ON THE CHARACTER OF CERTAIN TILTING MODULES

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1. Let G be an almost simple, simply connected algebraic group over $\mathbf{k}$, an algebraically closed field of characteristic $p>1$. We fix a maximal torus T of G; let $X=\mathrm{Hom}(T,\mathbf{k}^*)$ (with group operation written as $+$). Let Rep$G$ be the category of finite dimensional $\mathbf{k}$-vector spaces with a given rational linear action of G and let $\mathcal{T}G$ be the full subcategory of Rep$G$ consisting of tilting modules (see [Do]). Let X^+ be the set of dominant elements in X, defined in terms of a fixed choice of a set R^+ of positive roots in the set of roots of G with respect to T. For $V\in$ Rep$G$ and $\mu\in X$ let $n_\mu(V)$ be the dimension of the μ-weight space of V. For $V\in$ Rep$G$ we set $[V]=\sum_{\mu\in X}n_\mu(V)e^\mu\in\mathbf{Z}[X]$, where $\mathbf{Z}[X]$ is the group ring of X with standard basis $\{e^\mu;\mu\in X\}$. (Actually, we have $[V]\in\mathbf{Z}[X]^W$ where W is the Weyl group of G with respect to T, viewed as a subgroup of Aut(X) and $\mathbf{Z}[X]^W$ denotes the subring of W-invariants.) According to [Do], there is a well defined bijection $\lambda\leftrightarrow T_\lambda$ between X^+ and a set of representatives for the isomorphism classes of indecomposable objects of $\mathcal{T}G$ such that the following holds: $n_\lambda(T_\lambda)=1$; moreover, if $\mu\in X$, $n_\mu(T_\lambda)\ne 0$ then $\mu\le\lambda$ ($\le$ as in no.2). We set $X^{++}=X^++\rho$ where $\rho\in X^+$ is defined by $2\rho=\sum_{\alpha\in R^+}\alpha$. For $\zeta\in X^{++}$ we set $S_\zeta=[T_{\zeta-\rho}]$,

$$S^0_\zeta=\sum_{w\in W}\epsilon_w e^{w(\zeta)}/\sum_{w\in W}\epsilon_w e^{w(\rho)}\in\mathbf{Z}[X]^W.$$

(Here $w\mapsto\epsilon_w$ is the sign character of W.) By general principles, for $\zeta\in X^{++}$ we have

$$S_\zeta=\sum_{\mu\in X^{++}}y_{\mu,\zeta}S^0_\mu$$

where $y_{\mu,\zeta}\in\mathbf{N}$ is zero unless $\mu\le\zeta$. One of the mysteries of the subject is that there is no known explicit formula (even conjecturally and even for $p\gg 0$) for the coefficients $y_{\mu,\zeta}$ above. This in contrast with the situation for the quantum group at $\sqrt[p]{1}$ when, for $\lambda\in X^+$, the analogue T'_λ of T_λ and the corresponding element

G.L. supported in part by National Science Foundation grant DMS-1303060 and by a Simons Fellowship.

$[T'_\lambda] \in \mathbf{Z}[X]^W$ which records weight multiplicities in T'_λ are defined (see [A1]) and, setting $S^1_\zeta = [T'_{\zeta-\rho}]$ for $\zeta \in X^{++}$, we have

(a) $$S^1_\zeta = \sum_{\mu \in X^{++}} x_{\mu,\zeta} S^0_\mu$$

where $x_{\mu,\zeta} \in \mathbf{N}$ can be explicitly computed in terms of the polynomials $Q_{y,w}$ of [KL], see Soergel [S1,S2,S3] (it is zero unless $\mu \le \zeta$).

Now [Lu] gives a method to express directly the characters of simple objects of Rep$G$ (for $p \gg 0$) in terms of the characters of the irreducible representation of the quantum group, without the use of the Steinberg tensor product theorem. In this paper we show that a variation of the method of [Lu], combined with results of Donkin and Andersen, gives (at least conjecturally and for $p \gg 0$), a simple closed formula for the coefficients $y_{\mu,\zeta}$ (for a large set of $\zeta \in X^{++}$) in terms of the coefficients $x_{\mu,\zeta}$.

We thank H. H. Andersen for help with references.

2. Notation. For $\lambda, \mu \in X$ we write $\mu \le \lambda$ if $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbf{N}\alpha$.

Let $\{\check\alpha_i; i \in I\}$ be the basis of $\mathrm{Hom}(X, \mathbf{Z})$ consisting of simple coroots. We have $X^+ = \{\lambda \in X; \check\alpha_i(\lambda) \in \mathbf{N} \quad \forall i \in I\}$. For any $\xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in \mathbf{Z}[X]$ (with $c_\lambda \in \mathbf{Z}$) and any $h \ge 0$ we set $\xi^{(h)} = \sum_{\lambda \in X} c_\lambda e^{p^h \lambda} \in \mathbf{Z}[X]$. Let

$$X^+_{red} = \{\lambda \in X; \check\alpha_i(\lambda) \in [0, p-1] \quad \forall i \in I\}.$$

For any $\lambda \in X^+$ we define $\lambda^0, \lambda^1, \lambda^2, \dots$ in X^+_{red} by $\lambda = \sum_{k \ge 0} p^k \lambda^k$; note that $\lambda^k = 0$ for large k.

3. For $\lambda \in X^+, \nu \in X^+_{red}, \mu \in X^+$ such that $\lambda + \rho = \nu + p(\mu + \rho)$ we have the Donkin tensor product formula, see [Do]:

(a) $$[T_\lambda] = [T_{\nu+(p-1)\rho}][T_\mu]^{(1)},$$

and its quantum analogue, see [A1]:

(b) $$[T'_\lambda] = [T'_{\nu+(p-1)\rho}](S^0_{\mu+\rho})^{(1)}.$$

For $k \ge 0$ let $X^+_k = \{\zeta \in X^+; \zeta^k \in X^{++}, \zeta^{k+1} = \zeta^{k+2} = \dots = 0\}$. Note that $X^+_k \subset X^{++}$. For any $k \ge 0$ and any $\zeta \in X^+_k$ we show:

(c) $$S_\zeta = S_{\zeta^0+p\rho} S^{(1)}_{\zeta^1+p\rho} \dots S^{(k-1)}_{\zeta^{k-1}+p\rho} S^{(k)}_{\zeta^k}.$$

We argue by induction on k. For $k = 0$ the result is obvious: we have $z = z^0 \in X^{++}$ and $S_\zeta = S_{\zeta^0}$. Assume now that $k \ge 1$. We have $\zeta = \zeta^0 + p(\zeta^1 + p\zeta^2 + \dots)$ hence, by (a):

$$S_\zeta = S_{\zeta^0+p\rho} S^{(1)}_{\zeta^1+p\zeta^2+\dots}.$$

We have $\zeta^1 + p\zeta^2 + \cdots \in X^+_{k-1}$ hence by the induction hypothesis we have

$$S_{\zeta^1+p\zeta^2+\dots} = S_{\zeta^1+p\rho} \dots S^{(k-2)}_{\zeta^{k-1}+p\rho} S^{(k-1)}_{\zeta^k}.$$

The result follows.

Now, using (b), we see that, if $\zeta \in X^+$ satisfies $\zeta^1 + p\zeta^2 + \cdots \in X^{++}$ then

(d) $$S^1_\zeta = S^1_{\zeta^0+p\rho}(S^0_{\zeta^1+p\zeta^2+\dots})^{(1)}.$$

4. For any $h \ge 1$ and any $\zeta \in X^+$ such that $\zeta^{h-1} + p\zeta^h + p^2\zeta^{h+1} + \cdots \in X^{++}$ we define S^h_ζ by the inductive formula

(a) $$S^h_\zeta = \sum_{\mu \in X^{++}} x_{\mu,\zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S^{h-1}_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}.$$

If $h = 1$ this agrees with 0(a) (the condition on ζ becomes $\zeta \in X^{++}$). If $h \ge 2$, then $S^{h-1}_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}$ is defined by induction since

$$\zeta' := \zeta^0 + p\zeta^1 + \cdots + p^{h-2}\zeta^{h-2} + p^{h-1}\mu$$

satisfies

$$\zeta'^{h-2} + p\zeta'^{h-1} + p^2\zeta'^h + \cdots = \zeta^{h-2} + p\mu \in X^{++}.$$

For any $h \ge 1$ and any $\zeta \in X^+$ such that $\zeta^h + p\zeta^{h+1} + p^2\zeta^{h+2} + \cdots \in X^{++}$ we show:

(b) $$S^h_\zeta = S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{h-1}+p\rho})^{(h-1)}(S^0_{\zeta^h+p\zeta^{h+1}+\dots})^{(h)}.$$

Note that our assumption implies $\zeta^{h-1} + p\zeta^h + p^2\zeta^{h+1} + \cdots \in X^{++}$ hence S^h_ζ is defined. We argue by induction on h. Assume first that $h = 1$ and $\zeta^1 + p\zeta^2 + p^2\zeta^3 + \cdots \in X^{++}$. Then (b) reduces to $S^1_\zeta = S^1_{\zeta^0+p\rho}(S^0_{\zeta^1+p\zeta^2+\dots})^{(1)}$ which is known from 3(d). Now assume that $h \ge 2$. In the right hand side of (a) we replace (using the induction hypothesis) $S^{h-1}_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}$ by

$$S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{h-2}+p\rho})^{(h-2)}(S^0_\mu)^{(h-1)}.$$

(Note that, if ζ' is as above, then $\zeta'^{h-1} + p\zeta'^h + \cdots = \mu \in X^{++}$, hence the induction hypothesis is applicable.) Thus from (a) we obtain

$$S^h_\zeta = \sum_{\mu \in X^{++}} x_{\mu,\zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{h-2}+p\rho})^{(h-2)}(S^0_\mu)^{(h-1)}.$$

It remains to show

$$\sum_{\mu \in X^{++}} x_{\mu,\zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S^0_\mu = S^1_{\zeta^{h-1}+p\rho}(S^0_{\zeta^h+p\zeta^{h+1}+\dots})^{(1)}$$

that is,

$$S^1_{\zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} = S^1_{\zeta^{h-1}+p\rho}(S^0_{\zeta^h+p\zeta^{h+1}+\dots})^{(1)}.$$

This is known from 3(d). Thus, (a) is proved.

5. Now assume that $k \geq 1$ and $\zeta \in X_k^+$ that is, $\zeta^k \in X^{++}$, $\zeta^{k+1} = \zeta^{k+2} = \cdots = 0$. Then S_ζ^{k+1} is defined. We have

$$S_\zeta^{k+1} = \sum_{\mu \in X^{++}} x_{\mu,\zeta^k} S^k_{\zeta^0+p\zeta^1+\dots+p^{k-1}\zeta^{k-1}+p^k\mu}.$$

By 4(b), this becomes

$$S_\zeta^{k+1} = \sum_{\mu \in X^{++}} x_{\mu,\zeta^k} S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{k-1}+p\rho})^{(k-1)}(S^0_\mu)^{(k)}.$$

Hence, using 3(d), we have

(a) $$S_\zeta^{k+1} = S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{k-1}+p\rho})^{(k-1)}(S^1_{\zeta^k})^{(k)}.$$

6. According to [A2, 5.2(a)], for $\nu \in X^+_{red}$ we have

(a) $$S_{\nu+p\rho} = S^1_{\nu+p\rho} \text{ provided that } p \gg 0.$$

It is likely that for $\nu \in X^+_{red} \cap X^{++}$ we have

(b) $$S_\nu = S^1_\nu \text{ provided that } p \gg 0.$$

Note that (b) is a very special case of Conjecture 5.1 in [A2].

Proposition 7. *Assume that $k \geq 0$ and $\zeta \in X_k^+$. Assume that $p \gg 0$ and that 6(b) holds. Then $S_\zeta = S_\zeta^{k+1}$.*

If $k = 0$ this is just the assumption 6(b). Assume now that $k \geq 1$. Using 3(c) and 5(a) we see that it is enough to show

(a) $$\begin{aligned} &S_{\zeta^0+p\rho}S^{(1)}_{\zeta^1+p\rho} \dots S^{(k-1)}_{\zeta^{k-1}+p\rho}S^{(k)}_{\zeta^k} \\ &= S^1_{\zeta^0+p\rho}(S^1_{\zeta^1+p\rho})^{(1)} \dots (S^1_{\zeta^{k-1}+p\rho})^{(k-1)}(S^1_{\zeta^k})^{(k)}. \end{aligned}$$

This follows from the equalities $S_{\zeta^j+p\rho} = S^1_{\zeta^j+p\rho}$ for $j = 0, 1, \dots, k-1$ (see 6(a)) and $S_{\zeta^k} = S^1_{\zeta^k}$ which holds by the assumption 6(b).

Corollary 8. *In the setup of Proposition 7 we have*

$$S_\zeta = \sum_{\mu_0,\mu_1,\dots,\mu_k \in X^{++}} x_{\mu_k,\zeta^k} x_{\mu_{k-1},\zeta^{k-1}+p\mu_k} \dots x_{\mu_1,\zeta^1+p\mu_2} x_{\mu_0,\zeta^0+p\mu_1} S^0_{\mu_0}.$$

Using 4(a) repeatedly, we have

$$S^1_\zeta = \sum_{\mu_0 \in X^{++}} x_{\mu_0,\zeta^0+p\zeta^1+p^2\zeta^2+\dots} S^0_{\mu_0},$$

$$S^2_\zeta = \sum_{\mu_1 \in X^{++}} x_{\mu_1,\zeta^1+p\zeta^2+\dots} S^1_{\zeta^0+p\mu_1} = \sum_{\mu_0,\mu_1 \in X^{++}} x_{\mu_1,\zeta^1+p\zeta^2+\dots} x_{\mu_0,\zeta^0+p\mu_1} S^0_{\mu_0},$$

$$\begin{aligned} S^3_\zeta &= \sum_{\mu_2 \in X^{++}} x_{\mu_2,\zeta^2+p\zeta^3+\dots} S^2_{\zeta^0+p\zeta^1+p^2\mu_2} \\ &= \sum_{\mu_0,\mu_1,\mu_2 \in X^{++}} x_{\mu_2,\zeta^2+p\zeta^3+\dots} x_{\mu_1,\zeta^1+p\mu_2} x_{\mu_0,\zeta^0+p\mu_1} S^0_{\mu_0}. \end{aligned}$$

Continuing we get

$$S^{k+1}_\zeta = \sum_{\mu_0,\mu_1,\dots,\mu_k \in X^{++}} x_{\mu_k,\zeta^k} x_{\mu_{k-1},\zeta^{k-1}+p\mu_k} \dots x_{\mu_1,\zeta^1+p\mu_2} x_{\mu_0,\zeta^0+p\mu_1} S^0_{\mu_0}.$$

It remains to use Proposition 7.

9. For any $\zeta \in X^{++}$ we can write $S_\zeta = \sum_{\mu \in X^{++}} r_{\mu,\zeta} S^1_\mu$ where $r_{\mu,\zeta} \in \mathbf{N}$.

Corollary 10. *In the setup of Proposition 7 assume that $\mu \in X^{++}$ satisfies $r_{\mu,\zeta} \neq 0$. Then $\mu = \zeta \mod pX$.*

Using Corollary 8 and 4(a) we see that

$$r_{\mu,\zeta} = \sum_{\mu_1,\dots,\mu_k \in X^{++}; \mu=\zeta^0+p\mu_1} x_{\mu_k,\zeta^k} x_{\mu_{k-1},\zeta^{k-1}+p\mu_k} \dots x_{\mu_1,\zeta^1+p\mu_2}.$$

It follows that $\mu = \zeta^0 + p\mu_1$ for some $\mu_1 \in X^{++}$. Since $\zeta = \zeta^0 \mod pX$, it follows that $\mu = \zeta \mod pX$. The corollary is proved.

11. In view of Corollary 10, one could hope that for any $\zeta, \mu \in X^{++}$ such that $r_{\mu,\zeta} \neq 0$ we have $\mu = \zeta \mod pX$. Unfortunately, this is contradicted by example (i) in [Je].

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