On the centroids of fuzzy numbers

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Abstract

In a paper by Cheng [A new approach for ranking fuzzy numbers by distance method, Fuzzy Sets and Systems 95 (1998) 307–317], a centroid-based distance method was suggested for ranking fuzzy numbers, both normal and non-normal, where the fuzzy numbers are compared and ranked in terms of their Euclidean distances from their centroid points to the origin. It is found that the centroid formulae provided by the above paper are incorrect and have led to some misapplications. In this paper we present the correct centroid formulae for fuzzy numbers and justify them from the viewpoint of analytical geometry. A numerical example demonstrates that Cheng’s formulae can significantly alter the result of the ranking procedure.

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1. Introduction

In a paper by Cheng [2], a centroid-based distance method was suggested for ranking fuzzy numbers. The method utilizes the Euclidean distances from the origin to the centroid point of each fuzzy number to compare and rank the fuzzy numbers. Chu and Tsao [3] found that the distance method could not rank fuzzy numbers correctly if they are negative and therefore suggested using the area between the centroid point and the origin to rank fuzzy numbers. More recently, Abbasbandy and Asady [1] found that Chu and Tsao’s area method could sometimes lead to counterintuitive rankings and hence suggested a sign distance method.

In real applications, we find that the centroid formulae for fuzzy numbers provided by Cheng [2] are incorrect and have led to some misapplications such as by Chu and Tsao [3] and Pan and Yeh [7,8]. The incorrect formulae also appeared in [1] although Abbasbandy and Asady did not use them in their approach. To avoid possible more misapplications or spread in the future, we present in this paper the correct centroid formulae for fuzzy numbers and justify them from the viewpoint of analytical geometry.

The rest of the paper is organized as follows. Section 2 presents the correct centroid formulae for fuzzy numbers, derives the simplified formulae for triangular and trapezoidal fuzzy numbers, and justifies them using the knowledge of analytical geometry. Section 3 illustrates with a numerical example the fact that the incorrect formulae by Cheng...
can significantly alter the result of the ranking procedure and lead to a wrong ranking order. The paper is concluded in Section 4.

2. The correct centroid formulae for fuzzy numbers

A fuzzy number is a convex fuzzy subset of the real line $R$ and is completely defined by its membership function. Let $\tilde{A}$ be a fuzzy number, whose membership function $f_{\tilde{A}}(x)$ can generally be defined as

$$
\begin{cases}
  f_{\tilde{A}}^L(x), & a \leq x \leq b, \\
  \omega, & b \leq x \leq c, \\
  f_{\tilde{A}}^R(x), & c \leq x \leq d, \\
  0, & \text{otherwise},
\end{cases}
$$

(1)

where $0 < \omega \leq 1$ is a constant, $f_{\tilde{A}}^L : [a, b] \to [0, \omega]$ and $f_{\tilde{A}}^R : [c, d] \to [0, \omega]$ are two strictly monotonical and continuous mappings from $R$ to the closed interval $[0, \omega]$. If $\omega = 1$, then $\tilde{A}$ is a normal fuzzy number; otherwise, it is said to be a non-normal fuzzy number. If the membership function $f_{\tilde{A}}(x)$ is piecewise linear, then $\tilde{A}$ is referred to as a trapezoidal fuzzy number and is usually denoted by $\tilde{A} = (a, b, c, d; \omega)$ or $\tilde{A} = (a, b, c, d)$ if $\omega = 1$, which is plotted in Fig. 1. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $\tilde{A} = (a, b, d; \omega)$ or $\tilde{A} = (a, b, d)$ if $\omega = 1$. So, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.

Since $f_{\tilde{A}}^L(x)$ and $f_{\tilde{A}}^R(x)$ are both strictly monotonical and continuous functions, their inverse functions exist and should also be continuous and strictly monotonical. Let $g_{\tilde{A}}^L : [0, \omega] \to [a, b]$ and $g_{\tilde{A}}^R : [0, \omega] \to [c, b]$ be the inverse functions of $f_{\tilde{A}}^L(x)$ and $f_{\tilde{A}}^R(x)$, respectively. Then $g_{\tilde{A}}^L(y)$ and $g_{\tilde{A}}^R(y)$ should be integrable on the closed interval $[0, \omega]$. In other words, both $\int_0^w g_{\tilde{A}}^L(y) \, dy$ and $\int_0^w g_{\tilde{A}}^R(y) \, dy$ should exist. In the case of trapezoidal fuzzy number (see Fig. 1), the inverse functions $g_{\tilde{A}}^L(y)$ and $g_{\tilde{A}}^R(y)$ can be analytically expressed as

$$
g_{\tilde{A}}^L(y) = a + (b - a)y/\omega, \quad 0 \leq y \leq \omega. \quad (2)
$$

$$
g_{\tilde{A}}^R(y) = d - (d - c)y/\omega, \quad 0 \leq y \leq \omega. \quad (3)
$$

which are shown in Fig. 2.

In order to determine the centroid point $(\bar{x}_0, \bar{y}_0)$ of a fuzzy number $\tilde{A}$, Cheng [2] provided the following centroid formulae:

$$
\bar{x}_0(\tilde{A}) = \frac{\int_b^c (xf_{\tilde{A}}^L(x)) \, dx + \int_c^d (xf_{\tilde{A}}^R(x)) \, dx}{\int_b^c f_{\tilde{A}}^L(x) \, dx + \int_c^d f_{\tilde{A}}^R(x) \, dx}, \quad (4)
$$

$$
\bar{y}_0(\tilde{A}) = \frac{\omega \int_0^L (yg_{\tilde{A}}^L(y)) \, dy + \int_0^R (yg_{\tilde{A}}^R(y)) \, dy}{\int_0^L g_{\tilde{A}}^L(y) \, dy + \int_0^R g_{\tilde{A}}^R(y) \, dy}. \quad (5)
$$

Normal fuzzy numbers can be seen as special cases of non-normal fuzzy numbers with $\omega = 1$.

In Chu and Tsao [3], the centroid formulae were given as

$$
\bar{x}_0(\tilde{A}) = \frac{\int_b^c (xf_{\tilde{A}}^L(x)) \, dx + \int_c^d (xf_{\tilde{A}}^R(x)) \, dx}{\int_b^c f_{\tilde{A}}^L(x) \, dx + \int_c^d f_{\tilde{A}}^R(x) \, dx}, \quad (6)
$$

$$
\bar{y}_0(\tilde{A}) = \frac{\int_0^L (yg_{\tilde{A}}^L(y)) \, dy + \int_0^R (yg_{\tilde{A}}^R(y)) \, dy}{\int_0^L g_{\tilde{A}}^L(y) \, dy + \int_0^R g_{\tilde{A}}^R(y) \, dy}. \quad (7)
$$
The above formulae were also adopted by Pan and Yeh [7,8]. However, it is found that the above formulae (4)–(7) are incorrect despite the fact that formulae (4) and (6) are consistent with Yager’s ranking index [9,10] \( F(\tilde{A}) = \int_{-\infty}^{+\infty} g(x)f_{L}(x)dx / \int_{-\infty}^{+\infty} f_{L}(x)dx \) with the weight function \( g(x) \equiv x \) and \( \omega = 1 \) and the same as Murakami et al.’s ranking index for normal fuzzy sets [6] \( z_{0}(\tilde{A}) = \int_{-\infty}^{+\infty} xf_{L}(x)dx / \int_{-\infty}^{+\infty} f_{L}(x)dx \). The correct centroid formulae should be as follows:

\[
\bar{x}_{0}(\tilde{A}) = \frac{\int_{-\infty}^{+\infty} xf_{L}(x)dx}{\int_{-\infty}^{+\infty} f_{L}(x)dx} = \frac{\int_{a}^{b} xf_{L}(x)dx + \int_{b}^{c} (x-\omega)dx + \int_{c}^{d} xf_{R}(x)dx}{\int_{a}^{b} f_{L}(x)dx + \int_{b}^{c} (x-\omega)dx + \int_{c}^{d} f_{R}(x)dx},
\]

(8)

\[
\bar{y}_{0}(\tilde{A}) = \frac{\int_{0}^{\omega} y(g_{R}(y) - g_{L}(y))dy}{\int_{0}^{\omega} (g_{R}(y) - g_{L}(y))dy},
\]

(9)

where the denominator \( \int_{0}^{\omega} (g_{R}(y) - g_{L}(y))dy \) in (9) represents the area of the trapezoid in Fig. 2, while the numerator \( \int_{0}^{\omega} y(g_{R}(y) - g_{L}(y))dy \) is the weighed average of the area.

The main errors with formulae (4)–(7) are that \( \omega \) is missing from formulae (4) and (6), which makes them wrong when \( \omega \neq 1 \), and that both formulae (5) and (7) take a positive sign for the second item in both numerator and denominator, which is a fundamental error and makes the formulae wrong for any \( \omega \) values.

In particular, the natural properties of centroids are violated by formulae (4)–(7). Without loss of generality, we consider two general fuzzy numbers: \( \tilde{A} \) with the membership function \( f_{L}(x) \) defined by Eq. (1) and \( \tilde{B} \) with its
membership function \( f_{\tilde{B}}(x) \) defined as

\[
f_{\tilde{B}}(x) = \begin{cases} 
  f_{\tilde{A}}^{L}(x - \delta), & a + \delta \leq x \leq b + \delta, \\
  \omega, & b + \delta \leq x \leq c + \delta, \\
  f_{\tilde{A}}^{R}(x - \delta), & c + \delta \leq x \leq d + \delta, \\
  0 & \text{otherwise},
\end{cases}
\]

(10)

where \( \delta \) is a non-zero constant. It is evident that \( \tilde{B} \) is the right or left translation of \( \tilde{A} \) along horizontal axis. Such a translation should make the centroid point of \( \tilde{B} \) do exactly the same movement along the horizontal axis, but should not change the coordinate of its centroid on vertical axis. That is to say, there should exist \( \tilde{y}_{0}(\tilde{B}) = \tilde{x}_{0}(\tilde{A}) + \delta \) and \( \tilde{y}_{0}(\tilde{A}) \equiv \tilde{y}_{0}(\tilde{B}) \). This is one of the properties that correct centroid formulae have to possess. In addition, the changes of \( \omega \) should only affect \( \tilde{y}_{0}(\tilde{A}) \) and \( \tilde{y}_{0}(\tilde{B}) \), and should not result in any changes of \( \tilde{x}_{0}(\tilde{A}) \) and \( \tilde{x}_{0}(\tilde{B}) \). This is another property that correct centroid formulae should have.

Let \( g_{\tilde{A}}^{L}(y) \) and \( g_{\tilde{A}}^{R}(y) \) be the inverse functions of \( f_{\tilde{A}}^{L}(x) \) and \( f_{\tilde{A}}^{R}(x) \) respectively. Since the translation does not change the shape of fuzzy number \( \tilde{B} \), \( g_{\tilde{A}}^{L}(y) \) and \( g_{\tilde{A}}^{R}(y) \) can therefore be expressed as \( g_{\tilde{A}}^{L}(y) = g_{\tilde{A}}^{L}(y) + \delta \) and \( g_{\tilde{A}}^{R}(y) = g_{\tilde{A}}^{R}(y) + \delta \). By formula (5), it can be derived that

\[
\tilde{y}_{0}(\tilde{B})[\text{Cheng}] = \frac{\omega \int_{0}^{\omega}(yg_{\tilde{A}}^{L}(y)) \, dy + \int_{0}^{\omega}(yg_{\tilde{A}}^{R}(y)) \, dy + \delta}{\int_{0}^{\omega}(g_{\tilde{A}}^{L}(y)) \, dy + \int_{0}^{\omega}(g_{\tilde{A}}^{R}(y)) \, dy + 2\delta} = \tilde{y}_{0}(\tilde{A})[\text{Cheng}].
\]

It is very clear that the value of \( \tilde{y}_{0}(\tilde{B})[\text{Cheng}] \) varies with the value of \( \delta \) and \( \tilde{y}_{0}(\tilde{A})[\text{Cheng}] \neq \tilde{y}_{0}(\tilde{B})[\text{Cheng}] \). Such a result is counterintuitive and unacceptable. So, formula (5) cannot be correct. As such, it can be derived from formula (7) that

\[
\tilde{y}_{0}(\tilde{B})[\text{Chu and Tsao}] = \frac{\int_{0}^{\omega}(yg_{\tilde{A}}^{L}(y)) \, dy + \int_{0}^{\omega}(yg_{\tilde{A}}^{R}(y)) \, dy + \delta\omega^{2}}{\int_{0}^{\omega}(g_{\tilde{A}}^{L}(y)) \, dy + \int_{0}^{\omega}(g_{\tilde{A}}^{R}(y)) \, dy + 2\delta\omega} = \tilde{y}_{0}(\tilde{A})[\text{Chu and Tsao}],
\]

which leads to the conclusion that formula (7) cannot be correct either.

However, by formula (9), we have the following result:

\[
\tilde{y}_{0}(\tilde{B}) = \frac{\int_{0}^{\omega}y(g_{\tilde{B}}^{R}(y) - g_{\tilde{A}}^{L}(y)) \, dy}{\int_{0}^{\omega}(g_{\tilde{B}}^{R}(y) - g_{\tilde{A}}^{L}(y)) \, dy} = \frac{\int_{0}^{\omega}y(g_{\tilde{B}}^{R}(y) - g_{\tilde{A}}^{L}(y)) \, dy}{\int_{0}^{\omega}(g_{\tilde{B}}^{R}(y) - g_{\tilde{A}}^{L}(y)) \, dy} = \tilde{y}_{0}(\tilde{A}).
\]

It is obvious that formula (9) does not cause any changes of its centroid on the vertical axis when a fuzzy number moves along the horizontal axis.

To test whether or not the changes of \( \omega \) have any effect on the centroid of a fuzzy number on horizontal axis when it changes in vertical axis, we consider a general fuzzy number: \( \tilde{C} \) with its membership function \( f_{\tilde{C}}(x) \) defined as \( f_{\tilde{C}}(x) = \omega f_{\tilde{A}}^{U}(x) \), where \( 0 < \omega < 1 \) and \( \tilde{A} \) is assumed to be a normal fuzzy number. Theoretically, the fuzzy number \( \tilde{C} \) should have the same centroid coordinate on \( x \)-axis as the fuzzy number \( \tilde{A} \).

By formulae (4) or (6), we have the result below:

\[
\tilde{x}_{0}(\tilde{C})[\text{Cheng}] = \frac{\int_{a}^{b}(xf_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{b}(x/\omega) \, dx + \int_{a}^{d}(xf_{\tilde{A}}^{R}(x)) \, dx}{\int_{a}^{b}(f_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{b}(1/\omega) \, dx + \int_{a}^{d}(f_{\tilde{A}}^{R}(x)) \, dx} = \tilde{x}_{0}(\tilde{A})[\text{Cheng}].
\]

\[
\tilde{x}_{0}(\tilde{C})[\text{Chu and Tsao}] = \frac{\int_{a}^{b}(xf_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{b}(f_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{d}(xf_{\tilde{A}}^{R}(x)) \, dx}{\int_{a}^{b}(f_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{b}(f_{\tilde{A}}^{L}(x)) \, dx + \int_{a}^{d}(f_{\tilde{A}}^{R}(x)) \, dx} = \tilde{x}_{0}(\tilde{A}).
\]
So, formulae (4) and (6) cannot be correct. However, by formula (8), it is derived that

$$\bar{x}_0(\tilde{C}) = \frac{\int_a^b x f_L^C(x) \, dx + \int_b^c (x\omega) \, dx + \int_c^d x f_R^C(x) \, dx}{\int_a^b f_L^C(x) \, dx + \int_b^c (\omega) \, dx + \int_c^d f_R^C(x) \, dx}$$

It is found that formula (8) does not change the centroid coordinate of a fuzzy number on horizontal axis when it changes its membership degrees in vertical axis. This is consistent with our intuitive judgment.

To further justify formulae (8) and (9), we present in the following another derivation from the point of view of analytical geometry. Consider a general trapezoidal fuzzy number $\tilde{A} = [a, b, c, d; \omega]$, whose membership function is defined as

$$f_{\tilde{A}}(x) = \begin{cases} 
\omega(x - a)/(b - a), & a \leq x \leq b, \\
\omega, & b \leq x \leq c \text{ and } 0 \leq \omega \leq 1, \\
\omega(d - x)/(d - c), & c \leq x \leq d, \\
0 & \text{otherwise}.
\end{cases}$$

The corresponding inverse functions $g_{\tilde{A}}^L(y)$ and $g_{\tilde{A}}^R(y)$ are determined by Eqs. (2) and (3). For this trapezoidal fuzzy number, the following results have been derived from formulae (4) and (5):

$$\bar{x}_0(\tilde{A}) = \frac{\omega(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab) + 3(c^2 - b^2)}{3\omega(d - c + b - a) + 6(c - b)},$$

$$\bar{y}_0(\tilde{A}) = \omega \frac{1}{3} \left[ 1 + \frac{(b + c) - (a + d)(1 - \omega)}{(b + c - a - d) + 2(a + d)\omega} \right].$$

By formula (7), we obtain

$$\bar{y}_0(\tilde{A}) = \omega \frac{1}{3} \left[ 1 + \frac{b + c}{a + b + c + d} \right].$$

As such, by formulae (8) and (9), we derive the results below:

$$\bar{x}_0(\tilde{A}) = \frac{1}{3} \left[ a + b + c + d - \frac{dc - ab}{(d + c) - (a + b)} \right],$$

$$\bar{y}_0(\tilde{A}) = \omega \frac{1}{3} \left[ 1 + \frac{c - b}{(d + c) - (a + b)} \right].$$

To determine the centroid of the trapezoidal fuzzy number $\tilde{A} = [a, b, c, d; \omega]$ geometrically, Fig. 3 shows the geometric graph of $\tilde{A}$, which is a trapezoid, and its extension used for determining the centroid of the trapezoid.
The two straight lines EF and PQ intersect at point \( G \), which exactly represents the center of gravity of the trapezoid denoted by \( \tilde{A} = [a, b, c, d; \omega] \). In order to determine the coordinates of the point \( G \), we write the equations of the two straight lines EF and PQ as follows:

\[
\text{EF: } y = \frac{\omega(x - (a + b - c))}{(c + d - a) - (a + b - c)},
\]

\[
\text{PQ: } y = \frac{\omega(a + d - 2x)}{(a + d) - (b + c)}.
\]

Let

\[
\frac{\omega(x - (a + b - c))}{(c + d - a) - (a + b - c)} = \frac{\omega(a + d - 2x)}{(a + d) - (b + c)}.
\]

It follows that

\[
\bar{x}_0(\tilde{A}) = \frac{1}{3} \left[ \frac{1}{3} \right] \left[ a + b + c + d - \frac{dc - ab}{(d + c) - (a + b)} \right],
\]

which is exactly the same as Eq. (15) derived from formula (8). Substituting \( x = \bar{x}_0(\tilde{A}) \) into either Eq. (17) or (18) leads to the following result:

\[
\bar{y}_0(\tilde{A}) = \frac{1}{3} \left[ 1 + \frac{c - b}{(d + c) - (a + b)} \right],
\]

which is completely the same as Eq. (16) derived from formula (9). These facts verify the correctness and rationality of formulae (8) and (9).

For normal trapezoidal fuzzy number \( \tilde{A} = [a, b, c, d] \), formula (20) can be simplified as

\[
\bar{y}_0(\tilde{A}) = \frac{1}{3} \left[ 1 + \frac{1}{3} \frac{c - b}{(d + c) - (a + b)} \right].
\]

3. A numerical example

The correct formulae (8) and (9) applied to the numerical examples in Cheng [2] yield different numerical results but they do not alter the obtained rankings of the fuzzy numbers. Nevertheless, there are cases where the wrong formulae give wrong ranking orders. In this section, we illustrate this fact with a numerical example.
Table 1
The centroids of two triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ obtained by different formulae

<table>
<thead>
<tr>
<th>Fuzzy number</th>
<th>Centroids by Cheng's formulae</th>
<th>Centroids by Chu and Tsao's formulae</th>
<th>Centroids by formulae (22)–(24)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{x}_i$ $\bar{y}_i$ $\sqrt{(\bar{x}_i)^2 + (\bar{y}_i)^2}$</td>
<td>$\bar{x}_i$ $\bar{y}_i$ $\bar{x}_i\bar{y}_i$</td>
<td>$\bar{x}_i$ $\bar{y}_i$ $\sqrt{(\bar{x}_i)^2 + (\bar{y}_i)^2}$ $\bar{x}_i\bar{y}_i$</td>
</tr>
<tr>
<td>$\tilde{A}$</td>
<td>2 0.5 1.5811</td>
<td>2 0.5 1</td>
<td>2 1/3 1.5275 2/3</td>
</tr>
<tr>
<td>$\tilde{B}$</td>
<td>2 0.5114 1.5845</td>
<td>2 0.5105 1.021</td>
<td>2 9/28 1.5236 9/14</td>
</tr>
</tbody>
</table>

Example 1. Consider the two triangular fuzzy numbers $\tilde{A} = (1, 2, 3; 1)$ and $\tilde{B} = (0.5, 2.5, 3; 27/28)$, whose membership functions are respectively defined as

$$f_{\tilde{A}}(x) = \begin{cases} x - 1, & 1 \leq x < 2, \\ 1, & x = 2, \\ 3 - x, & 2 < x \leq 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_{\tilde{B}}(x) = \begin{cases} \frac{27}{28} \frac{x - 0.5}{2}, & 0.5 \leq x < 2.5, \\ \frac{27}{28}, & x = 2.5, \\ \frac{27}{28} \frac{3 - x}{0.5}, & 2.5 < x \leq 3, \\ 0 & \text{otherwise}. \end{cases}$$

Table 1 shows the results obtained by Cheng’s formulae (4) and (5), Chu and Tsao’s formulae (6) and (7) and the correct centroid formulae (8) and (9), respectively. It is very clear that both Cheng’s formulae and Chu and Tsao’s formulae lead to an incorrect ranking order: $\tilde{B} > \tilde{A}$, which is contrary to the ranking order: $\tilde{A} > \tilde{B}$ obtained by using the correct centroid formulae (22)–(24). This shows the fact that incorrect centroid formulae can lead to wrong ranking orders. So, using the correct centroid formulae is very important when comparing and ranking fuzzy numbers.

4. Conclusions

In this paper, the centroid formulae for fuzzy numbers provided by Cheng [2] were shown to be incorrect and to have led to some misapplications. To avoid possible further misapplications or spread in the future, we presented the correct centroid formulae for fuzzy numbers and derived their simplified expressions for trapezoidal and triangular fuzzy numbers. We also justified the presented formulae from the viewpoint of analytical geometry. A numerical example demonstrated that incorrect formulae could significantly alter the result of the ranking procedures and lead to a wrong ranking order. So, the correct centroid formulae are very crucial to the distance method [2] and the area method [3] for ranking fuzzy numbers. The simplified centroid formulae derived in this paper provide very useful computational support to the applications of the two ranking approaches.

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