}p(x, \xi)\in S^{m-\delta}_{0,\lambda}$. *Then for* $p^*(x, \xi)$ *in Corollary* 3.2 (ii) *we have*

$$\{p^*(x, \xi)-\overline{p(x, \xi)}\}\in S^{m-\delta}_{0,\lambda}. \tag{3.7}$$

**Corollary 3.5.** *For* $p(x, \xi)\in S^m_{0,\lambda}$, *there exists a symbol* $p_{L,m'}(x, \xi)$ *such that*

$$\{p_{L,m'}(x, \xi)-p(x, \xi)\lambda'(\xi)^{m'}\}\lambda'(\xi)^{-m'+1}\lambda(\xi)^{-m}\in B(R^{2n}), \tag{3.8}$$

$$p_{L,m'}(X, D_x)u = \lambda'(D_x)^{m'}\cdot p(X, D_x)u \qquad \text{for any } u\in S. \tag{3.9}$$

**Corollary 3.6.** *Let* $p_1(x, \xi)\in S^m_{0,\lambda}$ *and* $p_2(x, \xi)\in S^{m'}_{0,\lambda}$. *Assume that* $\partial_{\xi_j}p_1(x, \xi)\in S^{m-\delta}_{0,\lambda}$ *and* $\partial_{\xi_j}p_2(x, \xi)\in S^{m'-\delta}_{0,\lambda}$ $(j=1, \cdots, n)$. *Then there exists a symbol* $p(x, \xi)\in S^{m+m'-\delta}_{0,\lambda}$ *such that*

$$\begin{aligned} p(X, D_x)u &= [p_1(X, D_x), p_2(X, D_x)]u \\ &= \{p_1(X, D_x)\cdot p_2(X, D_x)-p_2(X, D_x)\cdot p_1(X, D_x)\}u \end{aligned} \tag{3.10}$$

*for any* $u\in S$.

The following $L^2$-estimate was proved in [1].

**Lemma 3.7.** *Let* $p(x, \xi)\in S^0_{0,\lambda}$. *Then it holds that*

$$\|p(X, D_x)u\|_0\leqq C\|u\|_0 \quad \text{for any } u\in S, \tag{3.11}$$

*where* $C=C_p=c\sum_{|\alpha|+|\beta|\leqq N}\sup_{(x,\xi)}|p^{(\alpha)}_{(\beta)}(x, \xi)|$ *for some positive integer* $N$.

Using Corollary 3.2 (i) and Lemma 3.7 we have

**Theorem 3.8.** *Let s be an arbitrary real number and* $p(x, \xi)\in S^m_{0,\lambda}$. *Then it holds that*

$$\|p(X, D_x)u\|_{s,\lambda}\leqq C\|u\|_{s+m,\lambda} \quad \text{for any } u\in S. \tag{3.12}$$

**Corollary 3.9.** *When* $p(x, \xi)\in S^m_{0,\lambda}$, *we have*

(3.13) $|(p(X, D_x)u, u)_0| \leqq C\|u\|^2_{m/2,\lambda}$ *for any* $u \in S$.

For any $p(x, \xi) \in S^m_{0,\lambda}$ we denote $|p|_m = \sup_{(x,\xi)} |p(x, \xi)\lambda(\xi)^{-m}|$.

Using the Friedrichs approximation (see [5], [10] and [13]) we have,

**Theorem 3.10.** *Assume that* $0 < \delta \leqq 1$ *and* $p^{(\alpha)}(x, \xi) \in S^{m-\delta|\alpha|}_{0,\lambda}$ *for* $|\alpha| \leqq 1$. *Then we have*

(3.14) $|Re\,(p(X, D_x)u, u)_0| \leqq |Re\,p|_m \|u\|^2_{m/2,\lambda} + C\|u\|^2_{(m-\delta/2)/2}$, *for any* $u \in S$.

**Corollary 3.11.** *Assume that* $p^{(\alpha)}(x, \xi) \in S^{m-|\alpha|}_{0,\lambda}$ *for* $|\alpha| \leqq 1$, *then we have*

(3.15) $\|p(X, D_x)u\|^2_{s,\lambda} \leqq |p|^2_m \|u\|^2_{m+s,\lambda} + C\|u\|^2_{m+s-\delta/4,\lambda}$ *for any* $u \in S$.

We note that all the theorems and corollaries of this section except for Corollary 3.6 remain valid when the symbols of operators are $k \times k$ matrix functions. But in the case of matrix symbols we must replace $|Re\,p|_m$ in (3.14) and $|p|^2_m$ in (3.15) by $k|Re\,\boldsymbol{p}|_m$ and $k|\boldsymbol{p}|^2_m$ respectively, where we mean that for $\boldsymbol{p}(x, \xi) = (p_{i,j}(x, \xi)) \in S^m_{0,\lambda}$, $Re\,\boldsymbol{p} = \frac{1}{2}\{\boldsymbol{p}(x, \xi) + \boldsymbol{p}(x, \xi)^*\}$ and $|\boldsymbol{p}|_m = \{\sum_{i,j=1}^{k} \sup_{(x,\xi)} |p_{i,j}(x, \xi)\lambda(\xi)^{-m}|^2\}^{1/2}$.

In the case of matrix symbols, Corollary 3.6 holds if matrix $\boldsymbol{p}_1(x, \xi)$ commutes with $\boldsymbol{p}_2(x, \xi)$.

By virtue of Corollary 3.2 (ii), we can define the pseudo-differential operators on the space $S'$ by $\langle p(X, D_x)u, v\rangle = \langle u, \overline{p^*(X, D_x)\bar{v}}\rangle$ for $u \in S'$ and $v \in S$. Then inequalities (3.11), (3.12), (3.13), (3.14) and (3.15) hold for functions in $H_{s,\lambda}$ spaces.

## 4. Spaces $H_{r,s}(\Omega)$

In what follows we fix a basic weight function $\lambda(\xi)$ in $R^n$ and a real number $m \geqq 1$. By Remark 2.7 (ii), there exists a basic weight function $\lambda_1(\tau, \xi)$ in $R^{n+1}$ such that $c_1\lambda_1(\tau, \xi) \leqq (\tau^2 + \lambda(\xi)^{2m})^{1/2m} \leqq c_2\lambda_1(\tau, \xi)$.

DEFINITION 4.1. For any real numbers $r$ and $s$, we define the space $H_{r,s}$ by $H_{r,s} = \{u \in S'(R^{n+1}); \tilde{u}(\tau, \xi) \in L^1_{loc}(R^{n+1}), \lambda_1(\tau, \xi)^r \lambda(\xi)^s \tilde{u}(\tau, \xi) \in L^2(R^{n+1})\}$ where $\tilde{u}(\tau, \xi)$ is the Fourier transform $\int e^{-i(t\tau + x\cdot\xi)} u(t, x)\,dt\,dx$ of $u(t, x)$.

The space $H_{r,s}$ is a Hilbert space with inner product

$$(4.1) \qquad (u, v)_{r,s} = \int \lambda_1(\tau, \xi)^{2r} \lambda(\xi)^{2s} \tilde{u}(\tau, \xi) \overline{\tilde{v}(\tau, \xi)}\,d\tau\,d\xi\,.$$

We can see that $S(R^{n+1})$ is a dense subset of $H_{r,s}$.

For $-\infty \leqq a < b \leqq +\infty$, we set $\Omega = \Omega_{a,b} = \{(t, x) \in R^{n+1};\ a < t < b,\ x \in R^n\}$.

DEFINITION 4.2. (i) $H_{r,s}(\Omega) = \{u \in D'(\Omega);\ v|_\Omega = u$ for some $v \in H_{r,s}\}$, where $v|_\Omega = u$ means that the restriction of $v$ to $\Omega$ coincides with $u$ and $D'(\Omega)$

denote the space of distributions on $\Omega$.

(ii) For any closed set $K$ in $R^{n+1}$, we set $H_{0,r,s}(K)=\{u\in H_{r,s};\ \text{supp}\ u\subset K\}$.

(iii) For any open set $G$ in $R^{n+1}$, we set $C_{(0)}^{\infty}(G)=\{\varphi|_G;\ \varphi\in C_0^{\infty}(R^{n+1})\}$.

For $u\in H_{r,s}(\Omega)$ we define the norm of $u$ by $||u||_{r,s,\Omega}=\inf\{||v||_{r,s};\ v\in H_{r,s},\ v|_{\Omega}=u\}$ where $||v||_{r,s}=\sqrt{(v,v)_{r,s}}$. The space $H_{r,s}(\Omega)$ is a Banach space with norm $||v||_{r,s,\Omega}$. We can see that $H_{0,r,s}(K)$ is a closed subspace of $H_{r,s}$.

Using a similar method in [6], [8] and [11], we can see that for any $r$ and $s$, the set $C_{(0)}^{\infty}(\Omega)$ is dense in $H_{r,s}(\Omega)$, $C_0^{\infty}(\Omega)$ is dense in $H_{0,r,s}(\bar{\Omega})$ and $C_0^{\infty}(\bar{\Omega}^c)$ is dense in $H_{0,r,s}(\Omega^c)$, where $\Omega^c$ means the complement of $\Omega$.

The following lemmas are stated in [13] and can be proved by the similar methods to those in [8] and [11].

**Lemma 4.3.** *Assume that* $u\in H_{r,s+m}(\Omega)$ *and* $\frac{\partial}{\partial t}u\in H_{r,s}(\Omega)$, *Then* $u\in H_{r+m,s}(\Omega)$ *and*

$$(4.2)\qquad ||u||_{r+m,s,\Omega}\leqq C\left\{||u||_{r,s+m,\Omega}+\left\|\frac{\partial}{\partial t}u\right\|_{r,s,\Omega}\right\}.$$

**Lemma 4.4.** *Assume that* $2r>m$ *and* $-\infty<a<b\leqq\infty$.

(i) *We can define the trace operator* $\gamma_a$: $H_{r,s}(\Omega)\to H_{r+s-m/2,\lambda}$ *such that* $(\gamma_a u)(x)=u(a,x)$ *for* $u(t,x)\in S(R^{n+1})$ *and*

$$(4.3)\qquad ||\gamma_a u||_{r+s-m/2,\lambda}\leqq C||u||_{r,s,\Omega}\,.$$

(ii) *There exists a bounded linear operator* $\gamma^a$: $H_{r+s-m/2,\lambda}\to H_{r,s}(\Omega)$ *such that* $\gamma_a\cdot\gamma^a u=u$ *for* $u\in H_{r+s-m/2,\lambda}$.

**Lemma 4.5.** *Assume that* $|r|<m/2$. *We put*

$$H_a\varphi(t,x)=\begin{cases}\varphi(t,x) & \text{for } t\geqq a\,,\\ 0 & \text{for } t<a\,,\end{cases}$$

*for* $\varphi(t,x)\in S(R^{n+1})$, *then it holds that* $||H_a\varphi||_{r,s}\leqq C||\varphi||_{r,s}$. *That is, the operator* $H_a$ *can be extended to a bounded linear operator on* $H_{r,s}$ *and the range of* $H_a$ *is* $H_{0,r,s}(\bar{\Omega}_{a,\infty})$.

When a function $p(t,x,\xi)$ satisfies that $|\partial_t^j\partial_x^\alpha\partial_\xi^\beta p(t,x,\xi)|\leqq C_{j,\alpha,\beta}\lambda(\xi)^l$ for any $j$, $\alpha$ and $\beta$, we write $p(t,x,\xi)\in S_{0,\lambda}^l$, by the same notation as in Definition 2.4. For $u(t,x)\in S(R^{n+1})$, we define

$$p(t,X,D_x)u(t,x)=\int e^{i(t\tau+x\cdot\xi)}p(t,x,\xi)\tilde{u}(\tau,\xi)d\tau d\xi$$
$$=\int e^{ix\cdot\xi}p(t,x,\xi)\hat{u}(t,\xi)d\xi \text{ where } \hat{u}(t,\xi)=\int e^{-ix\cdot\xi}u(t,x)dx\,.$$

**Proposition 4.6.** *Let r and s be arbitrary real numbers. For* $p(t,x,\xi)\in S_{0,\lambda}^l$, *it holds that*

$$(4.4)\qquad ||p(t,X,D_x)u||_{r,s}\leqq C||u||_{r,s+l} \text{ for } u\in S(R^{n+1})\,.$$

Proof. By the definitions,

$$||p(t, X, D_x)u||_{r,s} = ||\lambda_1(D_t, D_x)^r \cdot \lambda(D_x)^s \cdot p(t, X, D_x)u||_{L^2(R^{n+1})},$$

where $\lambda_1(D_t, D_x)^r v = \int e^{i(t\tau+x\cdot\xi)}\lambda_1(\tau, \xi)^r \tilde{v}(\tau, \xi)d\tau d\xi$.

Using Theorem 3.1 and Corollary 3.2 (i) we can write

$$\lambda_1(D_t, D_x)^r \cdot \lambda(D_x)^s \cdot p(t, X, D_x)u(t, x) = p_{r,s}(t, X, D_t, D_x)u(t, x)$$

where $p_{r,s}(t, x, \tau, \xi)\lambda_1(\tau, \xi)^{-r}\lambda(\xi)^{-s-l} \in B(R^{2(n+1)})$.

From Lemma 3.7, we have

$$||p(t, X, D_x)u||_{r,s} = ||p_{r,s}(t, X, D_t, D_z)\cdot\lambda_1(D_t, D_x)^{-r}\cdot\lambda(D_x)^{-s-l}$$
$$\cdot\lambda_1(D_t, D_x)^r\cdot\lambda(D_x)^{s+l}u||_{L^2(R^{n+1})} \leqq C||\lambda_1(D_t, D_x)^r\cdot\lambda(D_x)^{s+l}u||_{L^2(R^{n+1})}$$
$$= C||u||_{r,s+l}.$$ Q.E.D.

By Proposition 4.6, the pseudo-differential operator $p(t, X, D_x)$ with symbol $p(t, x, \xi) \in S^l_{0,\lambda}$ can be extended to a bounded linear operator from $H_{r,s+l}$ to $H_{r,s}$. In the above proof we used the fact that when $\lambda(\xi)$ is a basic weight function in $R^n$, $\lambda(\xi)$ is also a basic weight function in $R^{n+1}$.

For any $u \in H_{0,r,s}(\bar{\Omega})$, we take a sequence $\{u_j\}_{j=1}^{\infty}$ in $C_0^{\infty}(\Omega)$ such that $u_j \to u$ in $H_{r,s}$. Then by Proposition 4.6, $p(t, X, D_x)u_j \to p(t, X, D_x)u$ in $H_{r,s-l}$. Therefore we have $p(t, X, D_x)u \in H_{0,r,s-l}(\bar{\Omega})$ for $u \in H_{0,r,s}(\bar{\Omega})$. This fact permits us to extend the operator $p(t, X, D_x)$ from $H_{r,s}(\Omega)$ to $H_{r,s-l}(\Omega)$. Indeed, let $u \in H_{r,s}(\Omega)$, $v_1|_{\Omega} = v_2|_{\Omega} = u$ and $v_1, v_2 \in H_{r,s}$. Since $v_1 - v_2 \in H_{0,r,s}(\Omega^c)$, we have $p(t, X, D_x)(v_1 - v_2) \in H_{0,r,s-l}(\Omega^c)$. So we define $p(t, X, D_x)u$ by $p(t, X, D_x)u = p(t, X, D_x)v|_{\Omega}$ for $v \in H_{r,s}$ such that $v|_{\Omega} = u$. Furthermore, we have

$$||p(t, X, D_x)u||_{r,s-l,\Omega} = \inf\{||v||_{r,s-l};\ v|_{\Omega} = p(t, X, D_x)u,$$
$$v \in H_{r,s-l}\} \leqq \inf\{||p(t, X, D_x)v||_{r,s-l};\ v|_{\Omega} = u,\ v \in H_{r,s}\}$$
$$\leqq \inf\{C||v||_{r,s};\ v|_{\Omega} = u,\ v \in H_{r,s}\}| = C||u||_{r,s,\Omega}.$$

Thus we can extend the operator $p(t, X, D_x)$ to a bounded linear operator from $H_{r,s}(\Omega)$ to $H_{r,s-l}(\Omega)$.

For $\varphi(t, x), \psi(t, x) \in C_0^{\infty}(R^{n+1})$, we write $[\varphi, \psi] = \int_{R^{n+1}} \varphi(t, x)\overline{\psi(t, x)}\, dtdx$. Then we can see that $||\varphi||_{r,s} = \sup\left\{\dfrac{|[\varphi, \psi]|}{||\psi||_{-r,-s}};\ \psi \neq 0,\ \psi \in C_0^{\infty}(R^{n+1})\right\}$.

Thus, $H_{r,s}$ and $H_{-r,-s}$ are dual Hilbert spaces and the form $[\cdot, \cdot]$ can be extended to a sesqui-linear form defined on $H_{r,s} \times H_{-r,-s}$.

Let $\{\zeta_i(t, x)\}_{i=1}^{\infty}$ be a sequence of $C_0^{\infty}(R^{n+1})$ and $\{\psi_j(\xi)\}_{j=1}^{\infty}$ a sequence of $C_0^{\infty}(R^n)$ functions satisfying the following conditions:

(i) $\sum\zeta_i(t, x)^2 = 1$, $\sum\psi_j(\xi)^2 = 1$,

(ii) $\sum|\partial_t^l\partial_x^{\alpha}\zeta_i(t, x)| \leqq C_{l,\alpha}$, $\sum|\partial_{\xi}^{\alpha}\psi_j(\xi)| \leqq C_{\alpha}$ for any $l$ and $\alpha$,

(iii) there exists a positive integer $N$ such that for any $(t, x) \in R^{n+1}$, the number of supp $\zeta_i$ containing $(t, x)$ is at most $N$ and for any $\xi \in R^n$, the number of supp $\psi_j$ containing $\xi$ is at most $N$.

Let $\{c_{ij}\}_{i,j=1}^{\infty}$ be a bounded sequence of complex numbers.
Then,

$$\begin{aligned}&\sum_{i,j}[c_{ij}\zeta_i(t, x)\psi_j(D_x)\varphi(t, x),\ c_{ij}\zeta_i(t, x)\psi_j(D_x)\psi(t, x)]\\ &=\sum[\psi_j(D_x)|c_{ij}|^2\zeta_i(t, x)^2\psi_j(D_x)\varphi(t, x),\ \psi(t, x)]\\ &=[\sum\psi_j(D_x)|c_{ij}|^2\zeta_i(t, x)^2\psi_j(D_x)\varphi(t, x),\ \psi(t, x)]\,.\end{aligned}$$

By assumptions of $\{c_{ij}\}$, $\{\zeta_i(t, x)\}$ and $\{\psi_j(\xi)\}$, we can consider the operator $\sum\psi_j(D_x)|c_{ij}|^2\zeta_i(t, x)^2\psi_j(D_x)$ as a pseudo-differential operator with a double symbol $\sum\psi_j(\xi)|c_{ij}|^2\zeta_i(t, x')^2\psi_j(\xi')\in S_{0,\lambda}^{0,0}$.

Hence we have

$$\begin{aligned}&\sum[c_{ij}\zeta_i(t, x)\psi_j(D_x)\varphi(t, x),\ c_{ij}\zeta_i(t, x)\psi_j(D_x)\psi(t, x)]\\ &\leqq C\|\varphi\|_{r,s}\|\psi\|_{-r,-s}\,.\end{aligned}$$

From this inequality we obtain the following proposition.

**Proposition 4.7.** *The form* $\sum[c_{ij}\zeta_i(t, x)\psi_j(D_x)\varphi(t, x),\ c_{ij}\zeta_i(t, x)\psi_j(D_x)\psi(t, x)]$ *for* $\varphi, \psi\in C_0^\infty(R^{n+1})$ *can be extended uniquely to a continuous sesquilinear form defined on* $H_{r,s}\times H_{-r,-s}$.

Using Lemma 4.5 and Proposition 4.7, we obtain the similar proposition to Proposition 7 in [3].

**Proposition 4.8.** *Let* $\{c_{ij}\}$, $\{\zeta_i(t, x)\}$ *and* $\{\psi_j(\xi)\}$ *satisfy the above conditions. Let* $s_1, s_2, r_1$ *and* $r_2$ *be real numbers satisfying that* $r_1+r_2\geqq 0$, $r_1+r_2+s_1+s_2\geqq 0$, *min* $(r_1, r_2)>-m/2$ *and let* $-\infty\leqq a<b\leqq+\infty$.
*Then the form*

$$\sum\int_a^b(c_{ij}\zeta_i(t)\psi_j(D_x)\varphi(t),\ c_{ij}\zeta_i(t)\psi_j(D_x)\psi(t))_0\,dt$$

*for* $\varphi(t, x), \psi(t, x)\in C_{(0)}^\infty(\Omega)$ *can be extended uniquely to a continuous sesquilinear form on* $H_{r_1,s_1}(\Omega)\times H_{r_2,s_2}(\Omega)$.

## 5. Parabolic operators and energy inequalities

Consider the operator $L=D_t^k+\sum_{j=1}^k p_j(t, X, D_x)D_t^{k-j}$ where $D_t=(-i)\partial/\partial t$. We assume that the operator $L$ satisfies the following conditions:

(i) we can write $L=L_0+L_1$ where $L_0=D_t^k+\sum_{j=1}^k p_j^0(t, X, D_x)D_t^{k-j}$ and $L_1=\sum_{j=1}^k q_j(t, X, D_x)D_t^{k-j}$,

(ii) $p_j^0(t, x, \xi)\in S_{0,\lambda}^{mj}$ $(j=1, \cdots, k)$,

(iii) for some $0<\delta_1\leqq 1$, $\partial_{\xi_i}p_j^0(t, x, \xi)\in S_{0,\lambda}^{mj-\delta_1}$ $(i=1, \cdots, n, j=1, \cdots, k)$ and $q_j(t, x, \xi)\in S_{0,\lambda}^{mj-\delta_1}$,

(iv) roots $\tilde{p}_1(t, x, \xi), \cdots, \tilde{p}_k(t, x, \xi)$ of the equation $\sigma(\boldsymbol{L}_0)=\tau^k+\sum_{j=1}^{k} p_j^0(t, x, \xi)\tau^{k-j}=0$ satisfy the inequalities $\mathrm{Im}\ \tilde{p}_j(t, x, \xi)\geqq c_0\lambda(\xi)^m$ $(j=1, \cdots, k)$ where $c_0$ is a positive constant.

We can consider the operator $\boldsymbol{L}$ as an extended form for higher order parabolic differential operators.

For any $u\in S(R^{n+1})$, we put $u_j=\lambda(D_x)^{m(k-j)}D_t^{j-1}u$ for $j=1, \cdots, k$, and $\boldsymbol{U}={}^t(u_1, \cdots, u_k)$. Then we have $D_t u_j=\lambda(D_x)^m u_{j+1}$ for $j=1, \cdots, k-1$ and $D_t u_k=D_t^k u=\boldsymbol{L}u-\sum_{j=1}^{k} p_j^0(t, X, D_x)D_t^{k-j}u-\sum_{j=1}^{k} q_j(t, X, D_x)D_t^{k-j}u=\boldsymbol{L}u-\sum_{j=1}^{k} p_{k-j+1}^1(t, X, D_x)\lambda(D_x)^m u_j-\sum_{j=1}^{k} q_{k-j+1}^1(t, X, D_x)u_j$ where $p_{k-j+1}^1(t, x, \xi)=p_{k-j+1}^0(t, x, \xi)\lambda(\xi)^{-m(k-j+1)}\in S_{0,\lambda}^0$ and $q_{k-j+1}^1(t, x, \xi)=q_{k-j+1}(t, x, \xi)\lambda(\xi)^{-m(k-j)}\in S_{0,\lambda}^{m-\delta_1}$.

Hence we can write

$$D_t\boldsymbol{U}= h(t, X, D_x)\cdot\lambda(D_x)^m\boldsymbol{U}+\frac{1}{i}\boldsymbol{J}(t, X, D_x)\boldsymbol{U}+(\boldsymbol{L}u)e_k$$

where $e_k=\begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}$, $h(t, x, \xi)=\begin{pmatrix} 0 & 1 & & & & 0\\ \vdots & 0 & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & \vdots & \vdots & 0 & 1\\ -p_k^1 & -p_{k-1}^1 & \cdots & \cdots & \cdots & -p_1^1\end{pmatrix}$

and $\boldsymbol{J}(t, x, \xi)=\begin{pmatrix} & 0 & \\ -iq_k^1 & \cdots & -iq_k^1\end{pmatrix}$

Thus, $\partial/\partial t\ \boldsymbol{U}=\boldsymbol{H}\cdot\lambda(D_x)^m\boldsymbol{U}+\boldsymbol{J}\boldsymbol{U}+i(\boldsymbol{L}u)e_k$ and $\boldsymbol{H}=ih(t, X, D_x)$. We put $\boldsymbol{R}=\partial/\partial t-\boldsymbol{H}\cdot\lambda(D_x)^m-\boldsymbol{J}$.

From the assumptions of operator $\boldsymbol{L}$, we have

(i) $\sigma(\boldsymbol{H})=ih(t, x, \xi)\in S_{0,\lambda}^0$, $\partial_{\xi_j}\sigma(\boldsymbol{H})\in S_{0,\lambda}^{-\delta_1}$ $(j=1, \cdots, n)$ and $\sigma(\boldsymbol{J})=\boldsymbol{J}(t, x, \xi)\in S_{0,\lambda}^{m-\delta_1}$,

(ii) the eigenvalues of $\sigma(\boldsymbol{H})$ are contained in a fixed compact subset of the set $\{z\in\boldsymbol{C};\ \mathrm{Re}\ z\leqq -c_0\}$.

For a matrix $A=(a_{ij})$ we denote $|A|=\{\sum|a_{ij}|^2\}^{1/2}$.

The following lemma is shown in [3].

**Lemma 5.1.** *For any* $(t, x, \xi)$, *there exists a* $k\times k$ *matrix* $N(t, x, \xi)$ *such that*

(i) $|N(t, x, \xi)|+|N(t, x, \xi)^{-1}|\leqq C$,

(ii) $\mathrm{Re}\ (N(t, x, \xi)^{-1}\boldsymbol{H}(t, x, \xi)N(t, x, \xi)\zeta, \zeta)\leqq -\frac{c_0}{4}|\zeta|^2$ *for any* $\zeta={}^t(\zeta_1, \cdots, \zeta_k)\in\boldsymbol{C}^k$,

*where the constant* $C$ *is independent of* $(t, x, \xi)$.

**Lemma 5.2.** *We fix an arbitrary point* $(t_0, x_0, \xi_0)$ *and put* $N_0=N(t_0, x_0, \xi_0)$, $\boldsymbol{H}_0=\boldsymbol{H}(t_0, x_0, \xi_0)$ *and* $\boldsymbol{R}_0=\partial/\partial t-\boldsymbol{H}_0\lambda(D_x)^m-\boldsymbol{J}$. *Then we have*

(5.1) $c_1\|\boldsymbol{U}(b)\|_0^2-c_2\|\boldsymbol{U}(a)\|_0^2+\mu_1\int_a^b\|\boldsymbol{U}(t)\|_{m/2}^2\,dt$

$$-\mu_2\int_a^b \|U(t)\|_0^2\,dt \leqq Re\int_a^b (N_0^{-1}R_0U,\ N_0^{-1}U)_0\,dt$$

*for any* $U\in\{S(R^{n+1})\}^k$, *where* $c_1$, $c_2$, $\mu_1$ *and* $\mu_2$ *are constants which are independent of* $(t_0, x_0, \xi_0)$ *and*

$$\|U(t)\|_s^2=\int\lambda(\xi)^{2s}|\hat{U}(t,\xi)|^2d\xi\,.$$

Proof. Since $H_0$ and $N_0$ are constant matrices, we can write

$$\mathrm{Re}(N_0^{-1}R_0U,\ N_0^{-1}U)_0=\mathrm{Re}\Big(N_0^{-1}\frac{\partial U}{\partial t},\ N_0^{-1}U\Big)_0-\mathrm{Re}(N_0^{-1}H_0\lambda(D_x)^mU,\ N_0^{-1}U)_0$$

$$-\mathrm{Re}(N_0^{-1}JU,\ N_0^{-1}U)_0=\frac{1}{2}\frac{\partial}{\partial t}\|N_0^{-1}U(t)\|_0^2$$

$$-\mathrm{Re}(N_0^{-1}H_0\lambda(D_x)^{m/2}U,\ N_0^{-1}\lambda(D_x)^{m/2}U)_0-\mathrm{Re}(N_0^{-1}JU,\ N_0^{-1}U)_0\,.$$

Putting $N_0^{-1}\lambda(D_x)^{m/2}U=V$, we have

$$\mathrm{Re}\int_a^b(N_0^{-1}R_0U,\ N_0^{-1}U)_0dt\geqq\frac{1}{2}\|N_0^{-1}U(b)\|_0^2-\frac{1}{2}\|N_0^{-1}U(a)\|_0^2$$

$$-\mathrm{Re}\int_a^b(N_0^{-1}H_0N_0V,\ V)_0dt-C\int_a^b\|JU\|_{-(m-\delta_1)/2}\|U\|_{(m-\delta_1)/2}dt\,.$$

By Theorem 3.8, it holds that

$$\|JU\|_{-(m-\delta_1)/2}\leqq C\|U\|_{(m-\delta_1)/2}\,.$$

Using Lemma 5.1,

$$\mathrm{Re}(N_0^{-1}H_0N_0V,\ V)_0=\mathrm{Re}\int N_0^{-1}H_0N_0\hat{V}(t,\xi)\cdot\overline{\hat{V}(t,\xi)}d\xi$$

$$\leqq-\frac{c_0}{4}\int|\hat{V}(t,\xi)|^2d\xi\leqq-\mu_1'\int\lambda(\xi)^m|\hat{U}(t,\xi)|^2d\xi=-\mu_1'\|U(t)\|_{m/2}^2.$$

Hence we have

$$\mathrm{Re}\int_a^b(N_0^{-1}R_0U,\ N_0^{-1}U)_0dt\geqq c_1\|U(b)\|_0^2-c_2\|U(a)\|_0^2$$

$$+(\mu_1'-\varepsilon)\int_a^b\|U(t)\|_{m/2}^2dt-C_\varepsilon\int_a^b\|U(t)\|_0^2dt$$

for any $\varepsilon>0$. Taking $\varepsilon=\mu_1'/2$, we obtain (5.1). Q.E.D.

To obtain the similar energy inequalities to those of [3] or [4], we use the partition of unity of the space $R^{n+1}_{(t,x)}$ and $R^n{}_\xi$. Let $\varepsilon$ be a sufficiently small positive number which will be determined later.

Let $\zeta(t,x)\in C_0^\infty(R^{n+1})$satisfy $0\leqq\zeta(t,x)\leqq1$, $\mathrm{supp}\,\zeta\subset\{(t,x);\ |t|<1,\ |x_j|<1\ j=1,\cdots,n\}$ and $\zeta(t,x)=1$ for $|t|\leqq1/2$ and $|x_j|\leqq1/2$ $j=1,\cdots,n$.

Let $g=(g_0,g')=(g_0,g_1,\cdots,g_n)$ and $h=(h_0,h')$ denote $(n+1)$-tuples of integers.

We put $\zeta_g(t,x)=\dfrac{\zeta\Big(\dfrac{1}{\varepsilon}t-g_0,\ \dfrac{1}{\varepsilon}x-g'\Big)}{\Big\{\sum_h\zeta\Big(\dfrac{1}{\varepsilon}t-h_0,\ \dfrac{1}{\varepsilon}x-h'\Big)^2\Big\}^{1/2}}\,.$

Enumerating the points $\{\varepsilon g\}$ and the corresponding functions $\{\zeta_g\}$ in some order, we denote them by $(t_1, x_1)$, $(t_2, x_2)$, $\cdots$ and $\zeta_1, \zeta_2, \cdots$.

Then we have,

(i) $\sum_i \zeta_i(t, x)^2 \equiv 1$,

(ii) $\sum_i |\partial_t^l \partial_x^\alpha \zeta_i(t, x)| \leqq C_{l,\alpha,\varepsilon}$ for any $l$ and $\alpha$,

(iii) the supp $\zeta_i$ overlap in such a way that each fixed point in $R^{n+1}$ is contained in at most $2^{n+1}$ distinct ones of them,

(iv) $|\boldsymbol{H}(t, x, \xi) - \boldsymbol{H}(t_i, x_i, \xi)| \leqq C\{|t-t_i| + |x-x_i|\} \leqq C_1\varepsilon$ for any $(t, x) \in$ supp $\zeta_i$ and $\xi \in R^n$.

We take the set $\{\tilde{g}_{1,j}\}_{j=0}^\infty$ of points in $R^n$ as follows:

(i) $\tilde{g}_{1,0}=0$,

(ii) $\tilde{g}_{1,i} \neq \tilde{g}_{1,j}$ for $i \neq j$,

(iii) when $1+l(3^n-1) \leqq j \leqq (l+1)(3^n-1)$, $l=0, 1, \cdots$, writing $\tilde{g}_{1,j}=(a_1, \cdots, a_n)$, $a_i=2\cdot 3^l$ or $a_i=0$ or $a_i=-2\cdot 3^l$ $i=1, \cdots, n$. We put $\tilde{a}_{1,0}=2$ and $\tilde{a}_{1,j}=2\cdot 3^l$ for $1+l(3^n-1) \leqq j \leqq (l+1)(3^n-1)$, $l=0, 1, \cdots$. We put $\tilde{\Delta}_{1,j}=\left\{\xi \in R^n;\ |\xi_i - a_i| \leqq \frac{1}{2}\tilde{a}_{1,j},\ i=1, \cdots, n\right\}$ for $\tilde{g}_{1,j}=(a_1, \cdots, a_n)$.

Then it holds that $R^n = \bigcup_{j=0}^\infty \tilde{\Delta}_{1,j}$, $\bigcup_{j=1}^\infty \partial\tilde{\Delta}_{1,j}$ is a set of measure zero and for almost everywhere $\xi \in R^n$, there is a number $j$ uniquely such that $\xi \in \tilde{\Delta}_{1,j}$.

Enumerating the cubes which satisfy $\tilde{a}_{1,j} \leqq \varepsilon\lambda(\tilde{g}_{1,j})^{\delta_1}$, we denote them by $\Delta_{1,1}$, $\Delta_{1,2}$, $\cdots$ and their centers and the lengths of sides by $g_{1,1}, g_{1,2}, \cdots$ and $a_{1,1}$, $a_{1,2}$, $\cdots$ respectively.

Similarly we write $\Delta'_{1,1}, \Delta'_{1,2}, \cdots, g'_{1,1}, g'_{1,2}, \cdots$ and $a'_{1,1}, a'_{1,2}, \cdots$ for the cubes satisfying $\tilde{a}_{1,j} > \varepsilon\lambda(\tilde{g}_{1,j})^{\delta_1}$.

We devide each $\Delta'_{1,j}$ into $2^n$ congruent cubes and enumerate such cubes in some order: $\tilde{\Delta}_{2,1}, \tilde{\Delta}_{2,2}, \cdots$. We denote the center and length of side of each cube $\tilde{\Delta}_{2,j}$ by $\tilde{g}_{2,j}$ and $\tilde{a}_{2,j}$ respectively.

By the same way as above we write $\{\tilde{\Delta}_{2,j}\}_j = \{\Delta_{2,j}\}_j$, $\{\tilde{g}_{2,j}\}_j = \{g_{2,j}\}_j$ and $\{\tilde{a}_{2,j}\}_j = \{a_{2,j}\}_j$ if $\tilde{a}_{2,j} \leqq \varepsilon\lambda(\tilde{g}_{2,j})^{\delta_1}$ and $\{\tilde{\Delta}_{2,j}\}_j = \{\Delta'_{2,j}\}_j$ if $\tilde{a}_{2,j} > \varepsilon\lambda(\tilde{g}_{2,j})^{\delta_1}$.

Repeating this process, we obtain cubes $\{\Delta_{l,j}\}_{l,j}$ with centers $\{g_{l,j}\}_{l,j}$ and lengths of sides $\{a_{l,j}\}_{l,j}$.

**Lemma 5.3.** (i) $R^n = \bigcup_{l,j} \Delta_{l,j}$

(ii) *for sufficiently small* $\varepsilon > 0$, $\{\tilde{\Delta}_{1,j}\} = \{\Delta'_{1,j}\}$,

(iii) *for sufficiently small* $\varepsilon > 0$, *we have* $c_0\varepsilon\lambda(g_{l,j})^{\delta_1} \leqq a_{l,j} \leqq \varepsilon\lambda(g_{l,j})^{\delta_1}$ $(0 < c_0 < 1)$.

Proof of (i). We note that $R^n = \bigcup_{j=0}^\infty \tilde{\Delta}_{1,j}$. Assume that there exists a point $\xi \in R^n$ such that for any $l$, $\xi \in \Delta'_{l,j_l}$ for some $j_l$. Then by the definition of

$\Delta'_{l,j_l}$, $|\xi_i-a'_i|\leqq\frac{1}{2}a'_{l,j_l}(i=1,\cdots,n)$, $a'_{l,j_l}>\varepsilon\lambda(g'_{l,j_l})^{\delta_1}\geqq\varepsilon$ and $a'_{l,j_l}=\frac{1}{2^{l-1}}a'_{1,j_1}$ for some $j_1$, here $\xi=(\xi_1,\cdots,\xi_n)$ and $g'_{l,j_l}=(a'_1,\cdots,a'_n)$.

Taking sufficiently large $l$, we have a contradiction. Hence for any $\xi\in R^n$, there exists $l$ and $j_l$ such that $\xi\in\Delta_{l,j_l}$.

Proof of (ii). Taking $\varepsilon>0$ sufficiently small, we have $\varepsilon\lambda(0)^{\delta_1}<2=\tilde{a}_{1,0}$, hence $\tilde{\Delta}_{1,0}\in\{\Delta'_{1,j}\}$. For any $j_1\geqq 1$, by definitions, $2\leqq\tilde{a}_{1,j_1}\leqq|\tilde{g}_{1,j_1}|\leqq\sqrt{n}\,\tilde{a}_{1,j_1}$. By Lemma 2.2 (2.3), $\lambda(\tilde{g}_{1,j_1})^{\delta_1}\leqq C_1\langle\tilde{g}_{1,j_1}\rangle^{\delta_1}\leqq C_1\langle\tilde{g}_{1,j_1}\rangle\leqq 2C_1|\tilde{g}_{1,j_1}|\leqq(2C_1\sqrt{n}\,)$ $\tilde{a}_{1,j_1}$.

Hence, taking $0<\varepsilon<(2C_1\sqrt{n}\,)^{-1}$, we have $\varepsilon\lambda(\tilde{g}_{1,j_1})^{\delta_1}<\tilde{a}_{1,j_1}$. This means $\tilde{\Delta}_{1,j_1}\in\{\Delta'_{1,j}\}$.

Proof of (iii). By definitions we have $a_{l,j}\leqq\varepsilon\lambda(g_{l,j})^{\delta_1}$. By virtue of Lemma 2.2, we can take $\varepsilon>0$ sufficiently small such that

$$\frac{3}{4}\lambda(\xi)\leqq\lambda(\eta)\leqq\frac{4}{3}\lambda(\xi)\qquad\text{for } |\xi-\eta|\leqq 2\sqrt{n}\,\varepsilon\lambda(\xi)^{\delta_1}. \tag{5.2}$$

By definitions and (ii), $\Delta_{l,j}\subset\Delta'_{l-1,j_l}$. Then we have $a_{l,j}=\frac{1}{2}a'_{l-1,j_l}>\frac{1}{2}$ $\varepsilon\lambda(g'_{l-1,j_l})^{\delta_1}$.

Since $g'_{l-1,j_l}\in\Delta_{l,j}$, $|g'_{l-1,j_l}-g_{l,j}|\leqq\frac{1}{2}\sqrt{n}\,a_{l,j}\leqq\frac{1}{2}\sqrt{n}\,\varepsilon\lambda(g_{l,j})^{\delta_1}$ $\leqq 2\sqrt{n}\,\varepsilon\lambda(g_{l,j})^{\delta_1}$. Hence, we have $a_{l,j}>\frac{1}{2}\varepsilon\left(\frac{3}{4}\right)^{\delta_1}\lambda(g_{l,j})^{\delta_1}$. Q.E.D.

We put $\Delta^*_{l,j}=\{\xi;\ |\xi_i-a_i|\leqq\frac{5}{9}a_{l,j},\ i=1,\cdots,n\}$ where $g_{l,j}=(a_1,\cdots,a_n)$. It is clear that $\Delta_{l,j}\subset\Delta^*_{l,j}$.

**Lemma 5.4.** *We take* $\varepsilon>0$ *sufficiently small so that Lemma* 5.3 (ii) *and the inequality* (5.2) *hold. Then if* $\Delta^*_{l,j}\cap\Delta_{l',j'}\neq\phi$, *it holds that* $\frac{1}{3}a_{l,j}\leqq a_{l',j'}\leqq 3a_{l,j}$.

Proof. Assume that $\Delta^*_{l,j}\cap\Delta_{l',j'}\neq\phi$ and $a_{l',j'}<\frac{1}{3}a_{l,j}$. By definitions and Lemma 5.3 (ii), $\Delta_{l',j'}\subset\Delta'_{l'-1,j''}$ for some $\Delta'_{l'-1,j''}$. Taking $\xi\in\Delta^*_{l,j}\cap\Delta_{l',j'}$ we have

$$|g_{l,j}-g'_{l'-1,j''}|\leqq|g_{l,j}-\xi|+|\xi-g'_{l'-1,j''}|\leqq\frac{5}{9}\sqrt{n}\,a_{l,j}+\frac{1}{2}\sqrt{n}\,a'_{l'-1,j''}$$
$$=\frac{5}{9}\sqrt{n}\,a_{l,j}+\sqrt{n}\,a_{l',j'}\leqq\frac{8}{9}\sqrt{n}\,a_{l,j}\leqq 2\sqrt{n}\,\varepsilon\lambda(g_{l,j})^{\delta_1}.$$

From (5.2) we have $a'_{l'-1,j''}=2a_{l',j'}<\frac{2}{3}a_{l,j}\leqq\frac{2}{3}\varepsilon\lambda(g_{l,j})^{\delta_1}\leqq\frac{2}{3}\left(\frac{4}{3}\right)^{\delta_1}$ $\varepsilon\lambda(g'_{l'-1,j''})^{\delta_1}\leqq\varepsilon\lambda(g'_{l'-1,j''})^{\delta_1}$.

This contradicts to the definition of $\Delta'_{l'-1,j''}$.

Hence we have $a_{l',j'} \geqq \frac{1}{3} a_{l,j}$.

By the same way we can prove that $a_{l',j'} \leqq 3 a_{l,j}$. Q.E.D.

We denote the volume of cube $\Delta$ by $|\Delta|$.

**Lemma 5.5.** *There is a positive integer M such that for any l, j, the number of cubes $\Delta^*_{l',j'}$ which satisfy $\Delta_{l,j} \cap \Delta^*_{l',j'} \neq \phi$ is at most M.*

Proof. By Lemma 5.4, we have,
$\bigcup_{l',j'} \Delta_{l',j'} \subset \{\xi;\ |\xi_i - a_i| \leqq 4 a_{l,j}\}$ where $g_{l,j} = (a_1, \cdots, a_n)$ and the union is taken for the cubes satisfying $\Delta^*_{l',j'} \cap \Delta_{l,j} \neq \phi$.

We write the number of such cubes by $M_0$.

Consider the number $M_1$ of cubes which satisfy that $|\Delta| \geqq \left(\frac{1}{3} a_{l,j}\right)^n$ and $\Delta \subset \{\xi;\ |\xi_i - a_i| \leqq 4 a_{l,j},\ i=1, \cdots, n\}$.
Then we have,

$$M_1\left(\frac{1}{3} a_{l,j}\right)^n \leqq (8 a_{l,j})^n,$$

hence, $M_1 \leqq 24^n$.

Using Lemma 5.4, we obtain $M_0 \leqq M_1 \leqq 24^n$. Q.E.D.

Rearranging $\{\Delta_{l,j}\}$, $\{g_{l,j}\}$ and $\{a_{l,j}\}$, we denote them by $\{\Delta_j\}_{j=1}^\infty$, $\{g_j\}_{j=1}^\infty$ and $\{a_j\}_{j=1}^\infty$.

Let $\psi(\xi) \in C_0^\infty(R^n)$ satisfy that $\psi(\xi) = 1$ for $|\xi_i| \leqq \frac{1}{2}$ $(i=1, \cdots, n)$ $0 \leqq \psi(\xi) \leqq 1$ and supp $\psi(\xi) \subset \{\xi;\ |\xi_i| \leqq \frac{5}{9},\ i=1, \cdots, n\}$.

We put $\psi_j(\xi) = \psi\left(\frac{\xi - g_j}{a_j}\right)$, $\tilde{\psi}(\xi) = \{\sum_j \psi_j(\xi)^2\}^{1/2}$ and $\varphi_j(\xi) = \psi_j(\xi)/\tilde{\psi}(\xi)$.

**Theorem 5.6.** *For sufficiently small $\varepsilon > 0$, we have,*

(i) $\varphi_j(\xi) \in C_0^\infty(R^n)$, $0 \leqq \varphi_j(\xi) \leqq 1$,

(ii) $\sum_j \varphi_j(\xi)^2 \equiv 1$,

(iii) $\sum_j |\partial_\xi^\alpha \varphi_j(\xi)| \leqq C_{\alpha,\varepsilon} \lambda(\xi)^{-\delta_1|\alpha|}$ *for any* $\alpha$,

(iv) *there exists a positive integer M such that each $\xi \in R^n$ is contained in the supports of at most M of $\{\varphi_j\}$.*

Proof. We put $\Delta_j^* = \left\{\xi;\ |\xi_i - b_i| \leqq \frac{5}{9} a_j (i=1, \cdots, n)\right\}$ here $g_j = (b_1, \cdots, b_n)$.
Then by definitions supp $\varphi_j \subset \Delta_j^*$ and $\psi_j(\xi) = 1$ for $\xi \in \Delta_j$.

Using Lemma 5.3 (i) and Lemma 5.5, $\tilde{\psi}(\xi)$ is well-defined and $1 \leqq \tilde{\psi}(\xi) \leqq M$.

Therefore from the definitions of $\varphi_j(\xi)$, we obtain (i), (ii) and (iv).

Since $\partial_\xi^\alpha \psi_j(\xi) = \psi^{(\alpha)}\left(\frac{\xi - g_j}{a_j}\right) a_j^{-|\alpha|}$, using Lemma 5.3 (iii) and (5.2) we have,

$$|\partial_\xi^\alpha \psi_j(\xi)| \leqq |\psi^{(\alpha)}\Big(\frac{\xi-g_j}{a_j}\Big)|\, a_j{}^{-|\alpha|} \leqq C_\alpha(\varepsilon c_0)^{-|\alpha|}\lambda(g_j)^{-\delta_1|\alpha|}$$
$$\leqq C^1_{\varepsilon,\alpha}\lambda(\xi)^{-\delta_1|\alpha|}, \text{ for any } \alpha .$$

Hence $|\partial_\xi^\alpha \tilde{\psi}(\xi)| \leqq C^1_{\varepsilon,\alpha} M\lambda(\xi)^{-\delta_1|\alpha|}$ for any $\alpha$.
Using these inequalities we obtain (iii). Q.E.D.

We can see that for any $(t, x) \in R^n$ and $\xi \in \operatorname{supp} \varphi_j$,

$$(5.3) \quad |\boldsymbol{H}(t, x, \xi)-\boldsymbol{H}(t, x, g_j)| \leqq C|\xi-g_j| \sup_{0\leqq s\leqq 1} \lambda(g_j+s(\xi-g_j))^{-\delta_1} \leqq C_2\varepsilon .$$

Taking $\varepsilon>0$ sufficiently small, we have the following Theorem.

**Theorem 5.7.** *We put $\boldsymbol{N}_{ij}=\boldsymbol{N}(t_i, x_i, g_j)$. There exist positive constants $c_1$, $c_2$, $\mu_1$ and $\mu_2$ such that*

$$(5.4) \quad c_1\|\boldsymbol{U}(b)\|_0^2-c_2\|\boldsymbol{U}(a)\|_0^2+\mu_1\int_a^b \|\boldsymbol{U}(t)\|_{m/2}^2 dt$$
$$-\mu_2\int_a^b \|\boldsymbol{U}(t)\|_0^2 dt$$
$$\leqq \operatorname{Re}\int_a^b \sum_{i,j} (\boldsymbol{N}_{ij}^{-1}\zeta_i\Phi_j\boldsymbol{R}\boldsymbol{U}, \boldsymbol{N}_{ij}^{-1}\zeta_i\Phi_j\boldsymbol{U})_0 dt$$

*for any $\boldsymbol{U}\in\{S(R^{n+1})\}^k$ where $\Phi_j=\varphi_j(D_x)$.*

Proof. We put $\boldsymbol{H}_{ij}=\boldsymbol{H}(t_i, x_i, \xi_j)$ and $\boldsymbol{R}_{ij}=\partial/\partial t-\boldsymbol{H}_{ij}\lambda(D_x)^m$. By Lemma 5.2, there exist positive constants $c_1$, $c_2$, $\mu_1$ and $\mu_2$ such that

$$c_1\|\boldsymbol{U}(b)\|_0^2-c_2\|\boldsymbol{U}(a)\|_0^2+\mu_1\int_a^b \|\boldsymbol{U}(t)\|_{m/2}^2 dt$$
$$-\mu_2\int_a^b \|\boldsymbol{U}(t)\|_0^2 dt \leqq \operatorname{Re}\int_a^b (\boldsymbol{N}_{ij}^{-1}\boldsymbol{R}_{ij}\boldsymbol{U}, \boldsymbol{N}_{ij}^{-1}\boldsymbol{U})_0 dt .$$

Hence we have

$$c_1\|\zeta_i(b)\Phi_j\boldsymbol{U}(b)\|_0^2-c_2\|\zeta_i(a)\Phi_j\boldsymbol{U}(a)\|_0^2$$
$$+\mu_1\int_a^b \|\zeta_i(t)\Phi_j\boldsymbol{U}(t)\|_{m/2}^2 dt-\mu_2\int_a^b \|\zeta_i(t)\Phi_j\boldsymbol{U}(t)\|_0^2 dt$$
$$\leqq \operatorname{Re}\int_a^b (\boldsymbol{N}_{ij}^{-1}\boldsymbol{R}_{ij}\zeta_i\Phi_j\boldsymbol{U}, \boldsymbol{N}_{ij}^{-1}\zeta_i\Phi_j\boldsymbol{U})_0 dt .$$

We can see that

$$\sum_{i,j} \|\zeta_i\Phi_j\boldsymbol{U}(t)\|_s^2 = \sum_{i,j} \operatorname{Re}(\zeta_i\lambda(D_x)^{2s}\zeta_i\Phi_j\boldsymbol{U}, \Phi_j\boldsymbol{U})_0$$
$$= \operatorname{Re}\sum_j (\sum_i \zeta_i(t, X)\cdot\lambda(D_x)^{2s}\cdot\zeta_i(t, X')\Phi_j\boldsymbol{U}, \Phi_j\boldsymbol{U})_0 .$$

Since $\sum_i \zeta_i(t, x)\lambda(\xi)^{2s}\zeta_i(t, x')\in S^{2s}_{0,\lambda}$, from Theorem 3.3, we can write $\sum_i \zeta_i(t, X)\cdot\lambda(D_x)^{2s}\cdot\zeta_i(t, X')=\lambda(D_x)^{2s}\boldsymbol{I}+\boldsymbol{p}^1(t, X, D_x)$ where $\boldsymbol{p}^1(t, x, \xi)\in S^{2s-1}_{0,\lambda}$. Hence we obtain

$$(5.5)\quad \begin{cases} \sum_{i,j} \|\zeta_i\Phi_j U(t)\|_s^2 \leqq \|U(t)\|_s^2 + C\|U(t)\|_{s-1/2}^2 \\ \sum_{i,j} \|\zeta_i\Phi_j U(t)\|_s^2 \geqq \|U(t)\|_s^2 - C\|U(t)\|_{s-1/2}^2 , \end{cases}$$

in particular,

$$(5.6)\quad \sum_{i,j} \|\zeta_i\Phi_j U(t)\|_0^2 = \|U(t)\|_0^2 .$$

$$\begin{aligned} &|\sum_{i,j} (N_{ij}^{-1}\zeta_i\Phi_j JU,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0| \\ &\leqq C\sum_{i,j} \|\zeta_i\Phi_j JU(t)\|_{-(m-\delta_1)/2}\|\zeta_i\Phi_j U(t)\|_{(m-\delta_1)/2} \\ &\leqq C\sum_{i,j} \{\|\zeta_i\Phi_j JU(t)\|_{-(m-\delta_1)/2}^2 + \|\zeta_i\Phi_j U(t)\|_{(m-\delta_1)/2}^2\} \\ &\leqq C\{\|JU\|_{-(m-\delta_1)/2}^2 + \|U(t)\|_{(m-\delta_1)/2}^2\} \\ &\leqq C_\varepsilon\|U(t)\|_{(m-\delta_1)/2}^2 \\ &\leqq \frac{\mu_1}{N_0}\|U(t)\|_{m/2}^2 + C_{N_0,\varepsilon}\|U(t)\|_0^2 , \end{aligned}$$

for any positive number $N_0$.

By (5.5), $\sum_{i,j} \|\zeta_i\Phi_j U(t)\|_{m/2}^2$

$$\geqq \left(1-\frac{1}{N_0}\right)\|U(t)\|_{m/2}^2 - C_{N_0}\|U(t)\|_0^2 .$$

Hence we get the inequality

$$\begin{aligned} (5.7)\quad & c_1\|U(b)\|_0^2 - c_2\|U(a)\|_0^2 + \left(1-\frac{2}{N_0}\right)\mu_1\int_a^b \|U(t)\|_{m/2}^2 dt \\ & -C_{N_0,\varepsilon}\int_a^b \|U(t)\|_0^2 dt \\ & \leqq \mathrm{Re}\int_a^b \sum_{i,j} (N_{ij}^{-1}R_{ij}\zeta_i\Phi_j U,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0 dt \\ & -\mathrm{Re}\int_a^b \sum_{i,j} (N_{ij}^{-1}\zeta_i\Phi_j JU,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0 dt . \end{aligned}$$

The right hand side of this inequality can be written in the form:

$$\mathrm{Re}\int_a^b \sum_{i,j} \{(N_{ij}^{-1}\zeta_i\Phi_j RU,\ N_{ij}^{-1}\zeta_1\Phi_j U)_0 + A_{ij}\}dt$$

where

$$\begin{aligned} A_{ij} &= \left(N_{ij}^{-1}\left(\frac{\partial}{\partial t}\zeta_i\right)\Phi_j U,\ N_{ij}^{-1}\zeta_i\Phi_j U\right)_0 \\ &+(N_{ij}^{-1}\zeta_i[\Phi_j,\ H]\lambda(D_x)^m U,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0 \\ &-(N_{ij}^{-1}H_{ij}[\lambda(D_x)^m,\ \zeta_i]\Phi_j U,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0 \\ &+(N_{ij}^{-1}\zeta_i\{H-H_{ij}\}\lambda(D_x)^m\Phi_j U,\ N_{ij}^{-1}\zeta_i\Phi_j U)_0 \\ &= I_{ij}+II_{ij}+III_{ij}+IV_{ij} . \end{aligned}$$

We can see that

$$(5.8)\quad |\sum_{i,j} I_{ij}| \leqq C\sum_{i,j}\left\{\left\|\left(\frac{\partial}{\partial t}\zeta_i\right)\Phi_j U\right\|_0^2 + \|\zeta_i\Phi_j U\|_0^2\right\} \leqq C_\varepsilon \|U(t)\|_0^2 .$$

$$|\sum_{i,j} II_{ij}| \leqq |\sum_j ((\sum_i (N^*_{ij})^{-1} N_{ij}^{-1}\zeta_i^2)[\Phi_j, H]\lambda(D_x)^m U, \Phi_j U)| .$$

By Theorem 5.3, we get $[\Phi_j, H] = p_j^2(t, X, D_x)$ where $p_j^2(t, x, \xi) \in S_{0,\lambda}^{-\delta_1}$. Thus,

$$|\sum_{i,j} II_{ij}| \leqq |(\sum_j \Phi_j \zeta_j^{(1)} p_j^2(t, X, D_x)\lambda(D_x)^m U, U)|$$

where $\zeta_j^{(1)} = \sum_i (N_{ij}^*)^{-1} N_{ij}^{-1} \zeta_i(t, x)^2$. Since

$\sum_j \varphi_j(\xi)\zeta_j^{(1)}(t, x') p_j^2(t, x', \xi')\lambda(\xi')^m \in S_{0,\lambda}^{0, m-\delta_1}$, we have

$$(5.9)\quad |\sum_{i,j} II_{ij}| \leqq |(p^3(t, X, D_x)U, U)|$$

$$\leqq \|p^3(t, X, D_x)U\|_{-(m-\delta_1)/2} \|U\|_{(m-\delta_1)/2}$$

$$\leqq C_\varepsilon \|U\|_{(m-\delta_1)/2}^2 \leqq \frac{\mu_1}{N_0}\|U(t)\|_{m/2}^2 + C_{N_0,\varepsilon}\|U(t)\|_0^2$$

where $p^3(t, x, \xi) \in S_{0,\lambda}^{m-\delta_1}$.

By the similar way, we can obtain

$$\sum_{i,j} III_{ij} = (p^4(t, X, D_x)U, U)_0$$

where $p^4(t, x, \xi) \in S_{0,\lambda}^{m-1}$. Hence we get

$$(5.10)\quad |\sum_{i,j} III_{ij}| \leqq C_\varepsilon \|U\|_{(m-1)/2}^2 \leqq \frac{\mu_1}{N_0}\|U(t)\|_{m/2}^2 + C_{N_0,\varepsilon}\|U(t)\|_0^2 .$$

To estimate the term $\sum_{i,j} IV_{ij}$, we write

$$\begin{aligned} IV_{ij} &= (N_{ij}^{-1}\zeta_i\{H - H_{ij}\}\Phi_j\lambda(D_x)^{m/2}U, N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^{m/2}U)_0 \\ &+ (N_{ij}^{-1}\zeta_i\{H - H_{ij}\}\Phi_j\lambda(D_x)^{m/2}U, N_{ij}^{-1}[\lambda(D_x)^{m/2}, \zeta_i]\Phi_j U)_0 \\ &+ (N_{ij}^{-1}[\zeta_i, \lambda(D_x)^{m/2}]\{H - H_{ij}\}\Phi_j\lambda(D_x)^{m/2}U, N_{ij}^{-1}\zeta_i\Phi_j U)_0 \\ &+ (N_{ij}^{-1}\zeta_i[H, \lambda(D_x)^{m/2}]\Phi_j\lambda(D_x)^{m/2}U, N_{ij}^{-1}\zeta_i\Phi_j U)_0 \\ &= B_{ii} + C_{ij} + D_{ij} + E_{ij} . \end{aligned}$$

By the similar way to above estimates (5.9) and (5.10), we can obtain

$$(5.11)\quad |\sum_{i,j}(C_{ij} + D_{ij})| = |(p^5(t, X, D_x)U, U)| \leqq C_\varepsilon \|U\|_{(m-1)/2}^2$$

$$\leqq \frac{\mu_1}{N_0}\|U(t)\|_{m/2}^2 + C_{N_0,\varepsilon}\|U(t)\|_0^2 ,$$

where $p^5(t, x, \xi) \in S_{0,\lambda}^{m-1}$, and

$$(5.12)\quad |\sum_{i,j} E_{ij}| = |(p^6(t, X, D_x)U, U)| \leqq C_\varepsilon \|U\|_{(m-\delta_1)/2}^2$$

$$\leqq \frac{\mu_1}{N_0}\|U(t)\|_{m/2}^2 + C_{N_0,\varepsilon}\|U(t)\|_0^2,$$

where $p^6(t, x, \xi) \in S_{0,\lambda}^{m-\delta_1}$. Furthermore we have

$$|\sum_{i,j} B_{ij}| \leqq C_0\{\sum_{i,j} N_0 \|\zeta_i\{H - H_{ij}\}\Phi_j\lambda(D_x)^{m/2}U\|_0^2$$

$$+ \frac{1}{N_0}\sum_{i,j}\|\zeta_i\Phi_j\lambda(D_x)^{m/2}U\|_0^2\}$$

where the constant $C_0$ is independent of $N_0$ and $\varepsilon$.

Using Theorem 3.3 and Corollary 3.4 (i), (ii), we obtain

$$\sum_{i,j}\|\zeta_i\{\boldsymbol{H}-\boldsymbol{H}_{ij}\}\Phi_j\lambda(D_x)^{m/2}\boldsymbol{U}\|_0^2$$
$$=(\boldsymbol{p}^7(t, X, D_x)\lambda(D_x)^{m/2}\boldsymbol{U}, \lambda(D_x)^{m/2}\boldsymbol{U})_0$$
$$+(\boldsymbol{p}^8(t, X, D_x)\lambda(D_x)^{m/2}\boldsymbol{U}, \lambda(D_x)^{m/2}\boldsymbol{U})_0,$$

where $\boldsymbol{p}^7(t, x, \xi)=\sum_{i,j}\zeta_i(t, x)^2\{\boldsymbol{H}(t, x, \xi)-\boldsymbol{H}_{ij}\}^*$

$$\times\{\boldsymbol{H}(t, x, \xi)-\boldsymbol{H}_{ij}\}\varphi_j(\xi)^2$$

and $\boldsymbol{p}^8(t, x, \xi)\in S_{0,\lambda}^{-\delta_1}$.

By the assumptions of $\boldsymbol{H}$, $\zeta_i$, $\varphi_j$ and $\boldsymbol{H}_{jj}$,

$$|\boldsymbol{p}^7(t, x, \xi)|\leqq\sum_{i,j}|\zeta_i(t, x)|^2|\varphi_j(\xi)|^2\{C_1+C_2\}^2\varepsilon^2\leqq C_3\varepsilon,$$

where $C_1$ is the constant in (iv) of the definition of $\{\zeta_i\}$ and $C_2$ is the one in (5.3), and $\partial_{\xi_i}\boldsymbol{p}^7(t, x, \xi)\in S_{0,\lambda}^{-\delta_1}$ $i=1, \cdots, n$.

Hence by Theorem 3.10, we have

$$|(\boldsymbol{p}^7(t, X, D_x)\lambda(D_x)^{m/2}\boldsymbol{U}, \lambda(D_x)^{m/2}\boldsymbol{U})_0|$$
$$\leqq C_4\varepsilon\|\boldsymbol{U}(t)\|_{m/2}^2+C_\varepsilon\|\boldsymbol{U}(t)\|_{(m-\delta_1/2)/2}^2.\quad \text{Therefore,}$$
$$\sum_{i,j}\|\zeta_i\{\boldsymbol{H}-\boldsymbol{H}_{ij}\}\Phi_j\lambda(D_x)^{m/2}\boldsymbol{U}\|_0^2$$
$$\leqq C_4\varepsilon\|\boldsymbol{U}(t)\|_{m/2}^2+C_\varepsilon\|\boldsymbol{U}(t)\|_{(m-\delta_1/2)/2}^2+C_\varepsilon\|\boldsymbol{U}(t)\|_{(m-\delta_1)/2}^2.$$

Thus we obtain

$$(5.13)\quad |\sum_{i,j}B_{ij}|\leqq\{C_0C_4N_0\}\varepsilon\|\boldsymbol{U}(t)\|_{m/2}^2+\frac{C_0}{N_0}\|\boldsymbol{U}(t)\|_{m/2}^2$$
$$+C_{N_0,\varepsilon}\|\boldsymbol{U}(t)\|_{(m-\delta_1/2)/2}^2$$
$$\leqq\left(C_0C_4N_0\varepsilon+\frac{C_0}{N_0}\right)\|\boldsymbol{U}(t)\|_{m/2}^2+\frac{\mu_1}{N_0}\|\boldsymbol{U}(t)\|_{m/2}^2+C_{N_0,\varepsilon}\|\boldsymbol{U}(t)\|_0^2.$$

By virtue of the inequalities (5.7)~(5.13), we obtain

$$(5.14)\quad c_1\|\boldsymbol{U}(b)\|_0^2-c_2\|\boldsymbol{U}(a)\|_0^2+\left\{\left(1-\frac{6}{N_0}\right)\mu_1-C_0C_4N_0\varepsilon-\frac{C_0}{N_0}\right\}$$
$$\times\int_a^b\|\boldsymbol{U}(t)\|_{m/2}^2dt-C_{N_0,\varepsilon}\int_a^b\|\boldsymbol{U}(t)\|_0^2dt$$
$$\leqq\mathrm{Re}\int_a^b\sum_{i,j}(N_{ij}^{-1}\zeta_i\Phi_j\boldsymbol{R}\boldsymbol{U}, N_{ij}^{-1}\zeta_i\Phi_j\boldsymbol{U})_0dt$$

Taking $\varepsilon\leqq\dfrac{\mu_1}{N_0^2C_0C_4}$ and $N_0$ sufficiently large so that $\mu_1-\dfrac{7\mu_1+C_0}{N_0}\geqq\dfrac{\mu_1}{2}$, we complete the proof. Q.E.D.

Let $r$ and $s$ be real numbers satisfying $r>m/2$ and let $-\infty\leqq a<b\leqq+\infty$.

**Theorem 5.8.** *For sufficiently small $\varepsilon$ there exist positive constants $c_1$, $c_2$, $\mu_1$ and $\mu_2$ such that*

$$(5.15)\quad c_1\|U(b)\|_\rho^2-c_2\|U(a)\|_\rho^2+\mu_1\int_a^b\|U(t)\|_{\rho+m/2}^2dt$$
$$-\mu_2\int_a^b\|U(t)\|_\rho^2dt$$
$$\leqq \mathrm{Re}\int_a^b\sum_{i,j}(N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho RU,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)_0dt$$

*for any* $U\in\{H_{r,s}(\Omega)\}^k$, *where* $\rho=r+s-m/2$ *and* $U(t)=\gamma_tU$, *and* $\gamma_t$ *is the trace operator defined in Lemma* 4.4.

Proof. At first we assume $r+s-m/2=\rho=0$, then by Theorem 5.7, the inequality (5.14) holds for $U\in\{S(R^{n+1})\}^k$. Since $R$: $\{H_{r,s}(\Omega)\}^k\to\{H_{r-m,s}(\Omega)\}^k$ is a continuous linear operator, the form

$$\int_a^b\sum_{i,j}(N_{ij}^{-1}\zeta_i\Phi_jRU,\ N_{ij}^{-1}\zeta_i\Phi_jV)_0dt$$

is a continuous sesquilinear form defined on $\{H_{r,s}(\Omega)\}^k\times\{H_{r,s}(\Omega)\}^k$, because of Proposition 4.8. Using the continuity of the trace operator $\gamma_t$, we obtain the theorem for $\rho=0$.

Let $r+s-m/2=\rho$. We have that $R\lambda(D_x)^\rho=\lambda(D_x)^\rho R+\{R\lambda(D_x)^\rho-\lambda(D_x)^\rho R\}=\lambda(D_x)^\rho R+[\lambda(D_x)^\rho,\ H]\lambda(D_x)^m+[\lambda(D_x)^\rho,\ J]$. By assumptions of $H$ and $J$, we have $[\lambda(D_x)^\rho,\ H]\lambda(D_x)^m=p^1(t,X,D_x)$ and $[\lambda(D_x)^\rho,\ J]=p^2(t,X,D_x)$ where $p^1(t,x,\xi)$ and $p^2(t,x,\xi)$ belong to $S_{0,\lambda}^{m+\rho-\delta_1}$.

Thus we have

$$|\,\mathrm{Re}\int_a^b\sum_{i,j}(N_{ij}^{-1}\zeta_i\Phi_j[\lambda(D_x)^\rho,\ H]\lambda(D_x)^mU,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)_0dt\,|$$
$$\leqq C\int_a^b\sum_{i,j}\{\|\zeta_i\Phi_jp^1(t,X,D_x)U\|_{-(m-\delta_1)/2}^2+\|\zeta_i\Phi_j\lambda(D_x)^\rho U\|_{(m-\delta_1)/2}^2\}dt$$
$$\leqq C_\varepsilon\int_a^b\|U(t)\|_{\rho+m/2-\delta_1/2}^2dt$$
$$\leqq\frac{1}{N_0}\int_a^b\|U(t)\|_{\rho+m/2}^2dt+C_{N_0,\varepsilon}\int_a^b\|U(t)\|_0^2dt$$

for any $U\in\{H_{r,s}(\Omega)\}^k$. Similarly,

$$|\,\mathrm{Re}\int_a^b\sum_{i,j}(N_{ij}^{-1}\zeta_i\Phi_j[\lambda(D_x)^\rho,\ J]U,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)_0dt\,|$$
$$\leqq\frac{1}{N_0}\int_a^b\|U(t)\|_{\rho+m/2}^2dt+C_{N_0,\varepsilon}\int_a^b\|U(t)\|_0^2dt$$

for any $U\in\{H_{r,s}(\Omega)\}^k$.

Taking $N_0$ sufficiently large and using (5.4) for $\lambda(D_x)^\rho U$ in place of $U$ we obtain the theorem. Q.E.D.

## 6. The Cauchy problem for the operator $R$

In the proof of Lemma 4 in [3] (p. 193) replacing $|\xi|^{2k}$ by $\lambda(\xi)^m$, we have the following lemma.

**Lemma 6.1.** *We fix an arbitrary point* $(t_0, x_0, \xi_0)$, *and put* $\boldsymbol{H}_0=\boldsymbol{H}(t_0, x_0, \xi_0)$ *and* $\boldsymbol{R}_0=\partial/\partial t-\boldsymbol{H}_0\lambda(D_x)^m$. *Then there exists* $C>0$ *such that*

$$(6.1)\quad \int_{R^{n+1}}(\tau^2+\lambda(\xi)^{2m}+\eta^2)|\tilde{\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi \leqq C\|(\boldsymbol{R}_0+\eta\boldsymbol{I})\boldsymbol{U}\|_{0,0}^2$$

*for any* $\eta>0$ *and* $\boldsymbol{U}\in\{S(R^{n+1})\}^k$, *where* $\boldsymbol{I}$ *is the* $k\times k$ *identity matrix and* $C$ *is a constant independent of* $(t_0, x_0, \xi_0)$.

**Theorem 6.2.** *There exist constants* $C_1$, $C_2>0$ *such that*

$$(6.2)\quad \int_{R^{n+1}}(\tau^2+\lambda(\xi)^{2m}+\eta^2)|\tilde{\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi \leqq C_1\|(\boldsymbol{R}+\eta\boldsymbol{I})\boldsymbol{U}\|_{0,0}^2+C_2\|\boldsymbol{U}\|_{0,0}^2$$

*for any* $\boldsymbol{U}\in\{S(R^{n+1})\}^k$.

Proof. For sufficiently small $\varepsilon>0$, we take $\{\zeta_i\}_i$, $\{\varphi_j\}_j$ as in Section 5 and put $\boldsymbol{H}_{ij}=\boldsymbol{H}(t_i, x_i, \xi_j)$. By Lemma 6.1, we have

$$\int(\tau^2+\lambda(\xi)^{2m}+\eta^2)|\tilde{\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi \leqq C\|(\boldsymbol{R}_{ij}+\eta\boldsymbol{I})\boldsymbol{U}\|_{0,0}^2$$

for any $\boldsymbol{U}\in\{S(R^{n+1})\}^k$, where $\boldsymbol{R}_{ij}=\partial/\partial t-\boldsymbol{H}(t_i, x_i, \xi_j)\lambda(D_x)^m$

Taking $\zeta_i(t, x)\varphi_j(D_x)\boldsymbol{U}(t, x)$ in place of $\boldsymbol{U}(t, x)$, we have

$$\int(\tau^2+\lambda(\xi)^{2m}+\eta^2)|\widetilde{\zeta_i\Phi_j\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi \leqq C\|(\boldsymbol{R}_{ij}+\eta\boldsymbol{I})\zeta_i\Phi_j\boldsymbol{U}\|_{0,0}^2 .$$

Now we shall estimate various error terms to obtain (6.2). At first,

$$\begin{aligned}
&\sum_{i,j}\int\tau^2|\widetilde{\zeta_i\Phi_j\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi \\
&=\sum_{i,j}\int\left|\frac{\partial}{\partial t}\{\zeta_i\Phi_j\boldsymbol{U}(t, x)\}\right|^2 dtdx \\
&\geqq\int\left|\frac{\partial}{\partial t}\boldsymbol{U}(t, x)\right|^2 dtdx-C\int|\boldsymbol{U}(t, x)|^2 dtdx \\
&=\int\tau^2|\tilde{\boldsymbol{U}}(\tau, \xi)|^2 d\tau d\xi-C\|\boldsymbol{U}\|_{0,0}^2 .
\end{aligned}$$

By the same way as in Section 5, we have

$$\begin{aligned}
&\sum_{i,j}\int\lambda(\xi)^{2m}|\zeta_i\Phi_j\widetilde{\boldsymbol{U}(\tau, \xi)}|^2 d\tau d\xi \\
&=\sum_{i,j}\|\lambda(D_x)^m\{\zeta_i\Phi_j\boldsymbol{U}\}\|_{0,0}^2 \\
&=\mathrm{Re}(\sum_{i,j}\Phi_j\zeta_i\lambda(D_x)^{2m}\zeta_i\Phi_j\boldsymbol{U}, \boldsymbol{U}) \\
&=\mathrm{Re}(\boldsymbol{p}^1(t, X, D_x)\boldsymbol{U}, \boldsymbol{U}),
\end{aligned}$$

where $\boldsymbol{p}^1(t, x, \xi)=\lambda(\xi)^{2m}\boldsymbol{I}+\boldsymbol{p}^2(t, x, \xi)$ and $\boldsymbol{p}^2(t, x, \xi)\in S_{0,\lambda}^{2m-\delta_1}$.

So we get,

$$\sum_{i,j}\int\lambda(\xi)^{2m}|\widetilde{\zeta_i\Phi_j U}(\tau,\xi)|^2 d\tau d\xi$$
$$\geqq\int\lambda(\xi)^{2m}|\tilde{U}(\tau,\xi)|^2 d\tau d\xi - C\|U\|^2_{0,m-\delta_1/2}.$$

We can see easily that

$$\sum_{i,j}\int\eta^2|\widetilde{\zeta_i\Phi_j U}(\tau,\xi)|^2 d\tau d\xi=\eta^2\int|\tilde{U}(\tau,\xi)|^2 d\tau d\xi.$$

Now we can write,

$$\sum_{i,j}\|(R_{ij}+\eta I)\zeta_i\Phi_j U\|^2_{0,0}$$
$$\leqq C\sum_{i,j}\|\zeta_i\Phi_j(R+\eta I)U\|^2_{0,0}+C\sum_{i,j}\|\zeta_i\Phi_j(R-R_{ij})U\|^2_{0,0}$$
$$+C\sum_{i,j}\|[R_{ij},\zeta_i\Phi_j]U\|^2_{0,0}.$$

Using the method as in the proof of Theorem 5.7, we have

$$\sum_{i,j}\|\zeta_i\Phi_j(R-R_{ij})U\|^2_{0,0}\leqq 2\sum_{i,j}\|\zeta_i\Phi_j(H-H_{ij})\lambda(D_x)^m U\|^2_{0,0}$$
$$+2\sum_{i,j}\|\zeta_i\Phi_j JU\|^2_{0,0}$$
$$\leqq 2^2\sum_{i,j}\|\zeta_i(H-H_{ij})\Phi_j\lambda(D_x)^m U\|^2_{0,0}$$
$$+2^2\sum_{i,j}\|\zeta_i[\Phi_j, H]\lambda(D_x)^m U\|^2_{0,0}+2\|JU\|^2_{0,0}$$
$$\leqq 2^2\varepsilon C\|U\|^2_{0,m}+C_\varepsilon\|U\|^2_{0,m-\delta_1/2}$$
$$+C_\varepsilon\|U\|^2_{0,m-\delta_1/2}+C\|U\|^2_{0,m-\delta_1/2},$$

and we have,

$$\sum_{i,j}\|[R_{ij},\zeta_i\Phi_j]U\|^2_{0,0}\leqq 2\sum_{i,j}\left\|\left(\frac{\partial}{\partial t}\zeta_i\right)\Phi_j U\right\|^2_{0,0}$$
$$+2\sum_{i,j}\|H_{ij}[\lambda(D_x)^m,\zeta_i]\Phi_j U\|^2_{0,0}$$
$$\leqq C\|U\|^2_{0,0}+C\|U\|^2_{0,m-1/2}.$$

Summerizing these inequalities, we have,

$$\int\{\tau^2+\lambda(\xi)^{2m}+\eta^2\}|\tilde{U}(\tau,\xi)|^2 d\tau d\xi$$
$$\leqq C\sum_{i,j}\|\zeta_i\Phi_j(R+\eta I)U\|^2_{0,0}+C_\varepsilon\|U\|^2_{0,m-\delta_1/2}+C_3\varepsilon\|U\|^2_{0,m}$$
$$\leqq C\|(R+\eta I)U\|^2_{0,0}+C_3\varepsilon\|U\|^2_{0,m}+C_\varepsilon\|U\|^2_{0,m-\delta_1/2}.$$

Hence, taking $\varepsilon$ sufficiently small, we get,

$$\int(\tau^2+\lambda(\xi)^{2m}+\eta^2)|\tilde{U}(\tau,\xi)|^2 d\tau d\xi$$
$$\leqq C\|(R+\eta I)U\|^2_{0,0}+C_\varepsilon\|U\|^2_{0,m-\delta_1/2}$$
$$\leqq C\|(R+\eta I)U\|^2_{0,0}+\frac{1}{2}\|U\|^2_{0,m}+C\|U\|^2_{0,0}.$$

Thus we obtain (6.2) for some constants $C_1$, $C_2>0$. Q.E.D.

**Theorem 6.3.** *For any real numbers $r$ and $s$, there exist positive constants $\eta_0$ and $c_0$ such that for any $\eta>\eta_0$, it holds that*

(6.3) $c_0\|U\|_{r+m,s}\leqq\|(R+\eta I)U\|_{r,s}\leqq C_\eta\|U\|_{r+m,s}$ *for any* $U\in\{H_{r+m,s}\}^k$, *for some positive constant* $C_\eta$.

Proof. The inequality $\|(R+\eta I)U\|_{r,s}\leqq C_\eta\|U\|_{r+m,s}$ is clear. Because $\sigma(R)=i\tau I-H(t,x,\xi)\lambda(\xi)^m-J(t,x,\xi)\in S^m_{0,\lambda_1(\tau,\xi)}$, so

$$\|(R+\eta I)U\|_{r,s}\leqq\eta\|U\|_{r,s}+\|RU\|_{r,s}$$
$$\leqq\eta\|U\|_{r+m,s}+\|\lambda(D_x)^s\lambda_1(D_t,D_x)^rRU\|_{0,0},$$

and by Corollary 3.2 (i), we can write $\lambda(D_x)^s\lambda_1(D_t,D_x)^rR=p^1(t,X,D_t,D_x)$ where $p^1(t,x,\tau,\xi)\lambda(\xi)^{-s}\lambda_1(\tau,\xi)^{-r-m}\in B(R^{2n+2})$.

Hence, $\|\lambda(D_x)^s\lambda_1(D_t,D_x)^rRU\|_{0,0}=\|p^1(t,X,D_t,D_x)U\|$
$$\leqq C\|\lambda(D_x)^s\lambda_1(D_t,D_x)^{r+m}U\|_{0,0}=C\|U\|_{r+m,s}.$$

Thus we get $\|(R+\eta I)U\|_{r,s}\leqq(C+\eta)\|U\|_{r+m,s}$,
for any $U\in\{S(R^{n+1})\}^k$.

For any $U\in\{S(R^{n+1})\}^k$,

$$\|(R+\eta I)U\|^2_{r,s}=\|\lambda(D_x)^s\lambda_1(D_t,D_x)^r(R+\eta I)U\|^2_{0,0}$$
$$\geqq\frac{1}{2}\|(R+\eta I)\lambda(D_x)^s\lambda_1(D_t,D_x)^rU\|^2_{0,0}$$
$$-2\|[R,\lambda(D_x)^s\lambda_1(D_t,D_x)^r]U\|^2_{0,0}.$$

Now from Theorem 6.2, we have

$$\|(R+\eta I)\cdot\lambda(D_x)^s\cdot\lambda_1(D_t,D_x)^rU\|^2_{0,0}$$
$$\geqq c\int(\tau^2+\lambda(\xi)^{2m}+\eta^2)\lambda(\xi)^{2s}\lambda_1(\tau,\xi)^{2r}|\tilde{U}(\tau,\xi)|^2d\tau d\xi-C\|U\|^2_{r,s}$$
$$\geqq c\|U\|^2_{r+m,s}+(\eta^2-C)\|U\|^2_{r,s}.$$

Using Corollary 3.4 (i), we get

$$\|[R,\lambda(D_x)^s\lambda_1(D_t,D_x)^r]U\|^2_{0,0}=\|p^2(t,X,D_t,D_x)U\|^2_{0,0}$$

where $p^2(t,x,\tau,\xi)\lambda(\xi)^{-s+\delta_1-m}\lambda_1(\tau,\xi)^{-r}\in B(R^{2(n+1)})$.

*So*, $\|[R,\lambda(D_x)^s\lambda_1(D_t,D_x)^r]U\|^2_{0,0}\leqq C\|U\|^2_{r,s+m-\delta_1}$
$$\leqq\varepsilon_0\|U\|^2_{r,s+m}+C_{\varepsilon_0}\|U\|^2_{r,s}\leqq\varepsilon_0\|U\|^2_{r+m,s}+C_{\varepsilon_0}\|U\|^2_{r,s}$$

for any $\varepsilon_0>0$. Thus, we obtain,

$$\|(R+\eta I)U\|^2_{r,s}\geqq\left(\frac{1}{2}C-2\varepsilon_0\right)\|U\|^2_{r+m,s}+\left(\frac{1}{2}\eta^2-C-C_{\varepsilon_0}\right)\|U\|^2_{r,s}.$$

Taking $\varepsilon_0$ sufficiently small and $\eta_0$ such that $\frac{1}{2}\eta_0^2 - C - C_{\varepsilon_0} = 0$, we have (6.3) for any $U \in \{S(R^{n+1})\}^k$. Hence we have the theorem. Q.E.D.

Let $R^*$ be the formal adjoint operator of $R$, then we have

$$R^* = -\partial/\partial t - \{H \cdot \lambda(D_x)^m\}^* - J^*$$
$$= -\partial/\partial t - H^* \cdot \lambda(D_x)^m - J_1$$

where $\sigma(J_1) = J_1(t, x, \xi) \in S_{0,\lambda}^{m-\delta_1}$ and $\sigma(H^*) = H(t, x, \xi)^* = \overline{{}^t H(t, x, \xi)}$.

In fact, by Corollary 3.2 (ii) and Corollary 3.4 (ii), we have that

$$\sigma(\{H \cdot \lambda(D_x)^m\}^*) - H(t, x, \xi)^* \lambda(\xi)^m \in S_{0,\lambda}^{m-\delta_1}$$

and $\sigma(J^*) = J^*(t, X, \xi) \in S_{0,\lambda}^{m-\delta_1}$.

Hence we can write,

$$R^* = -\partial/\partial t - H^* \cdot \lambda(D_x)^m - J_1 .$$

Using the same way as the proof of Theorem 6.2 and Theorem 6.3, we have that for any real $r$ and $s$, there exist constant $\eta_0$ and $c_0$ such that for any $\eta > \eta_0$ it holds that

(6.4) $\quad c_0 \|U\|_{r+m,s} \leqq \|(R^* + \eta I)U\|_{r,s} \leqq C_\eta \|U\|_{r+m,s}$ for any $U \in \{H_{r+m,s}\}^k$.

Using (6.3) and (6.4), we have,

**Corollary 6.4.** *For any real numbers $r$ and $s$, there exists positive constant $\eta_0$ such that for any $\eta > \eta_0$, $R + \eta I$ is a topological isomorphism of $\{H_{r,s}\}^k$ onto $\{H_{r-m,s}\}^k$* (See Theorem 2 in [8]).

Using Theorem 5.8 and Corollary 6.4, we have

**Theorem 6.5.** *For any real numbers $r$, $s$ and $a$, there exists $\eta_0$ such that for any $\eta > \eta_0$, $R + \eta I$ is an isomorphism of $\{H_{0,r,s}(\bar{\Omega}_{a,\infty})\}^k$ onto $\{H_{0,r-m,s}(\bar{\Omega}_{a,\infty})\}^k$.*

**Theorem 6.6.** *Let real numbers $r$, $s$, $a$ and $b$ satisfy $r > \frac{m}{2}$ and $-\infty < a < b < \infty$. Then the mapping $U \rightsquigarrow \langle RU, \gamma_a U \rangle$ is a topological isomorphism of $\{H_{r,s}(\Omega_{a,b})\}^k$ onto $\{H_{r-m,s}(\Omega_{a,b})\}^k \oplus \{H_{r+s-m/2}\}^k$.*

This theorem can be shown by using Lemma, 4.3, 4.4, 4.5 and Theorem 6.5 (See [8] and [13]).

## 7. Cauchy problem for operator $L$

Let real numbers $r$, $s$, $a$ and $b$ satisfy $r > (k-1/2)m$ and $-\infty < a < b < +\infty$, and let $\Omega = \Omega_{a,b}$.

Then we have the following main theorems.

**Theorem 7.1.** *The mapping $u \rightsquigarrow \langle Lu, \gamma_a u, \gamma_a \frac{\partial}{\partial t} u, \cdots, \gamma_a \left(\frac{\partial}{\partial t}\right)^{k-1} u \rangle$*

*is a one to one mapping from* $H_{r,s}(\Omega)$ *into* $H_{r-mk,s}(\Omega)\oplus H_{r+s-m/2}\oplus H_{r+s-3m/2}\oplus\cdots\oplus H_{r+s-(k-1/2)m}$.

Proof. We can see that

$$(7.1)\quad \sum_{i,j}\int_a^b (N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)dt$$
$$\geqq C\int_a^b\sum_{i,j}\|\zeta_i\Phi_j\lambda(D_x)^\rho U\|_0^2dt = C\int_a^b\|U(t)\|_\rho^2dt\,.$$

By Theorem 5.8 and (7.1), it holds that for any $\eta>0$,

$$c_1\|U(b)\|_\rho^2-c_2\|U(a)\|_\rho^2+\mu_1\int_a^b\|U(t)\|_{\rho+m/2}^2dt$$
$$+c(\eta-\mu_2)\int_a^b\|U(t)\|_\rho^2dt$$
$$\leqq\sum_{i,j}\mathrm{Re}\int_a^b(N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho\cdot(R+\eta I)U,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)\,dt$$

for any $U\in\{H_{r-m(k-1),s}(\Omega)\}^k$, where $\rho=r+s-(k-1/2)m$.

Since $-\infty<a<b<+\infty$, $e^{-\eta t}U\in\{H_{r,s}(\Omega)\}^k$ for any $U\in\{H_{r,s}(\Omega)\}^k$. For each $u\in H_{r,s}(\Omega)$, let $U=\begin{pmatrix}u_1\\ \vdots\\ u_k\end{pmatrix}$ where $u_j=\lambda(D_x)^{m(k-j)}D_t^{j-1}u$. Then $U\in\{H_{r-m(k-1),s}(\Omega)\}^k$ and $RU\in\{H_{r-mk,s}(\Omega)\}^k$. In the above inequality, replacing $U$ by $e^{-\eta t}U$ and putting $Lu=f\in H_{r-mk,s}(\Omega)$, we have

$$(7.2)\quad c_1e^{-\eta b}\|U(b)\|_\rho^2-c_2e^{-\eta a}\|U(a)\|_\rho^2$$
$$+\mu_1e^{-\eta b}\int_a^b\|U(t)\|_{\rho+m/2}^2dt+c(\eta-\mu_2)e^{-\eta b}\int_a^b\|U(t)\|_\rho^2dt$$
$$\leqq\sum_{i,j}\mathrm{Re}\int_a^b e^{-2\eta t}(N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho\{i\,Lu\}e_k,\ N_{ij}^{-1}\zeta_i\Phi_j\lambda(D_x)^\rho U)dt$$

for $\eta>\mu_2$. Assume that $Lu=f=0$. Then,

$$c_1e^{-\eta b}\|U(b)\|_\rho^2-c_2e^{-\eta a}\|U(a)\|_\rho^2$$
$$+\mu_1e^{-\eta b}\int_a^b\|U(t)\|_{\rho+m/2}^2dt+c(\eta-\mu_2)e^{-\eta b}\int_a^b\|U(t)\|_\rho^2dt$$
$$\leqq 0\,.$$

If $\gamma_a u=0$, $\gamma_a\dfrac{\partial}{\partial t}u=0,\cdots,\gamma_a\left(\dfrac{\partial}{\partial t}\right)^{k-1}u=0$, we can see that $U(a)=0$. Thus we have

$$c_1e^{-\eta b}\|U(b)\|_\rho^2+\mu_1e^{-\eta b}\int_a^b\|U(t)\|_{\rho+m/2}^2dt$$
$$+c(\eta-\mu_2)e^{-\eta b}\int_a^b\|U(t)\|_\rho^2dt\leqq 0\,.$$

This inequality means $U=0$ and therefore $u=0$. Q.E.D.

**Theorem 7.2.** *Under the same assumptions as Theorem* 7.1, *the mapping*

$u \rightsquigarrow <Lu, \gamma_a u, \gamma_a \frac{\partial}{\partial t} u, \cdots, \gamma_a \left(\frac{\partial}{\partial t}\right)^{k-1} u>$ *is a topological isomorphism from* $H_{r,s}(\Omega)$ *onto* $H_{r-mk,s}(\Omega) \oplus H_{r+s-m/2} \oplus H_{r+s-3m/2} \oplus \cdots \oplus H_{r+s-(k-1/2)m}$.

Proof. We denote $\mathcal{L}u = \langle Lu, \gamma_a u, \gamma_a \frac{\partial}{\partial t} u, \cdots, \gamma_a \left(\frac{\partial}{\partial t}\right)^{k-1} u\rangle$. By Theorem 7.1, the operator $\mathcal{L}$ is a one to one mapping from $H_{r,s}(\Omega)$ to $H_{r-mk,s}(\Omega) \oplus H_{r+s-m/2} \oplus \cdots \oplus H_{r+s-(k-1/2)m}$.

So we have only to show that $\mathcal{L}$ is an onto mapping, due to the open mapping theorem. But the fact that $\mathcal{L}$ is onto can be shown by the same way as the proof of Theorem 8 in [3]. In this case we use the argument on Theorem 4.16 in [13], in place of Theorem 9 of [8]. Q.E.D.

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