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On the Centroids of Polygons and Polyhedra

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Abstract. In this paper we introduce the centroid of any finite set of points of the space and we find some general properties of centroids. These properties are then applied to different types of polygons and polyhedra.

1. Introduction

In elementary geometry the centroid of a figure in the plane or space (triangle, quadrilateral, tetrahedron, $\dots$) is introduced as the common point of some elements of the figure (medians or bimedians), once it has been proved that these elements are indeed concurrent. The proofs are appealing and have their own beauty in the spirit of Euclidean geometry. But they are different from figure to figure, and often use auxiliary elements. For example, the centroid of a triangle is defined as the common point of its three medians, after proving that they are concurrent. It is usually proved considering, as an auxiliary figure, the Varignon parallelogram of the quadrilateral whose vertices are the vertices of the triangle and the common point to two medians ([3, p. 10]). We can also define the centroid of a tetrahedron after proving that the four medians of the tetrahedron are concurrent (Commandino's Theorem, [1, p.57]). A natural question is: is it possible to characterize the properties of centroids of geometric figures with one unique and systematic method? In this paper we introduce the centroid of a finite set of points of the space, called a system, and find some of its general properties. These properties are then applied to different types of polygons and polyhedra. Then it is possible to obtain, in a simple and immediate way, old and new results of elementary geometry. At the end of the paper we introduce the notion of an extended system. This allows us to find some unexpected and charming properties of some figures, highlighting the great potential of the method that is used.

2. Systems and centroids

Throughout this paper, the ambient space is either a plane or a 3-dimensional space. Let $\mathcal{S}$ be a set of n points of the space. We call this an n-system or a system of order n. Let $\mathcal{S}'$ be a nonempty subset of S of k points, that we call a k-subsystem of $\mathcal{S}$ or a subsystem of order k of $\mathcal{S}$. There are $\binom{n}{k}$ different subsystems of order k. We say that two subsystems $\mathcal{S}'$ and $\mathcal{S}''$ of an n-system $\mathcal{S}$ are *complementary* if

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$\mathcal{S}' \cup S'' = S$ and $\mathcal{S}' \cap S'' = \emptyset$. We also say that $\mathcal{S}'$ is complementary to $\mathcal{S}''$ and $\mathcal{S}''$ is complementary to S'. If $\mathcal{S}'$ is a k-subsystem, $\mathcal{S}''$ is an $(n-k)$-subsystem. Let A_i, $i = 1, 2, \ldots, n$, be the points of an n-system S and $\mathbf{x}_i$ be the position vector of A_i with respect to a fixed point P. We call the *centroid* of $\mathcal{S}$ the point C whose position vector with respect to P is

$$\mathbf{x} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i.$$

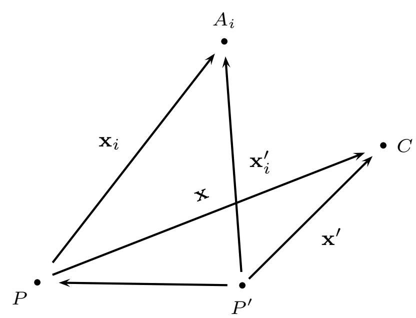

Figure 1

The point C does not depend on P. In fact, let P' be another point of the space and $\mathbf{x}'_i$ be the position vector of A_i with respect to P'. Since $\mathbf{x}'_i = \mathbf{x}_i + \overrightarrow{P'P}$, we have

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}'_i = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i + \overrightarrow{P'P}.$$

Every subsystem of $\mathcal{S}$ has its own centroid. The centroid of a 1-subsystem $\{A_i\}$ is A_i. The centroid of a 2-subsystem $\{A_i,\ A_j\}$ is the midpoint of the segment A_iA_j.

Let $\mathcal{S}'$ be a k-subsystem of $\mathcal{S}$ and C' its centroid. Let $\mathcal{S}''$ be the subsystem of $\mathcal{S}$ complementary to $\mathcal{S}'$ and C'' its centroid. We call the segment $C'C''$ the *median* of $\mathcal{S}$ relative to S'. The median relative to $\mathcal{S}''$ coincides with the one relative to $\mathcal{S}'$.

Let $\mathcal{S}$ be an n-system and C its centroid.

Theorem 1. *The medians of $\mathcal{S}$ are concurrent in C. Moreover, C divides the median $C'C''$ relative to a k-subsystem $\mathcal{S}'$ of $\mathcal{S}$ into two parts such that:*

$$\frac{C'C}{CC''} = \frac{n-k}{k}. \qquad (*)$$

Proof. In fact, let $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}''$ the position vectors of C, C', C'' respectively. It is easy to prove that

$$\mathbf{v} - \mathbf{v}' = \frac{n-k}{k}(\mathbf{v}'' - \mathbf{v}).$$

This relation means that $\overrightarrow{C'C} = \frac{n-k}{k}\overrightarrow{CC''}$. Hence, C, C', C'' are collinear and (*) holds. □

Here are some interesting consequences of Theorem 1.

Corollary 2. *The system of centroids of the $k-$subsystems of $\mathcal{S}$ is the image of the system of centroids of the $(n-k)$-subsystems of $\mathcal{S}$ in the dilatation with ratio $-\frac{n-k}{k}$ and center C. In this dilatation the centroid of a k-subsystem is the image of the centroid of its complementary.*

Corollary 3. *The segment $C_1'C_2'$ that joins the centroids of two k-subsystems $\mathcal{S}_1'$, $\mathcal{S}_2'$ of $\mathcal{S}$ is parallel to the segment $C_1''C_2''$ that joins the centroids of the $(n-k)$-subsystems complementary to $\mathcal{S}_1'$, $\mathcal{S}_2'$. Moreover,*

$$\frac{C_1'C_2'}{C_1''C_2''} = \frac{n-k}{k}.$$

Corollary 4. *If $n = 2k$, C is the center of symmetry of the system of centroids of the k-subsystems of $\mathcal{S}$. Moreover, the segment $C_1'C_2'$ that joins the centroids of two k-subsystems S_1', $\mathcal{S}_2'$ of $\mathcal{S}$ is parallel and equal to the segment $C_1''C_2''$ that joins the centroids of the k-subsystems complementary to $\mathcal{S}_1'$, S_2'.*

We conclude this section by the following theorem which is easily verified.

Theorem 5. *The centroid C of $\mathcal{S}$ is also the centroid of the system of centroids of the k-subsystems of $\mathcal{S}$.*

3. Applications

We propose here some applications to polygons and polyhedra. Let $\mathcal{P}$ be a polygon or a polyhedron. We associate with it the system $\mathcal{S}$ whose points are the vertices of $\mathcal{P}$.

3.1. *Triangles.* Let $\mathcal{T}$ be a triangle, with associated system $\mathcal{S}$ and centroid C. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{T}$, the 2-subsystems detect the sides. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides of $\mathcal{T}$ and detect the medial triangle of $\mathcal{T}$. The medians of $\mathcal{S}$ are the medians of $\mathcal{T}$.

As a consequence of Theorem 1, we have

Proposition 6 ([3, p.10], [4, p.8]). *The three medians of a triangle all pass through one point which divides each median into two segments in the ratio $2:1$.*

It follows that the centroid of $\mathcal{T}$ coincides with the centroid C of $\mathcal{S}$.

From Theorem 5 and Corollary 2, we deduce

Proposition 7 ([4, p.18], [5, p.11]). *A triangle $\mathcal{T}$ and its medial triangle have the same centroid C. Moreover, the medial triangle is the image of $\mathcal{T}$ in the dilatation with ratio $-\frac{1}{2}$ and center C.* See Figure 2.

Corollary 3 yields

Proposition 8 ([4, p.53])**.** *The segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as that third side.*

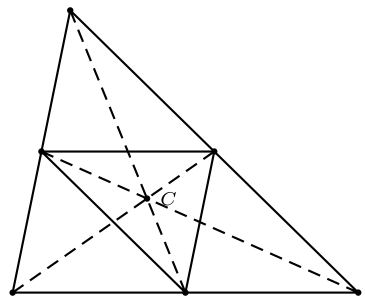

Figure 2.

3.2. *Quadrilaterals.* Let $A_1A_2A_3A_4$ be a quadrilateral which we denote by $\mathcal{Q}$. Let $\mathcal{S}$ be the system associated with $\mathcal{Q}$ and C its centroid. The 1-subsystems of $\mathcal{S}$ detect the vertices of $\mathcal{Q}$, the 2-subsystems detect the sides and the diagonals, the 3-subsystems detect the sub-triangles of $\mathcal{Q}$. The centroids of the 2-subsystems of $\mathcal{S}$ are the midpoints of the sides and of the diagonals of $\mathcal{Q}$. The centroids of the 3-subsystems are the centroids C_1, C_2, C_3, C_4 of the triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. We call $C_1C_2C_3C_4$ the quadrilateral of centroids and denote it by $\mathcal{Q}_c$ ([6]). The medians of $\mathcal{S}$ relative to the 2-subsystems are the *bimedians* of $\mathcal{Q}$ and the segment that joins the midpoints of the diagonals of $\mathcal{Q}$. The medians of $\mathcal{S}$ relative to the 1-subsystems are the segments A_iC_i, $i = 1, 2, 3, 4$.

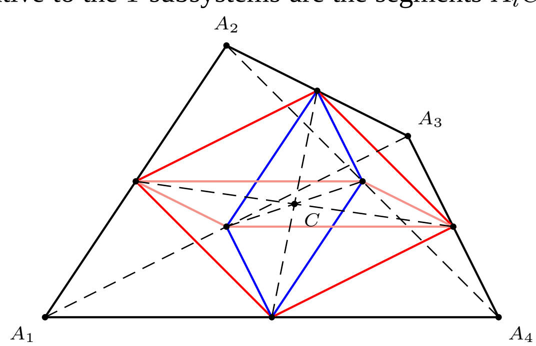

Figure 3

From Theorem 1 it follows that

Proposition 9 ([4, p.54])**.** *The bimedians of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent and bisect one another.* See Figure 3.

Thus, the centroid of the quadrilateral $\mathcal{Q}$, *i.e.*, the intersection point of the bimedians, coincides with the centroid C of $\mathcal{S}$. From Corollary 4, we obtain

Proposition 10 ([4, p.53])**.** *The quadrilateral whose vertices are the midpoints of the sides of a quadrilateral is a parallelogram* (Varignon's Theorem)*. Moreover, the quadrilateral whose vertices are the midpoints of the diagonals and of two opposite sides of a quadrilateral is a parallelogram.*

Thus, three parallelograms are naturally associated with a quadrilateral. These have the same centroid, which, by Theorem 1, coincides with the centroid of the quadrilateral.

Theorem 5 and Corollary 2 then imply

Proposition 11 ([6])**.** *The quadrilaterals $\mathcal{Q}$ and $\mathcal{Q}_c$ have the same centroid C. Moreover, $\mathcal{Q}_c$ is the image of $\mathcal{Q}$ in the dilatation with ratio $-\frac{1}{3}$ and center C.* See Figure 4.

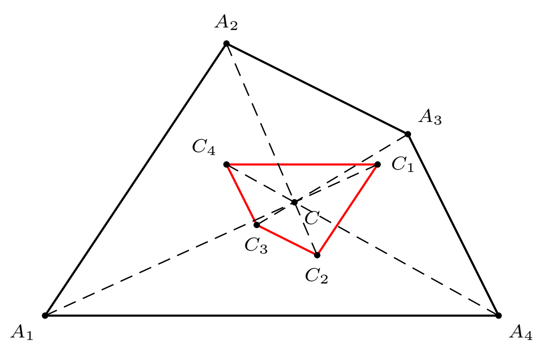

Figure 4

Some of these properties, with appropriate changes, hold also for polygons with more than four edges. For example, from Theorem 1 it follows that

Proposition 12. *The five segments that join the midpoint of a side of a pentagon with the centroid of the triangle whose vertices are the remaining vertices and the five segments that join a vertex of a pentagon with the centroid of the quadrilateral whose vertices are the remaining vertices are all concurrent in a point C that divides the first five segments in the ratio 3:2 and the other five in the ratio 4:1.*

The point C is the centroid of the system $\mathcal{S}$ associated with the pentagon. C will also be called the centroid of the pentagon.

3.3. *Tetrahedra.* Let $\mathcal{T}$ be a tetrahedron. Let $\mathcal{S}$ be the system associated with $\mathcal{T}$ and C its centroid. The subsystem of $\mathcal{S}$ of order 1, 2, and 3 detect the vertices, the edges and the faces of $\mathcal{T}$, respectively. The centroids of the 2-subsystems are the midpoints of the edges. Those of the 3-subsystems are the centroids of the faces of $\mathcal{T}$, which detect the medial tetrahedron of $\mathcal{T}$. The medians of $\mathcal{S}$ relative to the 2-subsystems are the bimedians of $\mathcal{T}$, *i.e.*, the segments that join the midpoints of two opposite sides. The medians of $\mathcal{S}$ relative to the 1-subsystems are the medians of $\mathcal{T}$, *i.e.*, the segments that join one vertex of $\mathcal{T}$ with the centroid of the opposite face.

From Theorem 1 follows Commandino's Theorem:

Proposition 13 ([1, p.57])**.** *The four medians of a tetrahedron meet in a point which divides each median in the ratio* $1:3$*.* See Figure 5.

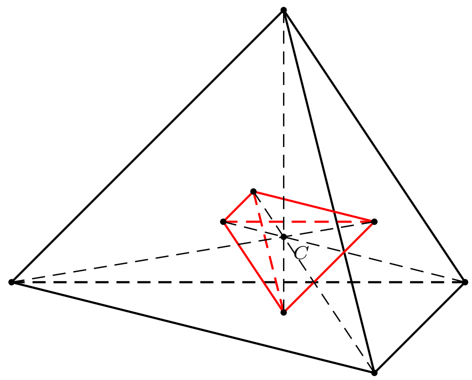

Figure 5

It follows that the centroid of the tetrahedron $\mathcal{T}$, intersection point of the medians, coincides with the centroid C of $\mathcal{S}$. From Theorem 5 and from Corollary 2 it follows that

Proposition 14 ([1, p.59])**.** *A tetrahedron* $\mathcal{T}$ *and its medial tetrahedron have the same centroid* C*. Moreover the medial tetrahedron is the image of* $\mathcal{T}$ *in the dilatation with ratio* $-\frac{1}{3}$ *and center* C*. The faces and the edges of the medial tetrahedron of a tetrahedron* $\mathcal{T}$ *are parallel to the faces and the edges of* $\mathcal{T}$*.*

Finally, Theorem 1 and Corollary 2 yield

Proposition 15 ([1, pp.54,58])**.** *The three bimedians of a tetrahedron are concurrent in the centroid of the tetrahedron and are bisected by it. Moreover, the midpoints of two pairs of opposite edges of tetrahedron are the vertices of a parallelogram.* See Figure 6.

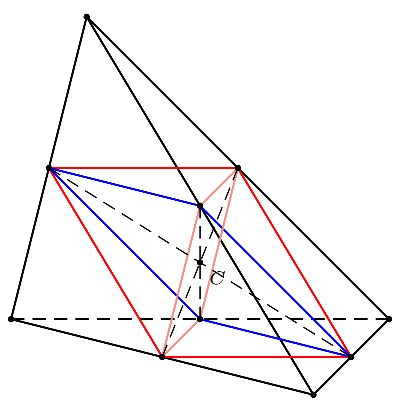

Figure 6

By using the theorems of the theory it is possible to find lots of interesting properties on polyhedra. For example, Corollary 4 gives

Proposition 16. *The centroids of the faces of an octahedron with triangular faces are the vertices of a parallelepiped. The centroids of the faces of a hexahedron with quadrangular faces are the vertices of an octahedron with triangular faces having a symmetry center C.* See Figures 7A and 7B.

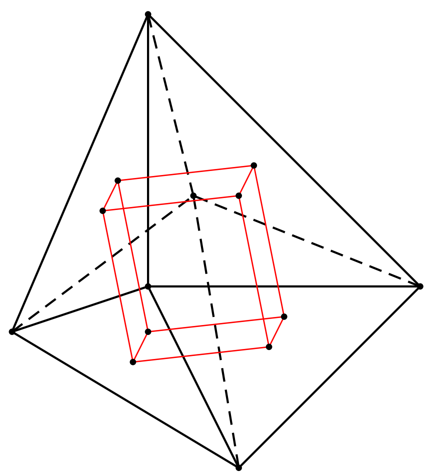

Figure 7A

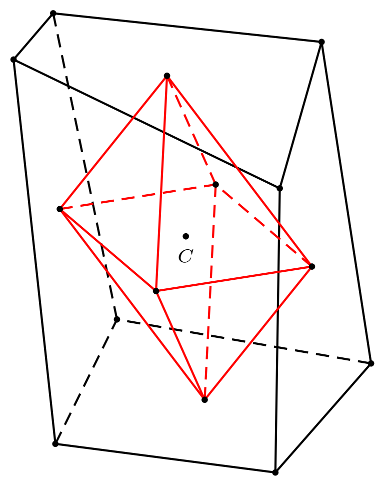

Figure 7B

The point C is the centroid of the system $\mathcal{S}$ associated with the hexahedron. This point is also called the centroid of the hexahedron.

4. Extended systems and applications

Let $\mathcal{S}$ be an n-system and h a fixed positive integer. Let H be a set of h points such that $\mathcal{S} \cap H = \emptyset$. We call h-extension of $\mathcal{S}$ the system $\mathcal{S}_H = \mathcal{S} \cup H$.

Let t be a fixed integer such that $1 \leq t < n$. Consider the system $\mathcal{C}_{H,t}$ of centroids of the subsystems of $\mathcal{S}_H$, of order $h+t$, that contain H. The complementary subsystems of these subsystems are the subsystems of $\mathcal{S}$ of order $n-t$ and we denote the system of their centroids by $\mathcal{C}'_{n-t}$.

Let us consider now two h-extensions of $\mathcal{S}$, $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$, and let C_1 and C_2 be their centroids. Consider the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$, and the system $\mathcal{C}'_{n-t}$.

From Corollary 2 applied to the system $\mathcal{S}_{H_1}$ (respectively $\mathcal{S}_{H_2}$) it follows that $\mathcal{C}_{H_{1,t}}$ (respectively $\mathcal{C}_{H_{2,t}}$) is the image of $\mathcal{C}'_{n-t}$ in the dilatation with ratio $-\frac{n-t}{h+t}$ and center C_1 (respectively C_2).

Thus, we have

Theorem 17. *If $\mathcal{S}_{H_1}$ and $\mathcal{S}_{H_2}$ are two h-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H_{1,t}}$ and $\mathcal{C}_{H_{2,t}}$ are correspondent in a translation.*

It is easy to see that the vector of the translation transforming $\mathcal{C}_{H_{1,t}}$ into $\mathcal{C}_{H_{2,t}}$ is $\frac{n+h}{h+t}\overrightarrow{C_1C_2}$.

The following theorem is also of interest.

Theorem 18. *If $\mathcal{S}$ is an n-system, $\mathcal{S}_H$ is a 1-extension of $\mathcal{S}$, $\mathcal{S}_K$ is a $(n-1)$-extension of $\mathcal{S}$, then the systems $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent is a half-turn.*

Proof. Let C and C_K be the centroids of $\mathcal{S}_H$ and K respectively. From Corollary 2 the system $\mathcal{C}_{H,n-1}$ is the image of the system $\mathcal{C}'_1 = \mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C that is, $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C.

Let $C' \in \mathcal{C}_{K,1}$ and suppose that C' is the centroid of the n-subsystem $\mathcal{S}' = K \cup \{A\}$ of $\mathcal{S}_K$, with $A \in \mathcal{S}$. From Theorem 1, C' lies on the median C_KA of $\mathcal{S}'$ and is such that $\frac{C_KC'}{C'A} = \frac{1}{n-1}$. It follows that $\frac{C_KC'}{C_KA} = \frac{1}{n}$, and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K.

Since $\mathcal{S}$ is the image of $\mathcal{C}_{H,n-1}$ in the dilatation with ratio $-n$ and center C and $\mathcal{C}_{K,1}$ is the image of $\mathcal{S}$ in the dilatation with ratio $\frac{1}{n}$ and center C_K, then $\mathcal{C}_{H,n-1}$ and $\mathcal{C}_{K,1}$ are correspondent in a dilatation with ratio -1, *i.e.*, in a half-turn. □

It is easy to see that the center $\overline{C}$ of the half-turn is the point of the segment CC_K such that $\frac{C\overline{C}}{\overline{C}C_K} = \frac{n-1}{n+1}$.

Now, we offer some applications of Theorems 17 and 18.

4.1. *Triangles.* Let $\mathcal{T}$ be a triangle and $\mathcal{S}$ its associated system. Let S_H be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and $\mathcal{S}_K$ be a 2-extension of $\mathcal{S}$, with $K = \{P_1, P_2\}$. The points of the system $C_{H,2}$ are vertices of a triangle $\mathcal{T}_H$ and the points of the system $C_{K,1}$ are vertices of a triangle $\mathcal{T}_K$. Theorem 18 gives

Proposition 19. *The triangles $\mathcal{T}_H$ and $\mathcal{T}_K$ are correspondent in a half-turn.* See Figure 8.

Let $\{\mathcal{T}_H\}$ be the family of triangles $\mathcal{T}_H$ obtained by varying the point P and $\{\mathcal{T}_K\}$ be the family of triangles $\mathcal{T}_K$ obtained by varying the points P_1 and P_2.

From Theorem 17 the triangles of the family $\{\mathcal{T}_H\}$ are all congruent and have corresponding sides that are parallel. The same property also holds for the triangles of the family $\{\mathcal{T}_K\}$. On the other hand, each triangle $\mathcal{T}_H$ and each triangle $\mathcal{T}_K$ are correspondent in a half-turn, then:

Proposition 20. *The triangles of the family $\{\mathcal{T}_H\} \cup \{\mathcal{T}_K\}$ are all congruent and have corresponding sides that are parallel.*

4.2. *Quadrilaterals.* Let $\mathcal{Q}$ be a quadrilateral $A_1A_2A_3A_4$ and $\mathcal{S}$ its associated system. Let $\mathcal{S}_H$ be a 1-extension of $\mathcal{S}$, with $H = \{P\}$, and let C be its centroid.

Let us consider the subsystems $\{P,\ A_1,\ A_2\}$, $\{P,\ A_2,\ A_3\}$, $\{P,\ A_3,\ A_4\}$, $\{P,\ A_4,\ A_1\}$ of $\mathcal{S}_H$ and their centroids C_1, C_2, C_3, C_4 respectively, that are points of $C_{H,2}$. From Corollary 3 applied to the system $\mathcal{S}_H$, the segments C_1C_2, C_2C_3, C_3C_4, C_4C_1 are parallel to the sides of the Varignon parallelogram of $\mathcal{Q}$ respectively. Thus, $C_1C_2C_3C_4$ is a parallelogram, that we denote by $\mathcal{Q}_H$. Moreover,

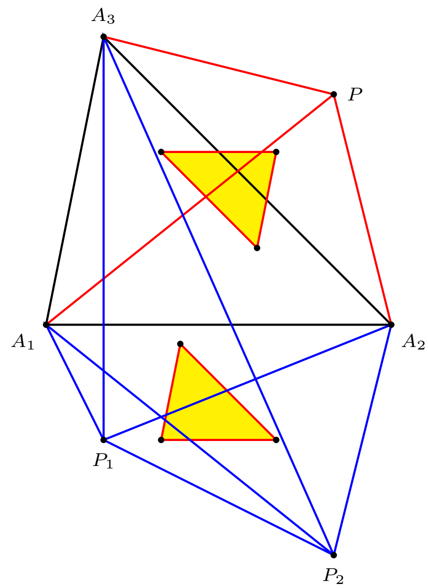

Figure 8.

from Corollary 2, $\mathcal{Q}_H$ is the image of the Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. In the case when P is the intersection point of the diagonals of Q, the existence of a dilatation between $\mathcal{Q}_H$ and the Varignon parallelogram of $\mathcal{Q}$ has already been proved ([2, p.424], [7, p.23]).

If we consider two 1-extensions of $\mathcal{S}$, the systems $\mathcal{C}_{H,2}$, for Theorem 17, are correspondent in a translation. Thus, if $\{\mathcal{Q}_H\}$ is the family of the parallelograms obtained as P varies, we obtain

Proposition 21. *The parallelograms of the family $\{\mathcal{Q}_H\}$ are all congruent and their corresponding sides are parallel.*

Moreover, taking P as the vertex of a pyramid with base $\mathcal{Q}$, we are led to

Proposition 22. *The centroids of the faces of a pyramid with a quadrangular base are vertices of the parallelogram that is the image to Varignon parallelogram of $\mathcal{Q}$ in the dilatation with ratio $-\frac{2}{3}$ and center C. Moreover, as P varies, the parallelograms whose vertices are the centroids of the faces are all congruent.* See Figure 9.

The point C is called the centroid of the pyramid.

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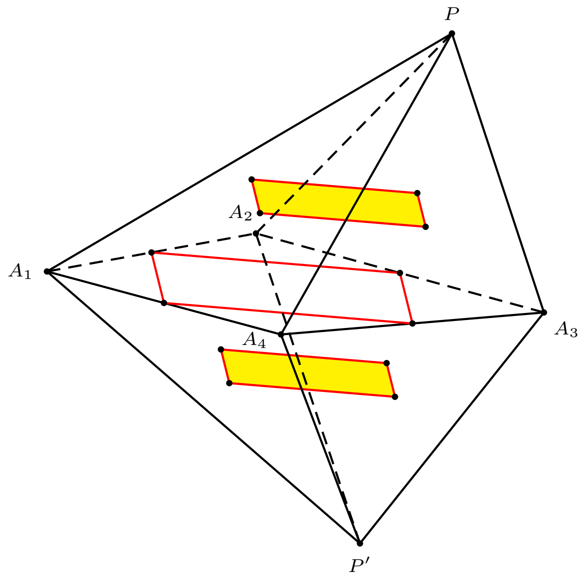

Figure 9

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