

ON THE CAUCHY PROBLEM FOR THE SYSTEM OF FUNDAMENTAL EQUATIONS DESCRIBING THE MOVEMENT OF COMPRESSIBLE VISCOUS FLUID

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Since the preceding century, a great many studies, classical and modern, have been made on the movement of non-viscous fluid, whether compressible or incompressible, (cf. Lichtenstein [29], Bers [2], Bergmann [1, bibliography], Goldstein [15]). As seen from a mathematical point of view, the boundary layer theory by Prandtl [34], [35] is only a variation of the above-mentioned studies.

In early thirties of the present century, Leray [26], [27], [28] began strictly mathematical discussions on the non-stationary problem for incompressible viscous fluid, inventing a concept “solution turbulente”, which corresponds to the concept “weak solution” in the present day. About twenty years later, Hopf [16], Kiselev-Ladyzhenskaya [22], and Lions [30], [31] proved the existence of a “generalized” or “weak” solution of this problem, each independently. (Kiselev-Ladyzhenskaya and Lions proved the uniqueness of the solution in a space of functions). In Japan, Ito [17] showed the existence and uniqueness of a classical solution of this problem, and Kato-Fujita [20], [12] improved these results above by the method of functional analysis, utilizing the theories of fractional powers and semi-groups of operators by Glushko-Krein [14], Krasnosel’sky-Sobolevsky [23] (also, cf. [36], [37]), Cattabriga [23], and Kato [18], [19]. They proved the existence and uniqueness of a solution in a certain space of functions. The stationary problem for incompressible viscous fluid was treated by Finn [7] [8], Fujita [11], etc.

As for the stationary and non-stationary problems for compressible viscous fluid, so far as the author knows, there have been very few studies on them, probably, because of the complexities enjoyed by the system of equations describing the movement of this kind of fluid.

In view of these circumstance, we dare to find a way of solving this problem firstly from a classical point of view, restricting the spatial domain to R^3 only. The whole discussion consists of two parts: §1~§4 and §5~§6. The former introduces the problem to be discussed and treats linear equations connected with it. The latter discusses on the original problem, trying to demonstrate the existence and uniqueness of a solution. In the last stage, Tikhonov’s fixed point theorem [39] plays a very important role. The proof of the uniqueness of the solution requires very delicate techniques and considerably lengthy calculations.

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CONTENTS

1. Introduction..... 61
 1. 1. Introduction and main theorems..... 61
 1. 2. Preliminary surveys..... 63
 2. The fundamental solution of a linear problem 66
 2. 1. Parametrix $Z(x-\xi, t; y, \tau)$ 66
 2. 2. Some calculations 69
 2. 3. Construction of the fundamental solution 71
 3. Estimates for the fundamental solution 73
 3. 1. Hölder-continuity in x of $\Phi(x, t; \xi, \tau)$ (I) 74
 3. 2. Hölder-continuity in x of $\Phi(x, t; \xi, \tau)$ (II) 75
 3. 3. Hölder-continuity in t of $\Phi(x, t; \xi, \tau)$ 79
 3. 4. Estimates for $q(x, t)$ 82
 3. 5. Estimates for $I(x, t; \xi, \tau)$ 85
 4. Estimates for the solution in $H_T^{2+\alpha}$ of a linear problem..... 89
 5. The existence of a bounded solution of (1. 9)-(1. 10) 94
 5. 1. Estimates for the characteristic curves 94
 5. 2. Operator G_{T_1} 97
 5. 3. The subset $S_{T'}$ 101
 5. 4. The continuity of $G_{T'}$ on $S_{T'}$ as a subset of $\dot{H}_T^{2+\alpha\beta} \times \dot{H}_T^{1+\alpha\beta}$ 103
 5. 5. The existence of a bounded solution of (1. 9)-(1. 10)111
 6. The problem of uniqueness112
 References119

§ 1. Introduction.

1. 1. **Introduction and main theorems.** The movement of fluid is described by a system of five differential equations corresponding to the laws of conservation of mass, momenta, and energy.

i) Conservation of mass.

$$(1. 1) \quad \frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0, \quad (\rho, \text{ density; } v, \text{ velocity}).$$

ii) Conservation of momenta.

$$(1. 2) \quad \rho \frac{Dv}{Dt} = \rho f + \operatorname{Div} P,$$

$$\left(\frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla); f, \text{ outer force; } P, \text{ stress tensor; } \operatorname{Div} P = \frac{\partial}{\partial x_i} P_{ik} \right).$$

In isotropic Newtonian fluid, P is expressed in the following way:

$$(1.3) \quad P_{ik} = -\left(p + \frac{2}{3}\mu\Theta\right)\delta_{ik} + \mu e_{ik}, \quad (p, \text{ pressure; } \mu, \text{ viscosity}),$$

where

$$(1.3)' \quad e_{ik} = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}, \quad \Theta = \frac{1}{2}e_{kk} = \text{div } v.$$

From (1.2), by (1.3), we obtain the so-called Navier-Stokes equation:

$$(1.4) \quad \rho \frac{Dv}{Dt} = \rho f - \text{grad} \left(p + \frac{2}{3}\mu \text{div } v \right) + (\nabla \cdot \mu \nabla) v + \text{Div} (\mu \text{grad } v).$$

a) If μ is a constant, (1.4) has the form:

$$(1.4)' \quad \rho \frac{Dv}{Dt} = \rho f - \nabla p + \frac{\mu}{3} \text{div } v + \mu \Delta v.$$

b) If μ and ρ are constants, (1.1) and (1.4) have the forms:

$$(1.1)' \quad \text{div } v = 0,$$

$$(1.4)'' \quad \rho \frac{Dv}{Dt} = \rho f - \nabla p + \mu \Delta v.$$

c) If $\mu=0$, (1.4) has the form:

$$(1.4)''' \quad \rho \frac{Dv}{Dt} = \rho f - \nabla p, \quad (\text{Euler's equation of perfect gas}).$$

iii) Conservation of energy.

$$(1.5) \quad \rho \frac{DE}{Dt} = \frac{1}{2} P_{ik} e_{ik} + \text{div} (\kappa \text{grad } \theta) = \Psi - p \text{div } v + \text{div} (\kappa \text{grad } \theta).$$

(E , internal energy; κ , heat conductivity; θ , absolute temperature),

where Ψ is the function of dissipation defined by

$$(1.6) \quad \Psi \equiv p\Theta + \frac{1}{2} P_{ik} e_{ik} = \mu \sum_{i < k} e_{ik}^2 + \frac{\mu}{6} \sum_{i < j} (e_{ii} - e_{jj})^2 (\geq 0).$$

$$(1.7) \quad \frac{DE}{Dt} = \frac{\partial E}{\partial \rho} \frac{D\rho}{Dt} + \frac{\partial E}{\partial \theta} \frac{D\theta}{Dt} = -\frac{1}{\rho^2} \left(\theta \frac{\partial p}{\partial \theta} - p \right) (-\rho \text{div } v) + C_v \left(\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta \right),$$

(C_v , specific heat at constant volume).

Thus, we have

$$(1.5)' \quad \rho C_v \frac{\partial \theta}{\partial t} = \text{div} (\kappa \text{grad } \theta) + \Psi - \theta \frac{\partial p}{\partial \theta} \text{div } v - C_v \rho v \cdot \nabla \theta.$$

If we assume that p is virially expanded, i.e.,

$$(1.8) \quad p = \theta \sum_{n=1}^{\infty} \hat{a}_n \rho^n, \quad (0 < \rho < \rho^* = \text{radius of convergence of the power-series} < +\infty),$$

then (1.5)' has a form,

$$(1.5)'' \quad \rho C_v \frac{\partial \theta}{\partial t} = \text{div}(\kappa \text{grad } \theta) + \Psi - p \text{div } v - C_v \rho v \cdot \nabla \theta.$$

When μ , κ , and C_v are constants and p is virially expanded, the system of equations to be considered is as follows:

$$(1.9)^1 \quad \begin{cases} \frac{\partial \rho}{\partial t} + \text{div } \rho v = 0, \\ \rho \frac{\partial v}{\partial t} = \rho f - \frac{\partial p}{\partial \theta} \nabla \theta - \frac{\partial p}{\partial \rho} \nabla \rho + \mu \Delta v + \frac{\mu}{3} \nabla \cdot \text{div } v - \rho(v \cdot \nabla)v, \\ \rho C_v \frac{\partial \theta}{\partial t} = \kappa \Delta \theta + \Psi - p \cdot \text{div } v - C_v \rho v \cdot \nabla \theta. \end{cases}$$

Hereafter, we shall consider a Cauchy problem of (1.9) in which the initial condition is given by

$$(1.10) \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x) (\geq 0), \quad \rho(x, 0) = \rho_0(x) (> 0).$$

As final results for the Cauchy problem (1.9)-(1.10), we have the following three main theorems. (As for notations, see 1.2, 5.1, 5.5, and §6).

THEOREM 1. For some $T' \in (0, T]$, there exists $(v, \theta, \rho) \in H_T^{2+\alpha} \times H_T^{2+\alpha} \times B_T^{1+\alpha}$ ($\theta \geq 0$ and $+\infty > \rho^* > \rho > 0$) such that (v, θ, ρ) satisfies (1.9)-(1.10), where $(v_0, \theta_0, \rho_0) \in H^{2+\alpha} \times H^{2+\alpha} \times H^{1+\alpha}$ ($0 < \bar{\rho}_0 \leq \rho_0 \leq \bar{\rho}_0 < \rho^*$, $\bar{\rho}_0$ and $\bar{\rho}_0$ are constants) and $f \in H_T^\alpha$.

THEOREM 2. For $(v_0, \theta_0, \rho_0) \in H^{3+L} \times H^{2+\alpha} \times H^{2+L}$ ($0 < \bar{\rho}_0 \leq \rho_0 \leq \bar{\rho}_0 < \rho^* < +\infty$) and $f \in H_T^\alpha$, if there exists a solution (v, θ, ρ) of (1.9)-(1.10) in

$$\left[H_T^{2+\alpha} \cap \left\{ u(x, t) : D_x^m u (|m|=3) \text{ are continuous, } \sum_{|m|=3} (|D_x^m u|_T^{(0)} + |D_x^m u|_{x,T}^{(L)}) < +\infty \right\} \right] \\ \times H_T^{2+\alpha} \times B_T^{1+\alpha},$$

then the solution is unique there. ("L" indicates that the Hölder exponent $\alpha=1$).

THEOREM 3. If $(v_0, \theta_0, \rho_0) \in H^{4+\alpha} \times H^{3+\alpha} \times H^{2+L}$ ($0 < \bar{\rho}_0 \leq \rho_0 \leq \bar{\rho}_0 < \rho^* < +\infty$) and $f \in H_T^{2+\alpha}$, then, for some $T' \in (0, T]$, there exists a unique solution (v, θ, ρ) of (1.9)-(1.10) in $H_T^{4+\alpha} \times H_T^{3+\alpha} \times B_T^{1+\alpha}$ ($\theta \geq 0$ and $\rho^* > \rho > 0$).

1.2. Preliminary surveys. In connection with (1.9)², we consider the following linear problem:

$$(1.11) \quad \begin{cases} \frac{\partial v}{\partial t} = \sigma(x, t) \left(\Delta + \frac{1}{3} \nabla \operatorname{div} \right) v + f, & ((x, t) \in R_T^3 \equiv R^3 \times (0, T)), \\ v(x, 0) = v_0(x), & (x \in R^3), \end{cases}$$

where

$$(1.11) \quad \sigma, f \in H_T^\alpha, \quad 0 < \sigma_0 \leq \sigma(x, t) \leq \sigma_1 \equiv |\sigma|_T^{(0)} < +\infty, \quad v_0(x) \in H^{2+\alpha}, \quad (\alpha \in (0, 1)).$$

For $\alpha \in (0, 1)$, $H_T^{n+\alpha}$ (n , integer ≥ 0) is a Banach space of real functions $g(x, t)$ defined on $\overline{R_T^3}$ such that

$$(1.12) \quad \|g\|_T^{(n+\alpha)} \equiv \sum_{2r+|m|=0}^n |D_t^r D_x^m g|_T^{(0)} + \sum_{2r+|m|=n-1}^n |D_t^r D_x^m g|_{t, T}^{(\alpha/2)} + \sum_{2r+|m|=n} |D_t^r D_x^m g|_{x, T}^{(\alpha)} < +\infty,$$

where, for $h(x, t)$,

$$(1.12)' \quad \begin{cases} |h|_T^{(0)} \equiv \sup_{(x, t) \in \overline{R_T^3}} |h(x, t)|, & |h|_{t, T}^{(\alpha/2)} \equiv \sup_{\substack{t, t' \in T \\ (t \neq t')}} \frac{|h(x, t) - h(x, t')|}{|t - t'|^{\alpha/2}}, \\ |h|_{x, T}^{(\alpha)} \equiv \sup_{\substack{x, x' \in T \\ (x \neq x')}} \frac{|h(x, t) - h(x', t)|}{|x - x'|^\alpha}, & |h|_T^{(\alpha)} \equiv |h|_{x, T}^{(\alpha)} + |h|_{t, T}^{(\alpha/2)}. \end{cases}$$

and, for $m = (m_i)$ ($m_i \geq 0$),

$$(1.12)'' \quad |m| \equiv \sum_i m_i, \quad D_x^m \equiv \frac{\partial^{|m|}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}.$$

For a vector function $g(x, t) = (g_i)$,

$$(1.13) \quad |g|_T^{(0)} \equiv \sum_i |g_i|_T^{(0)}, \quad |g|_T^{(\alpha)} \equiv \sum_i |g_i|_T^{(\alpha)}, \quad \|g\|_T^{(n+\alpha)} \equiv \sum_i \|g_i\|_T^{(n+\alpha)}, \quad \text{etc.}$$

$H^{n+\alpha}$ is a Banach space of functions $u(x)$ defined on R^3 such that

$$(1.13)' \quad \|u\|^{n+\alpha} \equiv \sum_{|m|=0}^n |D_x^m u|^{(0)} + \sum_{|m|=n} |D_x^m u|^{(\alpha)} < +\infty,$$

where

$$(1.13)'' \quad |u|^{(0)} \equiv \sup_{x \in R^3} |u(x)|, \quad |u|^{(\alpha)} \equiv \sup_{\substack{x, x' \in R^3 \\ (x \neq x')}} \frac{|u(x) - u(x')|}{|x - x'|^\alpha}.$$

We put

$$(1.14) \quad P_0(D_x) \equiv \Delta + \frac{1}{3} \nabla \operatorname{div}.$$

LEMMA 1.1. *The system (1.11) is uniformly parabolic in Petrowsky's sense, i.e., there exists a number $\delta > 0$ such that*

$$(1.15) \quad \max_j \sup_{|\xi|=1} \operatorname{Re} \{\lambda_j(\xi; x, t)\} \leq -\delta, \quad (\mathbf{v}(x, t) \in R_T^3),$$

(λ_j 's are the roots of $\det [\sigma(x, t)P_0(i\xi) - I\lambda]$).

Proof. For $|\xi|=1$, λ_j ($j=1, 2, 3$) satisfies

$$(1.16) \quad \det [\sigma(x, t)P_0(i\xi) - I\lambda] = \lambda^3 + \frac{4}{3} \sigma^2 \lambda + \sigma^3 \left(\frac{4}{3} - \frac{2}{27} \xi_1^2 \xi_2^2 \xi_3^2 \right) \equiv D(\lambda; \sigma; \xi) = 0.$$

$$(1.16)' \quad \begin{cases} \frac{d}{d\lambda} D(\lambda; \sigma; \xi) = 3\lambda^2 + \frac{8}{3} \sigma \lambda + \frac{11}{3} \sigma^2 = 3 \left(\lambda + \frac{4}{9} \sigma \right)^2 + \frac{7}{3} \sigma^2 \geq \frac{7}{3} \sigma^2, & (\text{for } \lambda \in R^1), \\ \frac{d}{d\lambda} D(0; \sigma; \xi) = \frac{11}{3} \sigma^2, \\ D(0; \sigma; \xi) = \sigma^3 \left(\frac{4}{3} - \frac{2}{27} \xi_1^2 \xi_2^2 \xi_3^2 \right) \geq \frac{35}{27} \sigma^3 > 0. \end{cases}$$

By (1.16)', $D(\lambda; \sigma; \xi)$ has one negative root λ_1 and two complex conjugate roots λ_2 and $\lambda_3 = \bar{\lambda}_2$. λ_1, λ_2 , and λ_3 satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{4}{3} \sigma.$$

$D_\lambda(\lambda; \xi; \sigma) \equiv (d/d\lambda)D(\lambda; \xi; \sigma)$ increases monotonically in $[-4\sigma/9, +\infty)$, and

$$D\left(-\frac{4}{9} \sigma; \sigma; \xi\right) = \frac{4}{3} \left(1 - \frac{361}{243}\right) \sigma^3 - \frac{2}{27} \sigma^3 (\xi_1^2 \xi_2^2 \xi_3^2) \leq -\frac{4}{3} \cdot \frac{118}{243} \sigma^3 < 0.$$

Therefore,

$$(1.17) \quad \begin{aligned} -\frac{4}{9} \sigma \leq \lambda_1 &\leq -\{D_\lambda(0; \sigma; \xi)\}^{-1} \cdot D(0; \sigma; \xi) \leq -\frac{3}{11} \sigma \left(\frac{4}{3} - \frac{2}{27} \xi_1^2 \xi_2^2 \xi_3^2 \right) \\ &\leq -\frac{3}{11} \sigma \left(\frac{4}{3} - \frac{2}{27^2} \right) \leq -\frac{35}{99} \sigma \leq -\frac{35}{99} \sigma_0 < 0, \\ \operatorname{Re} \lambda_2 = \operatorname{Re} \lambda_3 &= \frac{1}{2} \left[-\frac{4}{3} \sigma - \lambda_1 \right] \leq \frac{1}{2} \left[-\frac{4}{3} \sigma + \frac{4}{9} \sigma \right] = -\frac{4}{9} \sigma \leq -\frac{4}{9} \sigma_0 < 0. \end{aligned}$$

Thus,

$$(1.18) \quad \max_j \sup_{|\xi|=1} \operatorname{Re} \lambda_j(\xi; x, t) \leq -\frac{35}{99} \sigma_0.$$

This shows that the system (1.11) is uniformly parabolic in Petrowsky's sense. Q.E.D.

In the same way, we can relate (1.9)^s with the following linear equation:

$$(1.19) \quad \begin{cases} \frac{\partial \theta}{\partial t} = \bar{\sigma}(x, t) \Delta \theta + g, & (\bar{\sigma}, g \in H_T^{\alpha, \alpha/2}, 0 < \bar{\sigma}_0 \leq \bar{\sigma} \leq \bar{\sigma}_1 < +\infty), \\ \theta(x, 0) = \theta_0(x). \end{cases}$$

It is obvious that this equation is uniformly parabolic in Petrowsky's sense. So, we can treat (1.19) in parallel with (1.11).

§ 2. The fundamental solution of a linear problem.

2.1. **Parametrix** $Z(x-\xi, t; y, \tau)$. In connection with (1.11), we consider the following system of ordinary differential equations in t :

$$(2.1) \quad \begin{cases} \frac{dV}{dt} = \sigma(y, \tau) P_0(i\zeta) V(\zeta, t; y, \tau), & (\zeta \in C^3), \\ V(\zeta, t; y, \tau)|_{t=\tau} = I & \text{(unit matrix)}. \end{cases}$$

V can be solved directly from (2.1), i.e.,

$$(2.2) \quad V(\zeta, t; y, \tau) = \exp \{ (t-\tau) \sigma(y, \tau) P_0(i\zeta) \}.$$

We define the parametrix $Z = (Z^{ij}(x-\xi, t; y, \tau))$ of (2.1) in the following way:

$$(2.3) \quad Z^{ij}(x-\xi, t; y, \tau) \equiv \frac{1}{(2\pi)^3} \int_{R^3} e^{i\xi_0(x-\xi)} V^{ij}(\xi_0, t; y, \tau) d\xi_0, \quad (i, j=1, 2, 3).$$

We note that the following relation holds between the roots λ_j ($j=1, 2, 3$) of $\det [\sigma(x, t) P_0(i\zeta) - \lambda I]$ and the roots $\lambda_j^{(0)}$ of $\det [P_0(i\zeta) - \lambda I]$:

$$(2.4) \quad \lambda_j(\xi; x, t) = \sigma(x, t) \lambda_j^{(0)}(\xi), \quad (j=1, 2, 3).$$

For $\lambda_j^{(0)}$ ($i=1, 2, 3$), we have

$$(2.5) \quad \max_j \sup_{|\xi|=1} \operatorname{Re} \lambda_j^{(0)}(\xi) \leq -\frac{35}{99},$$

$$(2.5)' \quad \lambda_j^{(0)}(\nu\zeta) = \nu^2 \lambda_j^{(0)}(\zeta), \quad (\nu \geq 0, \zeta \in C^3).$$

The following two lemmas are taken from Friedman ([9], [10]) with some modifications.

LEMMA 2.1. *If $f(z)$ is a continuous real-valued function on C^n , satisfying*

$$(2.6) \quad \begin{cases} f(\nu z) = \nu^{2p} f(z), & (\nu \geq 0, z \in C^n; p, \text{ positive integer}), \\ f(x) \leq -\delta_0 |x|^{2p}, & (x \in R^n, \delta_0 > 0), \end{cases}$$

then, there exists a number $a_0 > 0$ such that

$$(2.6)' \quad f(x+iy) \leq -\frac{\delta_0}{2} |x|^{2p} + a_0 |y|^{2p}, \quad (x, y \in R^3), \quad (\text{cf. [9], CH. IX, § 2}).$$

LEMMA 2.2. For the matrix $e^{t\sigma(y, \tau)} P_0(i\zeta)$,

$$e^{t\sigma(y, \tau) P_0(i\zeta)} \leq 2\sqrt{3} \{1 + \sigma(y, \tau) |P_0(i\zeta)| t + \sigma(y, \tau)^2 |P_0(i\zeta)|^2 t^2\} \exp \left\{ t \max_i \operatorname{Re} \{ \sigma(y, \tau) \lambda_i^{(0)}(\zeta) \} \right\},$$

(cf. [10], CH. 7, 2),

where, for the matrix $A = (a_{ij})$,

$$(2.7)' \quad |A| \equiv \sum_{i,j} |a_{ij}|.$$

$$\left(\text{N.B.: } \frac{|A|}{2\sqrt{3}} \leq \|A\| \text{ (norm as an operator)} \leq (\sum |a_{ij}|^2)^{1/2} \leq |A| \right).$$

By Lemma 2.1, we have:

LEMMA 2.3. There exist δ and $a > 0$ such that

$$(2.8) \quad \operatorname{Re} \{ \sigma(y, \tau) \lambda_i^{(0)}(\zeta = \xi + i\eta) \} \leq \sigma(y, \tau) \left\{ -\frac{\delta}{2} |\xi|^2 + a |\eta|^2 \right\}, \quad (i=1, 2, 3).$$

Proof. We have only to put,

$$(2.8)' \quad f_i(\zeta) = \operatorname{Re} \{ \sigma(y, \tau) \lambda_i^{(0)}(\zeta) \}, \quad \delta_0 = \delta \equiv \frac{35}{99}. \quad \text{Q.E.D.}$$

If we define for $\zeta \in C^3$

$$(2.9) \quad |\zeta|^2 \equiv |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3|^2,$$

then

$$(2.9)' \quad |P_0(i\zeta)| \leq 4|\zeta|^2.$$

By the lemmas 2.2 and 2.3, we have:

LEMMA 2.4.

$$(2.10) \quad |V(\zeta, t; y, \tau)| = |e^{(t-\tau)\sigma(y, \tau) P_0(i\zeta)}| \\ \leq C_0 \exp \left\{ (t-\tau) \left[-\frac{\delta}{4} \sigma_0 |\xi|^2 + \left(\frac{a}{4} + \delta \right) \sigma_1 |\eta|^2 \right] \right\} \equiv C_0 e^{(t-\tau) Q_0(\xi, \eta)}.$$

Proof.

$$-\frac{\delta}{2} |\xi|^2 + a |\eta|^2 = -\frac{\delta}{4} |\zeta|^2 - \frac{\delta}{4} |\xi|^2 + \left(\frac{\delta}{4} + a \right) |\eta|^2.$$

$$|\sigma(y, \tau)(t-\tau)|\zeta|^2|^2 \exp \left\{ -\frac{\delta}{4}(t-\tau)\sigma(y, \tau)|\zeta|^2 \right\} \leq \left(\frac{4}{\delta} \right)^2 K(2, 0), \quad \left(K(\alpha, b) \equiv \left(\frac{\alpha}{e(1-b)} \right)^\alpha \right).$$

Hence,

$$|V(\zeta, t; y, \tau)| \leq C_0 \cdot \exp \left\{ (t-\tau)\sigma(y, \tau) - \frac{\delta}{4}|\xi|^2 + \left(a + \frac{\delta}{4} \right) |\eta|^2 \right\} \leq C_0 e^{(t-\tau)Q_0(\xi, \eta)}. \quad \text{Q.E.D.}$$

LEMMA 2.5. For the index-vector m ($|m| \geq 0$),

$$(2.11) \quad D_x^m Z^{ij}(x-\xi, t; y, \tau) \leq C_1^{|m|} (t-\tau)^{-(|m|+3)/2} \exp \left\{ -\frac{|x-\xi|^2}{2A_1\sigma_1(t-\tau)} \right\}, \quad \left(a_1 \equiv a + \frac{\delta}{4} \right).$$

Proof. By Cauchy's theorem, for $\eta \in R^3$,

$$(2.12) \quad Z^{ij}(x-\xi, t; y, \tau) = \frac{1}{(2\pi)^3} \int_{R^3} e^{i(\xi_0+i\eta)(x-\xi)} V^{ij}(\xi_0+i\eta, t; y, \tau) d\xi_0.$$

$$(2.13) \quad D_x^m Z^{ij}(x-\xi, t; y, \tau) = \frac{1}{(2\pi)^3} \int_{R^3} (i(\xi_0+i\eta))^m e^{i(\xi_0+i\eta)(x-\xi)} V^{ij}(\xi_0+i\eta, t; y, \tau) d\xi_0.$$

$$|(\xi_0+i\eta)^m| \leq (|\xi_0|+|\eta|)^{|m|} \leq 2^{|m|-1} (|\xi_0|^{|m|} + |\eta|^{|m|}).$$

Hence,

$$\begin{aligned} |D_x^m Z^{ij}(x-\xi, t; y, \tau)| &\leq \frac{C_0 2^{|m|-1}}{(2\pi)^3} \cdot e^{-\eta \cdot (x-\xi)} \cdot e^{a_1 \sigma(y, \tau)} |\eta|^2 (t-\tau) \\ &\quad \times \int_{R^3} (|\xi_0|^{|m|} + |\eta|^{|m|}) e^{-(t-\tau)(\delta/4)\sigma(y, \tau)|\xi_0|^2} d\xi_0 \end{aligned}$$

We choose η such that

$$(2.14) \quad \begin{cases} \eta_i = |\eta_1| \operatorname{sgn}(x_i - \xi_i), & (i=1, 2, 3), \\ |\eta| = \frac{|x-\xi|}{\sqrt{3} \cdot 2a_1\sigma(y, \tau)(t-\tau)}. \end{cases}$$

$$-\eta \cdot (x-\xi) \leq -|\eta| \cdot \frac{|x-\xi|}{\sqrt{3}},$$

$$-\eta \cdot (x-\xi) + a_1\sigma(y, \tau)(t-\tau)|\eta|^2$$

$$\leq -|\eta| \cdot \frac{|x-\xi|}{\sqrt{3}} + (t-\tau)a_1\sigma(y, \tau)|\eta|^2 = -\frac{|x-\xi|^2}{12a_1\sigma(y, \tau)(t-\tau)}$$

Thus,

$$\begin{aligned}
 & |D_x^m Z^{ij}(x-\xi, t; y, \tau)| \\
 (2.15) \quad & \leq C_0 \frac{2^{|m|-1}}{(2\pi)^3} \cdot \sigma(y, \tau)^{-\langle |m|+3 \rangle/2} \left\{ \left(\frac{4}{\delta} \right)^{\langle |m|+3 \rangle/2} \cdot I(|m|+2) \right. \\
 & \quad \left. + K \left(\frac{|m|}{2}, \frac{1}{2} \right) (2\sqrt{3})^{|m|} (12a_1)^{-|m|/2} \left(\frac{4\pi}{\delta} \right)^{3/2} \right\} (t-\tau)^{-\langle |m|+3 \rangle/2} \cdot \exp \left\{ \frac{|x-\xi|^2}{24a_1\sigma_1(t-\tau)} \right\} \\
 & \leq C_1^{\langle |m| \rangle} (C_0, \sigma_0, a_1) (t-\sigma)^{-\langle |m|+3 \rangle/2} \exp \left\{ -\frac{|x-\xi|^2}{24a_1\sigma_1(t-\tau)} \right\},
 \end{aligned}$$

$$(2.15)' \quad \left(I(\mu) \equiv \int_0^\infty r^\mu e^{-r^2} dr \right). \quad \text{Q.E.D.}$$

LEMMA 2.6. For m such that $|m| \geq 1$,

$$(2.16) \quad \int_{\mathbb{R}^3} D_x^m Z^{ij}(x-\xi, t; y, \tau) dx = 0, \quad (t > \tau).$$

Proof. Using the preceding lemma, we can prove this by means of direct integration; e.g.,

$$\begin{aligned}
 (2.17) \quad & \int_{\mathbb{R}^3} D_{x_1} Z^{ij}(x-\xi, t; y, \tau) dx = \lim_{N \rightarrow +\infty} \int_{\mathbb{R}^2} dx_2 dx_3 \int_{-N}^{+N} D_{x_1} Z^{ij}(x-\xi, t; y, \tau) dx_1 \\
 & = \int_{\mathbb{R}^2} dx_2 dx_3 \lim_{N \rightarrow +\infty} \{ Z^{ij}(x-\xi, t; y, \tau) |_{x_1=-N}^N \} = 0. \quad \text{Q.E.D.}
 \end{aligned}$$

COROLLARY of Lemma 2.6.

$$(2.18) \quad \int_{\mathbb{R}^3} D_i Z^{ij}(x-\xi, t; y, \tau) dx = 0.$$

Proof. This is obvious from the fact that Z satisfies

$$(2.18)' \quad D_t Z = \sigma(y, \tau) P_0(D_x) Z. \quad \text{Q.E.D.}$$

2.2. Some calculations. The difference

$$(2.19) \quad W(\zeta, t; y, \tau; h) \equiv V(\zeta, t; y, \tau) - V(\zeta, t; y+h, \tau)$$

satisfies

$$(2.20) \quad \begin{cases} \frac{dW}{dt} = \sigma(y, \tau) P_0(i\zeta) W + \{ \sigma(y, \tau) - \sigma(y+h, \tau) \} P_0(i\zeta) V(\zeta, t; y+h, \tau), \\ W|_{t=\tau} = 0, \quad (t > \tau \geq 0). \end{cases}$$

Hence

$$(2.20)' \quad W(\zeta, t; y, \tau; h) = \int_{\tau}^t e^{(t-\tau)\sigma(y, \tau)P_0(i\zeta)} \{\sigma(y, \tau) - \sigma(y+h, \tau)\} P_0(i\zeta) V(\zeta, \tau_0; y+h, \tau) d\tau_0.$$

From (2.20)', we have:

LEMMA 2.7.

$$(2.21) \quad \begin{aligned} |W(\zeta, t; y, \tau; h)| &\leq |h|^\alpha C_2 \exp \left\{ (t-\tau) - \frac{\delta}{8} \sigma_0 |\xi|^2 + \left(a + \frac{3}{8} \delta \right) \sigma_1 |\eta|^2 \right\} \\ &\equiv |h|^\alpha C_2 e^{(t-\tau)Q_1(\xi, \eta)}. \end{aligned}$$

Proof.

$$(2.22) \quad \begin{aligned} |W(\zeta; t; y, \tau; h)| &\leq \int_{\tau}^t C_0^2 \cdot \exp \{ (t-\tau_0) Q_0(\xi, \eta) \} \cdot |\sigma|_{T^{\sigma}} |h|^\alpha \cdot 4 |\zeta|^2 \cdot \exp \{ (\tau_0 - \tau) Q_0(\xi, \eta) \} d\tau_0 \\ &\leq |h|^\alpha \cdot (C_0)^2 \cdot 4 \frac{4}{\delta \sigma_0} K(1, 0) \exp \left\{ (t-\tau) \left\{ -\frac{\delta}{8} \sigma_0 |\xi|^2 + \left(a + \frac{3}{8} \delta \right) \sigma_1 |\eta|^2 \right\} \right\} \\ &\equiv |h|^\alpha C_2 e^{(t-\tau)Q_1(\xi, \eta)}. \end{aligned} \quad \text{Q.E.D.}$$

LEMMA 2.8. For m ($|m| \geq 0$),

$$(2.23) \quad \begin{aligned} &|D_x^m Z^{ij}(x-\xi, t; y, \tau) - D_x^m Z^{ij}(x-\xi, t; y+h, \tau)| \\ &\leq |h|^\alpha \bar{C}_1^{|m|} \exp \left\{ -\frac{1}{24a_2\sigma_1} \cdot \frac{|x-\xi|^2}{t-\tau} \right\} (t-\tau)^{-(|m|+3)/2}, \quad \left(a_2 = a + \frac{3}{8} \delta \right) \end{aligned}$$

Proof.

$$(2.23) \quad \begin{aligned} &D_x^m Z^{ij}(x-\xi, t; y, \tau) - D_x^m Z^{ij}(x-\xi, t; y+h, \tau) \\ &= (2\pi)^{-3} \int_{R^3} (i(\xi_0 + i\eta))^m e^{i(\xi_0 + i\eta)(x-\xi)} W^{ij}(\xi_0 + i\eta, t; y, \tau; h) d\xi_0. \end{aligned}$$

From this equality we obtain (2.23) as we did (2.11) from (2.13). Q.E.D.

In the same way we have the following lemmas.

LEMMA 2.9.

$$(2.24) \quad \begin{aligned} &|D_x^m Z^{ij}(x-\xi, t; y, \tau) - D_x^m Z^{ij}(x'-\xi, t; y, \tau)| \\ &\leq C_1^{|m|+1} (t-\tau)^{-(|m|+4)/2} \cdot \exp \left\{ -\frac{1}{24a_1\sigma_1} \cdot \frac{|\hat{x}-\xi|^2}{t-\tau} \right\} |x-x'|, \end{aligned}$$

($\hat{x} \in \overline{xx'}$ = convex hull of $\{x, x'\}$).

LEMMA 2.10. For $t > t' > \tau$

$$(2.25) \quad \begin{aligned} & |D_x^m Z^{ij}(x-\xi, t; y, \tau) - D_x^m Z^{ij}(x-\xi, t'; y, \tau)| \\ & \leq 4 \cdot \sigma_1 C_1^{(|m|+2)} (t-t')(t'-\tau)^{-\langle |m|+5 \rangle / 2} \cdot \exp \left\{ -\frac{1}{24a_1\sigma_1} \cdot \frac{|x-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

LEMMA 2.11.

$$(2.26) \quad \begin{aligned} & |D_t D_x^m Z^{ij}(x-\xi, t; y, \tau) - D_t D_x^m Z^{ij}(x-\xi, t; y+h, \tau)| \\ & \leq |h|^\alpha C_3^{(|m|)} (t-\tau)^{-\langle |m|+5 \rangle / 2} \exp \left\{ -\frac{1}{24a_2\sigma_1} \cdot \frac{|x-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

LEMMA 2.12.

$$(2.27) \quad \begin{aligned} & |D_t Z^{ij}(x-\xi, t; y, \tau) - D_t Z^{ij}(x-\xi, t; y', \tau)| \\ & \leq 2C_1^{(2)} |\sigma_T^{(\alpha)} + \sigma_1 \bar{C}_1^{(2)}| |y-y'|^\alpha (t-\tau)^{-5/2} \exp \left\{ -\frac{|x-\xi|^2}{24a_2\sigma_1(t-\tau)} \right\}. \end{aligned}$$

2.3. **Construction of the fundamental solution.** It is known that the system (1.11) has a unique solution in $H_T^{2+\alpha}$, (cf. [24], [6]). [More generally, (1.11) has a unique regular solution (cf. [6], Ch. 2 and 3)].

As shown in §4, $v(x, t)$ is expressed as

$$(2.28) \quad v(x, t) = v_0(x) + \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau) \{f(\xi, \tau) + \sigma(\xi, \tau)\} P_0(D_x) v_0(\xi) d\xi,$$

where $\Gamma(x, t; \xi, \tau)$ is defined by

$$(2.29) \quad \Gamma(x, t; \xi, \tau) \equiv Z(x-\xi, t; \xi, \tau) + \int_\tau^t d\tau_0 \int_{R^3} Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy, \quad (t > \tau \geq 0)$$

and Φ satisfies a Volterra-type integral equation,

$$(2.30) \quad \Phi(x, t; \xi, \tau) = K(x, t; \xi, \tau) + \int_\tau^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy, \quad (t > \tau \geq 0)$$

$$(2.31) \quad \begin{aligned} & (K(x, t; \xi, \tau) \equiv \{\sigma(x, t) P_0(D_x) - ID_t\} Z(x-\xi, t; \xi, \tau) \\ & = \{\sigma(x, t) - \sigma(\xi, \tau)\} P_0(D_x) Z(x-\xi, t; \xi, \tau)). \end{aligned}$$

$\Gamma(x, t; \xi, \tau)$ is the unique fundamental solution of (1.11) ($f=0$), so far as solutions in $H_T^{2+\alpha}$ (more generally, bounded regular solutions) are concerned. It will be known from the Hölder-continuity of Φ in x to be shown in §3 that $\Gamma(x, t; \xi, \tau)$ satisfies

$$(2.32) \quad \frac{\partial \Gamma}{\partial t}(x, t; \xi, \tau) = \sigma(x, t) P_0(D_x) \Gamma(x, t; \xi, \tau), \quad (t > \tau),$$

and it is obvious that

$$(2.33) \quad \lim_{t \searrow 0} \int_{\mathbb{R}^3} \Gamma(x, t; \xi, 0) v_0(\xi) d\xi = v_0(x).$$

Thus, Γ has all properties as a fundamental solution.

We give some formulas here.

LEMMA 2.13.

$$(2.34) \quad r^\mu e^{-r} \leq K(\mu, b) e^{-br}, \quad \left(K(\mu, b) \equiv \left[\frac{\mu}{e(1-b)} \right]^\mu, \mu \geq 0, b \in [0, 1) \right).$$

LEMMA 2.14.

$$(2.35) \quad \int_{\mathbb{R}^n} \exp \left\{ -\bar{a} \frac{|x-y|^2}{t-\tau_0} \right\} \cdot \exp \left\{ -\bar{a} \frac{|y-\xi|^2}{\tau_0-\tau} \right\} dy \\ = \left(\frac{\pi}{\bar{a}} \right)^{n/2} \left\{ \frac{(t-\tau_0)(\tau_0-\tau)}{t-\tau} \right\}^{n/2} \exp \left\{ -\bar{a} \frac{|x-\xi|^2}{t-\tau} \right\}, \quad (\bar{a} > 0, t > \tau_0 > \tau),$$

$$(2.35)' \quad \int_{\tau}^t (t-\tau_0)^{\alpha-1} (\tau_0-\tau)^{\beta-1} d\tau_0 = (t-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\alpha, \beta > 0).$$

LEMMA 2.15.

$$(2.36) \quad \int_{\tau}^t (t-\tau_0)^{-\mu/2} \exp \left\{ -h \frac{|x-\xi|^2}{t-\tau_0} \right\} d\tau_0 \leq \int_0^{\infty} 2h^{-(\mu-2)/2} \cdot |x-\xi|^{-\mu+2} \zeta^{\mu-3} e^{-\zeta^2} d\zeta \\ = 2h^{-(\mu-2)/2} |x-\xi|^2 I(\mu-3), \quad (\mu \geq 3, h > 0).$$

Φ is given in the form,

$$(2.37) \quad \Phi(x, t; \xi, \tau) = \sum_{m=0}^{\infty} K_m(x, t; \xi, \tau),$$

where

$$(2.37)' \quad \begin{cases} K_m(x, t; \xi, \tau) \equiv \int_{\tau}^t d\tau_0 \int_{\mathbb{R}^3} K_0(x, t; y, \tau_0) K_{m-1}(y, \tau_0; \tau) dy, \\ K_0 \equiv K. \end{cases}$$

$$(2.38) \quad |K(x, t; \xi, \tau)| = |\sigma(x, t) - \sigma(\xi, \tau)| \cdot |P_0(D_x)Z(x-\xi, t; \xi, \tau)| \\ \leq C_4 (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\}, \quad \left(h_1 \equiv \frac{1}{48a_1\sigma_1} \right).$$

Inductively,

$$\begin{aligned}
 |K_m(x, t; \xi, \tau)| &\leq \left[\left(\frac{\pi}{h_1} \right)^{-3/2} \frac{\left\{ \left(\frac{\pi}{h_1} \right)^{3/2} C_4 \Gamma\left(\frac{\alpha}{2}\right) \right\}^{m+1}}{\Gamma\left((m+1)\frac{\alpha}{2}\right)} \right] (t-\tau)^{-5/2+\alpha(m+1)/2} \\
 (2.38)' & \cdot \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (m=0, 1, 2, \dots).
 \end{aligned}$$

$$\begin{aligned}
 |K_m(x, t; \xi, \tau)| &\leq \left[\left(\frac{\pi}{h_1} \right)^{-3/2} \frac{T^{-\alpha/2} \left\{ \left(\frac{\pi}{h_1} \right)^{3/2} C_4 \Gamma\left(\frac{\alpha}{2}\right) T^{\alpha/2} \right\}^{m+1}}{\left((m+1)\frac{\alpha}{2}\right)} \right] (t-\tau)^{-5/2+\alpha/2} \\
 (2.38)'' & \cdot \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (m=0, 1, 2, \dots).
 \end{aligned}$$

By (2.38)', the righthand side of (2.37) converges uniformly for $t > \tau \geq 0$, and by (2.38)'' we have:

LEMMA 2.16.

$$\begin{aligned}
 |\Phi(x, t; \xi, \tau)| &\leq \left\{ \sum_{m=0}^{\infty} \left[\left(\frac{\pi}{h_1} \right)^{-3/2} T^{-\alpha/2} \frac{\left(\left(\frac{\pi}{h_1} \right)^{3/2} C_4 \Gamma\left(\frac{\alpha}{2}\right) T^{\alpha/2} \right)^{m+1}}{\Gamma\left(\frac{\alpha}{2}(m+1)\right)} \right] \right\} (t-\tau)^{-5/2+\alpha/2} \\
 (2.39) & \cdot \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\} \equiv C_5 (t-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}.
 \end{aligned}$$

REMARK. The linear equation connected with (1.9)³

$$(2.40) \quad \begin{cases} \frac{\partial \theta}{\partial t} = \tilde{\sigma}(x, t) \Delta \theta + g, & (\tilde{\sigma}, g \in H_T^\alpha, 0 < \tilde{\sigma}_0 \leq \sigma(x, t) \leq \tilde{\sigma}_1 < +\infty), \\ \theta(x, 0) = \theta_0(x) (\in H^{2+\alpha}), \end{cases}$$

is, of course, uniformly parabolic in Petrowsky's sense, so that we can treat (2.40) in parallel with (1.11). The corresponding constants and variables will be denoted, e.g., by 'C₁', 'Φ', etc.

§ 3. Estimates for the fundamental solution.

We want to know concrete ways of estimate the solution $v(x, t)$ of (1.11), for a later use in demonstrating the existence of a solution of the original problem (1.9)–(1.10). It will be shown in § 3 and § 4 that, by utilizing lemmas of § 2, we can estimate Γ and $u(x, t)$ almost along the line of [24], Ch. IV, which treats the

case of a scalar dependent variable. Practically, we need too lengthy calculations. For this reason, we make it our general principle to give rough outlines.

3.1. Hölder-continuity in x of $\Phi(x, t; \xi, \tau)$ (I).

$$\begin{aligned}
 & |K(x, t; \xi, \tau) - K(x', t; \xi, \tau)| \\
 (3.1) \quad & \leq |\{\sigma(x, t) - \sigma(x', t)\} P_0(D_x)Z(x - \xi, t; \xi, \tau)| \\
 & \quad + |\sigma(x', t) - \sigma(\xi, \sigma)| \cdot |P_0(D_x)Z(x - \xi, t; \xi, \tau) - P_0(D_{x'})Z(x' - \xi, t; \xi, \tau)|. \\
 (3.2) \quad & |\{\sigma(x, t) - \sigma(x', t)\}| \cdot |P_0(D_x)Z(x - \xi, t; \xi, \tau)| \\
 & \leq |\sigma|_T^{(\alpha)} |x - x'|^\alpha \cdot 36C_1^{(2)}(t - \tau)^{-5/2} \exp \left\{ -2h_1 \frac{|x - \xi|^2}{t - \tau} \right\}.
 \end{aligned}$$

We denote by x'' the nearer of x and x' to ξ if $|x - \xi| \neq |x' - \xi|$, and x' if $|x - \xi| = |x' - \xi|$. By Lemma 2.9, we have

$$\begin{aligned}
 (3.3) \quad & |P_0(D_x)Z(x - \xi, t; \xi, \tau) - P_0(D_{x'})Z(x' - \xi, t; \xi, \tau)| \\
 & \leq 36C_1^{(3)}(t - \tau)^{-3} \exp \left\{ -2h_1 \frac{|\hat{x} - \xi|^2}{t - \tau} \right\} \cdot |x - x'|, \quad (\hat{x} \in \overline{xx'}).
 \end{aligned}$$

i) If $4h_1(t - \tau)^{-1}|x - x'|^2 < 1$, then

$$(3.4) \quad \exp \left\{ -2h_1 \frac{|\hat{x} - \xi|^2}{t - \tau} \right\} \leq e^{-2h_1(t - \tau)^{-1}(|\hat{x} - x'| - |x' - \xi|)^2} \leq e \cdot e^{-h_1(t - \tau)^{-1}|x' - \xi|^2}.$$

Thus,

$$(3.5) \quad |P_0(D_x)Z - P_0(D_{x'})Z| \leq 36 \cdot eC_1^{(3)}(t - \tau)^{-3} \cdot |x - x'| e^{-h_1(t - \tau)^{-1}|x' - \xi|^2}.$$

ii) If $4h_1(t - \tau)^{-1}|x - x'|^2 > 1$, then

$$\begin{aligned}
 (3.6) \quad & |P_0(D_x)Z(x - \xi, t; \xi, \tau)| \\
 & \leq 36C_1^{(2)}\{4h_1(t - \tau)^{-1}|x - x'|^2\}^{1/2}(t - \tau)^{-5/2} e^{-2h_1(t - \tau)^{-1}|x - \xi|^2} \\
 & \leq 72C_1^{(2)}(h_1)^{1/2}(t - \tau)^{-3} e^{-2h_1(t - \tau)^{-1}|x - \xi|^2} \cdot |x - x'|.
 \end{aligned}$$

Hence,

$$(3.6)' \quad |P_0(D_x)Z - P_0(D_{x'})Z| \leq 44C_1^{(2)}(h_1)^{1/2}(t - \tau)^{-3}|x - x'| e^{-h_1(t - \tau)^{-1}|x'' - \xi|^2}.$$

From i) and ii), we have

$$\begin{aligned}
 (3.7) \quad & |P_0(D_x)Z(x - \xi, t; \xi, \tau) - P_0(D_{x'})Z(x' - \xi, t; \xi, \tau)| \\
 & \leq 36\{eC_1^{(3)} + 4C_1^{(2)}(h_1)^{1/2}\}|x - x'| (t - \tau)^{-3} \exp \left\{ -h_1 \frac{|x'' - \xi|^2}{t - \tau} \right\}.
 \end{aligned}$$

Thus, for $\gamma \in [0, 1]$,

$$(3.8) \quad |P_0(D_x)Z - P_0(D_{x'})Z| \leq C_6(t-\tau)^{-(5+\gamma)/2} |x-x'|^\gamma \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}.$$

Putting $\gamma = \alpha$ in (3.8), we obtain the inequality,

$$(3.9) \quad \begin{aligned} & |\sigma(x', t) - \sigma(\xi, \tau)| \cdot |P_0(D_x)Z - P_0(D_{x'})Z| \\ & \leq C_6 |\sigma|_T^{(\alpha)} \{ |x-\xi|^\alpha + |t-\tau|^{\alpha/2} \} |x-x'|^\alpha (t-\tau)^{-(5+\alpha)/2} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

When $x'' = x'$,

$$(3.10) \quad \begin{aligned} & |\sigma(x', t) - \sigma(\xi, \tau)| \cdot |P_0(D_x)Z - P_0(D_{x'})Z| \\ & \leq \bar{C}_6 (t-\tau)^{-5/2} |x-x'|^\alpha \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

For the case that $x'' = x$, we can estimate in an analogous way. Therefore, we have:

LEMMA 3.1.

$$(3.11) \quad |K(x, t; \xi, \tau) - K(x', t; \xi, \tau)| \leq C_{6,1} |x-x'|^\alpha (t-\tau)^{-5/2} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}.$$

LEMMA 3.2. For $\gamma \in [0, 1]$,

$$(3.12) \quad |K(x, t; \xi, \tau) - K(x', t; \xi, \tau)| \leq C_{6,2} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha(1-\gamma)/2} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}.$$

LEMMA 3.3. For $\beta \in [0, \alpha]$,

$$(3.13) \quad |\Phi(x, t; \xi, \tau) - \Phi(x', t; \xi, \tau)| \leq C_{6,3} |x-x'|^\beta (t-\tau)^{-5/2+(\alpha-\beta)/2} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\}.$$

Proof. This lemma follows directly from (2.30) and the lemmas 2.14, 2.16, and 3.2.

3.2. **Hölder-continuity in x of $\Phi(x, t; \xi, \tau)$ (II).** We show that we can improve (3.13), i.e., that Φ is Hölder-continuous in x with the exponent α .

$$(3.14) \quad \begin{aligned} & \int_\tau^t d\tau_0 \int_{\mathbb{R}^3} \{K(x, t; y, \tau_0) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \\ & = \int_\tau^t d\tau_0 \int_{\Sigma_0} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy - \int_\tau^t d\tau_0 \int_{\Sigma_0} K(x', t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\ & \quad + \int_\tau^t d\tau_0 \int_{\mathbb{R}^3 - \Sigma_0} \{K(x, t; y, \tau_0) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy = I_1 + I_2 + I_3, \end{aligned}$$

where $\Sigma_0 = \Sigma_1 - \Sigma_2$, Σ_1 is a closed ball with $x''(\xi)$ the center and with the radius $2|x-x'|$, and Σ_2 is a closed ball with ξ the center and with the radius $(1/2)|x''-\xi|$.

1° Estimate for I_1 .

$$|y-\xi| \cong \frac{|x''-\xi|}{2}, \quad (y \in \Sigma_0 \subset R^3 - \Sigma_2).$$

$$\begin{aligned} |I_1| &\cong C_4 C_5 \int_{\tau}^t (t-\tau_0)^{-(5-\alpha)/2} (\tau_0-\tau)^{-(5-\alpha)/2} d\tau_0 \int_{\Sigma_0} \exp \left\{ -h_1 \left[\frac{|x-y|^2}{t-\tau_0} + \frac{|y-\xi|^2}{\tau_0-\tau} \right] \right\} dy \\ &\cong C_4 C_5 \left[\int_{\Sigma_0} dy \int_{\tau}^{(t+\tau)/2} \left(\frac{t-\tau}{2} \right)^{-(5-\alpha)/2} (\tau_0-\tau)^{-(5-\alpha)/2} \exp \left\{ -h_1 \left[\frac{|y-\xi|^2}{2(\tau_0-\tau)} + \frac{|x''-\xi|^2}{4(t-\tau)} \right] \right\} d\tau_0 \right. \\ (3.15) \quad &+ \left. \int_{\Sigma_0} dy \int_{(t+\tau)/2}^t (t-\tau_0)^{-(5-\alpha)/2} \left(\frac{t-\tau}{2} \right)^{-(5-\alpha)/2} \exp \left\{ -h_1 \frac{|x-y|^2}{t-\tau_0} - \frac{h_1}{4} \frac{|x''-\xi|^2}{t-\tau} \right\} d\tau_0 \right] \\ &\cong C_{7,1} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -\frac{h_1}{4} \frac{|x''-\xi|^2}{t-\tau} \right\}. \end{aligned}$$

2° In the same way, we have

$$(3.16) \quad |I_2| \cong C_{7,2} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -\frac{h_1}{4} \frac{|x''-\xi|^2}{t-\tau} \right\}, \quad (C_{7,2} \cong C_{7,1}).$$

3° For clearness' sake, we assume that $x''=x'$.

$$\begin{aligned} I_3 &= \int_{\tau}^t d\tau_0 \int_{R^3-\Sigma_0} \{\sigma(x, t) - \sigma(x', t)\} P_0(D_x) Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\ &+ \int_{\tau}^t d\tau_0 \int_{R^3-\Sigma_0} \{\sigma(x', t) - \sigma(y, \tau_0)\} \{P_0(D_x) Z(x-y, t; y, \tau_0) \\ (3.17) \quad &- P_0(D_{x'}) Z(x'-y, t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \\ &\cong I_{3,1} + I_{3,2}. \end{aligned}$$

$$(3.18) \quad I_{3,1} = \int_{\tau}^{(t+\tau)/2} d\tau_0 \int_{R^3-\Sigma_0} \cdots dy + \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3-\Sigma_0} \cdots dy \cong I_{3,1,1} + I_{3,1,2}.$$

$$(3.19) \quad |I_{3,1,1}| \cong C_{7,3,1,1} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\}.$$

$$\begin{aligned}
I_{3,1,2} &= \{\sigma(x, t) - \sigma(x', t)\} \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3 - \Sigma_0} P_0(D_x) Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) \\
&\quad - \Phi(x, \tau_0; \xi, \tau)\} dy \\
(3.20) \quad &+ \{\sigma(x, t) - \sigma(x', t)\} \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3 - \Sigma_0} \{P_0(D_x) Z(x-y, t; y, \tau_0) \\
&\quad - P_0(D_x) Z(x-y, t; x_0, \tau_0)|_{x_0=x}\} \Phi(x, \tau_0; \xi, \tau) dy \\
&+ \{\sigma(x, t) - \sigma(x', t)\} \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3 - \Sigma_0} P_0(D_x) Z(x-y, t; x_0, \tau_0)|_{x_0=x} \cdot \Phi(x, \tau_0; \xi, \tau) dy \\
&= I_{3,1,2,1} + I_{3,1,2,2} + I_{3,1,2,3} \\
&\quad (P_0(D_x) Z(x-y, t; x_0, \tau)|_{x_0=x} = P_0(D_y) Z(x-y, t; x, \tau)).
\end{aligned}$$

By (3.13),

$$(3.21) \quad |I_{3,1,2,1}| \leq C_{7,3,1,2,1} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right\}.$$

$$(3.22) \quad |I_{3,1,2,2}| \leq C_{7,3,1,2,2} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (\text{cf. Lemma 2.8}).$$

For $P_0(D_x) = (a_{kl}^{ij} D_{x_k} D_{x_l})$, we define

$$(3.23) \quad \check{P}_0(D_x, \nu) \equiv (\nu_k a_{kl}^{ij} D_{x_l}), \quad (\nu, \text{ outer normal to the surface } S_{\Sigma_0} \text{ of } \Sigma_0).$$

$$I_{3,1,2,3} = \{\sigma(x, t) - \sigma(x', t)\} \int_{(t+\tau)/2}^t d\tau_0 \int_{S_{\Sigma_0}} \check{P}_0(D_y, \nu) Z(y-x, t; x, \tau_0) \Phi(x, \tau_0; \xi, \tau) dS_y.$$

Hence, by Lemma 2.5 and Lemma 2.15,

$$\begin{aligned}
(3.23)' \quad |I_{3,1,2,3}| &\leq |\sigma|_P^{(\alpha)} \cdot 36C_5 C_1^{(\nu)} |x-x'|^\alpha \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\} \int_{(t+\tau)/2}^t (t-\tau_0)^{-2} (\tau_0-\tau)^{-5/2+\alpha/2} d\tau_0 \\
&\quad \times \left\{ \int_{S_1} \exp\left\{-h_1 \frac{|x-y|^2}{t-\tau_0}\right\} dS_y + \int_{S_2} \exp\left\{-h_1 \frac{|x-y|^2}{t-\tau_0}\right\} dS_y \right\} \\
&\leq C_{7,3,1,2,3} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (S_1 = S_{\Sigma_1} - \Sigma_2, S_2 = S_{\Sigma_2} - \Sigma_1).
\end{aligned}$$

Thus,

$$(3.24) \quad |I_{3,1}| \leq C_{7,3,1} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right\}.$$

4° Estimate for $I_{3,2}$.

$$(3.25) \quad I_{3,2} = \int_{\tau}^t d\tau_0 \int_{R^3 - \Sigma_1} \cdots dy + \int_{\tau}^t d\tau_0 \int_{\Sigma_1 \cap \Sigma_2} \cdots dy \equiv I_{3,2,1} + I_{3,2,2}.$$

$$(3.26) \quad |I_{3,2,1}| \leq |\sigma|_T^{(\alpha)} C_6 C_8 \left[\int_{\tau}^{(t+\tau)/2} (t-\tau_0)^{-(5+\alpha)/2} (\tau_0-\tau)^{-5/2+\alpha/2} d\tau_0 \int_{R^3 - \Sigma_1} 2 \exp \left\{ -\frac{h_1}{2} \frac{|y-x'|^2}{t-\tau_0} \right\} \right. \\ \left. \cdot \exp \left\{ -h_1 \frac{|y-\xi|^2}{\tau_0-\tau} \right\} \{ |x'-y|^\alpha + |t-\tau_0|^{\alpha/2} |x-x'|^\alpha \} dy \right] \\ + |x-x'| \int_{(t+\tau)/2}^t (t-\tau_0)^{-3} (\tau_0-\tau)^{-5/2+\alpha/2} d\tau_0 \\ \cdot \int_{R^3 - \Sigma_1} 2 \{ |x'-y|^\alpha + |t-\tau_0|^{\alpha/2} \} \exp \left\{ -\frac{h_1}{2} \frac{|y-x'|^2}{t-\tau_0} \right\} \exp \left\{ -h_1 \frac{|y-\xi|^2}{\tau_0-\tau} \right\} dy \Big].$$

From (3.26), we have

$$(3.27) \quad |I_{3,2,1}| \leq C_{7,3,2,1} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -\frac{h_1}{16} \frac{|x'-\xi|^2}{t-\tau} \right\}, \\ \left(\text{N.B.: } \frac{1}{2} |x'-y| \leq |x-y| \leq \frac{3}{2} |x'-y|, \text{ for } y \in \Sigma_2 \right).$$

If $y \in \Sigma_2$, then

$$(3.28) \quad \frac{1}{3} |x'-y| \leq \frac{1}{2} |x'-\xi| \leq |x'-y|, \quad |x-y| \geq \frac{1}{3} |x'-y|.$$

$$(3.29) \quad |I_{3,2,2}| \leq |\sigma|_T^{(\alpha)} C_6 C_6 \cdot 2 \int_{\tau}^t (t-\tau_0)^{-5/2} (\tau_0-\tau)^{-5/2+\alpha/2} d\tau_0 \\ \cdot \int_{\Sigma_1 \cap \Sigma_2} \{ |x'-y|^\alpha + |t-\tau_0|^{\alpha/2} \} \exp \left\{ -\left[\frac{h_1}{9} \frac{|x'-y|^2}{t-\tau_0} + \frac{h_1}{9} \frac{|y-\xi|^2}{\tau_0-\tau} \right] \right\} dy \\ \leq C_{7,3,2,2} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -\frac{h_1}{72} \frac{|x'-\xi|^2}{t-\tau} \right\}.$$

From (3.26) and (3.29), we have

$$(3.30) \quad |I_{3,2}| \leq C_{7,3,2} |x-x'|^\alpha (t-\tau)^{-5/2+\alpha/2} \cdot \exp \left\{ -\frac{h_1}{72} \frac{|x'-\xi|^2}{t-\tau} \right\}, \\ (C_{7,3,2} \equiv C_{7,3,2,1} + C_{7,3,2,2}).$$

We obtain analogous results in the case when $x''=x$. Thus, we have:

LEMMA 3. 4.

$$(3. 31) \quad \left| \int_{\tau}^t d\tau_0 \int_{\mathbb{R}^3} \{K(x, t; y, \tau) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \right| \\ \leq C_7 |x - x'|^\alpha (t - \tau)^{-5/2 + \alpha/2} \exp \left\{ -\frac{h_1}{72} \frac{|x'' - \xi|^2}{t - \tau} \right\}, \quad (C_7 \equiv C_{7,1} + C_{7,2} + C_{7,3,1} + C_{7,3,2}).$$

From this lemma follows:

LEMMA 3. 5.

$$(3. 33) \quad |\Phi(x, t; \xi, \tau) - \Phi(x', t; \xi, \tau_0)| \\ \leq |K(x, t; \xi, \tau_0) - K(x', t; \xi, \tau_0)| \\ + \left| \int_{\tau}^t d\tau_0 \int_{\mathbb{R}^3} \{K(x, t; y, \tau_0) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \right| \\ \leq \bar{C}_7 |x - x'|^\alpha (t - \tau)^{-5/2} \exp \left\{ -\frac{h_1}{72} \frac{|x'' - \xi|^2}{t - \tau} \right\}, \quad (\bar{C}_7 \equiv C_{6,1} + C_7 T^{\alpha/2}).$$

3. 3. Hölder-continuity in t of $\Phi(x, t; \xi, \tau)$.

1°) We assume that $t \geq t' > \tau$ and $t - t' < (t - \tau)/3$, ($\tau \geq 0$).

LEMMA 3. 6. For $t \geq t' > \tau \geq 0$, $t - t' < (t - \tau)/3$,

$$(3. 34) \quad |K(x, t; \xi, \tau) - K(x, t'; \xi, \tau)| \leq C_8 (t - t')^{\alpha/2} (t' - \tau)^{-5/2} \exp \left\{ -h_1 \frac{|x - \xi|^2}{t - \tau} \right\}.$$

Proof.

$$|K(x, t; \xi, \tau) - K(x, t'; \xi, \tau)| \\ \leq |\sigma(x, t) - \sigma(x, t')| \cdot |P_0(D_x)Z(x - \xi, t; \xi, \tau)| \\ + |\sigma(x, t') - \sigma(\xi, \tau)| \cdot |P_0(D_x)Z(x - \xi, t; \xi, \tau) - P_0(D_x)Z(x - \xi, t'; \xi, \tau)|.$$

From this follows (3. 34).

Q.E.D.

LEMMA 3. 7. For $t \geq t' > \tau$, $t - t' < (t - \tau)/3$,

$$(3. 35) \quad |\Phi(x, t; \xi, \tau) - \Phi(x, t'; \xi, \tau)| \leq C_9 (t - t')^{\alpha/2} (t' - \tau)^{-5/2} \exp \left\{ -\frac{h_1}{144} \frac{|x - \xi|^2}{t - \tau} \right\}.$$

Proof. We estimate the difference

$$\begin{aligned}
& \int_{\tau}^t d\tau_0 \int_{\mathbb{R}^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy - \int_{\tau}^{t'} d\tau_0 \int_{\mathbb{R}^3} K(x, t'; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\
&= \int_{2\nu-t}^t d\tau_0 \int_{\mathbb{R}^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy - \int_{2\nu-t}^{t'} d\tau_0 \int_{\mathbb{R}^3} K(x, t'; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\
(3.36) \quad & + \{\sigma(x, t) - \sigma(x, t')\} \int_t^{2\nu-t} d\tau_0 \int_{\mathbb{R}^3} P_0(D_x) Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\
& + \int_{\tau}^{2\nu-t} d\tau_0 \int_{\mathbb{R}^3} \{\sigma(x, t) - \sigma(y, \tau_0)\} \{P_0(D_x) Z(x-y, t; y, \tau_0) \\
& \qquad \qquad \qquad - P_0(D_x) Z(x-y, t'; y, \tau)\} \Phi(y, \tau_0; \xi, \tau) dy \\
& \equiv \sum_{i=1}^4 I_i.
\end{aligned}$$

$$(3.37) \quad |I_1| \leq C_{9,1} (t-t')^{\alpha/2} (t-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\},$$

$$(3.37)' \quad |I_2| \leq C_{9,2} (t-t')^{\alpha/2} (t'-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t'-\tau} \right\}.$$

$$\begin{aligned}
I_3 &= \{\sigma(x, t) - \sigma(x, t')\} \int_{(\nu+\tau)/2}^{2\nu-t} d\tau_0 \int_{\mathbb{R}^3} P_0(D_x) Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(y, \tau_0; \xi, \tau)\} dy \\
& + \{\sigma(x, t) - \sigma(x, t')\} \int_{(\nu+\tau)/2}^{2\nu-t} d\tau_0 \int_{\mathbb{R}^3} P_0(D_x) Z(x-y, t; y, \tau_0) \Phi(x, \tau_0; \xi, t) dy \\
(3.38) \quad & + \{\sigma(x, t) - \sigma(x, t')\} \int_{\tau}^{(\nu+\tau)/2} d\tau_0 \int_{\mathbb{R}^3} P_0(D_x) Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\
& \equiv I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned}$$

$$(3.39) \quad |I_{3,1}| \leq C_{9,3,1} (t-t')^{\alpha/2} (t'-\tau)^{-5/2+\alpha/2} \exp \left\{ -\frac{h_1}{2 \cdot 72} \frac{|x-\xi|^2}{t-\tau} \right\}.$$

$$\begin{aligned}
|I_{3,2}| &\leq |\sigma|_T^{(\alpha)} (t-t')^{\alpha/2} \\
& \times \left| \int_{(\nu+\tau)/2}^{2\nu-t} d\tau_0 \int_{\mathbb{R}^3} \{P_0(D_x) Z(x-y, t; y, \tau_0) - P_0(D_x) Z(x-y, t; x_0, \tau_0)\}_{x_0=x} \right. \\
(3.40) \quad & \qquad \qquad \qquad \left. \cdot \Phi(x, \tau_0; \xi, \tau) dy \right| \\
& \leq C_{9,3,2} (t-t')^{\alpha/2} (t'-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\}, \quad (2k_1 \geq h_1).
\end{aligned}$$

$$(3.41) \quad |I_{3,3}| \leq C_{9,3,3} (t-t')^{\alpha/2} (t'-\tau)^{-5/2+\alpha/2} \exp \left\{ -h_1 \frac{|x-\xi|^2}{t-\tau} \right\}.$$

Thus,

$$(3.42) \quad |I_3| \leq C_{9,3}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2 \cdot 72} \frac{|x-\xi|^2}{t-\tau}\right\}, \quad \left((C_{9,3} \equiv T^{\alpha/2} \sum_i C_{9,3,1})\right).$$

$$(3.43) \quad I_4 = \int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{\mathbb{R}^3} \cdots dy + \int_{\tau}^{(t'+\tau)/2} d\tau_0 \int_{\mathbb{R}^3} \cdots dy \equiv I_{4,1} + I_{4,2}.$$

$$(3.44) \quad |I_{4,1}| \leq C_{9,4,1}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right\}.$$

$$(3.45) \quad \begin{aligned} |I_{4,2}| &\leq |\sigma| \frac{(\sigma)}{t} C_5 \int_{\tau}^{(t'+\tau)/2} d\tau_0 \int_{\mathbb{R}^3} \{|x-y|^\alpha + |t'-\tau_0|^{\alpha/2}\} |P_0(D_x)Z(x-y, t; y, \tau_0) \\ &\quad - P_0(D_x)Z(x-y, t'; y, \tau_0)|^{\alpha/2} \\ &\quad \times \{|P_0(D_x)Z(x-y, t; y, \tau_0)| \\ &\quad + |P_0(D_x)Z(x-y, t'; y, \tau_0)|\}^{1-(\alpha/2)} (\tau_0-\tau)^{-5/2+\alpha/2} \exp\left\{-h_1 \frac{|y-\xi|^2}{\tau_0-\tau}\right\} dy \\ &\leq C_{9,4,2}(t-t')^{\alpha/2}(t'-\tau)^{-5/8+\alpha/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}, \quad \left(t'-\tau \geq \frac{2}{3}(t-\tau)\right). \end{aligned}$$

$$(3.46) \quad |I_4| \leq C_{9,4}(t-t')^{\alpha/2}(t'-\tau)^{-5/2+\alpha/2} \exp\left\{-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (C_{9,4} \equiv C_{9,4,1} + C_{9,4,2}).$$

Hence, we have (3.35), where

$$C_9 \equiv C_8 + (C_{9,1} + C_{9,2} + C_{9,3} + C_{9,4})T^{\alpha/2}. \quad \text{Q.E.D.}$$

2°) For $t > t' > \tau$, $t-t' \geq (t-\tau)/3$,

$$(3.47) \quad \begin{cases} (t-\tau)^{-5/2+\alpha/2} \leq 3^{\alpha/2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2}, \\ (t'-\tau)^{-5/2+\alpha/2} \leq 2^{\alpha/2}(t-t')^{\alpha/2}(t'-\tau)^{-5/2}. \end{cases}$$

Thus, directly from Lemma 2.16 it follows that

$$(3.48) \quad |\Phi(x, t; \xi, \tau) - \Phi(x, t'; \xi, \tau)| \leq C_5(3^{\alpha/2} + 2^{\alpha/2})(t-t')^{\alpha/2}(t'-\tau)^{-5/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\}.$$

3°) After all, from Lemma 3.7 and (3.48) we have:

LEMMA 3.8. For $t \geq t' > \tau \geq 0$,

$$(3.49) \quad |\Phi(x, t; \xi, \tau) - \Phi(x, t'; \xi, \tau)| \leq \bar{C}_9(t-t')^{\alpha/2}(t'-\tau)^{-5/2} \exp\left\{-\frac{h}{2 \cdot 72} \frac{|x-\xi|^2}{t-\tau}\right\},$$

$$(\bar{C}_9 \equiv C_9 + C_5(3^{\alpha/2} + 2^{\alpha/2})).$$

3. 4. **Estimates for $q(x, t)$.** We define $q(x, t)$ as follows:

$$(3. 50) \quad q(x, t) \equiv \int_0^t d\tau \int_{R^3} \Phi(x, t; \xi, \tau) d\xi.$$

LEMMA 3. 9.

$$(3. 51) \quad |q(x, t) - q(x', t)| \leq C_{10} |x - x'|^\alpha.$$

Proof.

$$(3. 52) \quad \begin{aligned} q(x, t) - q(x', t) &= \int_{\Sigma_1(x')} d\xi \int_0^t \Phi(x, t; \xi, \tau) d\tau - \int_{\Sigma_1(x')} d\xi \int_0^t \Phi(x', t; \xi, \tau) d\xi \\ &+ \int_{R^3 - \Sigma_1} d\xi \int_0^t \{\Phi(x, t; \xi, \tau) - \Phi(x', t; \xi, \tau)\} d\tau = I_1 + I_2 + I_3, \\ &(\Sigma_1(x') \equiv \{\xi; |\xi - x'| \leq 2|x - x'|\}). \end{aligned}$$

$$(3. 53) \quad \begin{aligned} |I_1| &\leq C_5 \int_{\Sigma_1(x')} d\xi \int_0^t (t - \tau)^{-5/2 - \alpha/2} \exp \left\{ -h_1 \frac{|x - \xi|^2}{t - \tau} \right\} d\tau \\ &\leq C_5 \cdot 2h_1^{-(3-\alpha)/2} I(2-\alpha) \int_{\Sigma_1(x')} |x - \xi|^{-3+\alpha} d\xi \\ &\leq C_{10,1} |x - x'|^\alpha. \end{aligned}$$

$$(3. 53)' \quad |I_2| \leq C_{10,2} |x - x'|^\alpha.$$

$$(3. 54) \quad \begin{aligned} I_3 &= \int_{R^3 - \Sigma_1(x')} d\xi \int_0^t \{K(x, t; \xi, \tau) - K(x', t; \xi, \tau)\} d\tau \\ &+ \int_{R^3 - \Sigma_1(x')} d\xi \int_0^t d\tau \int_{R^3} \{K(x, t; y, \tau_0) - K(x', t; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \equiv I_{3,1} + I_{3,2}. \end{aligned}$$

$$(3. 55) \quad \begin{aligned} I_{3,1} &= \{\sigma(x, t) - \sigma(x', t)\} \int_0^t d\tau \int_{R^3 - \Sigma_1(x')} P_0(D_x) Z(x - \xi, t; \tau) d\xi \\ &+ \int_0^t d\tau \int_{R^3 - \Sigma_1(x')} \{\sigma(x', t) - \sigma(\xi, \tau)\} \{P_0(D_x) Z(x - \xi, t; \xi, \tau) \\ &\quad - P_0(D_x) Z(x' - \xi, t; \xi, \tau)\} d\xi \\ &\equiv I_{3,1,1} + I_{3,1,2}. \end{aligned}$$

$$\begin{aligned}
|I_{3,1,1}| &\leq |\sigma| \frac{(\sigma)}{T^2} |x-x'|^\alpha \left| \int_0^{t/2} d\tau \int_{R^3 - \Sigma_1(x')} P_0(D_x) Z(x-\xi, t; \xi, \tau) d\xi \right. \\
&\quad \left. + \int_{t/2}^t d\tau \int_{R^3 - \Sigma_1} P_0(D_x) Z(x-\xi, t; \xi, \tau) dy \right| \\
(3.56) \quad &\leq |\sigma| \frac{(\sigma)}{T^2} |x-x'|^\alpha \left[36C_1^{(2)} \cdot \log 2 \cdot (2h_1)^{-3/2} \pi^{3/2} \right. \\
&\quad \left. + \left| \int_{t/2}^t d\tau \int_{R^3 - \Sigma_1(x')} \{P_0(D_x) Z(x-\xi, t; x, \tau) - P_0(D_x) Z(x-\xi, t; x_0, \tau)\}_{x_0=x} d\xi \right| \right. \\
&\quad \left. + \left| \int_{t/2}^t d\tau \int_{R^3 - \Sigma_1} P_0(D_x) Z(x-\xi, t; x, \tau) d\xi \right| \right] \leq C_{10,3,1,1} |x-x'|^\alpha.
\end{aligned}$$

$$\begin{aligned}
|I_{3,1,2}| &\leq \bar{C}_6 \int_0^{t/2} d\tau \int_{R^3 - \Sigma_1} |x-x'|^\alpha (t-\tau)^{-5/2} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\} d\xi \\
(3.57) \quad &+ |\sigma| \frac{(\sigma)}{T^2} C_6 \int_{t/2}^t d\tau \int_{R^3 - \Sigma_1} |x-x'| (t-\tau)^{-3} \exp \left\{ -h_1 \frac{|x''-\xi|^2}{t-\tau} \right\} (|x'-\xi|^\alpha + |t-\tau|^{\alpha/2}) d\xi \\
&\leq C_{10,3,1,2} |x-x'|^\alpha.
\end{aligned}$$

Hence,

$$\begin{aligned}
(3.58) \quad |I_{3,1}| &\leq C_{10,3,1} |x-x'|^\alpha, \quad (C_{10,3,1} = C_{10,3,1,1} + C_{10,3,1,2}). \\
|I_{3,2}| &\leq C_7 |x-x'|^\alpha \int_0^t (t-\tau)^{-5/2+\alpha/2} d\tau \int_{R^3 - \Sigma_1} \exp \left\{ -\frac{h_1}{72} \frac{|x''-\xi|^2}{t-\tau} \right\} d\xi \\
(3.59) \quad &\leq C_{10,3,2} |x-x'|^\alpha, \quad (\text{cf. Lemma 3.4}).
\end{aligned}$$

From (3.53), (3.53)', (3.58), and (3.59) follows the lemma.

Q.E.D.

LEMMA 3.10. For $t \geq t' > 0$,

$$(3.60) \quad |q(x, t) - q(x, t')| \leq C_{11} (t-t')^{\alpha/2}.$$

Proof. For $t \geq t' > 0$,

$$\begin{aligned}
&q(x, t) - q(x, t') \\
&= \int_{0 \vee (2t-t)}^t d\tau \int_{R^3} \Phi(x, t; \xi, \tau) d\xi - \int_{0 \vee (2t'-t)}^{t'} d\tau \int_{R^3} \Phi(x, t; \xi, \tau) d\xi \\
&\quad + \int_0^{0 \vee (2t'-t)} d\tau \int_{R^3} d\xi \left[\int_\tau^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right. \\
(3.61) \quad &\quad \left. - \int_\tau^{t'} d\tau_0 \int_{R^3} K(x, t'; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right]
\end{aligned}$$

$$\begin{aligned}
& + \{\sigma(x, t) - \sigma(x, t')\} \int_0^{oV(2t'-t)} d\tau \int_{R^3} P_0(D_x) Z(x - \xi, t; \xi, \tau) d\xi \\
& + \int_0^{oV(2t'-t)} d\tau \int_{R^3} \{\sigma(x, t') - \sigma(\xi, \tau)\} \{P_0(D_x) Z(x - \xi, t; \xi, \tau) - P_0(D_x) Z(x - \xi, t'; \xi, \tau)\} d\xi \\
& \equiv \sum_{i=1}^5 J_i, \quad (oV(2t' - t) \equiv \max[0, 2t' - t]).
\end{aligned}$$

$$(3.62) \quad |J_1| \leq C_{11,1} (t - t')^{\alpha/2}.$$

$$(3.63) \quad |J_2| \leq C_5 (2^{1+(\alpha/2)} - 1) \alpha^{-1} \left(\frac{\pi}{h_1}\right)^{3/2} (t - t')^{\alpha/2} \equiv C_{11,2} (t - t')^{\alpha/2}.$$

Next, we note that, for $t \geq t' > \tau \geq 0$ and $t - t' < (t - \tau)/3$,

$$\begin{aligned}
& \left| \int_{\tau}^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy - \int_{\tau}^{t'} d\tau_0 \int_{R^3} K(x, t'; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right| \\
(3.64) \quad & \leq \left(\sum_{i=1}^4 C_{9,i} \right) (t - t')^{\alpha/2} (t' - \tau)^{-5/2} \exp \left\{ -\frac{h_1}{2 \cdot 72} \frac{|x - \xi|^2}{t - \tau} \right\},
\end{aligned}$$

(cf. (3.35), (3.36), (3.40), and (3.45)),

and that, for $t - t' \geq (t - \tau)/3$,

$$\begin{aligned}
(3.65) \quad & \left\{ \left| \int_{\tau}^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right| \leq 3^{\alpha/2} C_4 C_5 (t - t')^{\alpha/2} (t' - \tau)^{-5/2 + \alpha/2} \exp \left\{ -h_1 \frac{|x - \xi|^2}{t - \tau} \right\}, \right. \\
& \left. \left| \int_{\tau}^{t'} d\tau_0 \int_{R^3} K(x, t'; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right| \leq 2^{\alpha/2} C_4 C_5 (t - t')^{\alpha/2} (t' - \tau)^{-5/2 + \alpha/2} \exp \left\{ -h_1 \frac{|x - \xi|^2}{t - \tau} \right\}. \right.
\end{aligned}$$

Thus,

$$(3.66) \quad |J_3| \leq \left[\sum_{i=1}^4 C_{9,i} + C_4 C_5 (3^{\alpha/2} + 2^{\alpha/2}) \right] T^{\alpha/2} (t - t')^{\alpha/2} \equiv C_{11,3} (t - t')^{\alpha/2}.$$

$$\begin{aligned}
(3.67) \quad & |J_4| \leq |\sigma|_{\mathcal{F}}^{(\sigma)} (t - t')^{\alpha/2} \left| \int_0^{oV(2t'-t)} d\tau \int_{R^3} \{P_0(D_x) Z(x - \xi, t; \xi, \tau) - P_0(D_\xi) Z(x - \xi, t; x, \tau)\} d\xi \right| \\
& \leq C_{11,4} (t - t')^{\alpha/2},
\end{aligned}$$

$$\begin{aligned}
 |J_5| &\leq |\sigma|_T^{(\alpha)} \int_0^{oV(2t'-t)} d\tau \int_{R^3} \{|x-\xi|^\alpha + (t'-\tau)^{\alpha/2}\} |P_0(D_x)Z(x-\xi, t; \xi, \tau) \\
 &\quad - P_0(D_x)Z(x-\xi, t'; \xi, \tau)| d\xi \\
 (3.68) \quad &\leq |\sigma|_T^{(\alpha)} \left[2^{\alpha/2} K\left(\frac{\alpha}{2}, \frac{1}{2}\right) + 1 \right] 11^2 \sigma_1 C_1^{(4)} \int_0^{oV(2t'-t)} d\tau \int_{R^3} (t-t')(t'-\tau)^{-7/2+\alpha/2} \\
 &\quad \cdot \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\} d\xi \\
 &= \begin{cases} 0, & (0 \geq 2t'-t), \\ C_{11,5}(t-t')^{\alpha/2}, & (0 < 2t'-t), \end{cases} \\
 &\quad \left(C_{11,5} \equiv |\sigma|_T^{(\alpha)} \left[2^{\alpha/2} K\left(\frac{\alpha}{2}, \frac{1}{2}\right) + 1 \right] 11^2 \sigma_1 C_1^{(4)} 2^{5/2} \left(\frac{\pi}{h_1}\right)^{3/2} (2-\alpha)^{-1} \right).
 \end{aligned}$$

Thus,

$$(3.69) \quad |J_5| \leq C_{11,6}(t-t')^{\alpha/2}.$$

From the above follows the lemma, where $C_{11} \equiv \sum_{i=1}^5 C_{11,i}$.

Q.E.D.

3.5. Estimates for $\Gamma(x, t; \xi, \tau)$.

$$(3.70) \quad \Gamma(x, t; \xi, \tau) \equiv Z(x-\xi, t; \xi, \tau) + \int_\tau^t d\tau_0 \int_{R^3} Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \equiv Z + \Gamma_0.$$

LEMMA 3.11. For $0 \leq 2r + |m| \leq 2$,

$$(3.71) \quad |D_t^r D_x^m \Gamma(x, t; \xi, \tau)| \leq C_{12}^{(r, |m|)} (t-\tau)^{-(3+2r+|m|)/2} \exp\left\{-d(r, |m|) \frac{|x-\xi|^2}{t-\tau}\right\}.$$

Proof. i) $2r + |m| = 0$.

$$\begin{aligned}
 | \Gamma | &\leq |Z| + | \Gamma_0 | \leq 9C_1^{(0)} \left\{ 1 + C_5 \left(\frac{\pi}{h_1}\right)^{3/2} \cdot \frac{2}{\alpha} T^{\alpha/2} \right\} (t-\tau)^{-3/2} \exp\left\{-h_1 \frac{|x-\xi|^2}{t-\tau}\right\} \\
 (3.72) \quad &\equiv C_{12}^{(0,0)} (t-\tau)^{-3/2} \exp\left\{-d(0,0) \frac{|x-\xi|^2}{t-\tau}\right\}.
 \end{aligned}$$

ii) $r=0, |m|=1$.

$$\begin{aligned}
 |D_x^m \Gamma| &\leq |D_x^m Z| + \int_\tau^t d\tau_0 \int_{R^3} |D_x^m Z(x-y, t; y, \tau_0)| \cdot |\Phi(y, \tau_0; \xi, \tau)| dy \\
 (3.73) \quad &\leq C_{12}^{(0,1)} (t-\tau)^{-2} \exp\left\{-d(0,1) \frac{|x-\xi|^2}{t-\tau}\right\}, \quad (d(0,1) = h_1).
 \end{aligned}$$

iii) $|m|=2$.

$$\begin{aligned}
(3.74) \quad |D_x^m I| &\leq |D_x^m Z| + \left| \int_{\tau}^{(t+\tau)/2} d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \right| \\
&+ \left| \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x, \tau_0; \xi, \tau)\} dy \right| \\
&+ \left| \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_x^m Z(x-y, t; y, \sigma_0) \Phi(x, \tau_0; \xi, \tau) dy \right| \\
&\leq C_{12}^{(0,2)} (t-\tau)^{-5/2} \exp \left\{ -d(0, 2) \frac{|x-\xi|^2}{t-\tau} \right\}, \quad \left(d(0, 2) = \frac{h_1}{72} \right).
\end{aligned}$$

iv) $r=1, |m|=0$.

$$\begin{aligned}
(3.75) \quad D_t^r I(x, t; \xi, \tau) &= D_t Z + D_t I_0 = D_t Z + \Phi + \int_{\tau}^{(t+\tau)/2} d\tau_0 \int_{R^3} D_t Z(x-y, t; y, \tau_0) \Phi(y, \tau_0; \xi, \tau) dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} D_t Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x, \tau_0; \xi, \tau)\} dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} \{D_t Z(x-y, t; y, \tau_0) - D_t Z(x-y, t; x, \tau_0)\} \Phi(x, \tau_0; \xi, \tau) dy.
\end{aligned}$$

From (3.75), we have

$$(3.75)' \quad |D_t I| \leq C_{12}^{(1,0)} (t-\tau)^{-5/2} \exp \left\{ -d(1, 0) \frac{|x-\xi|^2}{t-\tau} \right\}, \quad \left(d(1, 0) \equiv \frac{h_1}{72} \right). \quad \text{Q.E.D.}$$

LEMMA 3.12. For $|m|=2$, and for ε and $\gamma \in [0, 1]$,

$$\begin{aligned}
(3.76) \quad &|D_x^m I(x, t; \xi, \tau) - D_x^m I(x, t; \xi, \tau)| \\
&\leq C_{13} \{ |x-x'|^{\gamma} (t-\tau)^{-(5+\gamma)/2} + |x-x'|^{\alpha\gamma} (t-\tau)^{-5/2+\alpha\gamma/2(1-\varepsilon)} \} \exp \left\{ -\bar{h}_1 \frac{|x''-\xi|^2}{t-\tau} \right\}, \\
&\quad \left(\bar{h}_1 \equiv \frac{h_1}{72} \right).
\end{aligned}$$

Proof.

$$\begin{aligned}
(3.77) \quad D_x^m I - D_x^m I' &= \{D_x^m Z - D_x^m Z\} + \int_{\tau}^{(t+\tau)/2} d\tau_0 \int_{R^3} \{D_x^m Z - D_x^m Z\} \Phi dy \\
&+ \int_{(t+\tau)/2}^t d\tau_0 \int_{R^3} \{D_x^m Z - D_x^m Z\} \Phi dy \equiv \{D_x^m Z - D_x^m Z\} + I_1 + I_2.
\end{aligned}$$

It is easy to estimate $\{D_x^m Z - D_x^m Z\}$ and I_1 . I_2 is transformed into:

$$\begin{aligned}
I_2 &= \int_{(t+\tau)/2}^t d\tau_0 \int_{\Sigma_1=\{y; y-x' < 2|x-x'\}} D_x^m Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x', \tau_0; \xi, \tau)\} dy \\
&\quad - \int_{(t+\tau)/2}^t d\tau_0 \int_{\Sigma_1} D_{x'}^m Z(x'-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x', \tau_0; \xi, \tau)\} dy \\
(3.77)' &+ \int_{(t+\tau)/2}^t d\tau_0 \int_{\mathbb{R}^3 - \Sigma_1} \{D_x^m Z(x-y, t; y, \tau_0) - D_{x'}^m Z(x'-y, t; y, \tau_0)\} \{\Phi(y, \tau_0; \xi, \tau) \\
&\quad \quad \quad - \Phi(x', \tau_0; \xi, \tau)\} dy \\
&\quad + \int_{(t+\tau)/2}^t d\tau_0 \int_{\Sigma_1} D_x^m Z(x-y, t; y, \tau_0) \{\Phi(x, \tau_0; \xi, \tau) - \Phi(x', \tau_0; \xi, \tau)\} dy \\
&\quad + \int_{(t+\tau)/2}^t d\tau_0 \int_{\Sigma_1} \{D_x^m Z(x-y, t; y, \tau_0) - D_{x'}^m Z(x'-y, t; y, \tau_0)\} \Phi(x', t; \xi, \tau) dy.
\end{aligned}$$

We have only to estimate I_2 on the basis of (3.77)', utilizing lemmas of §2 and the Hölder-continuity of Φ in x . As a result, we have (3.76). Q.E.D.

LEMMA 3.13. For $\beta \in [0, \alpha]$ and $\gamma \in [0, 1]$,

$$\begin{aligned}
(3.78) \quad & |D_t \Gamma(x, t; \xi, \tau) - D_t \Gamma(x', t; \xi, \tau)| \\
& \leq \bar{C}_{13} \{ |x-x'|^\beta (t-\tau)^{-5/2 + (\alpha-\beta)/2} + |x-x'|^\gamma (t-\tau)^{-(5+\gamma)/2} \exp \left\{ -\bar{h}_1 \frac{|x'-\xi|^2}{t-\tau} \right\} \}.
\end{aligned}$$

Proof. We have only to note that $D_t \Gamma$ satisfies

$$D_t \Gamma(x, t; \xi, \tau) = \sigma(x, t) P_0(D_x) \Gamma(x, t; \xi, \tau). \quad \text{Q.E.D.}$$

LEMMA 3.14. For $t \geq t' > \tau$ and $2 \geq |m| \geq 0$,

$$\begin{aligned}
(3.79) \quad & |D_x^m \Gamma(x, t; \xi, \tau) - D_x^m \Gamma(x, t'; \xi, \tau)| \\
& \leq C_{14}^{(|m|)} \{ (t-t')(t'-\tau)^{-(5+|m|)/2} + (t-t')^{(2-|m|+\alpha)/2} (t'-\tau)^{-5/2} \exp \left\{ -\bar{h}_1 \frac{|x-\xi|^2}{t-\tau} \right\}, \\
& \quad \quad \quad (|m|=1, 2),
\end{aligned}$$

$$|D_x^0 \Gamma(x, t; \xi, \tau) - D_x^0 \Gamma(x, t'; \xi, \tau)| \leq C_{14}^{(0)} (t-t')(t'-\tau)^{-5/2} \exp \left\{ -\frac{\bar{h}_1}{2} \frac{|x-\xi|^2}{t-\tau} \right\}.$$

Proof. i) For $t \geq t_i > \tau$ and $t-t' < (t'-\tau)/4$, we utilize the expression,

$$\begin{aligned}
& D_x^m \Gamma_0(x, t; \xi, \tau) - D_x^m \Gamma_0(x, t'; \xi, \tau) \\
&= \int_t^{(t'+\tau)/2} d\tau_0 \int_{\mathbb{R}^3} \{D_x^m Z(x-y, t; y, \tau_0) - D_x^m Z(x-y, t'; y, \tau_0)\} \Phi(y, \tau_0; \xi, \tau) dy \\
&\quad + \int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{\mathbb{R}^3} \{D_x^m Z(x-y, t; y, \tau_0) - D_x^m Z(x-y, t'; y, \tau_0)\} \\
&\qquad\qquad\qquad \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x, \tau_0; \xi, \tau)\} dy \\
(3.80) \quad &+ \int_{(t'+\tau)/2}^{2t'-t} d\tau_0 \int_{\mathbb{R}^3} \{D_x^m Z(x-y, t; y, \tau_0) - D_x^m Z(x-y, t'; y, \tau_0)\} \Phi(x, \tau_0; \xi, \tau) dy \\
&+ \int_{2t'-t}^t d\tau_0 \int_{\mathbb{R}^3} D_x^m Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x, \tau_0; \xi, \tau)\} dy \\
&- \int_{2t'-t}^{t'} d\tau_0 \int_{\mathbb{R}^3} D_x^m Z(x-y, t; y, \tau_0) \{\Phi(y, \tau_0; \xi, \tau) - \Phi(x, \tau_0; \xi, \tau)\} dy \\
&+ \int_{2t'-t}^t d\tau_0 \int_{\mathbb{R}^3} D_x^m Z(x-y, t; y, \tau_0) \Phi(x, \tau_0; \xi, \tau) dy \\
&- \int_{2t'-t}^{t'} d\tau_0 \int_{\mathbb{R}^3} D_x^m Z(x-y, t'; y, \tau_0) \Phi(x, \tau_0; \xi, \tau) dy.
\end{aligned}$$

After lengthy calculations, we have, for $t \geq t' > \tau$ and $t - t' < (t' - \tau)/4$,

$$\begin{aligned}
(3.81) \quad & |D_x^m \Gamma(x, t; \xi, \tau) - D_x^m \Gamma(x, t'; \xi, \tau)| \\
&\leq C_{14,1}^{(|m|)} (t-t')(t'-\tau)^{-(5+|m|)/2} + (t-t')^{(2-|m|+\alpha)/2} (t'-\tau)^{-5/2} \exp \left\{ -\frac{\bar{h}_1}{2} \frac{|x-\xi|^2}{t-\tau} \right\}, \\
&\qquad\qquad\qquad (|m|=1, 2),
\end{aligned}$$

$$|\Gamma(x, t; \xi, \tau) - \Gamma(x, t'; \xi, \tau)| \leq C_{14,1}^{(0)} (t-t')(t'-\tau)^{-5/2} \exp \left\{ -\frac{\bar{h}_1}{2} \frac{|x-\xi|^2}{t-\tau} \right\}.$$

ii) For $t > t' > \tau$, $t - t' \geq (t' - \tau)/4$, and $|m| \leq 2$, the following inequalities hold:

$$\begin{aligned}
(3.82) \quad & |D_x^m \Gamma_0| \leq \bar{C}_{12}^{(0,|m|)} (t-\tau)^{-(3+|m|-\alpha)/2} \exp \left\{ -d(0, |m|) \frac{|x-\xi|^2}{t-\tau} \right\} \\
&\leq \bar{C}_{12}^{(0,|m|)} 2^{2-|m|+\alpha} (t-t')^{(2-|m|+\alpha)/2} (t'-\tau)^{-5/2} \exp \left\{ -d(0, |m|) \frac{|x-\xi|^2}{t-\tau} \right\},
\end{aligned}$$

$$|D_x^m Z| \leq 36C_1^{(|m|)} (t-t')(t'-\tau)^{-(5+|m|)/2} \exp \left\{ -2h_1 \frac{|x-\xi|^2}{t-\tau} \right\}, \quad (\text{cf. Lemma 3.11}).$$

From i) and ii), we have the lemma, where

$$(3.83) \quad \begin{cases} C_{14}^{(|m|)} \equiv C_{14,1}^{(|m|)} + \bar{C}_{12}^{(0,|m|)}(2^{2-|m|+\alpha}+1)9C_1^{(|m|)}(2^2+1), & (|m|=1, 2), \\ C_{14}^{(0)} \equiv C_{14,1}^{(0)} + T^{\alpha/2}\bar{C}_{12}^{(0,0)}(2^\alpha+1)+9C_1^{(0)}(2^2+1). \end{cases} \quad \text{Q.E.D.}$$

In an analogous way, we have:

LEMMA 3.15. For $t \geq t' > \tau$,

$$(3.84) \quad \begin{aligned} & |D_t \Gamma(x, t; \xi, \tau) - D_{t'} \Gamma(x, t'; \xi, \tau)| \\ & \leq C_{15}(t-t')(t'-\tau)^{-7/2} + (t-t')^{\alpha/2}(t'-\tau)^{-5/2} \exp\left(-\frac{h_1}{2} \frac{|x-\xi|^2}{t-\tau}\right). \end{aligned}$$

Proof. This is obvious from the expression that Γ satisfies. Q.E.D.

§ 4. Estimates for the solution in $H_T^{2+\alpha}$ of a linear problem.

The unique solution in $H_T^{2+\alpha}$ of the Cauchy problem

$$(4.1) \quad \begin{cases} D_t v = \sigma(x, t)P_0(D_x)v + f, & (f, \sigma \in H_T^\alpha, 0 < \sigma_0 \leq \sigma \leq \sigma_1 < +\infty), \\ v(x, 0) = 0, \end{cases}$$

is given by

$$(4.2) \quad v(x, t) = \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi \equiv u(x, t),$$

for the righthand side of (4.2) certainly satisfies (4.1) and belongs to $H_T^{2+\alpha}$ as we shall see below.

LEMMA 4.1.

$$(4.3) \quad \begin{cases} |D_x^m u(x, t)| \leq C_{16}^{(|m|)} t^{(2-|m|+\alpha)/2} \|f\|_T^{(\sigma)}, & (|m|=1, 2), \\ |D_x^0 u(x, t)| \leq C_{16}^{(0)} \cdot t \cdot \|f\|_T^{(\sigma)}. \end{cases}$$

Proof. The second inequality is obvious. For the first one, we utilize the expression,

$$(4.4) \quad \begin{aligned} D_x^m u(x, t) &= \int_0^t d\tau \int_{R^3} D_x^m \Gamma(x, t; \xi, \tau) \{f(\xi, \tau) - f(x, \tau)\} \\ &+ \int_0^t d\tau \int_{R^3} \{D_x^m Z(x-\xi, t; \xi, \tau) + D_x^m \Gamma_0(x, t; \xi, \tau)\} d\xi. \end{aligned} \quad \text{Q.E.D.}$$

COROLLARY of Lemma 4.1.

$$(4.5) \quad |D_t u(x, t)| \leq |\sigma(x, t)P_0(D_x)u(x, t)| + |f| \leq \bar{C}_{16} \|f\|_T^{(\sigma)}.$$

LEMMA 4.2. For $t \geq t' > 0$,

$$(4.6) \quad \begin{cases} |D_x^m u(x, t) - D_x^m u(x, t')| \leq C_{17}^{(|m|)} (t-t')^{(2-|m|+\alpha)/2} \|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |D_x^0 u(x, t) - D_x^0 u(x, t')| \leq C_{17}^{(0)} (t-t') \|f\|_T^{(\alpha)}. \end{cases}$$

Proof. i) For $t > t' > 0$ and $t > 2t'$ (i.e., $t' < t-t'$, $t < 2(t-t')$), by Lemma 4.1,

$$(4.7) \quad \begin{cases} |D_x^m u(x, t) - D_x^m u(x, t')| \leq C_{16}^{(|m|)} (2^{(2-|m|+\alpha)/2} + 1) (t-t')^{(2-|m|+\alpha)/2} \|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |D_x^0 u(x, t) - D_x^0 u(x, t')| \leq 3C_{16}^{(0)} (t-t') \|f\|_T^{(\alpha)}. \end{cases}$$

ii) For $0 \leq t-t' \leq t'$, we utilize the following expression:

$$(4.8) \quad \begin{aligned} & D_x^m u(x, t) - D_x^m u(x, t') \\ &= \int_{2\nu-t}^t \int_{R^3} D_x^m \Gamma(x, t; \xi, \tau) \{f(\xi, \tau) - f(x, t)\} \\ &\quad - \int_{2\nu-t}^t d\tau \int_{R^3} D_x^m \Gamma(x, t'; \xi, \tau) \{f(\xi, \tau) - f(x, \tau)\} d\xi \\ &\quad + \int_0^{2\nu-t} d\tau \int_{R^3} \{D_x^m \Gamma(x, t; \xi, \tau) - D_x^m \Gamma(x, t'; \xi, \tau)\} \{f(\xi, \tau) - f(x, \tau)\} d\xi \\ &\quad + \int_{2\nu-t}^t d\tau \int_{R^3} D_x^m \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi - \int_{2\nu-t}^{t'} d\tau \int_{R^3} D_x^m \Gamma(x, t'; \xi, \tau) f(x, \tau) d\xi \\ &\quad + \int_0^{2\nu-t} d\tau \int_{R^3} \{D_x^m \Gamma(x, t; \xi, \tau) - D_x^m \Gamma(x, t'; \xi, \tau)\} \{f(x, \tau) - f(x, t)\} d\xi \\ &\quad + \int_0^{2\nu-t} d\tau \int_{R^3} \{D_x^m \Gamma(x, t; \xi, \tau) - D_x^m \Gamma(x, t'; \xi, \tau)\} f(x, t) d\xi \equiv \sum_{i=1}^7 I_i^{(|m|)}. \end{aligned}$$

$I_i^{(|m|)}$ ($i=1, 2, \dots, 6$) can easily be estimated, i.e.,

$$(4.9) \quad \begin{cases} |I_i^{(|m|)}| \leq C_{17,i}^{(|m|)} (t-t')^{(2-|m|+\alpha)/2} \|f\|_T^{(\alpha)}, & (|m|=1, 2; i=1, 2, \dots, 6), \\ |I_i^{(0)}| \leq C_{17,i}^{(0)} (t-t') \|f\|_T^{(\alpha)}, & (i=1, 2, \dots, 6). \end{cases}$$

In order to estimate $I_7^{(|m|)}$, we transform this into the form,

$$(4.10) \quad \begin{aligned} I_7^{(|m|)} &= \int_0^{2\nu-t} d\tau \int_{R^3} \{D_x^m Z(x-\xi, t; \xi, \tau) - D_x^m Z(x-\xi, t'; \xi, \tau)\} f(x, t) d\xi \\ &\quad - \int_{2\nu-t}^t d\tau \int_{R^3} D_x^m \Gamma(x, t; \xi, \tau) f(x, t) d\xi + \int_{2\nu-t}^t d\tau \int_{R^3} D_x^m \Gamma(x, t'; \xi, \tau) f(x, t) d\xi \\ &\quad + \left[\int_0^t d\tau \int_{R^3} D_x^m \Gamma_0(x, t; \xi, \tau) f(x, t) d\xi - \int_0^{t'} d\tau \int_{R^3} D_x^m \Gamma_0(x, t'; \xi, \tau) f(x, t) d\xi \right] \\ &\equiv \sum_{i=1}^4 I_{7,i}^{(|m|)}. \end{aligned}$$

For $i=1, 2, 3$, we have easily,

$$(4.11) \quad \begin{cases} |I_{7,t}^{(m)}| \leq C_{17,7,t}^{(m)}(t-t')^{(2-|m|+\alpha)/2} \|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |I_{7,t}^{(0)}| \leq C_{17,7,t}^{(0)}(t-t') \|f\|_T^{(\alpha)}. \end{cases}$$

$$(4.12) \quad \begin{aligned} I_{7,t}^{(m)} &= \left[\int_0^t d\tau \int_{R^3} D_x^m \Gamma_0(x, t, \xi, \tau) d\xi - \int_0^{t'} d\tau \int_{R^3} D_x^m \Gamma_0(x, t'; \xi, \tau) d\xi \right] f(x, t) \\ &= \left[D_x^m \int_0^t d\tau_0 \int_{R^3} Z(x-y, t; y, \tau_0) q(y, \tau_0) dy \right. \\ &\quad \left. - D_x^m \int_0^{t'} d\tau_0 \int_{R^3} Z(x-y, t'; y, \tau_0) q(y, \tau_0) dy \right] f(x, t) \\ &\equiv \hat{I}^{(m)} \cdot f(x, t), \\ &\quad \left(q(y, \tau_0) = \int_0^t d\tau_0 \int_{R^3} \Phi(y, \tau_0; \xi, \tau) d\xi, \quad \text{cf. (3.50)} \right). \end{aligned}$$

We note that

$$(4.13) \quad \hat{I}^{(m)} = [\text{the righthand side of (4.8) as obtained by replacing } \Gamma, f, \xi, \text{ and } \tau \text{ by } Z, q, y, \text{ and } \tau_0, \text{ respectively}] \equiv \sum_{i=1}^7 \hat{I}_i^{(m)}.$$

$$(4.14) \quad \begin{cases} |\hat{I}_i^{(m)}| \leq C_{17,7,4,t}^{(m)}(t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2; i=1, 2, \dots, 6), \\ |I_i^{(0)}| \leq C_{17,7,4,t}^{(0)}(t-t'), & (i=1, 2, \dots, 6), \end{cases}$$

$$(4.14)' \quad |\hat{I}_7^{(m)}| = [I_{7,t}^{(m)} \text{ with } f \text{ replaced by } q] \leq \begin{cases} C_{17,7,1}^{(m)} \cdot \|q\|_T^{(\alpha)} (t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2), \\ C_{17,7,1}^{(0)} \cdot \|q\|_T^{(\alpha)} (t-t'), & (|m|=0), \end{cases}$$

$$(\|q\|_T^{(\alpha)} \leq C_{10} + C_{11}; C_{17,7,4,t}^{(m)} \equiv C_{17,7,1}^{(m)}(C_{10} + C_{11})).$$

Thus,

$$(4.14)'' \quad |I_7^{(m)}| \leq \sum_{i=1}^3 C_{17,7,t}^{(m)} + \sum_{i=1}^7 C_{17,7,4,t}^{(m)} \|f\|_T^{(\alpha)} \times \begin{cases} (t-t')^{(2-|m|+\alpha)/2}, & (|m|=1, 2), \\ (t-t'), & (|m|=0). \end{cases}$$

Hence, for $0 \leq t-t' \leq t'$, we have

$$(4.15) \quad \begin{cases} |D_x^m u(x, t) - D_x^m u(x, t')| \leq \left(\sum_{i=1}^7 C_{17,t}^{(m)} \right) (t-t')^{(2-|m|+\alpha)/2} \|f\|_T^{(\alpha)}, & (|m|=1, 2), \\ |D_x^0 u(x, t) - D_x^0 u(x, t')| \leq \left(\sum_{i=1}^7 C_{17,t}^{(0)} \right) (t-t') \|f\|_T^{(\alpha)}, & (m=0), \end{cases}$$

$$\left(C_{17,t} \equiv \sum_{i=1}^7 C_{17,7,4,t} + \sum_{i=1}^3 C_{17,7,t} \right).$$

From (4. 7) and (4. 15) follows (4. 6), where

$$(4. 16) \quad \begin{cases} C_{17}^{(|m|)} \equiv C_{16}^{(|m|)}(2^{(2-|m|+\alpha)/2} + 1) + \sum_{i=1}^7 C_{17,i}^{(|m|)}, & (|m|=1, 2), \\ C_{17}^{(0)} \equiv 3C_{16}^{(0)} + \sum_{i=1}^7 C_{17,i}^{(0)}. \end{cases} \quad \text{Q.E.D.}$$

LEMMA 4. 3. For $|m|=2$,

$$(4. 17) \quad |D_x^m u(x, t) - D_{x'}^m u(x', t)| \leq C_{18} |x - x'|^\alpha \|f\|_T^\alpha.$$

Proof.

$$(4. 18) \quad \begin{aligned} & D_x^m u(x, t) - D_{x'}^m u(x', t) \\ &= \int_0^t d\tau \int_{\Sigma_1 = \{\xi; |\xi - x| \leq 2|x - x'|\}} D_x^m \Gamma(x, t; \xi, \tau) [f(\xi, \tau) - f(x, \tau)] + [f(x, \tau) - f(x', \tau)] d\xi \\ & - \int_0^t d\tau \int_{\Sigma_1} D_{x'}^m \Gamma(x', t; \xi, \tau) [f(\xi, \tau) - f(x', \tau)] + [f(x', \tau) - f(x', t)] d\xi \\ & + \int_0^t d\tau \int_{R^3 - \Sigma_1} \{D_x^m \Gamma(x, t; \xi, \tau) - D_{x'}^m \Gamma(x', t; \xi, \tau)\} [f(\xi, \tau) - f(x', t)] d\xi \\ & + \left[\int_0^t d\tau \int_{R^3} \{D_x^m \Gamma(x, t; \xi, \tau) - D_{x'}^m \Gamma(x', t; \xi, \tau)\} d\xi \right] f(x', t) \equiv \sum_{i=1}^4 J_i. \end{aligned}$$

For $i=1, 2, 3$, we have easily

$$(4. 19) \quad |J_i| \leq C_{18,i} |x - x'|^\alpha \|f\|_T^\alpha, \quad (i=1, 2, 3).$$

$$(4. 20) \quad \begin{aligned} J_4 &= \left[\int_0^t d\tau \int_{R^3} \{D_x^m Z(x - \xi, t; \xi, \tau) - D_{x'}^m Z(x' - \xi, t; \xi, \tau)\} d\xi \right] f(x', t) \\ & + \left[\int_0^t d\tau \int_{R^3} \{D_x^m \Gamma_0(x - \xi, t; \xi, \tau) - D_{x'}^m \Gamma_0(x' - \xi, t; \xi, \tau)\} d\xi \right] f(x', t) \\ & \equiv J_{4,1} + J_{4,2}. \end{aligned}$$

$$(4. 21) \quad |J_{4,1}| \leq C_{18,4,1} |x - x'|^\alpha \|f\|_T^\alpha.$$

$$(4. 22) \quad \begin{aligned} J_{4,2} &= \left[D_x^m \int_0^t d\tau \int_{R^3} Z(x - y, t; y, \tau) q(y, \tau) dy - D_{x'}^m \int_0^t d\tau \int_{R^3} Z(x' - y, t; y, \tau) q(y, \tau) \right] f(x', t) \\ & = [\text{the righthand side of (4. 18) with } \Gamma \text{ and } f \text{ replaced by } Z \text{ and } q, \\ & \quad \text{respectively}] f(x', t) \end{aligned}$$

$$\equiv \sum_{i=1}^4 J_{4,2,i}.$$

For $i=1, 2, 3$,

$$(4.23) \quad |J_{4,2,i}| \leq C_{18,4,2,i} |x-x'|^\alpha \|f\|_T^{(\alpha)}, \quad (i=1, 2, 3).$$

$$J_{4,2,4} = [J_{4,1} \text{ with } f \text{ replaced by } q] f(x', t).$$

Thus,

$$(4.24) \quad |J_{4,2,4}| \leq C_{18,4,2,4} |x-x'|^\alpha \|f\|_T^{(\alpha)}.$$

Therefore,

$$(4.25) \quad |J_4| \leq C_{18,4,1} + \sum_{i=1}^4 C_{18,4,2,i} |x-x'|^\alpha \|f\|_T^{(\alpha)} \equiv C_{18,4} |x-x'|^\alpha \|f\|_T^{(\alpha)}.$$

From (4.19) and (4.25) follows the lemma, where

$$(4.26) \quad C_{18} \equiv \sum_{i=1}^4 C_{18,i}. \quad \text{Q.E.D.}$$

The discussion above shows that $u(x, t) \in H_T^{2+\alpha}$. Therefore, from the uniqueness of the solution in $H_T^{2+\alpha}$ of (4.1) follows that $u(x, t) = v(x, t)$. Next, we consider the linear problem,

$$(1.11) \quad \begin{cases} D_t v = \sigma P_0(D_x)v + f, & (\sigma, f \in H_T^\alpha), \\ v(x, 0) = v_0(x), & (v_0 \in H^{2+\alpha}). \end{cases}$$

The solution $v(x, t)$ in $H_T^{2+\alpha}$ is written,

$$(4.27) \quad v(x, t) = v_0(x) + \{v(x, t) - v_0(x) - u(x, t)\} + u(x, t) \equiv v_0(x) + \bar{u}(x, t) + u(x, t),$$

where $\bar{u}(x, t)$ satisfies

$$(4.28) \quad \begin{cases} D_t \bar{u} = D_t v - D_t u = \sigma P_0 v + f - (\sigma P_0 u + f) = \sigma P_0(v - v_0 - u) + \sigma P_0 v_0 = \sigma P_0 \bar{u} + \sigma P_0 v_0 \\ \bar{u}(x, 0) = 0. \end{cases}$$

Hence, by (4.2),

$$(4.28)' \quad \bar{u}(x, t) = \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau) \sigma(\xi, \tau) P_0(D_\xi) v_0(\xi) d\xi, \quad (\sigma P_0 v_0 \in H_T^\alpha).$$

Thus,

$$(4.29) \quad v(x, t) = v_0(x) + \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau) \{f(\xi, \tau) + \sigma(\xi, \tau) P_0(D_\xi) v_0(\xi)\} d\xi.$$

From (4.29), we have the following four lemmas.

LEMMA 4.4. *For the solution in $H_T^{2+\alpha}$ of the problem (1.11),*

$$(4.30) \quad |D_x^m v(x, t)| \leq |D_x^m v_0(x)| + C_{16}^{(|m|)} t^{(2-|m|+\alpha)/2} \|f + \sigma P_0 v_0\|_T^{(\alpha)}, \quad (|m|=1, 2).$$

$$(4.30)' \quad |D_t v(x, t)| \leq |\sigma P_0 v + f| \leq \frac{4}{3} \sigma_1 \sum_{|m|=2} \{|D_x^m v_0(x)| + C_{16}^{(|m|)} t^{\alpha/2} \|f + \sigma P_0 v_0\|_T^{(\alpha)}\} + \|f\|_T^{(\alpha)},$$

$$(4.30)'' \quad |D_x^\alpha v(x, t)| \leq |v_0(x)| + t \cdot C_{16}^{(0)} \|f + \sigma P_0 v_0\|_T^{(0)}.$$

LEMMA 4.5. For $T \geq t$, $t' > 0$,

$$(4.31) \quad \begin{cases} |D_x^m v(x, t) - D_x^m v(x, t')| \leq C_{17}^{(|m|)} |t - t'|^{(2-|m|+\alpha)/2} \cdot \|f + \sigma P_0 v_0\|_T^{(\alpha)}, & (|m|=1, 2), \\ |D_x^0 v(x, t) - D_x^0 v(x, t')| \leq C_{17}^{(0)} |t - t'| \cdot \|f + \sigma P_0 v_0\|_T^{(0)}, \end{cases}$$

$$(|D_t v(x, t) - D_t v(x, t')| \leq |\sigma(x, t) P_0 v(x, t) - \sigma(x, t') P_0 v(x, t')| + |f(x, t) - f(x, t')|).$$

LEMMA 4.6. For $|m|=2$,

$$(4.32) \quad |D_x^m v(x, t) - D_x^m v(x', t)| \leq |x - x'|^\alpha \{ |D_x^m v_0|^\alpha + C_{18} \|f + \sigma P_0 v_0\|_T^{(\alpha)} \},$$

$$(|D_t v(x, t) - D_t v(x', t)| \leq |\sigma(x, t) P_0(D_x)v(x, t) - \sigma(x', t) P_0(D_x)v(x', t)| + |f(x, t) - f(x', t)|).$$

LEMMA 4.7. $C_{16}^{(|m|)}$, $C_{17}^{(|m|)}$, and $C_{18}^{(1)}$ are positive functions continuous in σ_0^{-1} , σ_1 , $|\sigma|_T^{(\alpha)}$, and T , and monotonically increasing in each argument.

Proof. We obtain this by tracing the lengthy calculations made in §2~§4.

§5. The existence of a bounded solution of (1.9)–(1.10).

5.1. **Estimates for the characteristic curves.** We define:

$$(5.1) \quad \begin{cases} \langle v \rangle_T \equiv \sum_{|m|=0}^2 |D_x^m v|_T^{(0)} + \sum_{|m|=0}^1 |D_x^m v|_{t, T}^{(\alpha/2)}, \\ \langle v \rangle'_T \equiv \sum_{|m|=2} \{ |D_x^m v|_{x, T}^{(\alpha)} + |D_x^m v|_{t, T}^{(\alpha/2)} \}, & \langle \langle v \rangle \rangle_T \equiv \langle v \rangle_T + \langle v \rangle'_T, \\ (\theta)_T \equiv \sum_{|m|=0}^1 \{ |D_x^m \theta|_T^{(0)} + |D_x^m \theta|_{t, T}^{(\alpha/2)} \} + \sum_{|m|=1} |D_x^m \theta|_{x, T}^{(\alpha)}, \end{cases}$$

$$(5.1)' \quad \begin{cases} \hat{H}_T^{2+\alpha}: \text{Banach space of functions } v(x, t) \text{ defined on } \bar{R}_T^3 \text{ such that} \\ \quad \langle \langle v \rangle \rangle_T < +\infty, \quad (\text{the norm is } \langle \langle \cdot \rangle \rangle_T); \\ \hat{H}_T^{2+\alpha}: \text{Banach space of functions } \theta(x, t) \text{ defined on } \bar{R}_T^3 \text{ such that} \\ \quad (\theta)_T < +\infty, \quad (\text{the norm is } (\cdot)_T). \end{cases}$$

If $\bar{v} \in \hat{H}_T^{2+\alpha}$, (1.9)¹ is written in the following way: (v is replaced by \bar{v})

$$(5.2) \quad D_t \rho + \bar{v} \cdot \nabla \rho = -\rho \operatorname{div} \bar{v}.$$

1) Particularly speaking, $C_{16}^{(|m|)}$, $C_{17}^{(|m|)}$, and C_{18} are bounded above by polynomials with non-negative coefficients in $T^{\alpha i}$, $(|\sigma|_T^{(\alpha)})^{\beta j}$, $\sigma_0^{-\gamma k}$, $\sigma_1^{\delta l}$, and $\exp \{S(T^{\alpha/2}, |\sigma|_T^{(\alpha)}, \sigma_0^{-(6/2)}, \sigma_0^{-(6-\alpha)/2}, \sigma_1^{3/2})\}$ ($\alpha, \beta, \gamma, \delta$, and $\delta_i > 0$, and the index sets $\{i\}$, etc., are finite), where S is a polynomial with positive coefficients in the five arguments.

LEMMA 5.1.

$$(5.15) \quad |D_x^m \bar{x}_i(\tau; x, t)| \leq B_1^{(|m|)}(t-\tau; \langle \bar{v} \rangle_T), \quad (|m|=1, 2; 0 \leq \tau \leq t \leq T),$$

$$[B_1^{(1)} \equiv e^{\langle \bar{v} \rangle_T(t-\tau)}, B_1^{(2)} \equiv e^{\langle \bar{v} \rangle_T(t-\tau)} \{e^{\langle \bar{v} \rangle_T(t-\tau)} - 1\}],$$

$$(5.15)' \quad |D_x^m D_t \bar{x}_i(\tau; x, t)| \leq B_2^{(|m|)}(t-\tau; \langle \bar{v} \rangle_T), \quad (|m|=0, 1; 0 \leq \tau \leq t \leq T),$$

$$[B_2^{(0)} \equiv \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T(t-\tau)}, B_2^{(1)} \equiv \langle \bar{v} \rangle_T e^{\langle \bar{v} \rangle_T(t-\tau)} \{e^{\langle \bar{v} \rangle_T(t-\tau)} - 1\}],$$

$$(5.16) \quad \begin{cases} |D_x^m x_{0,i}(x, t)| \leq B_1^{(|m|)}(t; \langle \bar{v} \rangle_T), & (|m|=1, 2), \\ |D_x^m D_t x_{0,i}(x, t)| \leq B_2^{(|m|)}(t; \langle \bar{v} \rangle_T), & (|m|=0, 1). \end{cases}$$

5.2. **Operator** G_{T_1} . For the system (1.9)–(1.10),

$$(1.9) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0, \\ \rho \frac{\partial v}{\partial t} - \frac{\mu}{3} (3\Delta + \nabla \operatorname{div}) v = -\rho(v \cdot \nabla)v - \nabla \rho \frac{\partial p}{\partial \rho} - \nabla \theta \frac{\partial p}{\partial \theta} + \rho f \equiv N_1(x, t), \\ C_v \rho \frac{\partial \theta}{\partial t} = \kappa \Delta \theta + \Psi - \theta \frac{\partial p}{\partial \theta} \operatorname{div} v - C_v \rho v \cdot \nabla \theta \equiv C_v \rho N_2(x, t), \end{cases}$$

$$(1.10) \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x), \quad \rho(x, 0) = \rho_0(x),$$

we assume that

$$(5.17) \quad \theta_0(x), v_0(x) \in H^{2+\alpha}, \quad \rho_0(x) \in H^{1+\alpha}, \quad \text{and } f(x, t) \in H_T^\alpha,$$

$$(0 < \bar{\rho}_0 \leq \rho_0(x) \leq \bar{\bar{\rho}}_0 \equiv |\rho_0|^{(0)} < \rho^* < +\infty).$$

For $\theta < 0$, we define p by the righthand side of (1.8).

We take an arbitrary number $M_1 > \|v_0\|^{(2)}$ and choose T_1 such that

$$(5.18) \quad 0 < T_1 < M_1^{-1} \log \frac{\rho^*}{\bar{\rho}_0}, \quad \left(\|v_0\|^{(2)} = \sum_{|m|=0}^2 |D_x^m v_0(x)|^{(0)} \right).$$

Next, we define a mapping G_{T_1} from

$$(5.19) \quad S_{T_1}^0 \equiv \{(v, \theta) : (v, \theta) \in \hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}\}, \quad \langle v \rangle_{T_1} \leq M_1, \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x)$$

into $\hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}$ in the following way:

$$(5.20) \quad \begin{cases} \hat{v}(x, t) = v_0(x) + \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau; \rho_{(v)}) \left\{ N_1 + \frac{\mu}{\rho_{(v)}} P_0 v_0 \right\}(\xi, \tau) d\xi, \\ \hat{\theta}(x, t) = \theta_0(x) + \int_0^t d\tau \int_{R^3} \Gamma(x, t; \xi, \tau; \rho_{(v)}) \left\{ N_2 + \frac{\kappa}{C_v \rho_{(v)}} \Delta \theta_0 \right\}(\xi, \tau) d\xi, \end{cases}$$

where $(\hat{v}, \hat{\theta}) = G_{T_1}(v, \theta)$. As for $'I$, cf. § 2, Remark. The norm in $\hat{H}_{T_1}^{2+\alpha} \times \hat{H}_{T_1}^{1+\alpha}$ is defined by

$$(5.21) \quad \|(v, \theta)\|_{T_1} \equiv \langle v \rangle_{T_1} + \langle \theta \rangle_{T_1}.$$

LEMMA 5.2. G_{T_1} is well-defined.

Proof. It suffices to prove that

$$(5.22) \quad N_1 + \frac{\mu}{\rho_{(v)}} P_0 v_0 \quad \text{and} \quad N_2 + \frac{\kappa}{C_{\nu} \rho_{(v)}} \Delta \theta_0 \in H_{T_1}^{\alpha},$$

(especially, $\rho_{(v)}$ and $\nabla \rho_{(v)} \in H_{T_1}^{\alpha}$).

i)

$$(5.23) \quad \begin{aligned} & |\rho_{(v)}(x, t) - \rho_{(v)}(x', t')| \\ & \leq |\rho_{(v)}(x, t) - \rho_{(v)}(x', t)| + |\rho_{(v)}(x', t) - \rho_{(v)}(x', t')| \\ & \leq \{|\nabla \rho_{(v)}(\bar{x}, \bar{t})| |x - x'|\}^{\alpha} \{|\rho_{(v)}(x, t)| + |\rho_{(v)}(x', t)|\}^{1-\alpha} \\ & \quad + \left\{ \left| \frac{\partial \rho_{(v)}}{\partial t}(x', \bar{t}) \right| |t - t'| \right\}^{\alpha/2} \{|\rho_{(v)}(x', t)| + |\rho_{(v)}(x', t')|\}^{1-\alpha/2}. \end{aligned}$$

$$(5.24) \quad \left| \frac{\partial \rho_{(v)}}{\partial t} \right| \leq |v| |\nabla \rho| + |\rho| |\operatorname{div} v| \leq \{|\rho| + |\nabla \rho|\} \langle v \rangle_{T_1},$$

$$|\nabla \rho_{(v)}(x, t)| = \left| \rho(x, t) \left\{ \frac{D_{x_0, i} \rho_0(x_0(x, t))}{\rho_0(x_0(x, t))} - \nabla_{x_0, i}(x, t) \right. \right.$$

$$(5.24)' \quad \begin{aligned} & \left. - \int_0^t \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) \nabla_{x_0, i} \bar{x}_i(\tau; x, t) d\tau \right| \\ & \leq \bar{\rho}_0 e^{\langle v \rangle_{T_1} T_1} \{(\bar{\rho}_0)^{-1} \|\rho_0\|^{(1+\alpha)} B_1^{(1)}(T_1; \langle v \rangle_{T_1}) + \langle v \rangle_{T_1} B_1^{(1)}(T_1; \langle v \rangle_{T_1}) T_1\}, \\ & \quad (|\rho| \leq \bar{\rho}_0 e^{\langle v \rangle_{T_1} T_1}). \end{aligned}$$

Thus,

$$(5.25) \quad \begin{aligned} |\rho_{(v)}|_{T_1}^{(\alpha)} & \leq 2^{1-\alpha} \|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} T_1} \left\{ \left(\frac{\|\rho_0\|^{(1+\alpha)}}{\bar{\rho}_0} + \langle v \rangle_{T_1} \cdot T_1 \right) B_1^{(1)}(T_1; \langle v \rangle_{T_1}) \right\}^{\alpha} \\ & \quad + [2 \|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} T_1}]^{1+\alpha/2} [\|\rho_0\|^{(1+\alpha)} e^{\langle v \rangle_{T_1} T_1} + (\bar{\rho}_0)^{-1} \|\rho_0\|^{(1+\alpha)} \\ & \quad \quad \quad + \langle v \rangle_{T_1} T_1] B_1^{(1)}(T_1; \langle v \rangle_{T_1})]^{\alpha/2} \end{aligned}$$

$$(5.25)' \quad \left| \frac{1}{\rho_{(v)}} \right|_{T_1}^{(\alpha)} \leq (\bar{\rho}_0)^{-2} |\rho_{(v)}|_{T_1}^{(\alpha)}.$$

ii)

$$(5.26) \quad \frac{\mathcal{F}\rho_{(v)}}{\rho_{(v)}}(x, t) = \mathcal{F}_x \log \rho_{(v)} = \mathcal{F}_x \left\{ \log \rho_0(x_0(x, t)) - \int_0^t \operatorname{div} v(\bar{x}(\tau; x, t), \tau) d\tau \right\}.$$

$$(5.27) \quad \begin{aligned} & \mathcal{F}_x \log \rho_{(v)}(x, t) - \mathcal{F}_{x'} \log \rho_{(v)}(x', t) \\ &= \left\{ \frac{D_{x_0, i} \rho_0(x_0(x, t))}{\rho_0(x_0(x, t))} \mathcal{F}_{x_0, i} x_0(x, t) - \frac{D_{x_0, i} \rho_0(x_0(x', t))}{\rho_0(x_0(x', t))} \mathcal{F}_{x', i} x_0(x', t) \right\} \\ & \quad - \int_0^t \{ \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) \mathcal{F}_x \bar{x}_i(\tau; x, t) - \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x', t), \tau) \mathcal{F}_{x'} \bar{x}_i(\tau; x', t) \} d\tau. \end{aligned}$$

Hence,

$$(5.28) \quad \begin{aligned} & | \mathcal{F} \log \rho_{(v)}(x, t) - \mathcal{F} \log \rho_{(v)}(x', t) | \\ & \leq | D_{x_0, i} \rho_0(x_0(x, t)) - D_{x_0, i} \rho_0(x_0(x', t)) | \cdot \left| \frac{\mathcal{F}_x x_0, i(x, t)}{\rho_0(x_0(x, t))} \right| \\ & \quad + \left| D_{x_0, i} \rho_0(x_0(x, t)) \left\{ \frac{1}{\rho_0(x_0(x, t))} - \frac{1}{\rho_0(x_0(x', t))} \right\} \mathcal{F}_x x_0(x, t) \right| \\ & \quad + \left| \frac{D_{x_0, i} \rho_0(x_0(x', t))}{\rho_0(x_0(x', t))} \{ \mathcal{F}_x x_0, i(x, t) - \mathcal{F}_{x'} x_0, i(x', t) \} \right| \\ & \quad + \int_0^t | \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) - \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x', t), \tau) | \cdot | \mathcal{F}_x \bar{x}_i(\tau; x, t) | d\tau \\ & \quad + \int_0^t | \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x', t), \tau) | \cdot | \mathcal{F}_x \bar{x}_i(\tau; x, t) - \mathcal{F}_{x'} \bar{x}_i(\tau; x', t) | d\tau. \end{aligned}$$

$$(5.29) \quad | D_{x_0, i} \rho_0(x_0(x, t)) - D_{x_0, i} \rho_0(x_0(x', t)) | \leq \| \rho_0 \|^{(1+\alpha)} 9B_1^{(1)}(T_1; \langle v \rangle_{T_1})^\alpha |x - x'|^\alpha,$$

$$(5.29)' \quad \left| \frac{1}{\rho_0(x_0(x, t))} - \frac{1}{\rho_0(x_0(x', t))} \right| \leq (\rho_0)^{-2} \| \rho_0 \|^{(1+\alpha)} \{ 3B_1^{(1)}(T_1; \langle v \rangle_{T_1}) \}^\alpha 2^{1+\alpha} |x - x'|^\alpha.$$

$$(5.29)'' \quad \begin{aligned} & | \mathcal{F}_x x_0, i(x, t) - \mathcal{F}_{x'} x_0, i(x', t) | \\ & \leq 3^{1+\alpha} \{ B_1^{(2)}(T_1; \langle v \rangle_{T_1}) \}^\alpha \{ 2B_1^{(2)}(T_1; \langle v \rangle_{T_1}) \}^{1-\alpha} |x - x'|^\alpha \\ & \quad \cdot \int_0^t | \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) - \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x', t), \tau) | \cdot | \mathcal{F}_x \bar{x}_i(\tau; x, t) | d\tau. \end{aligned}$$

$$(5.30) \quad \leq 3^{1+2\alpha} \langle v \rangle_{T_1}' T_1 \{ B_1^{(3)}(T_1; \langle v \rangle_{T_1}) \}^{1+\alpha} |x - x'|^\alpha$$

$$(5.30)' \quad \cdot \int_0^t | \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x', t), \tau) \{ \mathcal{F}_x \bar{x}_i(\tau; x, t) - \mathcal{F}_{x'} \bar{x}_i(\tau; x', t) \} | d\tau$$

$$(5.30)'' \quad \leq T_1 \langle v \rangle_{T_1} 3^{1+\alpha} \{ B_1^{(2)}(T_1; \langle v \rangle_{T_1}) \}^\alpha \{ 2B_1^{(3)}(T_1; \langle v \rangle_{T_1}) \}^{1-\alpha} |x - x'|^\alpha.$$

Thus, there exist non-negative constants D_1 and D_2 ($< +\infty$) depending on $\langle v \rangle_{T_1}$, T_1 , $(\bar{\rho}_0)^{-1}$, and $\|\rho_0\|^{(1+\alpha)}$ such that

$$(5.31) \quad \begin{aligned} & |\nabla \log \rho|(x, t) - \nabla \log \rho(x', t)| \\ & \leq \{D_1(\langle v \rangle_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|\rho_0\|^{(1+\alpha)}) + \langle v \rangle'_{T_1} T_1 D_2(\langle v \rangle_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|\rho_0\|^{(1+\alpha)})\} |x - x'|^\alpha. \end{aligned}$$

In an analogous way, we have an inequality, ($t \geq t'$)

$$(5.32) \quad \begin{aligned} & |\nabla \log \rho_{(v)}(x, t) - \nabla \log \rho_{(v)}(x, t')| \\ & \leq \left| \left\{ D_{x_0, i} \rho_0(x_0(x, t)) - D_{x_0, i} \rho_0(x_0(x, t')) \right\} \frac{\nabla_x x_{0, i}(x, t)}{\rho_0(x_0(x, t))} \right| \\ & \quad + \left| D_{x_0, i} \rho_0(x_0(x, t')) \left\{ \frac{1}{\rho_0(x_0(x, t))} - \frac{1}{\rho_0(x_0(x, t'))} \right\} D_x x_{0, i}(x, t) \right| \\ & \quad + \left| \frac{D_{x_0, i} \rho_0(x_0(x, t))}{\rho_0(x_0(x, t))} \{ \nabla_x x_{0, i}(x, t) - \nabla_x x_{0, i}(x, t') \} \right| \\ & \quad + \int_{t'}^t |\operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) \nabla_x \bar{x}_i(\tau; x, t)| d\tau \\ & \quad + \int_0^{t'} |\{\operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t), \tau) - \operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t'), \tau)\} \nabla_x \bar{x}_i(\tau; x, t)| d\tau \\ & \quad + \int_0^{t'} |\operatorname{div} D_{\bar{x}_i} v(\bar{x}(\tau; x, t'), \tau) \{ \nabla_x \bar{x}_i(\tau; x, t) - \nabla_x \bar{x}_i(\tau; x, t') \}| d\tau. \end{aligned}$$

From this follows that there exist non-negative constants D_3 and D_4 depending on $\langle v \rangle_{T_1}$, T_1 , $(\bar{\rho}_0)^{-1}$, and $\|\rho_0\|^{(1+\alpha)}$ such that

$$(5.32) \quad \begin{aligned} & |\nabla \log \rho_{(v)}(x, t) - \nabla \log \rho_{(v)}(x, t')| \\ & \leq \{D_3(\langle v \rangle_{T_1}, T_1) + D_4(\langle v \rangle_{T_1}, T_1) T_1 \langle v \rangle'_{T_1}\} |t - t'|^{\alpha/2}. \end{aligned}$$

From (5.31) and (5.32)' we have

$$(5.33) \quad |\nabla \log \rho_{(v)}|_{T_1}^{\alpha} \leq D_1(\langle v \rangle_{T_1}, T_1) + D_3(\langle v \rangle_{T_1}, T_1) + \langle v \rangle'_{T_1} T_1 \{D_2(\langle v \rangle_{T_1}, T_1) + D_4(\langle v \rangle_{T_1}, T_1)\}.$$

E_i ($i=1, 2, 3, 4$) are continuous functions of $\langle v \rangle_{T_1}$ and T_1 , monotonically increasing in each argument.

iii) It is easily shown that $\partial P / \partial \rho$, $\partial p / \partial \rho$, and $\Psi \in H_{T_1}^\alpha$, because p is assumed to be virially expanded and $\rho_{(v)} \in H_{T_1}^\alpha$.

From i), ii), and iii) follows the result that

$$(5.34) \quad \left\{ \begin{array}{l} \left\| N_1 + \frac{\mu}{\rho(v)} P_0 v_0 \right\|_{T_1}^{(\alpha)} \leq A_1(\langle v \rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_{T'}^{(\alpha)}) \\ \quad + T_1 A_2(\langle v \rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_{T'}^{(\alpha)}) \langle v \rangle'_{T_1} < +\infty, \\ \left\| N_2 + \frac{\kappa}{C_v \rho(v)} \Delta \theta_0 \right\|_{T_1}^{(\alpha)} \leq A_3(\langle v \rangle_{T_1}, (\theta)_{T_1}, T_1; (\bar{\rho}_0)^{-1}, \|v_0\|^{(2+\alpha)}, \|f\|_{T'}^{(\alpha)}) < +\infty, \end{array} \right.$$

where A_i ($i=1, 2, 3$) are positive constants continuous in the six parameters and increasing monotonically in each parameter. Q.E.D.

5.3. The subset $S_{T'}$.

LEMMA 5.3. *There exist $T' \in (0, T_1]$, $M_2 (> 0)$, and $M_3 (> 0)$ such that, for a subset $S_{T'}$ of $S_{T_0}^0$, defined by*

$$(5.35) \quad S_{T'} \equiv \{(v, \theta) : (v, \theta) \in S_{T_0}^0, (\theta)_{T'} \leq M_2, \langle v \rangle'_{T'} \leq M_3\},$$

the following holds:

$$(5.35)' \quad G_{T', S_{T'}} \subset S_{T'} \subset S_{T_0}^0 \subset \hat{H}_{T'}^{2+\alpha} \times \hat{H}_{T'}^{1+\alpha}.$$

Proof. The lemmas 4.4~4.7 and the procedure of the proof of Lemma 5.2 guarantee that, for an arbitrary $T_0 \in (0, T_1]$ and for an arbitrary $(v, \theta) \in S_{T_0}$, we have

$$(5.36) \quad \left\{ \begin{array}{l} \langle \hat{v} \rangle_{T_0} \leq \|v_0\|^{(2)} + C(\langle v \rangle_{T_0}, T_0) \{A_1(\langle v \rangle_{T_0}, (\theta)_{T_0}, T_0) + T_0 A_2(\langle v \rangle_{T_0}, (\theta)_{T_0}, T_0) \langle v \rangle'_{T_0}\}, \\ \langle \hat{\theta} \rangle_{T_0} \leq \|\theta_0\|^{(1+\alpha)} + 'C(\langle v \rangle_{T_0}, T_0) A_3(\langle v \rangle_{T_0}, (\theta)_{T_0}, T_0), \\ \langle \hat{v} \rangle'_{T_0} \leq |\nabla \nabla v_0|^{(\alpha)} + \hat{C}(\langle v \rangle_{T_0}, T_0) A_1(\langle v \rangle_{T_0}, (\theta)_{T_0}, T_0) + T_0 A_2(\langle v \rangle_{T_0}, (\theta)_{T_0}, T_0) \langle v \rangle'_{T_0}, \end{array} \right.$$

where $C(\cdot, T_0)$ and $'C(\cdot, T_0) \searrow 0$, and $\hat{C}(\cdot, T_0) \searrow C_0 > 0$, as $T_0 \searrow 0$. Taking an arbitrary $M_2 > \|\theta_0\|^{(1+\alpha)}$, we can choose $T_2 \in (0, T_1]$ such that

$$(5.37) \quad \left\{ \begin{array}{l} \|v_0\|^{(2)} + C(M_1, T_2) A_1(M_1, M_2, T_2) < M_1, \\ \|\theta_0\|^{(1+\alpha)} + 'C(M_1, T_2) A_3(M_1, M_2, T_2) \leq M_2, \\ \hat{C}(M_1, T_2) T_2 A_2(M_1, M_2, T_2) < 1. \end{array} \right.$$

Moreover, there exists $T' \in (0, T_2]$ such that

$$(5.37)' \quad \begin{aligned} & |\nabla \nabla v_0|^{(\alpha)} + C(M_1, T') A_1(M_1, M_2, T') \\ & \leq \frac{1 - C(M_1, T') T' A_2(M_1, M_2, T')}{C(M_1, T') T' A_2(M_1, M_2, T')} \{M_1 - \|v_0\|^{(2)} - C(M_1, T') A_1(M_1, M_2, T')\}. \end{aligned}$$

Hence, there exists $M_3 > 0$ such that

(5.37)''

$$\frac{|V \nabla v_0|^{(c)} + \hat{C}(M_1, T') A_1(M_1, M_2, T')}{1 - \hat{C}(M_1, T') T' A_2(M_1, M_2, T')} \leq M_3 \leq \frac{M_1 - \{\|v_0\|^{(2)} + C(M_1, T') A_1(M_1, M_2, T')\}}{C(M_1, T') T' A_2(M_1, M_2, T')},$$

$$(C(M_1, T'), A_1(M_1, M_2, T'), C(M_1, T'), A_2(M_1, M_2, T')) > 0).$$

From (5.36), (5.37), and (5.37)'' follows (5.35)'.

Q.E.D.

We note that

(5.38)
$$G_{T', S_{T'}} \subset S_{T'} \cap \{H_{T'}^{2+\alpha} \times H_{T'}^{2+\alpha}\} \subset S_{T'} \cap \{\hat{H}_{T'}^{2+\alpha} \times \hat{H}_{T'}^{2+\alpha}\} \subset S_{T'}, \quad (\text{cf. } \S 4),$$

and that

(5.38)'
$$G_{T', S_{T'}} \subset S_{T'} \subset \hat{H}_{T'}^{2+\alpha} \times \hat{H}_{T'}^{1+\alpha} \subset \dot{H}_{T'}^{2+\alpha} \times \dot{H}_{T'}^{1+\alpha} \subset \dot{H}_{T'}^{2+\alpha, \beta} \times \dot{H}_{T'}^{1+\alpha, \beta},$$

($\beta \in (0, 1)$ is arbitrary),

where $\dot{H}_{T'}^{\gamma, \beta}$ ($\gamma \in (0, 1)$) is a Fréchet space defined by a countable system of seminorms

(5.39)
$$[(v, \theta)]_{N, T'}^{(q)} \equiv \langle\langle v \rangle\rangle_{N, T'}^{(q)} + (\theta)_{N, T'}^{(q)}, \quad (N=1, 2, 3, \dots),$$

[the suffix “ N, T' ” indicates that the supremum is taken on $R_{N, T'}^3 \equiv \{(x, t); |x| \leq N + M_1(T' - t), 0 \leq t \leq T', x \in R^3\}$ instead of $R_{T'}^3$].

LEMMA 5.4. $S_{T'}$ is a convex compact subset in $\dot{H}_{T'}^{2+\alpha, \beta} \times \dot{H}_{T'}^{1+\alpha, \beta}$.*Proof.* i) Convexity. For (v_1, θ_1) and $(v_2, \theta_2) \in S_{T'}$, and for $\lambda \in [0, 1]$,

(5.40)
$$(v_1, \theta_1) + (1 - \lambda)(v_2, \theta_2) = (\lambda v_1 + (1 - \lambda)v_2, \lambda \theta_1 + (1 - \lambda)\theta_2) \in S_{T'},$$

because

(5.40)'
$$\begin{cases} \langle \lambda v_1 + (1 - \lambda)v_2 \rangle_{T'}^{(q)} \leq M_1, & [\lambda \theta_1 + (1 - \lambda)\theta_2]_{T'}^{(q)} \leq M_2, & \langle \lambda v_1 + (1 - \lambda)v_2 \rangle_{T'}^{(q)} \leq M_3, \\ (\lambda v_1 + (1 - \lambda)v_2, \lambda \theta_1 + (1 - \lambda)\theta_2)|_{t=0} = (v_0, \theta_0). \end{cases}$$

ii) Compactness. By a simple calculation it is shown that

(5.41)
$$[(v, \theta)]_{N, T'}^{(q, \beta)} \leq 3[(v, \theta)]_{N, T'}^{(q)} \leq 3\langle\langle v, \theta \rangle\rangle_{T'}^{(q)}.$$

Next, for an arbitrary sequence $\{(v_i, \theta_i)\}_{i=1}^\infty \subset S_{T'}$, by Ascoli-Arzelà's theorem, there exist a subsequence $\{(v_{i_k}, \theta_{i_k}) \in S_{T'}\}$ and $(\bar{v}, \bar{\theta}) \in S_{T'}$, such that, in a compact uniform way,

(5.42)
$$\begin{cases} D_x^m v_{i_k} \rightarrow D_x^m \bar{v}, & (k \rightarrow +\infty; |m| \leq 2), \\ D_x^m \theta_{i_k} \rightarrow D_x^m \bar{\theta}, & (k \rightarrow +\infty; |m| \leq 1). \end{cases}$$

It remains to show that (v_{i_k}, θ_{i_k}) converges to $(\bar{v}, \bar{\theta})$ in the topology of $\dot{H}_{T'}^{2+\alpha, \beta} \times \dot{H}_{T'}^{1+\alpha, \beta}$.

Firstly, in $R_{N,T'}^3$,

$$\begin{aligned}
& \{|D_x^m v_{i_k}(x, t) - D_x^m \bar{v}(x, t)\} - \{|D_x^m v_{i_k}(x, t') - D_x^m \bar{v}(x, t')\}| \\
(5.43) \quad & \leq \{|D_x^m v_{i_k}(x, t) - D_x^m v_{i_k}(x, t')\} - \{|D_x^m \bar{v}(x, t) - D_x^m \bar{v}(x, t')\}|^\beta \\
& \quad \times \{|D_x^m v_{i_k}(x, t) - D_x^m \bar{v}(x, t)\} + \{|D_x^m v_{i_k}(x, t') - D_x^m \bar{v}(x, t')\}|^{1-\beta} \\
& \leq \{|D_x^m v_{i_k}|_{i,N,T'}^{(\alpha/2)} + |D_x^m \bar{v}|_{i,N,T'}^{(\alpha/2)}\}^\beta |t - t'|^{\alpha\beta/2} \{2|D_x^m v_{i_k} - D_x^m \bar{v}|_{N,T'}^{(0)}\}^{1-\beta}, \quad (|m| \leq 2).
\end{aligned}$$

Hence,

$$\begin{aligned}
(5.43)' \quad & |D_x^m v_{i_k} - D_x^m \bar{v}|_{i,N,T'}^{(\alpha\beta/2)} \leq \{|D_x^m v_{i_k}|_{i,N,T'}^{(\alpha/2)} + |D_x^m \bar{v}|_{i,N,T'}^{(\alpha/2)}\}^\beta \{2|D_x^m v_{i_k} - D_x^m \bar{v}|_{N,T'}^{(0)}\}^{1-\beta}, \\
& \quad (|m| \leq 2).
\end{aligned}$$

Therefore,

$$(5.43)'' \quad |D_x^m v_{i_k} - D_x^m \bar{v}|_{i,N,T'}^{(\alpha\beta/2)} \rightarrow 0 \quad (k \rightarrow +\infty; N=1, 2, \dots; |m| \leq 2).$$

In the same way, as $k \rightarrow +\infty$,

$$\begin{aligned}
(5.44) \quad & |D_x^m v_{i_k} - D_x^m \bar{v}|_{i,N,T'}^{(\alpha\beta)} \quad (|m|=2), \quad |D_x^{m'} \theta_{i_k} - D_x^{m'} \bar{\theta}|_{i,N,T'}^{(\alpha\beta/2)} \quad (|m'|=0, 1), \\
& \text{and} \quad |D_x^{m''} \theta_{i_k} - D_x^{m''} \bar{\theta}|_{i,N,T'}^{(\alpha\beta)} \quad (|m''|=1) \rightarrow 0.
\end{aligned}$$

Thus,

$$(5.44)' \quad d_T^{(\alpha\beta)}((v_{i_k}, \theta_{i_k}), (\bar{v}, \bar{\theta})) \rightarrow 0 \quad (k \rightarrow +\infty),$$

where $d_T^{(\alpha\beta)}(\cdot, \cdot)$ is a distance defined by

$$(5.44)'' \quad d_T^{(\alpha\beta)}((v_1, \theta_1), (v_2, \theta_2)) \equiv \sum_{N=1}^{\infty} \left(\frac{1}{2}\right)^N \frac{[(v_1, \theta_1) - (v_2, \theta_2)]_{N,T'}^{(\alpha\beta)}}{1 + [(v_1, \theta_1) - (v_2, \theta_2)]_{N,T'}^{(\alpha\beta)}}. \quad \text{Q.E.D.}$$

5.4. The continuity of $G_{T'}$ on $S_{T'}$ as a subset of $\dot{H}_{T'}^{2+\alpha\beta} \times \dot{H}_{T'}^{1+\alpha\beta}$.

LEMMA 5.5. $G_{T'}$ is continuous as an operator from $S_{T'}$ as a subset of $\dot{H}_{T'}^{2+\alpha\beta} \times \dot{H}_{T'}^{1+\alpha\beta}$ into itself.

Proof. We put

$$(\hat{v}, \hat{\theta}) \equiv G_{T'}(v, \theta), \quad ((v, \theta) \in S_{T'}).$$

We show that, for an arbitrary sequence $\{(v_n, \theta_n) \in S_{T'}\}$ that converges to an arbitrary fixed element $(v, \theta) \in S_{T'}$ in the topology of $\dot{H}_{T'}^{2+\alpha\beta} \times \dot{H}_{T'}^{1+\alpha\beta}$,

$$(5.45) \quad \begin{cases} G_{T'}(v_n, \theta_n) = (\hat{v}_n, \hat{\theta}_n) \rightarrow (\hat{v}, \hat{\theta}) = G_{T'}(v, \theta) & (\text{in } \dot{H}_{T'}^{2+\alpha\beta} \times \dot{H}_{T'}^{1+\alpha\beta}), \quad (n \rightarrow +\infty), \\ \text{i.e., for } \forall \varepsilon > 0, \end{cases}$$

$$[(\hat{v}_n, \hat{\theta}_n) - (\hat{v}, \hat{\theta})]_{N,T'}^{(\alpha\beta)} < \varepsilon, \quad (n \geq n_0(\varepsilon, N)).$$

In the first place, we estimate $\|\tilde{N}_1\|_{\tilde{N}, T'}^{(\alpha, \beta)}$ and $\|\tilde{N}_2\|_{\tilde{N}, T'}^{(\alpha, \beta)}$ ($N=1, 2, \dots$). We note that

$$(5.49) \quad \|\tilde{N}_i\|_{\tilde{N}, T'}^{(\alpha, \beta)} \leq \|\tilde{N}_i\|_{\tilde{N}, T'}^{(\alpha, \beta)} \leq \dots \leq \|\tilde{N}_i\|_{\tilde{N}, T'}^{(\alpha, \beta)} \leq \text{const} \|\tilde{N}_i\|_{\tilde{N}, T'}^{(\alpha)}, \quad (i=1, 2).$$

i)

$$\|\rho_{(v)} - \rho_{(v_n)}\|_{\tilde{N}, T'}^{(\alpha, \beta)} = |\rho_{(v)} - \rho_{(v_n)}|_{\tilde{N}, T'}^{(0)} + |\rho_{(v)} - \rho_{(v_n)}|_{\tilde{N}, T'}^{(\alpha, \beta)} + |\rho_{(v)} - \rho_{(v_n)}|_{\tilde{N}, T'}^{(\alpha, \beta)}.$$

For $(\xi, \tau) \in R_{\tilde{N}, T'}^{\ddagger}$,

$$(5.50) \quad \begin{aligned} & |\rho_{(v)}(\xi, \tau) - \rho_{(v_n)}(\xi, \tau)| \\ &= \left| \rho_0(x_0(\xi, \tau)) \exp \left\{ - \int_0^\tau \text{div } v(\bar{x}(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \right. \\ & \quad \left. - \rho_0(x_{n,0}(\xi, \tau)) \exp \left\{ - \int_0^\tau \text{div } v_n(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \right| \\ & \leq |\nabla \rho_0|^{(0)} |(x_0 - x_{n,0})(\xi, \tau)| e^{M_1 T'} \\ & \quad + |\rho_0|^{(0)} e^{M_1 T'} \left\{ \int_0^\tau |\text{div } v(\bar{x}(\tau_0; \xi, \tau), \tau_0) - \text{div } v(\bar{x}_n(\tau_0; \xi, \tau), \tau_0)| d\tau_0 \right\} \\ & \quad + \int_0^\tau |\text{div } v(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) - \text{div } v_n(\bar{x}_n(\tau_0; \xi, \tau), \tau_0)| d\tau_0 \\ & \leq \left[|\nabla \rho_0|^{(0)} |(x_0 - x_{n,0})(\xi, \tau)| + |\rho_0|^{(0)} \left\{ |\nabla v|_{\tilde{N}, T'}^{(0)} \int_0^\tau |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)| d\tau_0 \right. \right. \\ & \quad \left. \left. + \int_0^\tau |\nabla v - \nabla v_n|_{\tilde{N}, T'}^{(0)} d\tau_0 \right\} \right] e^{M_1 T'}. \end{aligned}$$

Now, we note that

$$(5.51) \quad \begin{aligned} (\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau) &= - \int_0^\tau \{ (v(x(\tau_1; \xi, \tau), \tau_1) - v_n(x(\tau_1; \xi, \tau), \tau_1)) \\ & \quad + (v_n(\bar{x}(\tau_1; \xi, \tau), \tau_1) - v_n(\bar{x}_n(\tau_1; \xi, \tau), \tau_1))) d\tau_1, \end{aligned}$$

and, therefore, that

$$(5.51)' \quad \left\{ \begin{aligned} |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)| &\leq |v - v_n|_{\tilde{N}, T'}^{(0)} (\tau - \tau_0) + M_1 \int_{\tau_0}^\tau |(\bar{x} - \bar{x}_n)(\tau_1; \xi, \tau)| d\tau_1, \\ |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)| &\leq \frac{1}{M_1} |v - v_n|_{\tilde{N}, T'}^{(0)} (e^{M_1(\tau - \tau_0)} - 1). \end{aligned} \right.$$

Thus,

$$(5.52) \quad |\rho_{(v)} - \rho_{(v_n)}|_{\tilde{N}, T'}^{(0)} \leq E_1(M_1, T'; \|\rho_0\|^{(1)}) \langle v - v_n \rangle_{\tilde{N}, T'}^{(\alpha, \beta)},$$

where E_1 is a constant depending on M_1 , T' , and $\|\rho_0\|^{(1)}$. Next, we have

$$\begin{aligned}
& |(\rho_{(v)} - \rho_{(v_n)})(\xi, \tau) - (\rho_{(v)} - \rho_{(v_n)})(\xi', \tau)| \\
(5.53) \quad & \leq \{ |(\rho_{(v)} - \rho_{(v_n)})(\xi, \tau)| + |(\rho_{(v)} - \rho_{(v_n)})(\xi', \tau)| \}^{1-\beta} \cdot \{ |\rho_{(v)}(\xi, \tau) - \rho_{(v)}(\xi', \tau)| \\
& \quad + |\rho_{(v_n)}(\xi, \tau) - \rho_{(v_n)}(\xi', \tau)| \}^\beta \\
& \leq \{ 2|\rho_{(v)} - \rho_{(v_n)}|_{\bar{N}, T'}^{(0)} \}^{1-\beta} \{ |\rho_{(v)}|_{\bar{x}, T'}^{(\alpha)} + |\rho_{(v_n)}|_{\bar{x}, T'}^{(\alpha)} \}^\beta |\xi - \xi'|^{\alpha\beta}.
\end{aligned}$$

Hence,

$$(5.53)' \quad |\rho_{(v)} - \rho_{(v_n)}|_{\bar{x}, \bar{N}, T'}^{(\alpha\beta)} \leq (2E_1)^{1-\beta} \{ |\rho_{(v)}|_{\bar{x}, T'}^{(\alpha)} + |\rho_{(v_n)}|_{\bar{x}, T'}^{(\alpha)} \}^\beta \{ \langle v - v_n \rangle_{\bar{N}, T'}^{(\alpha\beta)} \}^{1-\beta}.$$

In the same way, we have

$$(5.53)'' \quad |\rho_{(v)} - \rho_{(v_n)}|_{\bar{t}, \bar{N}, T'}^{(\alpha\beta)} \leq (2E_1)^{1-\beta} \{ |\rho_{(v)}|_{\bar{t}, T'}^{(\alpha/2)} + |\rho_{(v_n)}|_{\bar{t}, T'}^{(\alpha/2)} \}^\beta \{ \langle v - v_n \rangle_{\bar{N}, T'}^{(\alpha\beta)} \}^{1-\beta}.$$

ii)

$$\begin{aligned}
& \| \mathcal{F} \log \rho_{(v)} - \mathcal{F} \log \rho_{(v_n)} \|_{\bar{N}, T'}^{(\alpha\beta)} \\
& = | \mathcal{F} \log \rho_{(v)} - \mathcal{F} \log \rho_{(v_n)} |_{\bar{N}, T'}^{(0)} + | \mathcal{F} \log \rho_{(v)} - \mathcal{F} \log \rho_{(v_n)} |_{\bar{x}, T'}^{(\alpha\beta)} \\
& \quad + | \mathcal{F} \log \rho_{(v)} - \mathcal{F} \log \rho_{(v_n)} |_{\bar{t}, \bar{N}, T'}^{(\alpha\beta/2)}. \\
(5.54) \quad & (\mathcal{F} \log \rho_{(v)} - \mathcal{F} \log \rho_{(v_n)})(\xi, \tau) \\
& = \frac{(\mathcal{F} \rho_{(v)} - \mathcal{F} \rho_{(v_n)})(\xi, \tau)}{\rho_{(v)}(\xi, \tau)} + \frac{\rho_{(v_n)}(\xi, \tau)(\rho_{(v_n)} - \rho_{(v)})(\xi, \tau)}{\rho_{(v)}(\xi, \tau)\rho_{(v_n)}(\xi, \tau)}.
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{F} \rho_{(v)} - \mathcal{F} \rho_{(v_n)})(\xi, \tau) \\
(5.55) \quad & = \mathcal{F}_\xi \rho_0(x_0(\xi, \tau)) \exp \left\{ - \int_0^\tau \operatorname{div} v(\bar{x}(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \\
& \quad + \rho_0(x_0(\xi, \tau)) \exp \left\{ - \int_0^\tau \operatorname{div} v(\bar{x}(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \cdot \mathcal{F}_\xi \int_0^\tau \operatorname{div} v(\bar{x}(\tau_0; \xi, \tau), \tau_0) d\tau_0 \\
& \quad - \left\{ \mathcal{F}_\xi \rho_0(x_{n,0}(\xi, \tau)) \exp \left\{ - \int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \right. \\
& \quad \left. + \rho_0(x_{n,0}(\xi, \tau)) \exp \left\{ - \int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\} \cdot \int_0^\tau \operatorname{div} v_n(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) d\tau_0 \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
 & |(\mathcal{F}\rho_{(v)} - \mathcal{F}\rho_{(v_n)})(\xi, \tau)| \\
 \leq & \{|\mathcal{F}_{x_0}\rho_0(x_0(\xi, \tau))| \cdot |(\mathcal{F}_\xi x_0 - \mathcal{F}_\xi x_{n,0})(\xi, \tau)| \\
 & + |\mathcal{F}_{x_0}\rho_0(x_0(\xi, \tau)) - \mathcal{F}_{x_{n,0}}\rho_0(x_{n,0}(\xi, \tau))| \cdot |\mathcal{F}_\xi x_{n,0}(\xi, \tau)|\} e^{M_1 T'} \\
 (5.55)' & + |\mathcal{F}\rho_0|^{(0)} |\mathcal{F}_\xi x_0|^{(0)} \left\{ |\mathcal{F}\mathcal{F}v|_{T'}^{(0)} \int_0^\tau |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)| d\tau_0 + |\mathcal{F}v - \mathcal{F}v_n|_{N, T'}^{(0)} \cdot T' \right\} \\
 & + |\rho_{(v)} - \rho_{(v_n)}|_{N, T'}^{(0)} |\mathcal{F}\mathcal{F}v|_{T'}^{(0)} \int_0^\tau |\mathcal{F}_\xi \bar{x}(\tau_0; \xi, \tau)| d\tau_0 \\
 & + |\rho_{(v_n)}|_{T'}^{(0)} \int_0^\tau \{|\mathcal{F}\mathcal{F}v|_{\bar{x}, T'}^{(0)} |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)|^\alpha + |\mathcal{F}\mathcal{F}v - \mathcal{F}\mathcal{F}v_n|_{N, T'}^{(0)}\} |\mathcal{F}_\xi \bar{x}(\tau_0; \xi, \tau)| d\tau_0 \\
 & + |\rho_{(v_n)}|_{T'}^{(0)} \int_0^\tau |\mathcal{F}\mathcal{F}v_n|_{T'}^{(0)} |\mathcal{F}_\xi \bar{x} - \mathcal{F}_\xi \bar{x}_n(\tau_0; \xi, \tau)| d\tau_0.
 \end{aligned}$$

There is a necessity to estimate $|\mathcal{F}_\xi \bar{x} - \mathcal{F}_\xi \bar{x}_n(\tau_0; \xi, \tau)|$. The matrices

$$(5.56) \quad \bar{X}(\tau_0; \xi, \tau) = \left(\frac{\partial \bar{x}_i(\tau_0; \xi, \tau)}{\partial \xi_j} \right), \quad \bar{X}_n(\tau_0; \xi, \tau) \equiv \left(\frac{\partial \bar{x}_{n,i}(\tau_0; \xi, \tau)}{\partial \xi_j} \right),$$

satisfy, respectively,

$$(5.56)' \quad \left\{ \begin{aligned} \frac{d}{d\tau_0} \bar{X}(\tau_0; \xi, \tau) &= \left(\frac{\partial v_i}{\partial \bar{x}_j}(\bar{x}(\tau_0; \xi, \tau), \tau_0) \bar{X}(\tau_0; \xi, \tau) \right) \\ &\equiv \hat{\mathcal{V}}(\tau_0; \xi, \tau) \bar{X}(\tau_0; \xi, \tau), \quad \bar{X}(\tau; \xi, \tau) = I, \\ \frac{d}{d\tau_0} \bar{X}_n(\tau_0; \xi, \tau) &= \left(\frac{\partial v_{n,i}}{\partial \bar{x}_j}(\bar{x}_n(\tau_0; \xi, \tau), \tau_0) \bar{X}_n(\tau_0; \xi, \tau) \right) \\ &\equiv \hat{\mathcal{V}}_n(\tau_0; \xi, \tau) \bar{X}_n(\tau_0; \xi, \tau), \quad \bar{X}_n(\tau; \xi, \tau) = I. \end{aligned} \right.$$

From this follows that, for $\tau_0 \in [0, \tau]$,

$$\begin{aligned}
 & |(\bar{X} - \bar{X}_n)(\tau_0; \xi, \tau)| \\
 (5.57) & \leq \int_{\tau_0}^\tau |(\bar{X} - \bar{X}_n)(\tau_1; \xi, \tau)| |\hat{\mathcal{V}}_n(\tau_1; \xi, \tau)| d\tau_1 + \int_{\tau_0}^\tau |(\hat{\mathcal{V}} - \hat{\mathcal{V}}_n)(\tau_1; \xi, \tau)| |\bar{X}(\tau_1)| d\tau_1.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & |(\bar{X} - X_n)(\tau_0; \xi, \tau)| \\
 (5.58) & \leq \{ |v - v_n|_{N, T'}^{(0)} (e^{M_1 T'} - 1) + M_1 |\mathcal{F}v - \mathcal{F}v_n|_{N, T'}^{(0)} \} (e^{M_1 T'} - 1) 9B_1^{(0)}(T'; M_1).
 \end{aligned}$$

From the above, we have

$$(5.59) \quad |\mathcal{F}\rho_{(v)} - \mathcal{F}\rho_{(v_n)}|_{N, T'}^{(0)} \leq E_{2,1}(M_1, T') \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} + E_{2,2}(M_1, T') \{\langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)}\}^\alpha,$$

where $E_{2,1}$ and $E_{2,2}$ are non-negative constants depending on M_1 and T' . Therefore, from (5.54) we obtain

$$(5.60) \quad \begin{aligned} & |\mathcal{F}\log \rho_{(v)} - \mathcal{F}\log \rho_{(v_n)}|_{N, T'}^{(0)} \\ & \leq E_{3,1}(M_1, T') \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} + E_{3,2}(M_1, T') \{\langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)}\}^\alpha, \end{aligned}$$

where $E_{3,1}$ and $E_{3,2}$ are non-negative constants depending on M_1 and T' . Next,

$$(5.61) \quad \begin{aligned} & | \{ \mathcal{F}\log \rho_{(v)}(\xi, \tau) - \mathcal{F}\log \rho_{(v_n)}(\xi, \tau) \} - \{ \mathcal{F}\log \rho_{(v)}(\xi', \tau) - \mathcal{F}\log \rho_{(v_n)}(\xi', \tau) \} | \\ & \leq (| \mathcal{F}\log \rho_{(v)}(\xi, \tau) - \mathcal{F}\log \rho_{(v)}(\xi', \tau) | + | \mathcal{F}\log \rho_{(v_n)}(\xi, \tau) - \mathcal{F}\log \rho_{(v_n)}(\xi', \tau) |)^\beta \\ & \quad \times \{ | \mathcal{F}\log \rho_{(v)}(\xi, \tau) - \mathcal{F}\log \rho_{(v_n)}(\xi, \tau) | + | \mathcal{F}\log \rho_{(v)}(\xi', \tau) - \mathcal{F}\log \rho_{(v_n)}(\xi', \tau) | \}^{1-\beta} \\ & \leq | \xi - \xi' |^{\alpha\beta} \{ | \mathcal{F}\log \rho_{(v)} |_{x, T'}^{(\alpha)} + | \mathcal{F}\log \rho_{(v_n)} |_{x, T'}^{(\alpha)} \}^\beta \cdot \{ 2 | \mathcal{F}\log \rho_{(v)} - \mathcal{F}\log \rho_{(v_n)} |_{N, T'}^{(0)} \}^{1-\beta}. \end{aligned}$$

Hence, we have

$$(5.61)' \quad \begin{aligned} & | \mathcal{F}\log \rho_{(v)} - \mathcal{F}\log \rho_{(v_n)} |_{x, N, T'}^{(\alpha\beta)} \\ & \leq 2^{1-\beta} \{ | \mathcal{F}\log \rho_{(v)} |_{x, T'}^{(\alpha)} + | \mathcal{F}\log \rho_{(v_n)} |_{x, T'}^{(\alpha)} \}^\beta \times \{ E_{3,1} \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} + E_{3,2} \{ \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} \}^\alpha \}^{1-\beta}. \end{aligned}$$

In an analogous way, we have

$$(5.61)'' \quad \begin{aligned} & | \mathcal{F}\log \rho_{(v)} - \mathcal{F}\log \rho_{(v_n)} |_{t, N, T'}^{(\alpha\beta)} \\ & \leq 2^{1-\beta} \{ | \mathcal{F}\log \rho_{(v)} |_{t, T'}^{(\alpha/2)} + | \mathcal{F}\log \rho_{(v_n)} |_{t, T'}^{(\alpha/2)} \}^\beta \times \{ E_{3,1} \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} + E_{3,2} \{ \langle\langle v - v_n \rangle\rangle_{N, T'}^{(\alpha\beta)} \}^\alpha \}^{1-\beta}. \end{aligned}$$

iii) From i) and ii) it follows that

$$(5.62) \quad \begin{cases} | \tilde{N}_1 |_{N, T'}^{(\alpha\beta)} \leq E_{4,1} [(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\alpha\beta)} + E_{4,2} \{ [(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\alpha\beta)} \}^{1-\beta}, \\ | \tilde{N}_2 |_{N, T'}^{(\alpha\beta)} \leq E_{5,1} [(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\alpha\beta)} + E_{5,2} \{ [(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\alpha\beta)} \}^{1-\beta}, \end{cases}$$

where $E_{4,1}$ and $E_{4,2}$ depend on M_1, M_2, M_3 , and T' , and $E_{5,1}$ and $E_{5,2}$ depend on M_1, M_2 , and T' .

iv) Finally we estimate $(\bar{w}_n, \bar{\zeta}_n)$.

If $(x, t) \in R_{N', T'}^3$, then, for an arbitrary $\varepsilon > 0$, there exists a number $n_0(N', \varepsilon, N(\varepsilon))$ such that, for $n \geq n_0$,

$$\begin{aligned}
 |\bar{w}_n(x, t)| &\leq \int_0^t d\tau \int_{R^3} |F_{(v)}(x, t; \xi, \tau) \tilde{N}_1(\xi, \tau; v, \theta; v_n, \theta_n)| d\xi \\
 &\leq \int_0^t d\tau \int_{R_N^3} |F_{(v)}(x, t; \xi, \tau) \tilde{N}_1(\xi, \tau)| d\xi + \int_0^t d\tau \int_{R^3 - R_N^3} |F_{(v)} \cdot \tilde{N}_1| d\xi \\
 &\quad (R_N^3 \equiv \{x: x \in R^3, |x| \leq N\}) \\
 (5.63) \quad &\leq \{E_{4,1}[(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\sigma \beta)} + (E_{4,2}[(v, \theta) - (v_n, \theta_n)]_{N, T'}^{(\sigma \beta)})^{1-\beta}\} \\
 &\quad \times C_{12}^{(0,0)}(M_1, T') \left(\frac{\pi}{h_1(v)} \right)^{3/2} \\
 &\quad + \int_0^t d\tau \int_{R^3 - R_N^3} |F_{(v)} \tilde{N}_1| d\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
 \end{aligned}$$

Hence,

$$(5.63)' \quad |\bar{w}_n|_{N', T'}^{(0)} < \varepsilon, \quad (\text{for } n \geq n_0(N', \varepsilon, N(\varepsilon)); N(\varepsilon) \geq N' + M_1 T').$$

For $|m|=1, 2$,

$$\begin{aligned}
 &|D_x^m \bar{w}_n(x, t)| \\
 &\leq \int_0^t d\tau \int_{R_N^3} |D_x^m F_{(v)}(x, t; \xi, \tau)| \cdot |\tilde{N}_1(\xi, \tau) - \tilde{N}_1(x, t)| d\xi \\
 &\quad + \int_0^t d\tau \int_{R^3 - R_N^3} |D_x^m F_{(v)} \cdot \{\tilde{N}_1(\xi, \tau) - \tilde{N}_1(x, t)\}| d\xi \\
 (5.63)'' \quad &+ \int_0^t d\tau \int_{R_N^3} D_x^m Z_{(v)}(x - \xi, t; \xi, \tau) + D_x^m F_{(v), 0}(x, t; \xi, \tau) \tilde{N}_1(x, t) d\xi \\
 &\quad + \int_{R_N^3} D_x^m Z_{(v)}(x - \xi, t; x, \tau) \tilde{N}_1(x, t) d\xi + \int_{R^3 - R_N^3} D_x^m Z_{(v)}(x - \xi, t; x, \tau) \tilde{N}_1(x, t) d\xi \\
 &\quad + \int_{R^3 - R_N^3} D_x^m Z_{(v)}(x - \xi, t; \xi, \tau) + D_x^m F_{(v), 0}(x, t; \xi, \tau) \tilde{N}_1(x, t) d\xi, \\
 &\quad \left(\int_{R^3} D_x^m Z_{(v)}(x - \xi, t; x, \tau) \tilde{N}_1(x, t) d\xi = 0 \right).
 \end{aligned}$$

From this, we have, for an arbitrary number $\varepsilon > 0$,

$$\begin{aligned}
& |D_x^m \bar{w}_n(x, t)| \\
& \leq |\tilde{N}_1(x, t; v, \theta; v_n, \theta_n)|_{N, T}^{(\alpha\beta)} C t^{(2-|m|+\alpha)/2} + |\tilde{N}_1|_{T'}^{(\alpha\beta)} C t^{(2-|m|+\alpha)/2} \int_{h_1(N-N')/\sqrt{T'}}^{+\infty} 4\pi r^2 e^{-r^2} dr \\
(5.64) \quad & + |\tilde{N}_1(n)|_{N, T}^{(0)} C' t^{(2-|m|+\alpha)/2} + |\tilde{N}_1|_{T'}^{(0)} C' t^{(2-|m|+\alpha)/2} \int_{h_1(N-N')/\sqrt{T'}}^{+\infty} 4\pi r^2 e^{-r^2} dr \\
& \leq t^{(2-|m|+\alpha)/2} (C+C') \{ \|\tilde{N}_1\|_{N, T}^{(\alpha\beta)} + \|\tilde{N}_1\|_{T'}^{(\alpha\beta)} \cdot \varepsilon(N(\varepsilon), N') \} < \varepsilon, \\
& (n \geq n_0(N', \varepsilon, N(\varepsilon)); \varepsilon \searrow 0 \text{ as } N \nearrow +\infty; C \text{ and } C' \text{ depend on } M_1, T', \text{ and } \alpha\beta).
\end{aligned}$$

Thus,

$$(5.64)' \quad |D_x^m \bar{w}_n|_{N, T'}^{(0)} < \varepsilon, \quad (n \geq n_0(N', \varepsilon, N(\varepsilon))).$$

By (5.64), for $x \in R_{N'}^3$, and $t-t' \geq t' > 0$ and for an arbitrary number $\varepsilon > 0$, there exists a number $n_0(N', \varepsilon, N(\varepsilon))$ such that, for $n \geq n_0$,

$$\begin{aligned}
(5.65) \quad & |D_x^m \bar{w}_n(x, t) - D_x^m \bar{w}_n(x, t')| \\
& \leq (C+C')(2^{(2-|m|+\alpha\beta)/2} + 1)(t-t')^{(2-|m|+\alpha\beta)/2} \{ \|\tilde{N}_1(n)\|_{N, T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)} \varepsilon(N(\varepsilon), N') \}.
\end{aligned}$$

For $t-t' \leq t'$, we estimate along the line of (4.8). After tedious calculations, we have

$$\begin{aligned}
(5.65)' \quad & |D_x^m \bar{w}_n(x, t) - D_x^m \bar{w}_n(x, t')| \\
& \leq C_{17}^{(m)} (\alpha\beta)(t-t')^{(2-|m|+\alpha\beta)/2} \{ \|\tilde{N}_1(n)\|_{N, T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)} \varepsilon(N(\varepsilon), N') \}.
\end{aligned}$$

Therefore, by (5.65) and (5.65)', for an arbitrary number $\varepsilon > 0$

$$(5.65)'' \quad |D_x^m \bar{w}_n|_{t, N', T'}^{(\alpha\beta/2)} < \varepsilon, \quad (n \geq n_1(N', \varepsilon, N(\varepsilon))).$$

For $|m|=2$ and $x, x' \in R_{N'+M_1(T'-t)}^3$ ($0 \leq t \leq T'$),

$$(5.66) \quad |D_x^m \bar{w}_n(x, t) - D_x^m \bar{w}_n(x', t)|$$

is estimated along the line of (4.18). We have only to note that, for an arbitrary function $\varphi(x, t; \xi, \tau)$,

$$\begin{aligned}
(5.66)' \quad & \int_0^t d\tau \int_{R^3 - \Sigma_1} \varphi(\cdot; \xi, \tau) d\xi \\
& = \int_0^t d\tau \int_{R^3 - \Sigma_1^k} \varphi(\cdot; \xi, \tau) d\xi + \int_0^t d\tau \int_{\Sigma_1^k - \Sigma_1} \varphi(\cdot; \xi, \tau) d\xi,
\end{aligned}$$

where

$$(5.66)'' \quad \Sigma_1^k \equiv \{\xi: |x - \xi| \leq 2k|x - x'|, k \text{ is an integer } \geq 1\}.$$

Obviously,

$$\Sigma_1 \subset \Sigma_1^k \subset R_{(2k+1)(N'+M_1T')}.$$

For example, the integral (cf. I_s of (4.18)),

$$(5.67) \quad \int_{R^3 - \Sigma_1^k} d\xi \int_0^t |x - x'| |x' - \xi|^{\alpha\beta} (t - \tau)^{-3} \exp \left\{ -h_1 \frac{|x' - \xi|^2}{t - \tau} \right\} d\tau \equiv \hat{I}_k$$

is estimated as follows:

$$\begin{aligned} \hat{I}_k &\leq |x - x'| 2h_1^{-2} I(3) \int_{R^3 - \Sigma_1^k} |x' - \xi|^{-4 + \alpha\beta} d\xi \\ &\leq 2^{5 - \alpha\beta} |x - x'| \cdot h_1^{-2} I(3) \int_{R^3 - \Sigma_1^k} |x - \xi|^{-4 + \alpha\beta} d\xi \\ (5.67)' \quad &\left(\frac{1}{2} \leq \frac{x' - \xi}{x - \xi} \leq \frac{3}{2}, \quad \text{for } \xi \in (\Sigma_1)^c \right) \\ &\leq 2^{5 - \alpha\beta} |x - x'| h_1^{-2} I(3) \int_{2k|x-x'|}^{+\infty} 4\pi r^{-2 + \alpha\beta} dr = 2^6 \pi I(3) (1 - \alpha\beta)^{-1} k^{-1 + \alpha\beta} |x - x'|^{\alpha\beta}. \end{aligned}$$

Therefore,

$$(5.67)'' \quad I_k / |x - x'|^{\alpha\beta} \rightarrow 0 \quad (k \rightarrow +\infty).$$

In this way follows the result that, for $|m|=2$ and an arbitrary number $\varepsilon > 0$, there exists a number $n_0(N', \varepsilon, k(\varepsilon))$ such

$$(5.68) \quad |D_x^m \bar{w}_n|_{x, N', T'}^{(\alpha\beta)} \leq C_{18}^{(\alpha\beta)} \|N_1(n)\|_{N(k), T'}^{(\alpha\beta)} + \|\tilde{N}_1(n)\|_{T'}^{(\alpha\beta)} \varepsilon_1(k(\varepsilon), N') < \varepsilon, \\ (\varepsilon_1 \searrow 0 \text{ as } \varepsilon \searrow 0).$$

$|D_{x \setminus n}^m|_{N, T'}^{(0)}$, etc., are estimated in an analogous way. From iii) and iv) follows the assertion of this lemma. Q.E.D.

5.5. **The existence of a bounded solution of (1.9)-(1.10).** Now, to complete the proof of Theorem 1, it remains just to apply the following

THEOREM (Tikhonov [39]). *A continuous operator from a compact convex subset in a locally convex linear topological space into itself has at least one fixed point.*

Proof of Theorem 1. By Tikhonov's theorem, $G_{T'}$ has a fixed point such that

$$(5.69) \quad G_{T'}(v, \theta) = (\hat{v}, \hat{\theta}) = (v, \theta), \quad ((v, \theta) \in S_{T'}).$$

Moreover,

$$(5.70) \quad \rho_{(v)} \in \tilde{B}_{T'}^{1+\alpha} \equiv \{\rho: |\rho|_{T'}^{(0)} + |\nabla \rho|_{T'}^{(\alpha)} + |D_t \rho|_{T'}^{(\alpha)}\} < +\infty.$$

We can consider (1.9)³ as a linear parabolic equation in θ . The fact that

$(v, \theta, \rho) \in H_T^{2+\alpha} \times H_T^{2+\alpha} \times B_T^{1+\alpha}$, and that $\theta(x, 0) = \theta_0(x) \geq 0$ and $\Psi \geq 0$, guarantees the non-negativity of $\theta(x, t)$, (cf. [9], CH. II, § 4). Thus, we have Theorem 1. Q.E.D.

§ 6. The problem of uniqueness.

Proof of Theorem 2. Let (v_1, θ_1, ρ_1) and (v_2, θ_2, ρ_2) satisfy (1.9)–(1.10) and the assumption of this theorem. Then, the differences,

$$(6.1) \quad w = v_1 - v_2, \quad \theta = \theta_1 - \theta_2,$$

satisfy

$$(6.2) \quad \begin{cases} \frac{\partial w}{\partial t} - \frac{\mu}{\rho_{(v_1)}} P_0(D_x)w = \tilde{N}_1(x, t; v_1, \theta_1; v_2, \theta_2), & w(x, 0) = 0; \\ \frac{\partial \theta}{\partial t} - \frac{\kappa}{C_v \rho_{(v_1)}} \Delta \theta = \tilde{N}_2(x, t; v_1, \theta_1; v_2, \theta_2), & \theta(x, 0) = 0. \end{cases}$$

It is obvious that

$$(6.3) \quad (w, \theta) \in H_T^{2+\alpha} \times H_T^{2+\alpha}, \quad \tilde{N}_1 \text{ and } \tilde{N}_2 \in H_T^2.$$

Therefore, w and θ can be expressed by

$$(6.4) \quad \begin{cases} w(x, t) = \int_0^t d\tau \int_{R^3} \Gamma_{(v_1)}(x, t; \xi, \tau) \tilde{N}_1(\xi, \tau) d\xi, \\ \theta(x, t) = \int_0^t d\tau \int_{R^3} \Gamma'_{(v_1)}(x, t; \xi, \tau) \tilde{N}_2(\xi, \tau) d\xi. \end{cases}$$

Hence, if $T_0 \in [0, T]$, then

$$(6.5) \quad \begin{cases} \langle w \rangle_{T_0} \leq C \langle v_1 \rangle_T, T_0 \|\tilde{N}_1\|_{T_0}^{(\infty)}, & (C \searrow 0 \text{ as } T_0 \searrow 0, \text{ cf. (5.36)}), \\ (\theta)_{T_0} \leq C \langle v_1 \rangle_T, T_0 \|\tilde{N}_2\|_{T_0}^{(\infty)}, & (C \searrow 0 \text{ as } T_0 \searrow 0), \\ \langle w \rangle'_{T_0} \leq C \langle v_1 \rangle_T, T_0 \|\tilde{N}_1\|_{T_0}^{(\infty)}, & (C \searrow C_0(>0) \text{ as } T_0 \searrow 0). \end{cases}$$

i) We put

$$(6.6) \quad M \equiv \max [\langle v_1 \rangle_T, \langle v_2 \rangle_T].$$

In such a way as we obtained (5.52), we have

$$(6.7) \quad \begin{aligned} & |\rho_{(v_1)} - \rho_{(v_2)}|_{T_0}^{(\infty)} \\ & \leq E_1(M, T_0) \langle v_1 - v_2 \rangle_{T_0}, \quad (E_1 \text{ decreases monotonically as } T_0 \searrow 0). \end{aligned}$$

$$\begin{aligned}
& |(\rho_{(v_1)} - \rho_{(v_2)})(x, t) - (\rho_{(v_1)} - \rho_{(v_2)})(x', t)| \\
(6.8) \quad &= \left| \int_0^1 \frac{d}{d\lambda} \{ \rho_{(v_1)}(x_\lambda, t) - \rho_{(v_2)}(x_\lambda, t) \} d\lambda \right| \\
&\leq \left| \int_0^1 \left\{ \frac{\partial \rho_{(v_1)}}{\partial x_{\lambda, i}}(x_\lambda, t) - \frac{\partial \rho_{(v_2)}}{\partial x_{\lambda, i}}(x_\lambda, t) \right\} d\tau \right| \cdot |x_i - x'_i| \\
&\leq |\mathcal{V}\rho_{(v_1)} - \mathcal{V}\rho_{(v_2)}|_{T_0}^{(0)} \cdot |x - x'|, \\
&\quad (x_\lambda \equiv \lambda x + (1-\lambda)x'; t \in [0, T_0]).
\end{aligned}$$

Therefore, noting that, for a and $b \geq 0$ and for $\gamma \in [0, 1]$,

$$(6.8)' \quad a^\gamma b^{1-\gamma} \leq \max[a, b] \leq a + b,$$

we have

$$\begin{aligned}
(6.8)'' \quad & |(\rho_{(v_1)} - \rho_{(v_2)})(x, t) - (\rho_{(v_1)} - \rho_{(v_2)})(x', t)| \\
&\leq 2^{1-\alpha} |x - x'|^\alpha \{ |\mathcal{V}\rho_{(v_1)} - \mathcal{V}\rho_{(v_2)}|_{T_0}^{(0)} + |\rho_{(v_1)} - \rho_{(v_2)}|_{T_0}^{(0)} \}.
\end{aligned}$$

We can estimate $|\mathcal{V}\rho_{(v_1)} - \mathcal{V}\rho_{(v_2)}|_{T_0}^{(0)}$ almost in the same way as in (5.55)', if we replace $v, v_n, \bar{x}, \bar{x}_n, T'$, and $|\mathcal{V}\mathcal{V}v|_{x, T'}^{(\alpha)} |(\bar{x} - \bar{x}_n)(\tau_0; \xi, \tau)|$, in (5.55)' by $v_1, v_2, \bar{x}_1, \bar{x}_2, T_0$, and $|\mathcal{V}\mathcal{V}\mathcal{V}v_1|_{T_0}^{(0)} |(\bar{x}_1 - \bar{x}_2)(\tau_0; \xi, \tau)|$, respectively. As a result, we have

$$\begin{aligned}
(6.9) \quad & |\mathcal{V}\rho_{(v_1)} - \mathcal{V}\rho_{(v_2)}|_{T_0}^{(0)} \leq F_1(M, \bar{M}, T_0) \langle v_1 - v_2 \rangle_{T_0}, \\
& (F_1 \searrow 0 \text{ as } T_0 \searrow 0, \bar{M} = \max[|\mathcal{V}\mathcal{V}\mathcal{V}v_1|_{T_0}^{(0)}, |\mathcal{V}\mathcal{V}\mathcal{V}v_2|_{T_0}^{(0)}]).
\end{aligned}$$

Thus,

$$(6.9)' \quad |\rho_{(v_1)} - \rho_{(v_2)}|_{x, T_0}^{(\alpha)} \leq 2^{1-\alpha} \{ E_1(T_0) + F_1(T_0) \} \langle v_1 - v_2 \rangle_{T_0}.$$

Next, for t and $t' \in [0, T_0]$,

$$\begin{aligned}
(6.10) \quad & |(\rho_{(v_1)} - \rho_{(v_2)})(x, t) - (\rho_{(v_1)} - \rho_{(v_2)})(x, t')| \\
&\leq \left| \int_0^1 \left\{ \frac{\partial}{\partial t_\lambda} \rho_{(v_1)}(x, t_\lambda) - \frac{\partial}{\partial t_\lambda} \rho_{(v_2)}(x, t_\lambda) \right\} d\lambda \right| \cdot |t - t'| \\
&\leq \int_0^1 | \{ \operatorname{div} \rho_{(v_1)}(x, t_\lambda) v_1(x, t_\lambda) - \operatorname{div} \rho_{(v_2)}(x, t_\lambda) v_2(x, t_\lambda) \} | d\lambda \cdot |t - t'| \\
&\leq T_0^{1-(\alpha/2)} |t - t'|^{\alpha/2} \{ M(E_1(T_0) + F_1(T_0)) + |\rho_{(v_2)}|_{T_0}^{(0)} \} \langle v_1 - v_2 \rangle_{T_0}, \\
&\quad (t_\lambda \equiv \lambda t + (1-\lambda)t').
\end{aligned}$$

ii) We put

$$\rho_1 = \rho_{(v_1)}, \quad \rho_2 = \rho_{(v_2)}.$$

$$\begin{aligned}
& \left(\frac{\mathcal{F}_x \rho_1}{\rho_1} - \frac{\mathcal{F}_x \rho_2}{\rho_2} \right)(x, t) - \left(\frac{\mathcal{F}_x \rho_1}{\rho_1} - \frac{\mathcal{F}_x \rho_2}{\rho_2} \right)(x', t) \\
&= \left[\left\{ \frac{\mathcal{F}_x \rho_0(x_{1,0}(x, t))}{\rho_0(x_{1,0}(x, t))} - \mathcal{F}_x \int_0^t \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) d\tau \right\} \right. \\
&\quad \left. - \left\{ \frac{\mathcal{F}_x \rho_0(x_{1,0}(x, t))}{\rho_0(x_{2,0}(x, t))} - \mathcal{F}_x \int_0^t \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) d\tau \right\} \right] \\
&\quad - \left[\left\{ \frac{\mathcal{F}_{x'} \rho_0(x_{1,0}(x', t))}{\rho_0(x_{1,0}(x', t))} - \mathcal{F}_{x'} \int_0^t \operatorname{div} v_1(\bar{x}_1(\tau; x', t), \tau) d\tau \right\} \right. \\
(6.11) \quad &\quad \left. - \left\{ \frac{\mathcal{F}_{x'} \rho_0(x_{2,0}(x', t))}{\rho_0(x_{2,0}(x', t))} - \mathcal{F}_{x'} \int_0^t \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) d\tau \right\} \right] \\
&= \left[\left\{ \frac{\mathcal{F}_x \rho_0(x_{1,0}(x, t)) - \mathcal{F}_x \rho_0(x_{2,0}(x, t))}{\rho_0(x_{1,0}(x, t))} \right. \right. \\
&\quad \left. \left. + \mathcal{F}_x \rho_0(x_{2,0}(x, t)) \left(\frac{1}{\rho_0(x_{1,0}(x, t))} - \frac{1}{\rho_0(x_{2,0}(x, t))} \right) \right\} \right]_I - \{x \rightarrow x'\}_I \Big]_A \\
&\quad - \left[\int_0^t \{ \mathcal{F}_{\bar{x}_1} \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) \cdot \mathcal{F}_x \bar{x}_1(\tau; x, t) \right. \\
&\quad \left. - \mathcal{F}_{\bar{x}_2} \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) \mathcal{F}_x \bar{x}_2(\tau; x, t) \} \Big]_{II} d\tau - \int_0^t \{x \rightarrow x'\}_{II} d\tau \Big]_B.
\end{aligned}$$

$$\begin{aligned}
[\]_A &= \{ \mathcal{F}_x \rho_0(x_{1,0}(x, t)) - \mathcal{F}_x \rho_0(x_{2,0}(x, t)) \} \\
&\quad - \{ \mathcal{F}_{x'} \rho_0(x_{1,0}(x', t)) - \mathcal{F}_{x'} \rho_0(x_{2,0}(x', t)) \} \left[\frac{1}{\rho_0(x_{1,0}(x, t))} \right. \\
&\quad \left. - \{ \mathcal{F}_{x'} \rho_0(x_{1,0}(x', t)) - \mathcal{F}_{x'} \rho_0(x_{2,0}(x', t)) \} \left\{ \frac{1}{\rho_0(x_{1,0}(x, t))} - \frac{1}{\rho_0(x_{1,0}(x', t))} \right\} \right] \\
(6.12) \quad &\quad + \{ \mathcal{F}_x \rho_0(x_{2,0}(x, t)) - \mathcal{F}_x \rho_0(x_{2,0}(x', t)) \} \\
&\quad \times \{ \rho_0(x_{2,0}(x, t)) - \rho_0(x_{1,0}(x, t)) \} \frac{1}{\rho_0(x_{1,0}(x, t)) \rho_0(x_{2,0}(x, t))} \\
&\quad + \mathcal{F}_{x'} \rho_0(x_{2,0}(x', t)) \left[\frac{\rho_0(x_{2,0}(x, t)) - \rho_0(x_{1,0}(x, t))}{\rho_0(x_{1,0}(x, t)) \rho_0(x_{2,0}(x, t))} - \frac{\rho_0(x_{2,0}(x', t)) - \rho_0(x_{1,0}(x', t))}{\rho_0(x_{1,0}(x', t)) \rho_0(x_{2,0}(x', t))} \right] \\
&\equiv S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

For example, the first term is estimated in the following way:

$$\begin{aligned}
|S_1| &\leq (\bar{\rho}_0)^{-1} \{ |\mathcal{F}_x \rho_0(x_{1,0}(x, t)) - \mathcal{F}_x \rho_0(x_{2,0}(x, t)) \}_C - \{x \rightarrow x'\}_C |^{(1-\alpha)+\alpha} \\
&\leq (\bar{\rho}_0)^{-1} 2^{1-\alpha} \{ |\mathcal{F} x_{1,0}|_{\mathcal{T}}^{(0)} |\mathcal{F} \mathcal{F} \rho_0|^{(0)} |x_{1,0} - x_{2,0}|_{\mathcal{T}_0}^{(0)} + |\mathcal{F} \rho_0|^{(0)} |\mathcal{F} x_{1,0} - \mathcal{F} x_{2,0}|_{\mathcal{T}_0}^{(0)} \}^{1-\alpha} \\
(6.12)' \quad &\times \{ |\mathcal{F} x_{1,0}|_{\mathcal{T}}^{(0)} |\mathcal{F} \rho_0|^{(0)} |\mathcal{F} \mathcal{F} (x_{1,0} - x_{2,0})|_{\mathcal{T}_0}^{(0)} + |\mathcal{F} \mathcal{F} x_{2,0}|_{\mathcal{T}}^{(0)} |\mathcal{F} \rho_0|^{(0)} |\mathcal{F} (x_{1,0} - x_{2,0})|_{\mathcal{T}_0}^{(0)} \\
&+ |\mathcal{F} \mathcal{F} x_{2,0}|_{\mathcal{T}}^{(0)} |\mathcal{F} x_{2,0}|_{\mathcal{T}}^{(0)} |\mathcal{F} \mathcal{F} \rho_0|^{(0)} |x_{1,0} - x_{2,0}|_{\mathcal{T}_0}^{(0)} + (|\mathcal{F} x_{1,0}|_{\mathcal{T}}^{(0)})^2 |\mathcal{F} \mathcal{F} \rho_0|^{(L)} |x_{1,0} - x_{2,0}|_{\mathcal{T}_0}^{(0)} \\
&+ |\mathcal{F} \mathcal{F} \rho_0|^{(0)} (|\mathcal{F} x_{1,0}|_{\mathcal{T}}^{(0)} + |\mathcal{F} x_{2,0}|_{\mathcal{T}}^{(0)}) |\mathcal{F} (x_{1,0} - x_{2,0})|_{\mathcal{T}_0}^{(0)} \}^\alpha |x - x'|^\alpha.
\end{aligned}$$

In this way, by making use of (6.8)', we have

$$\begin{aligned}
(6.12)'' \quad |I_A| &\leq F_2(M, T; (\bar{\rho}_0)^{-1}, \|\rho_0\|^{(2+L)}) \{ |x_{1,0} - x_{2,0}|_{\mathcal{T}_0}^{(0)} + |\mathcal{F} (x_{1,0} - x_{2,0})|_{\mathcal{T}_0}^{(0)} \\
&+ |\mathcal{F} \mathcal{F} (x_{1,0} - x_{2,0})|_{\mathcal{T}_0}^{(0)} \} |x - x'|^\alpha,
\end{aligned}$$

where F_2 is a non-negative constant depending on $M, T, (\bar{\rho}_0)^{-1}$, and $\|\rho_0\|^{(2+L)}$. Next,

$$\begin{aligned}
(6.13) \quad |I_B| &= \int_0^t \{ |\mathcal{F} \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) - \mathcal{F} \operatorname{div} v_1(\bar{x}_1(\tau; x', t), \tau) \}_D - \{v_1 \rightarrow v_2\}_D |\mathcal{F}_x \bar{x}_1(\tau; x, t) d\tau \\
&+ \int_0^t \{ \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) - \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) \} \times \{ \mathcal{F}_x \bar{x}_1(\tau; x, t) - \mathcal{F}_x \bar{x}_2(\tau; x, t) \} d\tau \\
&- \int_0^t \{ \mathcal{F} \operatorname{div} v_1(\bar{x}_1(\tau; x', t), \tau) - \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) \} \times \{ \mathcal{F}_x \bar{x}_1(\tau; x, t) - \mathcal{F}_x \bar{x}_2(\tau; x', t) \} d\tau \\
&- \int_0^t \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) [\{ \mathcal{F}_x \bar{x}_1(\tau; x, t) - \mathcal{F}_x \bar{x}_1(\tau; x', t) \}_E - \{ \bar{x}_1 \rightarrow \bar{x}_2 \}_E] d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
(6.14) \quad |I_B| &\leq \int_0^t \{ |\mathcal{F} \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) - \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) \}_E - \{x \rightarrow x'\}_E \cdot |\mathcal{F}_x \bar{x}_1(\tau; x, t)| d\tau \\
&+ \int_0^t \{ |\mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) - \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) \}_G - \{x \rightarrow x'\}_G |G|^{1-\alpha} \\
&\times \{ |\mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) - \mathcal{F} \operatorname{div} v_2(\bar{x}_2(\tau; x', t), \tau) \}_G - \{x \rightarrow x'\}_G |G|^\alpha \cdot |\mathcal{F}_x \bar{x}_1(\tau; x, t)| d\tau \\
&+ \int_0^t |\mathcal{F} \operatorname{div} v_2|_{\mathcal{E}, \mathcal{T}}^{(\alpha)} (|\mathcal{F}_x \bar{x}_2|_{\mathcal{T}_0}^{(0)} |x - x'|)^\alpha \cdot |\mathcal{F}_x \bar{x}_1 - \mathcal{F}_x \bar{x}_2|_{\mathcal{T}_0}^{(0)} d\tau \\
&+ \int_0^t \{ |\mathcal{F} \mathcal{F} v_1|_{\mathcal{T}}^{(0)} |\bar{x}_1 - \bar{x}_2|_{\mathcal{T}_0}^{(0)} + |\mathcal{F} \mathcal{F} (v_1 - v_2)|_{\mathcal{T}_0}^{(0)} \} \times (2 |\mathcal{F}_x \bar{x}_1|_{\mathcal{T}_0}^{(0)})^{1-\alpha} (|\mathcal{F} \mathcal{F} \bar{x}_1|_{\mathcal{T}_0}^{(0)})^\alpha |x - x'|^\alpha d\tau \\
&+ \int_0^t |\mathcal{F} \mathcal{F} v_2|_{\mathcal{T}}^{(0)} \left\{ \int_0^1 |\mathcal{F} \mathcal{F} \bar{x}_1(\tau; x_2, t) - \mathcal{F} \mathcal{F} \bar{x}_2(\tau; x_2, t)| \cdot |x - x'|^\alpha d\lambda \right\}^\alpha \{ 2 |\mathcal{F}_x (\bar{x}_1 - \bar{x}_2)|_{\mathcal{T}_0}^{(0)} \}^{1-\alpha} d\tau
\end{aligned}$$

$$\begin{aligned}
& (|\bar{x}_1|_{T_0}^{(0)} \equiv \sup_{\substack{0 \leq \tau \leq t \leq T_0 \\ x \in \mathbb{R}^3}} |\bar{x}_1(\tau; x, t)|, \text{ etc.}) \\
& \leq [|\mathcal{F}\mathcal{F}(v_1 - v_2)|_{x, T}^{(0)} (|\mathcal{F}\bar{x}_1|_{T_0}^{(0)})^{1+\alpha} \cdot T_0 + T_0 |\mathcal{F}\bar{x}_1|_{T_0}^{(0)} \{2|\mathcal{F}\mathcal{F}\mathcal{F}v_2|_{T_0}^{(0)} |\bar{x}_1 - \bar{x}_2|_{T_0}^{(0)}\}^{1-\alpha} \\
& \quad \cdot \{|\mathcal{F}\mathcal{F}\mathcal{F}v_2|_{x, T}^{(L)} |\bar{x}_1 - \bar{x}_2|_{T_0}^{(0)} |\mathcal{F}\bar{x}_1|_{T_0}^{(0)} + |\mathcal{F}\mathcal{F}\mathcal{F}v_2|_{T_0}^{(0)} |\mathcal{F}(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)}\}^\alpha \\
& \quad + T_0 \{|\mathcal{F}\mathcal{F}v_2|_{x, T}^{(0)} (|\mathcal{F}_x \bar{x}_2|_{T_0}^{(0)})^\alpha |\mathcal{F}_x(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)} + T_0 \{|\mathcal{F}\mathcal{F}\mathcal{F}v_2|_{T_0}^{(0)} |\bar{x}_1 - \bar{x}_2|_{T_0}^{(0)} + |\mathcal{F}\mathcal{F}(v_1 - v_2)|_{T_0}^{(0)}\} \\
& \quad \times (2|\mathcal{F}_x \bar{x}_1|_{T_0}^{(0)})^{1-\alpha} (|\mathcal{F}\mathcal{F}\bar{x}_1|_{T_0}^{(0)})^\alpha + 2^{1-\alpha} |\mathcal{F}\mathcal{F}v_2|_{T_0}^{(0)} \{|\mathcal{F}\mathcal{F}(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)} + |\mathcal{F}(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)}\} |x - x'|^\alpha].
\end{aligned}$$

We need to estimate $|\mathcal{F}\mathcal{F}(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)}$. We use the notations of (5.56) and (5.56)'.

$$(6.15) \quad \begin{cases} \frac{d}{d\tau} \left(\frac{\partial}{\partial x_k} \bar{X}_1 \right) = \frac{\partial}{\partial x_k} (\hat{\mathcal{V}}_1 \bar{X}_1) = \left(\frac{\partial}{\partial x_k} \hat{\mathcal{V}}_1 \right) \bar{X}_1 + \hat{\mathcal{V}}_1 \left(\frac{\partial}{\partial x_k} \bar{X}_1 \right), & \frac{\partial}{\partial x_k} \bar{X}_1 \Big|_{\tau=t} = 0, \\ \frac{d}{d\tau} \left(\frac{\partial}{\partial x_k} \bar{X}_2 \right) = \frac{\partial}{\partial x_k} (\hat{\mathcal{V}}_2 \bar{X}_2) = \left(\frac{\partial}{\partial x_k} \hat{\mathcal{V}}_2 \right) \bar{X}_2 + \hat{\mathcal{V}}_2 \left(\frac{\partial}{\partial x_k} \bar{X}_2 \right), & \frac{\partial}{\partial x_k} \bar{X}_2 \Big|_{\tau=t} = 0 \end{cases}$$

Hence,

$$\begin{aligned}
(6.16) \quad \left| \frac{\partial}{\partial x_k} (\bar{X}_1 - \bar{X}_2)(\tau) \right| & \leq \int_\tau^t |\hat{\mathcal{V}}_1| \cdot \left| \frac{\partial}{\partial x_k} (\bar{X}_1 - \bar{X}_2)(\tau) \right| d\tau \\
& + \int_\tau^t \left[\left| \frac{\partial}{\partial x_k} (\hat{\mathcal{V}}_1 - \hat{\mathcal{V}}_2) \right| \cdot |\bar{X}_1| + \left| \frac{\partial}{\partial x_k} \hat{\mathcal{V}}_2 \right| \cdot |\bar{X}_1 - \bar{X}_2| \right. \\
& \left. + |\hat{\mathcal{V}}_1 - \hat{\mathcal{V}}_2| \left| \frac{\partial}{\partial x_k} \bar{X}_2 \right| \right] dt.
\end{aligned}$$

From this, we have

$$(6.16)' \quad |\mathcal{F}\mathcal{F}(\bar{x}_1 - \bar{x}_2)|_{T_0}^{(0)} \leq F_3(M, M', T_0) \langle v_1 - v_2 \rangle_{T_0}, \quad (F_3 \searrow 0 \text{ as } T_0 \searrow 0).$$

Thus,

$$(6.17) \quad \begin{cases} [|\]_A \leq \bar{F}_2(M, T_0) \langle v_1 - v_2 \rangle_{T_0} |x - x'|^\alpha, & (\bar{F}_2 \searrow 0 \text{ as } T_0 \searrow 0), \\ [|\]_B \leq [F_{4.1}(M, \bar{M}, M', T_0) \langle v_1 - v_2 \rangle_{T_0} + F_{4.2}(M, T_0) \langle v_1 - v_2 \rangle_{T_0}] |x - x'|^\alpha, \\ F_{4.1} \text{ and } F_{4.2} \searrow 0 \text{ as } T_0 \searrow 0; M' \equiv \max [|\mathcal{F}\mathcal{F}\mathcal{F}v_1|_{x, T_0}^{(L)}, |\mathcal{F}\mathcal{F}\mathcal{F}v_2|_{x, T_0}^{(L)}]. \end{cases}$$

Accordingly, we have

$$(6.18) \quad |\mathcal{F} \log \rho_{(v_1)} - \mathcal{F} \log \rho_{(v_2)}|_{x, T_0}^{(0)} \leq \{\bar{F}_2(T_0) + F_{4.1}(T_0)\} \langle v_1 - v_2 \rangle_{T_0} + F_{4.2} \langle v_1 - v_2 \rangle_{T_0}'.$$

iii) For $T_0 \geq t \geq t' > 0$,

$$\begin{aligned}
& \{(\nabla_x \log \rho_1 - \nabla_x \log \rho_2)(x, t)\}_{H_0} - \{t \rightarrow t'\}_{H_0} \\
= & \left[\frac{\nabla_x \rho_0(x_{1,0}(x, t)) - \nabla_x \rho_0(x_{2,0}(x, t))}{\rho_0(x_{1,0}(x, t))} + \nabla_x \rho_0(x_{2,0}(x, t)) \left\{ \frac{1}{\rho_0(x_{1,0}(x, t))} - \frac{1}{\rho_0(x_{2,0}(x, t))} \right\} \right. \\
& - \left. \left\{ \frac{\nabla_x \rho_0(x_{1,0}(x, t')) - \nabla_x \rho_0(x_{2,0}(x, t'))}{\rho_0(x_{1,0}(x, t'))} \right. \right. \\
(6.19) \quad & \left. \left. + \nabla_x \rho_0(x_{2,0}(x, t')) \left(\frac{1}{\rho_0(x_{1,0}(x, t'))} - \frac{1}{\rho_0(x_{2,0}(x, t'))} \right) \right\} \right]_H \\
& - \left[\int_0^t \{ \nabla \bar{x}_1 \operatorname{div} v_1(\bar{x}_1(\tau; x, t), \tau) \nabla_x \bar{x}_1(\tau; x, t) - \nabla \bar{x}_2 \operatorname{div} v_2(\bar{x}_2(\tau; x, t), \tau) \nabla_x \bar{x}_2(\tau; x, t) \}_I d\tau \right. \\
& \left. - \int_0^{t'} \{t \rightarrow t'\}_I d\tau \right]_J.
\end{aligned}$$

We can calculate in a way analogous to that in ii). As a result, we have

$$(6.20) \quad \|\cdot\|_H \leq F_{5,1}(M, T_0; \|\rho_0\|^{(2+L)}) \langle v_1 - v_2 \rangle_{T_0} (t-t')^{\alpha/2}, \quad (F_{5,1} \searrow 0 \text{ as } T_0 \searrow 0),$$

$$\begin{aligned}
(6.20)' \quad & \|\cdot\|_J \leq F_{5,2}(M, \bar{M}, M', T_0) \langle v_1 - v_2 \rangle_{T_0} (t-t')^{\alpha/2} \\
& + F_{5,3}(M, T_0) \|\nabla \nabla (v_1 - v_2)\|_{\mathcal{D}, T_0}^{(\infty)} (t-t')^{\alpha/2}, \quad (F_{5,2} \text{ and } F_{5,3} \searrow 0 \text{ as } T_0 \searrow 0).
\end{aligned}$$

From (6.18), (6.20), and (6.20)', we have

$$\begin{aligned}
(6.21) \quad & \|\nabla \log \rho_{\langle v_1 \rangle} - \nabla \log \rho_{\langle v_2 \rangle}\|_{T_0}^{(\infty)} \\
& \leq \{\bar{F}_2(T_0) + F_{4,1}(T_0) + F_{5,2}(T_0)\} \langle v_1 - v_2 \rangle_{T_0} + \{F_{4,2}(T_0) + F_{5,3}(T_0)\} \langle v_1 - v_2 \rangle'_{T_0} \\
& \equiv F_6(T_0) \langle v_1 - v_2 \rangle_{T_0} + \hat{F}_6(T_0) \langle v_1 - v_2 \rangle'_{T_0}, \\
& \quad (F_6 \text{ and } \hat{F}_6(T_0) \searrow 0 \text{ as } T_0 \searrow 0).
\end{aligned}$$

iv) By making use of the results of i), ii), and iii), it is easy to show that

$$(6.22) \quad \begin{cases} \|\tilde{N}_1\|_{T_0}^{(\infty)} \leq F_{7,1}(T_0) \langle w \rangle'_{T_0} + F_{7,2} \langle w \rangle_{T_0} + (\theta)_{T_0}, \\ \|\tilde{N}_2\|_{T_0}^{(\infty)} \leq F_{7,3}(T_0) \langle w \rangle_{T_0} + (\theta)_{T_0}, \end{cases}$$

where $F_{7,1}(T_0) \searrow 0$, $F_{7,2}(T_0) \searrow \operatorname{const}(>0)$, $F_{7,3}(T_0) \searrow \operatorname{const}(>0)$, as $T_0 \searrow 0$. Therefore, from (6.5) we obtain:

$$(6.23) \quad \begin{cases} \langle w \rangle_{T_0} + (\theta)_{T_0} \leq C(T_0) F_{7,2}(T_0) + {}'C(T_0) F_{7,3}(T_0) (\langle w \rangle_{T_0} + (\theta)_{T_0}) + F_{7,1}(T_0) C(T_0) \langle w \rangle'_{T_0}, \\ \langle w \rangle'_{T_0} \leq C(T_0) F_{7,2}(T_0) (\langle w \rangle_{T_0} + (\theta)_{T_0}) + C(T_0) F_{7,1}(T_0) \langle w \rangle'_{T_0}, \end{cases}$$

(C and $'C \searrow 0$ as $T_0 \searrow 0$).

Next, there exists $T_1 \in (0, T]$ such that

$$(6.24) \quad 0 \leq C(T_1)F_{7,1}(T_1) < 1.$$

Thus, for $T_0 \in (0, T_1]$

$$(6.25) \quad \langle w \rangle'_{T_0} \leq \frac{C(T_0)F_{7,2}(T_0)}{1 - C(T_0)F_{7,1}(T_0)} (\langle w \rangle_{T_0} + (\theta)_{T_0}).$$

Hence,

$$(6.26) \quad \begin{aligned} & (\langle w \rangle_{T_0} + (\theta)_{T_0}) \\ & \leq \left\{ C(T_0)F_{7,2}(T_0) + C(T_0)F_{7,3}(T_0) + \frac{F_{7,1}(T_0)C(T_0)C(T_0)F_{7,2}(T_0)}{1 - C(T_0)F_{7,1}(T_0)} \right\} (\langle w \rangle_{T_0} + (\theta)_{T_0}) \\ & \equiv F_8(T_0)(\langle w \rangle_{T_0} + (\theta)_{T_0}). \end{aligned}$$

There exists $T_2 \in (0, T_1]$ such that

$$(6.27) \quad 0 \leq F_8(T_2) < 1.$$

Therefore,

$$(6.28) \quad 0 \leq (\langle w \rangle_{T_2} + (\theta)_{T_2})(1 - F_8(T_2)) \leq 0.$$

Thus,

$$(6.29) \quad (w, \theta)(x, t) = 0, \quad (t \in [0, T_2]).$$

We can repeat this procedure, and by virtue of the assumption of the theorem, after a finite number of repetitions, we arrive at the conclusion of the theorem.

Q.E.D.

An outline of the proof of Theorem 3. By the hypothesis of the theorem, for some $T' \in (0, T]$, we can find an operator $G_{T'}$ from a subset of $H_{T'}^{4+\alpha} \times H_{T'}^{3+\alpha}$ into itself. Next, we follow the same procedure as we did in Theorem 1. Finally we apply Tikhonov's fixed point theorem to the results obtained. The uniqueness of the solution is obvious.

The definitions of the notations in Theorem 3 are as follows:

$$(6.30) \quad \left\{ \begin{aligned} H_{T'}^{n+\alpha} & \equiv \left\{ g: \sum_{2r+|m|=0}^n |D_t^r D_x^m g|_{T'}^{(0)} + \sum_{2r+|m|=n-1}^n |D_t^r D_x^m g|_{t, T'}^{(\alpha/2)} \right. \\ & \quad \left. + \sum_{2r+|m|=n} |D_t^r D_x^m g|_{x, T'}^{(\alpha)} < +\infty \right\}, \quad (n=3, 4), \\ H_{T'}^{n+\alpha} & \equiv \left\{ g: \sum_{|m|=0}^n |D_x^m g|_{T'}^{(0)} + \sum_{|m|=0}^n |D_x^m g|_{t, T'}^{(\alpha/2)} + \sum_{|m|=n} |D_x^m g|_{x, T'}^{(\alpha)} < +\infty \right\}, \quad (n=3, 4). \end{aligned} \right.$$

In studies to follow, the boundary value and global problems and related theories of abstract analysis will be treated.

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