ON THE CENTRAL LIMIT THEOREM FOR THETA SERIES

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To the memory of David L. Williams

1. Introduction. In this paper we consider the sums

$$S_N(x) = \tfrac{1}{2} + \sum_{n=1}^{N-1} \exp(i\pi n^2 x) + \tfrac{1}{2}\exp(i\pi N^2 x)$$

for real x and positive integers N and study the behavior of the distribution functions $D_N(\lambda) = |\{x \in [0,1] : \lambda \le N^{-1/2}|S_N(x)|\}|$ as N tends to infinity, where λ is a non-negative real, and $|A|$ denotes the Lebesgue measure of the set A. One result is that for all N there are constants c_0, c_1, c_2 such that if $0 \le \lambda \le c_0\sqrt{N}$ then $c_1(1+\lambda^4)^{-1} \le D_N(\lambda) \le c_2(1+\lambda^4)^{-1}$. We conjecture that the limit of $D_N(\lambda)$ as N tends to ∞ exists, which would be the central limit theorem for theta series. This however appears rather difficult. We prove a somewhat related statement on the theta series

$$\theta_0(z) = \sum_{n=-\infty}^{\infty} \exp(i\pi n^2 z)$$

where z is a complex number with positive imaginary part. Then

$$|\{x \in [0,1] : \lambda \le N^{-1/2}|\theta_0(x+iN^{-2})|\}|$$

converges for all $\lambda \ge 0$ as N tends to infinity. The form of the limit is complicated but may be given explicitly.

The methods used to obtain these results are different from those used to treat similar questions regarding sums of the form $\sum_{k=0}^{N}\exp(i\pi n_k x)$ in place of S_N, where n_k is a rapidly increasing sequence of integers. In the case of Hadamard gaps, for example, $n_k = 2^k$, the summands are sufficiently independent for probabilistic methods to show that their distributions converge to a normal distribution [5], [6], [8]. We rely upon a general form of the functional equation for theta series and the asymptotic expansion of S_N in neighborhoods of rational points due to Fiedler, Jurkat and Körner [2]. We partition [0, 1] into subintervals on which the behavior of the sums S_N and θ_0 can be described by an associated function in a neighborhood of 0. The distribution function can then be expressed as a sum similar to a Riemann sum involving these associated functions.

The following notation will be used. $E(x)$ denotes $e^{i\pi x}$. Given a natural number q, $\sum_{h \bmod q}$ means summation over a complete residue system modulo q. If $g(x)$ is a complex valued measurable function of $x \in [0,1]$, and if $\lambda \ge 0$ then $D(\lambda; g) = |\{x \in [0,1] : \lambda \le |g(x)|\}|$. If $P(x, y, \ldots, z)$ is a property of variables $x, y, \ldots, z$,

$$\chi(x, y, \ldots, z; P) = \begin{cases} 1 & \text{if } P(x, y, \ldots, z) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

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If $f(x)>0$ and $h(x)$ are defined for $x\in D$, and there is a constant B such that for all $x\in D$, $|h(x)|\le Bf(x)$, then we say $h(x)=O(f(x))$ for $x\in D$.

2. A convergence theorem. Suppose N is a positive integer and $\mathfrak{F}_N$ is the Farey series of order N; that is, $\mathfrak{F}_N$ is the set of all rationals p/q for which $p/q\in[0,1]$, $(p,q)=1$ and $1\le q\le N$. If $1\ne p/q\in\mathfrak{F}_N$ and p'/q' is the successor of p/q in $\mathfrak{F}_N$, then the *mediant* of p/q and p'/q' is $(p+p')/(q+q')$. It is known (see [1] or [4]) that

$$p'q-pq'=1 \tag{2.1}$$

so that $(p+p')/(q+q')=p/q+1/(q(q+q'))$. Partition $[0,1]$ into subintervals $I_{p,q,N}\ni p/q$ using as endpoints all the mediants, 0, and 1. Then for $q>1$,

$$I_{p,q,N}=\frac{p}{q}+\frac{1}{N^2}J\left(\frac{N}{q},\frac{r}{q}\right) \tag{2.2}$$

where

$$J(y,z)=\left[\frac{-y^2}{[y+z]+1-z},\frac{y^2}{[y-z]+1+z}\right]$$

for $1\le y$, $-1/2\le z\le 1/2$, and $r=r(p,q)$ is uniquely defined by

$$pr\equiv -1\,(\operatorname{mod} q),\quad -q/2<r\le q/2. \tag{2.3}$$

To see (2.2) observe that because of (2.3) for some n, $nq-pr=1$. This with (2.1) shows that $p'=kp+n$ and $q'=kq+r$ for some integer k. Because $q'\le N$ and $q+q'>N$ we must have $k=[(N/q)-(r/q)]$. Hence the right endpoint of $I_{p,q,N}$ is

$$\begin{aligned}\frac{p}{q}+\frac{1}{q(q+q')}&=\frac{p}{q}+\frac{1}{q([(N/q)-(r/q)]q+q+r)}\\&=\frac{p}{q}+\frac{1}{N^2}\frac{1}{(q^2/N^2)([(N/q)-(r/q)]+1+(r/q))}\end{aligned}$$

which is precisely the right endpoint of the interval on the right side of (2.2). The argument for the left endpoint is similar.

From $x<[x]+1$ for all real x one sees that $J(y,z)\subseteq[-y,y]$. Also observe that $J(y,-z)=-J(y,z)$.

Let $c(q,r)$ be an array of complex numbers defined for certain integers q and r satisfying $(q,r)=1$, and let B be a Riemann measurable set of positive measure in the plane. We say $c(q,r)$ has *average density* L in B if when we extend the definition of $c(q,r)$ by zero to all pairs (q,r) then

$$\lim_{t\to\infty}\frac{1}{t^2|A|}\sum_{(q,r)\in tA}c(q,r)=L \tag{2.4}$$

for all bounded Riemann measurable sets A in B with positive measure.

To formulate our basic convergence result we assume K is a fixed nonnegative integer, $k=0,\dots,K$, and

(i) if $q\ge 3$, $-q/2<r<q/2$, and $(r,q)=1$, then $\nu=\nu(q,r)$ is an integer from 0 to K. Further, for such q and r, $c_k(q,r)=\delta(k,\nu(q,r))+\delta(k,\nu(q,-r))$ has average

density L_k in $\{(y,z): 0 \le z \le \frac{1}{2}y\}$, where $\delta(m,n)$ is the Kronecker delta.

(ii) $a(p,q)$ is an array of complex numbers of modulus 1 defined for integers $3 \le q$, $1 \le p \le q$ and $(p,q)=1$.

(iii) For $1 \le y$, $-1/2 \le z \le 1/2$ and for all real t, $f_k(y,z,t)$ is a complex valued Lebesgue measurable function with respect to t for fixed (y,z) such that

$$|f_k(y,-z,t)| = |f_k(y,z,-t)|. \tag{2.5}$$

(iv) Finally $h_N(x)$ is a sequence of functions for $x \in [0,1]$ such that for $3 \le q \le N$, $1 \le p \le q$, $-q/2 < r < q/2$ satisfying $pr \equiv -1 \pmod q$ and $t \in [-N^2p/q, N^2(q-p)/q]$,

$$h_N\left(\frac{p}{q} + \frac{t}{N^2}\right) = a(p,q) f_\nu\left(\frac{N}{q}, \frac{r}{q}, t\right).$$

Observe that from (iii) $h_N(x)$ is measurable. Now for $\lambda \ge 0$ let

$$F_{\lambda,k}(y,z) = \int_{J(y,z)} \chi(y,z,t: \lambda \le |f_k(y,z,t)|)\, dt.$$

Observe that $F_{\lambda,k}(y,z) \le 2y$ and $F(y,-z) = F(y,z)$.

THEOREM 2.1. *If* $\lambda \ge 0$ *and assumptions* (i)-(iv) *are satisfied then*

$$\liminf_{N\to\infty} D(\lambda; h_N) \ge \sum_{k=0}^{K} L_k \underline{\int_1^\infty \int_0^{1/2}} F_{\lambda,k}(y,z) y^{-3}\, dy\, dz$$

$$\limsup_{N\to\infty} D(\lambda; h_N) \le \sum_{k=0}^{K} L_k \overline{\int_1^\infty \int_0^{1/2}} F_{\lambda,k}(y,z) y^{-3}\, dy\, dz$$

where $\underline{\iint}$ *and* $\overline{\iint}$ *denote the lower and upper double Riemann integrals; that is, the limits of the lower and upper double Riemann integrals over* $[1,R] \times [0,1/2]$ *as* $R \longrightarrow \infty$.

Proof. Let $N = 3, 4, \ldots$ and $\lambda \ge 0$ be given, and use the Farey dissection $\{I_{p,q,N}\}$ of $[0,1]$ to obtain

$$D(\lambda; h_N) = \sum_{q=1}^{N} \sum_{\substack{p=0 \\ (p,q)=1}}^{q} |\{x \in I_{p,q,N} : \lambda \le |h_N(x)|\}|. \tag{2.6}$$

Because $|I_{p,q,N}| = (N+1)^{-1}$ for $q=1$ and $|I_{1,2,N}| \le (N+1)^{-1}$ the contribution of these terms is $O(N^{-1})$. For the remaining terms we apply hypothesis (iv). Replace x by $p/q + t/N^2$ so that the condition $x \in I_{p,q,N}$ becomes $t \in J(N/q, r(p,q)/q)$ where $r(p,q)$ satisfies $-q/2 < r(p,q) < q/2$ and $pr(p,q) \equiv -1 \pmod q$. Then with $\nu = \nu(q,r)$ (2.6) becomes

$$D(\lambda; h_N) = \sum_{q=3}^{N} \sum_{\substack{p=1 \\ (p,q)=1}}^{q} \frac{1}{N^2} \left| \left\{ t \in J\left(\frac{N}{q}, \frac{r(p,q)}{q}\right) : \lambda \le \left| f_\nu\left(\frac{N}{q}, \frac{r(p,q)}{q}, t\right) \right| \right\} \right|$$
$$+ O\left(\frac{1}{N}\right)$$

$$= \sum_{q=3}^{N} \sum_{\substack{-q/2<r<q/2 \\ (r,q)=1}} \frac{1}{N^2} F_{\lambda,\nu}\left(\frac{N}{q}, \frac{r}{q}\right) + O\left(\frac{1}{N}\right)$$

$$= \sum_{q=3}^{N} \sum_{\substack{r=1 \\ (r,q)=1}}^{q/2} \frac{1}{N^2} \left(F_{\lambda,\nu(q,r)}\left(\frac{N}{q}, \frac{r}{q}\right) + F_{\lambda,\nu(q,-r)}\left(\frac{N}{q}, \frac{r}{q}\right)\right) + O\left(\frac{1}{N}\right)$$

$$= \sum_{k=0}^{K} \sum_{q=3}^{N} \sum_{\substack{r=1 \\ (r,q)=I}}^{q/2} \frac{c_k(q,r)}{N^2} F_{\lambda,k}\left(\frac{N}{q}, \frac{r}{q}\right) + O\left(\frac{1}{N}\right). \tag{2.7}$$

We consider k fixed and estimate the sum over q and r from below. Let $0<\epsilon<1$ and $1 \le \eta \le N/2$ be given. By a *half-open rectangle* we mean a nonempty open rectangle plus possibly any of its open edges and/or vertices. Let $\{A_j\}$ be a finite collection of nonintersecting half-open rectangles such that

$$\bigcup_j A_j = \{(y,z) : 1 \le y \le \eta,\ 0 \le z \le \tfrac{1}{2}\}$$

and such that for each j, $\operatorname{diam} A_j \le \epsilon$. Define a transformation G by $G(y,z) = (1/y, z/y)$ and let $B_j = G(A_j)$. Then

$$\frac{1}{N^2} \sum_{(q,r) \in NB_j} c_k(q,r) F_{\lambda,k}\left(\frac{N}{q}, \frac{r}{q}\right) \ge \frac{1}{N^2} \sum_{(q,r) \in NB_j} c_k(q,r) \inf_{(y,z) \in A_j} F_{\lambda,k}(y,z)$$

$$= L_k |B_j| \inf_{(y,z) \in A_j} F_{\lambda,k}(y,z) + \delta_{\lambda,k}(N,j)$$

where $\delta_{\lambda,k}(N,j) \longrightarrow 0$ as $N \longrightarrow \infty$. The Jacobian of G is y^{-3}, and one can find $M>0$ an absolute constant such that $|B_j| \ge |A_j| (\sup_{(y,z) \in A_j} y^{-3} - M\epsilon)$. Therefore

$$\frac{1}{N^2} \sum_{(q,r) \in NB_j} c_k(q,r) F_{\lambda,k}\left(\frac{N}{q}, \frac{r}{q}\right)$$

$$\ge L_k |A_j| \inf_{(y,z) \in A_j} F_{\lambda,k}(y,z) \Big(\sup_{(y,z) \in A_j} y^{-3} - M\epsilon\Big) + \delta_{\lambda,k}(N,j)$$

$$\ge L_k |A_j| \Big(\inf_{(y,z) \in A_j} F_{\lambda,k}(y,z) y^{-3} - 2\eta M\epsilon\Big) + \delta_{\lambda,k}(N,j).$$

Because there are a finite number of j,

$$\frac{1}{N^2} \sum_j \sum_{(q,r) \in NB_j} c_k(q,r) F_{\lambda,k}\left(\frac{N}{q}, \frac{r}{q}\right)$$

$$\ge L_k \sum_j |A_j| \Big(\inf_{(y,z) \in A_j} F_{\lambda,k}(y,z) y^{-3} - 2\eta M\epsilon\Big) + \delta_{\lambda,k}(N). \tag{2.8}$$

where $\delta_{\lambda,k}(N) \longrightarrow 0$ as $N \longrightarrow \infty$. The summation on the left in (2.8) ranges over $N/\eta \le q \le N$, $0 \le r \le q/2$, which is a portion of the summation in (2.7) approximating

$$\liminf_{N\to\infty} D(\lambda; h_N) \geq \sum_{k=0}^{K} L_k \underline{\int_1^\eta \int_0^{1/2}} F_{\lambda,k}(y,z) y^{-3}\,dy\,dz,$$

since there is a partition $\{A_j\}$ for which the lower Riemann sums of the functions $F_{\lambda,k}y^{-3}$ are simultaneously close to their respective lower Riemann integrals for $k=0,1,\ldots,K$. The left side is independent of η, so take η to ∞ to get

$$\liminf_{N\to\infty} D(\lambda; h_N) \geq \sum_{k=0}^{K} L_k \underline{\int_1^\infty \int_0^{1/2}} F_{\lambda,k}(y,z) y^{-3}\,dy\,dz.$$

In a similar manner we can obtain an upper estimate corresponding to (2.8). The remaining terms appearing in (2.7) can be estimated by $F_{\lambda,k}(N/q,\ r/q) \leq 2N/q$ so that

$$\frac{1}{N^2} \sum_{q=3}^{N/\eta} \sum_{\substack{r=0\\(r,q)=1}}^{q/2} c_k(q,r) F_{\lambda,k}\left(\frac{N}{q}, \frac{r}{q}\right) = O\left(\frac{1}{\eta}\right).$$

Since this error vanishes when η goes to ∞, we have

$$\limsup_{N\to\infty} D(\lambda; h_N) \leq \sum_{k=0}^{K} L_K \overline{\int_1^\infty \int_0^{1/2}} F_{\lambda,k}(y,z) y^{-3}\,dy\,dz. \qquad \square$$

It is clear that the limit will exist if the functions $F_{\lambda,k}(y,z)$ are locally Riemann integrable. A sufficient condition for this is

LEMMA 2.2. *Let $f(y,z,t)$ be a complex valued function defined for $y \geq 1$, $0 \leq z \leq 1/2$ and all real t, and assume f is continuous in (y,z) for fixed t and measurable in t for fixed (y,z). Then $F_\lambda(y,z) = |\{t \in J(y,z) : \lambda \leq |f(y,z,t)|\}|$ is locally Riemann integrable in (y,z) for all $\lambda \geq 0$ if $|\{t \in \mathbf{R} : \lambda = |f(y,z,t)|\}| = 0$ for all y and z in the domain of f.*

Proof. By the hypothesis no discontinuities can arise from the condition $\lambda \leq |f(y,z,t)|$. The endpoints of $J(y,z)$ vary continuously except at the lines $y+z=m$ and $y-z=n$, for m and n integral, where the discontinuities are caused by the jumps of $[y+z]$ and $[y-z]$. Because these lines have measure zero in the plane, $F_\lambda(y,z)$ is locally Riemann integrable.

3. The distribution of theta sums. The generalized Gaussian sums are defined for integers k as

$$g_k(p,q) = \sum_{h \bmod 2q} E\left(\frac{ph^2+kh}{q}\right).$$

It was shown in [2] that if p' is a solution of $pp' \equiv 1 \pmod q$ which also satisfies $4 \mid p'$ if p is odd and if $p^* = -p(p')^2$ then

$$g_k(p,q) = \begin{cases} 0 & \text{if } 2 \nmid pq+k \\ E\left(\dfrac{p^*k^2}{4q}\right) g_0(p+\delta q, q) & \text{if } 2 \mid pq+k \end{cases}$$

where $\delta = 0$ or 1 as pq is even or odd. We require

LEMMA 3.1. *When* $2 \mid pq+k$ *and* r *and* n *are determined by*

$$nq - pr = 1, \quad -q/2 < r \leq q/2 \tag{3.1}$$

then

$$g_k(p,q) = \begin{cases} E\left(\frac{-rn}{4}k^2\right)E\left(\frac{r}{4q}k^2\right)g_0(p,q) & \text{if } 2 \mid pq \\ E\left(\frac{r}{2} - \frac{rn}{4}\right)E\left(\frac{r}{4q}k^2\right)g_0(p+q,q) & \text{if } 2 \nmid pq, \quad 2 \mid r \\ E\left(\frac{pq-rn}{4}\right)E\left(\frac{r}{4q}k^2\right)g_0(p+q,q) & \text{if } 2 \nmid pq, \quad 2 \mid n. \end{cases}$$

Proof. Note that because of (3.1), $r \equiv -p' \pmod q$, so $r - nqr = -pr^2 \equiv p^* \pmod q$.

Case I. $2 \mid q$.

Here we may take $p' = -r$ so that $p^* = r - nqr$ and therefore $(p^*/4q)k^2 = (-rn/4)k^2 + (r/4q)k^2$.

Case II. $2 \mid p$.

We now have $p^* \equiv 0 \equiv -pr^2 \pmod 2$ so that $p^* \equiv r - nqr \pmod{2q}$. Since $2 \mid k$, $k^2/4$ is integral and $(p^*/4q)k^2 \equiv (r/4q)k^2 - (nr/4)k^2 \pmod 2$.

Case III. $2 \nmid pq$.

In this case $p^* \equiv 0 \pmod 8$ and because $q^2 \equiv 1 \pmod 8$, $pr^2q^2 - pr^2 \equiv 0 \pmod 8$. Hence $p^* \equiv pr^2q^2 - pr^2 \pmod{8q}$, and since $k^2 \equiv 1 \pmod 8$, $(p^*/4q)k^2 \equiv (pr^2q - nr)/4 + (r/4q)k^2 \pmod 2$ If $2 \mid r$, then $pr^2q/4 \equiv r/2 \pmod 2$, and if $2 \nmid r$, $pr^2q/4 \equiv pq/4 \pmod 2$. □

We wish to apply this to Lemma 2 of [2], which states that if we extend our definition of S_N to real $N>0$ by $S_N(x) = 1/2 + \sum_{1 \leq n \leq N} E(n^2x)$ when N is not an integer, then for $(p,q) = 1$ and real t

$$S_N\left(\frac{p}{q} + \frac{t}{N^2}\right) = \sum_{k=-\infty}^{\infty} \frac{g_k(p,q)}{2q} \int_0^N E\left(\frac{s^2t}{N^2} - \frac{ks}{q}\right) ds, \tag{3.2}$$

and the series converges, For $y>0$, z rational and t real define

$$T_0(y,z,t) = \sum_{k=-\infty}^{\infty} E(zk^2) \int_0^1 E(s^2t - 2ksy)\, ds$$

$$T_1(y,z,t) = \sum_{k=-\infty}^{\infty} (-1)^k E(zk^2) \int_0^1 E(s^2t - 2ksy)\, ds$$

$$T_2(y,z,t) = \sum_{k=-\infty}^{\infty} E(z(k+\tfrac{1}{2})^2) \int_0^1 E(s^2t - 2(k+\tfrac{1}{2})sy)\, ds$$

when these series converge. When one applies Lemma 3.1 to (3.2) one obtains

PROPOSITION 3.2. *Let* p, q, r *and* n *be integers satisfying* $2 \leq q$, $1 \leq p \leq q$, $-q/2 < r \leq q/2$ *and* $nq - pr = 1$,

$$\nu = \nu(q,r) = \begin{cases} 0 & \text{if } 2 \mid pq \text{ and } 2 \mid nr \\ 1 & \text{if } 2 \mid pq \text{ and } 2 \nmid nr \\ 2 & \text{if } 2 \nmid pq \end{cases}$$

and

$$a(p,q) = \begin{cases} \frac{1}{2\sqrt{q}}\, g_0(p,q) & \text{if } 2 \mid pq \\ \frac{1}{2\sqrt{q}}\, E\left(\frac{r}{2} - \frac{rn}{4}\right) g_0(p+q,q) & \text{if } 2 \nmid pq \text{ and } 2 \mid r \\ \frac{1}{2\sqrt{q}}\, E\left(\frac{pq-rn}{4}\right) g_0(p+q,q) & \text{if } 2 \nmid pq \text{ and } 2 \mid n. \end{cases}$$

Then for all real t and $N>0$,

$$S_N\left(\frac{p}{q} + \frac{t}{N^2}\right) = a(p,q)\, \frac{N}{\sqrt{q}}\, T_\nu\left(\frac{N}{q}, \frac{r}{q}, t\right),$$

and $(N/q,\ r/q,\ t)$ is within the set where $T_{\nu(q,r)}$ converges.

In the same way one can show a similar result for the theta series $\theta_0(\zeta) = \sum_{k=-\infty}^{\infty} E(\zeta k^2)$, for $\operatorname{Im}\zeta>0$, which is actually a reformulation of the usual transformation formula (see for example [7]). Letting $\theta_1(\zeta) = \sum_{k=-\infty}^{\infty} (-1)^k E(\zeta k^2)$ and $\theta_2(\zeta) = \sum_{k=-\infty}^{\infty} E(\zeta(k+1/2)^2)$ for $\operatorname{Im}\zeta>0$ we have

PROPOSITION 3.3. *Under the assumptions of Proposition* 3.2,

$$\theta_0\left(\frac{p}{q} + \frac{t+i}{N^2}\right) = a(p,q)\, \frac{N}{\sqrt{q}} \left(\frac{i}{t+i}\right)^{1/2} \theta_\nu\left(\frac{r}{q} - \frac{N^2}{q^2(t+i)}\right)$$

where we take the principal value of the square root.

Next we discuss the average densities of the arrays we will be needing. It is clear by simple measure theoretic considerations that to show an array $c(q,r)$ has average density L in $Q = \{y, z : 0 \le z,\ 0 \le y\}$ it suffices to show (2.4) for

$$A = \{y, z : 0 \le y \le a, \quad 0 \le z \le b\},$$

where a and b are arbitrary positive reals.

First we consider

$$\alpha(q,r) = \begin{cases} 1 & \text{if } (q,r) = 1, \quad q \ge 1, \quad r \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 3.4. *The array $\alpha(q,r)$ has average density $6/\pi^2$ in Q.*

Proof. Let a and b be given positive numbers. Note that for $q \ge 1$ and $r \ge 1$,

$$\alpha(q,r) = \sum_{d \mid (q,r)} \mu(d) = \sum_{\substack{d \mid q \\ d \mid r}} \mu(d).$$

Hence for real N sufficiently large

$$\sum_{q=0}^{aN}\sum_{r=0}^{bN}\alpha(q,r)=\sum_{q=1}^{aN}\sum_{r=1}^{bN}\sum_{\substack{d\mid q\\ d\mid r}}\mu(d)$$

$$=\sum_{d=1}^{cN}\mu(d)\left[\frac{aN}{d}\right]\left[\frac{bN}{d}\right]\qquad (c=\min(a,b))$$

$$=abN^2\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O(N\log N)$$

$$=\frac{6}{\pi^2}abN^2+O(N\log N)$$

as desired. □

Next let

$$\beta(q,r)=\begin{cases}1 & \text{if}\quad (q,r)=1,\quad 2\mid q,\quad q\neq 0,\quad r\neq 0\\ 0 & \text{otherwise}\end{cases}$$

LEMMA 3.5. *The array* $\beta(q,r)$ *has average density* $2/\pi^2$ *in* Q.

Proof. Let a and b be arbitrary positive numbers. Then for real positive N

$$\sum_{q=0}^{aN}\sum_{r=0}^{bN}\beta(q,r)=\sum_{\substack{q=1\\ 2\mid q}}^{aN}\sum_{r=1}^{bN}\sum_{\substack{d\mid q\\ d\mid r}}\mu(d)$$

$$=\sum_{d=1}^{cN}\mu(d)\left[\frac{aN}{[2,d]}\right]\left[\frac{bN}{d}\right]$$

where again $c=\min(a,b)$ and $[m,n]$ denotes the least common multiple of m and n. For N sufficiently large one calculates

$$\sum_{q=0}^{aN}\sum_{r=0}^{bN}\beta(q,r)=abN^2\sum_{d=1}^{\infty}\frac{\mu(d)}{[2,d]d}+O(N\log N)$$

$$=\frac{2}{\pi^2}abN^2+O(N\log N).$$ □

COROLLARY 3.6. *The array*

$$\gamma(q,r)=\begin{cases}1 & \textit{if}\quad (q,r)=1,\quad 2\nmid q,\quad q\neq 0,\quad r\neq 0\\ 0 & \textit{otherwise}\end{cases}$$

has average density $4/\pi^2$ *in* Q.

Proof. Clear from the previous two lemmas. □

We turn now to the local Riemann integrability of the $F_{\lambda,k}$ associated with $\theta_0(x+iN^{-2})$.

LEMMA 3.7. *For* $k=0,1,2$ *and* $\lambda \geq 0$,

$$F_{\lambda,k}(y,z) = \left|\left\{t \in J(y,z) : \lambda \leq \left|\left(\frac{y}{t+i}\right)^{1/2} \theta_k\left(z - \frac{y^2}{t+i}\right)\right|\right\}\right|$$

is locally Riemann integrable in y *and* z *for* $1 \leq y$ *and* $0 \leq z \leq 1/2$.

Proof. Let $k=0$, 1, or 2 and $\lambda \geq 0$ be given. By Lemma 2.2 it suffices to show

$$\left|\left\{t \in \mathbf{R} : \lambda = \left|\left(\frac{y}{t+i}\right)^{1/2} \theta_k\left(z - \frac{y^2}{t+i}\right)\right|\right\}\right| = 0. \tag{3.3}$$

Because

$$\left|\left(\frac{y}{t+i}\right)^{1/2} \theta_k\left(z - \frac{y^2}{t+i}\right)\right|^2 = \left(\frac{y}{t+i}\right)^{1/2} \left(\frac{y}{t-i}\right)^{1/2} \theta_k\left(z - \frac{y^2}{t+i}\right) \overline{\theta_k\left(z - \frac{y^2}{t+i}\right)}$$
$$= \frac{y}{\sqrt{t^2+1}} \theta_k\left(z - \frac{y^2}{t+i}\right) \theta_k\left(-z - \frac{y^2}{-t+i}\right),$$

(3.3) is equivalent to

$$\left|\left\{t \in \mathbf{R} : \lambda^2 = \frac{y}{\sqrt{t^2+1}} \theta_k\left(z - \frac{y^2}{t+i}\right) \theta_k\left(-z - \frac{y^2}{-t+i}\right)\right\}\right| = 0 \tag{3.4}$$

However since $y(\zeta^2+1)^{-1/2}$, $\theta_k(z-y^2(\zeta+i)^{-1})$ and $\theta_k(-z-y^2(-\zeta+i)^{-1})$ are each analytic in the region $-1 < \operatorname{Im} \zeta < 1$, either their product is constant or (3.4) is true. It is known [3] that for a dense set of t-values, $\theta_k(z-y^2(t+i\epsilon)^{-1})$ approaches infinity as ϵ tends to 0. Therefore the product cannot be constant and (3.4) is true. □

We now present our main result.

THEOREM 3.8. *For all* $\lambda \geq 0$,

$$D(\lambda; \theta_0(x+iN^{-2})N^{-1/2}) \longrightarrow$$
$$\frac{4}{\pi^2} \sum_{k=0}^{2} \int_1^\infty \frac{dy}{y^3} \int_0^{1/2} dz \int_{J(y,z)} dt\, \chi\left(y,z,t : \lambda \leq \left|\left(\frac{y}{t+i}\right)^{1/2} \theta_k\left(z - \frac{y^2}{t+i}\right)\right|\right)$$

as $N \longrightarrow \infty$.

Proof. It is clear from Proposition 3.3 that we take

$$f_k(y,z,t) = (y/(t+i))^{1/2}\theta_k(z - y^2/(t+i)) \quad \text{for} \quad k = 0,1,2.$$

Hypothesis (iii) is satisfied because a simple check shows (2.5). For integers p, q, n, and r satisfying $3 \leq q$, $1 \leq p \leq q$, $-q/2 < r < q/2$ and $nq - pr = 1$, let $a(p,q)$ and $\nu(q,r)$ be as in Proposition 3.2. It remains only to show that

$$c_k(q,r) = \delta(k, \nu(q,r)) + \delta(k, \nu(q,-r))$$

has average density $4/\pi^2$ in $\{y, z : 0 \leq z \leq y/2\}$. From the definition of ν and the relation $nq - pr = 1$ we conclude that

(i) if $2\mid q$, so that $2\nmid r$, then either $\nu(q,r)=0$ and $\nu(q,-r)=1$ or $\nu(q,r)=1$ and $\nu(q,-r)=0$;

(ii) if $2\nmid q$ and $2\mid r$ then either $\nu(q,r)=0$ and $\nu(q,-r)=2$ or $\nu(q,r)=2$ and $\nu(q,-r)=0$;

(iii) if $2\nmid q$ and $2\nmid r$ then either $\nu(q,r)=1$ and $\nu(q,-r)=2$ or $\nu(q,r)=2$ and $\nu(q,-r)=1$.

Therefore for $q\geq 3$ and $-q/2<r<q/2$, $(r,q)=1$, we have $c_0(q,r)=\beta(q,r)+\beta(r,q)$, $c_1(q,r)=\gamma(r,q)$, $c_2(q,r)=\gamma(q,r)$. We now extend the definition of $c_k(q,r)$ to $q\geq 3$, $r\neq 0$, $(r,q)=1$ by means of these formulas. Since the values of $c_k(q,r)$ for $q=1,2$ do not affect the average density, it follows from Lemma 3.5 and Corollary 3.6 that for $k=0,1,2$ the extended $c_k(q,r)$ has average density $4/\pi^2$ in Q and hence in $\{y,z:0\leq z\leq y/2\}$, where it is the original c_k. □

Convergence of $D(\lambda;N^{-1/2}S_N)$ will be shown when the local Riemann integrability of the corresponding $F_{\lambda,k}$ has been demonstrated. Now we can obtain an order estimate using the approximation of $S_N(x)$ by Fiedler, Jurkat, and Körner [2]. This says that for integers p,q,N and real t such that $1\leq q\leq 4N$, $(p,q)=1$, $0\leq p\leq q$ and $p/q+t/16N^2\in I_{p,q,4N}$, then

$$\text{(3.5)}\qquad S_N\left(\frac{p}{q}+\frac{t}{16N^2}\right)=a(p,q)\frac{N}{\sqrt{q}}\int_0^1 E\left(\frac{ts^2}{16}\right)ds+\delta(p,q,N,t),$$

where $a(p,q)$ is such that $|a(p,q)|=0$ or 1 as pq is odd or even, and $\delta(p,q,N,t)$ is bounded by $K_0\sqrt{q}$ for an absolute constant $K_0>0$. We denote by $H(t)$ the integral $\int_0^1 E(ts^2/16)\,ds$ and construct a function using the normalized main term of (3.5). That is, for each N let

$$A_N(x)=\frac{a(p,q)}{2}\left(\frac{N}{q}\right)^{1/2}H(t);\qquad \left(x=\frac{p}{q}+\frac{t}{N^2}\in I_{p,q,N}\right).$$

Then $D(\lambda;A_N)$ and $D(\lambda;N^{-1/2}S_N)$ are related through the following:

PROPOSITION 3.9. *Let* $\lambda>0$ *and* $\epsilon>0$ *be given such that*

(i) $1+4K_0<\lambda$

(ii) $(1+4K_0)K_0\lambda^{-1}<\epsilon$

Then $D(\lambda+\epsilon;A_{4N})\leq D(\lambda;N^{-1/2}S_N)\leq D(\lambda-\epsilon;A_{4N})$.

Proof. Let N be given and suppose ϵ and λ satisfy the hypotheses. Consider p and q such that $1\leq q\leq 4N$, $0\leq p\leq q$ and $(p,q)=1$.

Suppose pq is odd. Then if $x\in I_{p,q,4N}$,

$$|N^{-1/2}S_N(x)|\leq N^{-1/2}|\delta(p,q,N,t)|\leq K_0(q/N)^{1/2}\leq 2K_0<\lambda$$

by (i), so that $\{x\in I_{p,q,4N}:\lambda\leq N^{-1/2}|S_N(x)|\}=\emptyset$. Clearly in this case also $A_{4N}=0$ on $I_{p,q,4N}$.

Now suppose pq is even, so for $x\in I_{p,q,4N}$,

$$N^{-1/2}|S_N(x)|\leq (N/q)^{1/2}|H(t)|+K_0(q/N)^{1/2}\leq (1+4K_0)(N/q)^{1/2}$$

and $|A_{4N}(x)|\leq (N/q)^{1/2}$ since $|H(t)|\leq 1$ for all real t. If $q>(1+4K_0)^2\lambda^{-2}N$ then

$\{x\in I_{p,q,4N}:\lambda\le N^{-1/2}|S_N(x)|\}=\emptyset$ and $\{x\in I_{p,q,4N}:\lambda\le|A_{4N}(x)|\}=\emptyset$. For $q\le(1+4K_0)^2\lambda^{-2}N$ and $x\in I_{p,q,4N}$,

$$|N^{-1/2}S_N(x)-A_{4N}(x)|\le K_0(q/N)^{1/2}\le K_0(1+4K_0)\lambda^{-1}<\epsilon.$$

Therefore

$$\begin{aligned}&\{x\in I_{p,q,4N}:\lambda+\epsilon\le|A_{4N}(x)|\}\\ &\subseteq\{x\in I_{p,q,4N}:\lambda\le N^{-1/2}|S_N(x)|\}\\ &\subseteq\{x\in I_{p,q,4N}:\lambda-\epsilon\le|A_{4N}(x)|\}.\end{aligned}\tag{3.6}$$

Taking the measures of the sets in (3.6) and adding for all p, q in the permitted range one reaches the conclusion. □

Observe that $H(\zeta)H(-\zeta)$ is a nonconstant analytic function of complex ζ and therefore $\{t\in R:\lambda=|H(t)|\}=\{t\in R:\lambda^2=H(t)H(-t)\}$ has measure zero. Hence $F_\lambda(y,z)=|\{t\in J(y,z):\lambda\le\frac{1}{2}\sqrt{y}|H(t)|\}|$ is locally Riemann integrable, and when we apply Theorem 2.1 to A_N we obtain

PROPOSITION 3.10. For $\lambda>0$,

$$\lim_{N\to\infty}D(\lambda;A_N)=\frac{8}{\pi^2}\int_1^\infty dy\int_0^{1/2}dz\int_{J(y,z)}dt\,y^{-3}\chi(y,z,t:\lambda\le\tfrac{1}{2}\sqrt{y}|H(t)|).$$

Proof. Take $K=1$, $f_0(y,z,t)=\sqrt{y}H(t)/2$ and $f_1(y,z,t)=0$. For integers p,q,r, and n satisfying $3\le q$, $1\le p\le q$, $-q/2<r<q/2$ and $nq-pr=1$ let $a(p,q)$ be as in Proposition 3.2 and

$$\nu(q,r)=\begin{cases}0 & \text{if } 2\,|\,pq\\ 1 & \text{otherwise}\end{cases}$$

One determines that $c_0(q,r)=2\beta(q,r)+\gamma(q,r)$ which has average density $8/\pi^2$ in $\{y,z:0\le z\le y/2\}$. □

Our next step is to rewrite the integral in Proposition 3.10.

LEMMA 3.11. *For* $\lambda>0$,

$$\lim_{N\to\infty}D(\lambda;A_N)=\frac{8}{\pi^2}\int_0^\infty dt\int_M^\infty\frac{\min(1,\,(y^2/t)-y)}{y^3}\,dy$$

where $M=\max(1,t,4\lambda^2/|H(t)|^2)$.

Proof. Denote by I the triple integral in Proposition 3.10, and let $a_{y,z}=y^2/([y-z]+1+z)$ so $J(y,z)=[-a_{y,-z},a_{y,z}]$. On the range $-a_{y,-z}\le t\le 0$ substitute $-t$ and $-z$ for t and z and use $|H(-t)|=|H(t)|$ to obtain

$$\begin{aligned}&\int_0^{1/2}dz\int_{-a_{y,-z}}^{a_{y,z}}\chi(y,z,t:\lambda\le\tfrac{1}{2}\sqrt{y}|H(t)|)\,dt\\ &\qquad=\int_{-1/2}^{1/2}dz\int_0^{a_{y,z}}\chi(y,z,t:\lambda\le\tfrac{1}{2}\sqrt{y}|H(t)|)\,dt.\end{aligned}$$

The integrand is independent of z, and $a_{y,-z}$ is periodic in z with period 1, so we may instead integrate over $y-1<z\le y$ where $a_{y,z}=y^2/(1+z)$. Rearranging we get

$$I=\int_0^\infty dt\int_{\max(1,t)}^\infty \min(1,y^2/t-y)y^{-3}\chi(y,t:\lambda^2\le(1/4)y|H(t)|^2)\,dy$$

$$=\int_0^\infty dt\int_M^\infty \min(1,y^2/t-y)y^{-3}\,dy. \qquad \square$$

THEOREM 3.12. *There is an absolute constant K such that for $\lambda>0$,*

$$\liminf_{N\to\infty} D(\lambda;N^{-1/2}S_N)\ge\frac{1}{4\pi^2}\int_0^\infty |H(t)|^4\,dt\frac{1}{\lambda^4}-\frac{K}{\lambda^6}$$

$$\limsup_{N\to\infty} D(\lambda;N^{-1/2}S_N)\le\frac{1}{4\pi^2}\int_0^\infty |H(t)|^4\,dt\frac{1}{\lambda^4}+\frac{K}{\lambda^6}$$

Proof. Because $\int_0^\infty E(s^2)\,ds=\sqrt{i}/2$ we may find a constant K_1 such that $\sqrt{t}|H(t)|\le K_1$ for all $t>0$.

Consider $\lambda\ge\max(K_1/\sqrt{2},\ 1+4K_0,\ \sqrt{(1+\sqrt{5})/8}\ )$. Then

$$4\lambda^2|H(t)|^{-2}\ge 2K_1^2|H(t)|^{-2}\ge 2t \qquad\text{and}\qquad 4\lambda^2|H(t)|^{-2}\ge\lambda^2\ge 1$$

since $|H(t)|\le 1$ for all real t. Therefore $\max(1,t,4\lambda^2|H(t)|^{-2})=4\lambda^2|H(t)|^{-2}$. Now suppose $y\ge 4\lambda^2|H(t)|^{-2}$. If $0\le t\le 1$ then $y^2/t-y\ge y^2-y\ge 1$ since $y\ge(1+\sqrt{5})/2$. If $t>1$ then $4\lambda^2|H(t)|^{-2}\ge 2K_1^2|H(t)|^{-2}\ge 2t$ so that $y^2/t-y\ge 2y-y=y\ge 1$. Therefore

$$\lim_{N\to\infty} D(\lambda;A_N)=\frac{8}{\pi^2}\int_0^\infty dt\int_{4\lambda^2/|H(t)|^2}^\infty y^{-3}\,dy=\frac{1}{4\pi^2}\int_0^\infty |H(t)|^4\,dt\frac{1}{\lambda^4}.$$

Let $K_2>(1+4K_0)K_0$ and apply Proposition 3.9 to see

$$\liminf_{N\to\infty} D(\lambda;N^{-1/2}S_N)\ge\lim_{N\to\infty} D(\lambda+K_2/\lambda;A_{4N})=\frac{1}{4\pi^2}\int_0^\infty |H(t)|^4\,dt\frac{1}{(\lambda+K_2/\lambda)^4}$$

$$\limsup_{N\to\infty} D(\lambda;N^{-1/2}S_N)\le\lim_{N\to\infty} D(\lambda-K_2/\lambda;A_{4N})=\frac{1}{4\pi^2}\int_0^\infty |H(t)|^4\,dt\frac{1}{(\lambda-K_2/\lambda)^4}.$$

Using $(\lambda-K_2/\lambda)^{-4}\le\lambda^{-4}+K_3\lambda^{-6}$ and $(\lambda+K_2/\lambda)^{-4}\ge\lambda^{-4}-K_3\lambda^{-6}$ for an absolute constant K_3, we obtain the desired estimates for this range of λ. It is clear, though, that by increasing the constant K we may take the estimates valid for all $\lambda>0$. $\square$

REMARK. In a similar way one could show that there are constants c_0, c_1, c_2 such that if $0\le\lambda\le c_0\sqrt{N}$ then

$$c_1(1+\lambda^4)^{-1}\le D(\lambda;N^{-1/2}S_N)\le c_2(1+\lambda^4)^{-1}$$

by first proving a corresponding estimate for A_N.

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