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ON THE CHARACTER OF WEIL'S REPRESENTATION[1]

BY

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ABSTRACT. The importance of certain representations of symplectic groups, usually called Weil representations, for the general problem of finding representations of certain group extensions is made explicit. Some properties of the character of Weil's representation for a finite symplectic group are given and discussed, again in the context of finding representations of group extensions. As a by-product, the structure of anisotropic tori in symplectic groups is given.

I. In the title above a pun is intended, for this paper is concerned with two aspects of the celebrated Weil representation—first with its character in the sense of character theory in group representations, then with its character in the more everyday sense of its nature. I think also that the character (in the technical sense) of the Weil representation says something about the character in the general sense. In any case, both facts presented here seem to me rather striking. We proceed to describe them.

Let F be a field of characteristic not 2, and let $\mathcal{H}_n(F)$ be the group of $(n+2)\times(n+2)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & 1 & & 0 & y_1 \\ & & \ddots & & \vdots \\ & 0 & & 1 & y_n \\ & & & & 1 \end{pmatrix}$$

with entries in F. $\mathcal{H}_n$ is a two-step nilpotent, unipotent algebraic group over F. It is called the nth order Heisenberg group of F. $\mathcal{Z}$, the center of $\mathcal{H}_n$, is one dimensional. In the above realization, $\mathcal{Z}$ consists of those elements for which $x_1 = x_2 = \cdots = x_n = 0 = y_1 = y_2 = \cdots = y_n$. $\mathcal{H}_n/\mathcal{Z}$ is isomorphic to F^{2n}. If $\mathcal{Z}$ is identified to F, then the commutator operation induces a symplectic form $\langle\ ,\ \rangle$ on

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$\mathcal{H}_n/\mathcal{Z}$. Using this form, $\mathcal{H}_n$ may also be realized as $F^{2n} \times F$, with group law $(v_1, x_1)\cdot(v_2, x_2) = (v_1 + v_2, x_1 + x_2 + \frac{1}{2}\langle v_1, v_2\rangle)$. From this, we see that $Sp_{2n}(F)$, the group of linear transformations of F^{2n} preserving $\langle\ ,\ \rangle$, acts on $\mathcal{H}_n(F)$ as a group of automorphisms, preserving F^{2n} (in the decomposition $F^{2n} \times F$), and leaving $\mathcal{Z}$ pointwise fixed.

Now suppose F is locally compact. It is well known (Stone-Von Neumann Theorem when $F = \mathbf{R}$ or $\mathbf{C}$, and easier to show in all other cases) that any irreducible unitary representation of $\mathcal{H}_n(F)$ which is not one dimensional is determined by the character χ it determines on $\mathcal{Z}$. Call it U_χ. Since the action of Sp_{2n} is trivial on $\mathcal{Z}$, the dual action on $\hat{\mathcal{H}}_n$, the space of irreducible representations, fixes U_χ. Thus one obtains, in the usual way [2], a projective representation of Sp_{2n} on the space of U_χ. In fact, this projective representation is often an actual representation (it always is if F is finite), and, in general, it is "almost" an actual representation (the "Mackey obstruction" is of order 2). These facts were first exposed and exploited by Weil [7], and the projective representations thus found, and ones derived from them, are known collectively as "the Weil representation".

The Weil representation has proved most useful for the concrete realization of representations for various symplectic groups, or subgroups thereof, perhaps most notably for Sl_2 (also Gl_2; see [3], and [6]).

On the other hand, the Weil representation seems not to have a context beyond itself, and has not been made to fit into a framework encompassing (for example) all semisimple groups. Thus we have the tantalizing prospect of generalizing it in some way, to remedy this gap. Such a generalization would probably have implications not only for group representations, but also for number theory, particularly the theory of θ-functions, which was Weil's original motivation.

One scheme for the generalization of the Weil representation might go as follows. Given a (say) semisimple algebraic group G over F, find a unipotent algebraic group N on which G acts by automorphisms, and find a representation U of N, fixed by the action of G on $\hat{N}$, the space of irreducible representations of N, and look to see if the corresponding projective representation of G is interesting. It is the burden of our first result that such an approach always essentially reduces to the situation of the usual Weil representation. That this is so is by no means an accident, and in some sense is one of the underpinnings of Kirillov theory [1], [4], particularly in the solvable case, or whenever other than strictly nilpotent groups are involved. Thus, there is offered here some metaphysical evidence to supplement C. Moore's cryptic comment [5] that the Weil representation is in some way special.

There are several possible formulations of the first result, depending on the context of its application. We give two. One is in terms of Lie algebras, and will be relevant to groups which have a good Kirillov theory, and the context of algebraic groups. The second is directly in terms of representations, and goes beyond the context of algebraic groups. We state it only for finite groups in order to avoid technical complications, but it is true very generally.

Let L be a Lie algebra over F. Take $v \in L^*$ (the dual space). Then $v([l_1, l_2])$ (where $[\,,]$ is the bracket operation on L) is an antisymmetric bilinear form on L. The set of l such that $v([l, l']) = 0$ for all $l' \in L$ (the radical of the form) is a subalgebra of L, which we will denote by R. A subalgebra $L_1 \subseteq L$ is subordinate to v, if $v([L_1, L_1]) = 0$. If $P \subseteq L$ is a subalgebra subordinate to v, and if $\dim P = \frac{1}{2}(\dim L + \dim R)$, then we will say P polarizes v, or is maximal subordinate to v. If P polarizes v, then necessarily $R \subseteq P$.

Proposition 1A. *Let L be a nilpotent Lie algebra over F. Let G be the group of automorphisms of L which fix some $v \in L^*$. Then there are subalgebras $L_1 \subseteq L_2 \subseteq P \subseteq L_3$, such that L_1, L_2 and L_3 are invariant by G; P polarizes v; $L_1 = L_2 \cap \ker v$; $\dim L_2 = \dim L_1 + 1$; L_3/L_1 is isomorphic to the Lie algebra of $\mathcal{H}_n$ for some n; L_2/L_1 is the center of L_3/L_1; and P/L_1 is maximal abelian in L_3/L_1.*

For a finite group N, let $\hat{N}$ denote the set of irreducible representations of N. Since a representation is determined by its character, which is a class function, the automorphism group of N, through its action on conjugacy classes of N, acts on $\hat{N}$.

Proposition 1B. *Let N be a finite p-group. Let G be a group of automorphisms of N which leave fixed a given $U \in \hat{N}$. Then there are groups $N_1 \subseteq N_2 \subseteq P \subseteq N_3$ in N such that N_1, N_2, and N_3 are invariant by a subgroup G' of the semidirect product $G \times_s N$, such that $\pi(G') = G$, where $\pi\colon G \times_s N \to G$ is the natural projection; there is a linear character ϕ of P such that ϕ induces U; ϕ is nontrivial on N_2 and $N_1 = \ker \phi \cap N_2$; N_1, N_2 are normal in N_3; N_3/N_1 is a two-step nilpotent p-group; N_2/N_1 is the center and contains the commutator subgroup of N_3/N_1; and P/N_1 is a maximal abelian subgroup of N_3/N_1.*

In order not to interrupt the present discussion, we will prove Propositions 1A and 1B in II.

Proposition 1B shows that in investigating representations of groups which have normal p-subgroups (which would happen, for instance, if one were to compare representations of a group with representations of the normalizer of a Sylow p-subgroup, and which also happens in analyzing compact p-adic groups), the Weil representation and mild variants of it will play a pivotal role. Hence the character of the Weil representation is important to know in this context.

Now let F be a finite field. We will regard the Weil representation W as a representation of the semidirect product $G_n = Sp_{2n}(F) \times_s \mathcal{H}_n(F)$, gotten by extending U_χ, for a given character χ of $\mathcal{Z}$, from $\mathcal{H}_n$ to G. (This extension is unique, since Sp_{2n} has no linear characters.) G_n contains as a subgroup the direct product $Sp_{2n}(F) \times_s \mathcal{Z}$. The restriction of W to this group will also be denoted W. It is of course just the (outer) tensor product of a representation of $Sp_{2n}(F)$ with the character χ of $\mathcal{Z}$.

We denote the character of W by χ_W.

Proposition 2. (i) *χ_W vanishes except on conjugacy classes intersecting $Sp_{2n}(F) \times \mathcal{Z}$. Hence it is determined by χ and by its restriction to Sp_{2n}.*

(ii) *If q is the cardinality of F, and $g \in Sp_{2n}$, then $\chi_W(g)\overline{\chi_W(g)} = |\chi_W(g)|^2 = q^{r(g)}$, where $r(g) = \dim \ker (g - 1)$. (g is regarded as a linear map of F^{2n}.) Hence $|\chi_W(g)|$ is a power of $q^{1/2}$. (Here $|\ |$ denotes absolute value and $\overline{}$ denotes complex conjugate.)*

(iii) *Suppose $F^{2n} = V_1 \oplus V_2$ (direct sum), and g_i acts trivially on V_i, $i = 1, 2$. Then $\chi_W(g_1 g_2) = q^{-2n}\chi_W(g_1)\chi_W(g_2)$.*

(iv) *If $g \in Sp_{2n}$ is semisimple (in purely group theoretical terms, if g is p-regular), then $\chi_W(g)$ is a rational integer.*

We will prove this in II. Now we discuss its significance. The main fact is (i). From it, (ii) follows, as we will explain. Parts (iii) and (iv) are complements.

Suppose G is a finite group, and $c_1, c_2, \cdots, c_l$ are the conjugacy classes of G. Let c_i have $^{\#}(c_i)$ elements. Let $\chi_1, \chi_2, \cdots, \chi_l$ be the irreducible characters of G. Then a standard way to write the Schur orthogonality relations is

$$(\chi_i, \chi_j) = {}^{\#}(G)^{-1} \sum_{k=1}^{l} {}^{\#}(c_k)\chi_i(c_k)\overline{\chi_j(c_k)} = \delta_{ij},$$

where δ_{ij} is Kronecker's δ.

Now suppose $G = H \times_s N$ is a semidirect product of a subgroup H and a normal subgroup N. Let U be a representation of N which is fixed by the action of H on $\hat{N}$. Suppose furthermore that U may be extended to a representation U' of G. Then it is well known that there is a bijection between those representations of G which yield a multiple of U when restricted to N, and representations of H. If $V \in \hat{H}$, define $V'' \in \hat{G}$ by lifting through $\pi: G \longrightarrow H \simeq G/N$. Then the desired bijection is the correspondence $V \longrightarrow V'' \otimes U'$. On the level of characters, this sends a character χ_V of V to the character $\chi_{V''} \cdot \chi_{U'}$. Since this sends irreducible characters to irreducible characters, it may be regarded as an isometric injection of the class functions of H (with their usual inner product, as in the Schur orthogonality relations) into the class functions on G (with their inner product).

Let $\alpha_1, \cdots, \alpha_m$ be the conjugacy classes of H, and let $\{\beta_{ij}: 1 \leq i \leq m, 1 \leq j \leq n_i\}$ be the classes of G, such that, in the natural isomorphism $H \simeq G/N$, β_{ij} is sent to α_i. Then the fact that $\chi_V \longrightarrow \chi_{V''} \cdot \chi_{U'}$ is an isometry is expressed by the equation

$$\#(H)^{-1}\left(\sum_{i=1}^{m} \#(\alpha_i)\chi_{V_1}(\alpha_i)\overline{\chi}_{V_2}(\alpha_i)\right)$$
$$= \#(G)^{-1}\left(\sum_{i=1}^{m} \chi_{V_1}(\alpha_i)\overline{\chi}_{V_2}(\alpha_i)\left(\sum_{j=1}^{n_i} \chi_{U'}(\beta_{ij})\overline{\chi}_{U'}(\beta_{ij})\,\#(\beta_{ij})\right)\right).$$

From this we see that

$$\#(H)^{-1}\#(\alpha_i) = \#(G)^{-1}\sum_{j=1}^{n_i} |\chi_{U'}(\beta_{ij})|^2\#(\beta_{ij})$$

or

$$\sum_{j=1}^{n_i} |\chi_{U'}(\beta_{ij})|^2\#(\beta_{ij}) = \#(G/H)\,\#(\alpha_i).$$

Now look at our particular $G_n = Sp_{2n}\times_s \mathcal{H}_n$, and $U' = W$. From (i) of Proposition 2, we have $\chi_W(\beta_{ij}) = 0$ unless β_{ij} intersects $Sp_{2n}\times \mathcal{Z}$. Moreover, if $z\in\mathcal{Z}$, then $z\beta_{ij} = \beta_{ik}$ is also a conjugacy class of G_n, whose projection onto Sp_{2n} is the same as β_{ij}. Moreover, $\chi_W(z\beta_{ij}) = \chi(z)\chi_W(\beta_{ij})$. Let α_i' be the unique conjugacy class of G_n which intersects Sp_{2n} and projects onto the class α_i of Sp_{2n}. Then we see that the above formula for general G reduces in our case to $\sum_{z\in\mathcal{Z}} |\chi_W(z\alpha_i')|^2\#(z\alpha_i') = \#(\mathcal{H}_n)\#(\alpha_i)$, or $q|\chi_W(\alpha_i')|^2\#(\alpha_i') = q^{2n+1}\#(\alpha_i)$, or $\#(\alpha_i')/\#(\alpha_i) = q^{2n}/|\chi_W(\alpha_i)|^2$. (Here we write interchangeably $\chi_W(\alpha_i)$ or $\chi_W(\alpha_i')$ depending on whether W is considered as a representation of G_n or Sp_{2n}.) Thus, χ_W, or more precisely its absolute value, simply expresses the difference in volume between conjugacy classes of Sp_{2n} and of the larger group G_n. One may draw an analogy with the Weyl character formula for semisimple groups. There the characters, as functions on the maximal torus, are simple expressions, which vary in a simple way with the representations, multiplied by the Weyl denominator, which essentially expresses the volume element of the group in terms of the volume element of the torus. In fact, there is evidence that this comparison is not too far-fetched.

II. Here we give the proofs of the propositions stated in I.

Both Propositions 1A and 1B are proved by induction. Start with 1A. Let $\mathcal{Z}(L)$ be the center of L. Since L is nilpotent $\mathcal{Z}(L)\neq 0$. Now $\dim\mathcal{Z}(L) - \dim(\ker v\cap\mathcal{Z}(L)) \leq 1$. If $\ker v\cap\mathcal{Z}(L)\neq 0$, then it is invariant by G. We may therefore divide out by it, and the proposition is reduced to a case of smaller degree, where we may assume it is true.

Thus we are reduced to the case when $\dim\mathcal{Z}(L) = 1$, and $\mathcal{Z}(L)\not\subseteq\ker v$. Let $\mathcal{Z}^{(2)}(L)$ be the ideal of elements of L which map into $\mathcal{Z}(L/\mathcal{Z}(L))$ under the natural projection. The action of G preserves $\mathcal{Z}^{(2)}(L)$, and $\mathcal{Z}(L)$, so the quotient action of G on $\mathcal{Z}^{(2)}(L)/\mathcal{Z}(L)$ is defined. Let S be the inverse image in L of an irreducible supspace for the G-action. The bracket operation defines on $S/\mathfrak{Z}(L)$ an antisymmetric bilinear form, invariant by G since G acts by automorphisms. Hence this form is either trivial or nondegenerate, since its radical must be G-invariant. Thus S is either abelian or a Heisenberg Lie algebra.

In any case, let $C(S)$ be the centralizer of S. Since $\mathfrak{Z}(L) \subseteq S \subseteq \mathfrak{Z}^{(2)}(S)$, we have $\dim C(S) + \dim(S) = \dim L + 1$. In particular, $\dim C(S) < \dim L$, and $C(S)$ is G-invariant, so we may assume the proposition holds for $C(S)$. Let L_1', L_2', P', L_3' be the appropriate subalgebras of $C(S)$.

If S is abelian, then $S \subseteq \mathfrak{Z}(C(S))$. Thus $S \subseteq R'$, the radical of $\nu([\,,\,])$ on $C(S)$. On the other hand, it is clear by the construction of S that $S \cap R = \mathfrak{Z}(L)$; R being the radical of $\nu([\,,\,])$ on L. Since obviously $\nu([\,,\,])$ defines a nondegenerate pairing between $S/\mathfrak{Z}(L)$ and $L/C(S)$, we see $R \subseteq R'$. Hence $\dim R' \geq \dim R + \dim S - 1$. But we have also $2 \dim P' = \dim R' + \dim C(S)$. Hence $2 \dim P' \geq \dim R + \dim L$. Since P' is still subordinate to ν, whether considered as a subalgebra of L or $C(S)$, we must also have $2 \dim P' \leq \dim R + \dim L$. Therefore, we see P' polarizes ν on L, and L_1', L_2', P' and L_3' satisfy the conditions of the proposition for L as well as for $C(S)$.

The other possibility is that S is Heisenberg. Then $C(S) \cap S = \mathfrak{Z}(L)$. Let $T \subseteq S$ be a maximal abelian subalgebra, so $2 \dim T = \dim S + 1$. If L_1', L_2', P', L_3' satisfy the proposition for $C(S)$, put $L_1 = L_1'$, $L_2 = L_2'$, $P = P' + T$, and $L_3 = L_3' + S$. Then L_1, L_2 and L_3 are certainly G-invariant, and since P' and T commute, and T is abelian, P is certainly subordinate to ν. We check

$$2 \dim P = 2(\dim P' + \dim T - 1) = \dim C(S) + \dim R' + \dim S - 1 = \dim R' + \dim L.$$

But since S is Heisenberg and ν is nonzero on $\mathfrak{Z}(L) = \mathfrak{Z}(S)$, $\nu([\,,\,])$ is nondegenerate on $S/\mathfrak{Z}(L)$. Since $C(S)$ commutes with S, and is the annihilator of S with respect to $\nu([\,,\,])$, we see $R' = R$. Therefore P polarizes ν. We have checked the main properties of L_1, L_2, P, L_3; the others are easily verified. This concludes the proof of Proposition 1A.

We proceed to Proposition 1B. If the group G of automorphisms leaves $U \in \hat{N}$ invariant, then it leaves $\ker U \subseteq N$ invariant. If $\ker U$ is nontrivial, we may divide out by it and reduce the problem to an N of lower cardinality. Therefore, we might as well assume U is faithful on N. Let $\mathfrak{Z}(N)$ be the center of N. If U is faithful, the character ψ which U defines on $\mathfrak{Z}(N)$ must be faithful. Hence $\mathfrak{Z}(N)$ is cyclic. Let $\mathfrak{Z}^{(2)}(N)$ be the inverse image in N of $\mathfrak{Z}(N/\mathfrak{Z}(N))$ by the natural projection. Let H be a subgroup of $\mathfrak{Z}^{(2)}(N)$ which contains $\mathfrak{Z}(N)$ and is invariant by G, and is minimal with respect to those two properties. We note then that all elements of $H/\mathfrak{Z}(N)$ have order p, and so $H/\mathfrak{Z}(N)$ may be regarded as a vector space over the prime field of characteristic p, and G then acts as linear transformations on $H/\mathfrak{Z}(N)$. As before, a commutator induces an antisymmetric bilinear form on $H/\mathfrak{Z}(N)$, and G preserves this form. By minimality of H, G acts irreducibly, so the form is either nondegenerate, or trivial and H is abelian. In any case, let $C(H)$ be the centralizer of H in N.

First suppose H is abelian, so $H \subseteq C(H)$. Then in fact $H \subseteq \mathfrak{Z}(C(H))$. We see that commutator induces a nondegenerate pairing between $H/\mathfrak{Z}(N)$ and $N/C(H)$ into $\mathfrak{Z}(N)$. This implies $^{\#}(N/C(H)) = {}^{\#}(H/\mathfrak{Z}(N))$. Clearly $C(H)$ is normal in N and is G-invariant.

Let V be an irreducible component of the restriction of U to $C(H)$. Since $H \subseteq \mathfrak{Z}(C(H))$, the restriction of V to H is a multiple of a linear character ψ' which extends ψ on $\mathfrak{Z}(N)$. Now since $C(H)$ is G-invariant and normal, it is normal in the semidirect product $G \times_s N$. Hence $G \times_s N$ acts on $\widehat{C(H)}$. Since the action of G on $\hat{N}$ fixes U, the $G \times_s N$ orbit of V in $\widehat{C(H)}$ must be contained in the set of representations of $C(H)$ which appear as irreducible components of the restriction of U to H. On the other hand, standard representation theory ("Clifford's theorem"; see [2]) shows N acts transitively on this set. Therefore, we see that the isotropy group $G' \subseteq G \times_s N$ of V projects onto G under the natural map $\pi\colon G \times_s N \longrightarrow G$.

We may now by induction assert the existence of subgroups $N_1' \subseteq N_2' \subseteq P' \subseteq N_3'$ of $C(H)$ which satisfy the theorem with respect to V. We see that necessarily $H \subseteq P$, and that if ϕ is the character of P which induces V, then ϕ must agree with ψ' on H. But now since the commutator pairing of $H/\mathfrak{Z}(N)$ and $N/C(H)$ is nondegenerate, and since ψ is faithful on $\mathfrak{Z}(N)$, we see that $N/C(H)$ acts faithfully and transitively on the characters of H which extend ψ. It follows that ϕ on P induces an irreducible representation of N; this representation obviously is U. Hence we see that N_1', N_2', P', and N_3' satisfy the proposition for N as well as for $C(H)$.

Therefore we are left with the case when H is nonabelian. Then we see that $C(H) \cap H = \mathfrak{Z}(N)$, and that the restriction of U to $C(H)$ is a sum of $^{\#}(H/\mathfrak{Z}(H))^{1/2}$ copies of a fixed representation V. Let $N_1' \subseteq N_2' \subseteq P' \subseteq N_3'$ be groups satisfying the proposition for $C(H)$ and V. Let J be a maximal abelian subgroup of H, so that $^{\#}(J/\mathfrak{Z}(N)) = {}^{\#}(H/J)$. Put $N_1 = N_1'$, $N_2 = N_2'$, $P = P' \cdot J$, $N_3 = N_3' \cdot H$. Then it may easily be checked that these groups satisfy the conditions of the proposition for N and U. Proposition 1B is now proved.

Remarks. (a) The similarity between the proofs of Propositions 1A and 1B of course forces itself upon one.

(b) In fact, the above induction shows that one may always choose N_3/N_2 to have exponent p—that is, so that all elements of N_3/N_2 have order p. Then one is indeed very close to Weil's representation.

(c) It also follows from the inductive step that if G is a p-group, then we can take $N_2 = P = N_3$.

Now we turn to the proof of Proposition 2. As we have seen (i) is the important part. We see $Sp_{2n}(F) \times_s F^{2n}$ is the quotient of $G_n = Sp_{2n}(F) \times_s \mathcal{H}_n(F)$ by $\mathcal{Z}$, and a conjugacy class in G_n intersects $Sp_{2n}(F) \times \mathcal{Z}$ if and only if its image in $Sp_{2n}(F) \times_s F^{2n}$ intersects Sp_{2n}. If $g_0 \in Sp_{2n}$, and $g_1 \cdot v \in Sp_{2n}(F) \times_s F^{2n}$, we have $(g_1 v)^{-1} g_0 (g_1 v) = g_2 \cdot (v - g_2(v))$ where $g_2 = g_1^{-1} g_0 g_1$, and $g_2(v) = g_2 v g_2^{-1}$. (We do not distinguish between g as an element of Sp_{2n} and as a linear transformation on F^{2n}.) Thus in general we see that gv is in a conjugacy class coming from Sp_{2n} if and only if $v = (g-1)(u)$ for some $u \in F^{2n}$. This shows immediately that the ratio of the cardinality of the conjugacy class $\alpha \subseteq Sp_{2n}$ containing g to the cardinality of the class $\alpha' \subset Sp_{2n} \times_s F^{2n}$ containing g is $q^{s(g)}$ where $s(g) = \dim \operatorname{im}(g-1)$. Putting this in the formula developed in the discussion in §I, and using $\dim \operatorname{im}(g-1) + \dim \ker (g-1) = 2n$, we see (ii) follows immediately from (i).

We see also that if $g - 1$ is nonsingular, then (i) is trivial for g. Hence we need only worry when g has 1 as an eigenvalue. Write $F^{2n} = V_1 \oplus V_2$ where V_1 is the 1-eigenspace of g (that is $g - 1$ is nilpotent on V_1), and $g - 1$ is nonsingular on V_2. Then on V_1, we have $(g-1)^{p^a} = g^{p^a} - 1 = 0$ for some a. Thus, if $v_1 \in V_1$, $v_2 \in V_2$, we have $v_2 = (g-1)^{p^a} u_2$ for some $u_2 \in V_2$, and so if $\langle\ ,\ \rangle$ is the symplectic form on F^{2n}, $\langle v_1, v_2 \rangle = \langle v_1, (g-1)^{p^a} u_2 \rangle = \langle v_1, (g^{p^a} - 1) u_2 \rangle = \langle v_1, g^{p^a} u_2 \rangle - \langle v_1, u_2 \rangle = \langle g^{-p^a} v_1, u_2 \rangle - \langle v_1, u_2 \rangle = \langle (g^{-p^a} - 1) v_1, u_2 \rangle = \langle 0, u_2 \rangle = 0$. Thus V_1 and V_2 are orthogonal with respect to $\langle\ ,\ \rangle$.

Let S_1 be the subgroup of Sp_{2n} which fixes V_1 and V_2, and acts trivially on V_2. Let S_2 fix V_1 and V_2 and act trivially on V_1. Let H_i be the inverse image of V_i in $\mathcal{H}_n$. Then $S_i \times_s H_i = Sp_{2m_i}(F) \times_s \mathcal{H}_{m_i}(F)$ for some m_i, such that $m_1 + m_2 = n$. Moreover, the product of the inclusions gives a map j: $(S_1 \times_s H_1) \times (S_2 \times_s H_2) \longrightarrow Sp_{2n} \times_s \mathcal{H}_n$. It is clear that the pullback by j of W is just the (outer) tensor product of the Weil representations of degrees m_1 and m_2. This, incidentally, establishes (iii) of Proposition 2. Moreover, we observe that if $v_i \in V_i$, then $v_1 + v_2 \in (g-1)(F^{2n})$ if and only if $v_1 \in (g-1)V_1$. Thus, to prove (i) we are reduced to the case when $g \in Sp_{2n}$ is unipotent.

So take g unipotent. Put $X = \ker (g-1) \subseteq F^{2n}$, and put $Y = (g-1)F^{2n}$. Then if $x \in X$, $y \in Y$, $\langle x, y \rangle = \langle x, (g-1)v \rangle = \langle (g^{-1} - 1)x, v \rangle = \langle 0, v \rangle = 0$. Here $v \in F^{2n}$. Since $\dim X + \dim Y = 2n$, Y is the orthogonal complement of X with respect to $\langle\ ,\ \rangle$. Therefore $\langle\ ,\ \rangle$ induces a nondegenerate pairing between X and F^{2n}/Y. In particular $X/X \cap Y$ must be paired with itself. Hence, if X_1 is a complement in X to $X \cap Y$, $\langle\ ,\ \rangle$ must be nondegenerate on X_1. Let X_2 be the orthogonal complement of X_1 in F^{2n}. Then $F^{2n} = X_1 \oplus X_2$. Let I_1, I_2 be the inverse images of X_1, X_2 in $\mathcal{H}_n$. Let $S_1 \leq Sp_{2n}$ fix I_1, and fix I_2 pointwise, and let S_2 do the same, with I_1, I_2 reversed. Then again we have the product of the inclusions j: $(S_1 \times_s I_1) \times (S_2 \times_s I_2) \longrightarrow G_n$, and clearly $g \in S_2$. By the same principal as before we are reduced to two cases: $g = 1$, or $\ker (g-1) \subseteq \operatorname{im}(g-1)$. The first case is all right, since it is well known and easy to show that the character of U_χ on $\mathcal{H}_n$ vanishes off $\mathcal{Z}$. (More generally, the character of a faithful representation of a finite p-group N always vanishes on $\mathcal{Z}^{(2)}(N) - \mathcal{Z}(N)$ and off the centralizer of $\mathcal{Z}^{(2)}(N)$.) Thus we are finally left with g unipotent and $\ker(g-1) \subseteq \operatorname{im}(g-1)$. Let K be the cyclic group generated by g. Then the restriction to $K \times_s \mathcal{H}_n$ of the Weil representation is realizable as a representation induced from a linear character on $K \times_s J$, where $J \subseteq \mathcal{H}_n$ is a maximal abelian subgroup normalized by K. (This is essentially the reason that the Weil representation is an actual representation for a finite field.) Let Z be the image of J in $F^{2n} = \mathcal{H}_n/\mathcal{Z}$. The fact that K normalizes J translates to the statement that Z is g-invariant. On the other hand, any g-invariant n-dimensional subspace of F^{2n} which is isotropic with respect to $\langle\ ,\ \rangle$ may be Z. Since $\ker(g-1)$ is isotropic, we may take $\ker(g-1) \subseteq Z$. Then, since J is abelian, $Z \subseteq Y = (g-1)F^{2n}$. Now it follows that if $h \in \mathcal{H}_n$, and the image of h mod $\mathcal{Z}$ is $v \in F^{2n}$, and $v \notin Y$, then $g \cdot h$ cannot be conjugate in $K \times_s \mathcal{H}_n$ to any element of $K \times_s J$. Therefore χ_W, the character of the Weil representation, must vanish on $g \cdot h$, by the formula for induced characters. Thus part (i) of Proposition 2 is proved.

To finish completely, we only have to show (iv) holds. We have already essentially shown that $|\chi_W(g)|$ is an integer if g is semisimple. Because $|\chi_W(g)|$ is always a power of $q^{1/2}$, and if $F^{2n} = V_1 \oplus V_2$ where V_1 is the 1-eigenspace of g, then V_1 and V_2 are both even dimensional since $\langle\ ,\ \rangle$ is nondegenerate on each, and if g is semisimple, then $V_1 = \ker(g-1)$, so $|\chi_W(g)|$ is an even power of $q^{1/2}$. Thus we only need show $\chi_W(g)$ is real. This follows easily from the fact that $\overline{\chi_W(g)} = \chi_W(g^{-1})$, and g and g^{-1} are conjugate. That g and g^{-1} are conjugate for g semisimple in a symplectic group over an algebraically closed field has a simple proof based on the general theory of semisimple algebraic groups, and using the fact that Sp_{2n} has no outer automorphisms. This was pointed out to me by Professor A. Borel. Here we shall follow a more elementary route, which also gives some information on the structure of tori in Sp_{2n}, and will also indicate how to compute the signs of $\chi_W(g)$.

If $g \in Sp_{2n}$ is semisimple, we may write $F^{2n} = \bigoplus_{i=1}^m V_i$, where the V_i are irreducible subspaces for g. Either $\langle\ ,\ \rangle$ is nondegenerate on a given V_i, or it is completely trivial, since V_i is irreducible for g. If $\langle\ ,\ \rangle$ is trivial on V_i, then there must be some V_j such that $\langle\ ,\ \rangle$ is nondegenerate on $V_i \oplus V_j$. Since then g preserves the orthogonal complement of $V_i \oplus V_j$, we may assume a decomposition of the form $F^{2n} = (\bigoplus_{i=1}^l V_i) \oplus (\bigoplus_{j=1}^k Y_j)$ where $l + 2k = m$, $Y_j = V_{l+2j-1} \oplus V_{l+2j}$, and all summands V_i, $i < l$, and Y_j are mutually orthogonal. By (iii) it is enough to analyze the situation when there is only one summand V_1 or one summand Y_1.

First, take the case of $Y_1 = V_1 \oplus V_2$. Then $V_i \simeq F^n$, and $\langle\ ,\ \rangle$ defines a nondegenerate pairing between V_1 and V_2. If $\{x_i\}_{i=1}^n$ is a basis for V_1, and $\{z_i\}_{i=1}^n$ is the dual basis to $\{x_i\}$ with respect to $\langle\ ,\ \rangle$, then it is evident that the transformation r which takes x_i to z_i, z_i to $-x_i$ preserves $\langle\ ,\ \rangle$. It is also clear that g preserves $\langle\ ,\ \rangle$ if and only if the matrix with respect to $\{z_i\}$ of the restriction of g to V_2 is the inverse transpose of the matrix with respect to $\{x_i\}$ of the restriction of g to V_1. It follows that the subgroup Γ of Sp_{2n} leaving V_1 and V_2 invariant is isomorphic to $Gl_n(F)$. Also, by r, we see g is conjugate to its inverse transpose. But g and the transposed matrix have the same characteristic polynomial, and so are conjugate in Gl_n (since they are both irreducible). Thus g and g^{-1} are conjugate.

Moreover, if H_i is the inverse image in $\mathcal{H}_n$ of V_i, and ϕ is a linear character of $\Gamma \times_s H_1$, trivial on Γ, and agreeing with χ on $\mathcal{Z}$, then ϕ induced to $\Gamma \times_s \mathcal{H}_n$ defines an extension W_1 of U_χ on $\mathcal{H}_n$ to $\Gamma \times_s \mathcal{H}_n$. Moreover, it is easy to compute, by the usual formula for induced characters, that the character of W_1 is positive on semisimple elements. Thus W_1 agrees up to a real-valued character with the restriction of W to $\Gamma \times_s \mathcal{H}_n$. In particular, W_1 and W agree on all elements of Γ, the determinant of whose action on V_1 is a square in F. Also, the computation of the sign of $\chi_W(g)$ is reduced to the computation of the character of the Weil representation for Sl_2 (and even for Sl_2 over the prime field). With these remarks we turn to the case when g acts irreducibly on V_1, on which $\langle\ ,\ \rangle$ is nondegenerate. This is covered by the following lemma.

Lemma. *Let k be a field, not of characteristic 2. Let V be a vector space over k, and let $\langle\ ,\ \rangle$ be a symplectic form on V. Let G be an abelian group, acting irreducibly on V, and preserving $\langle\ ,\ \rangle$. Suppose that the action of G splits over a Galois extension k_1 of k. Then there is a separable extension k' of k, and a quadratic extension k'' of k', and an embedding of G into the norm units of k'' over k', and an isomorphism $t: k'' \to V$, which is G-equivariant (with G acting on k'' by left multiplication), such that the pullback of $\langle\ ,\ \rangle$ to k'' by t has the form $\langle t(x), t(y)\rangle = \mathrm{tr}\,(k''/k)(c(r(x)y - r(y)x))$, where r is the Galois automorphism of k'' over k', and $\mathrm{tr}\,(k''/k)$ denotes the trace of k'' over k, and $c \in \ker \mathrm{tr}\,(k''/k')$ is chosen appropriately.*

Remark. Notice that for $\langle\ ,\ \rangle$ of the above form, the entire group of norm units of k'' over k' fixes $\langle\ ,\ \rangle$. Thus the lemma describes the anisotropic tori of Sp_{2n}. Moreover, if k is a finite field, then -1 is a norm from k'' to k'. If -1 is the norm of a, then $x \to ar(x)$ preserves $\langle\ ,\ \rangle$, and if b is a norm unit of k'' over k', then, $(ar)^{-1}b(ar) = r(b) = b^{-1}$, so b and b^{-1} are conjugate. Since it follows from our general formula that if g acts irreducibly on F^{2n}, $|\chi_W(g)| = 1$, we see $\chi_W(g) = \pm 1$. Since the unique field of dimension $2n$ over F has a norm unit group (over the field of dimension n) of degree $q^n + 1$, and the degree of W is q^n, it follows that all characters but one of the norm unit group must appear in W, and that one character must be real-valued, so there are only two choices for it. In particular, if g is a square in the norm unit group, then $\chi_W(g) = -1$, and the entire computation is reduced to determining the missing character.

Proof. Since G is abelian and irreducible, the subgroup of $\mathrm{Hom}(V)$ generated by G will be a field by Schur's lemma. This field will be k''. G is naturally included in k''. For any $v \in V$, the map $t: x \longrightarrow x(v)$ for $x \in k''$ defines a G-equivariant isomorphism of k'' onto V.

Since the action of G splits over the Galois extension k_1, k'' must be separable over k, and the split action must be completely reducible. Therefore, putting $V_1 = V \otimes_k k_1$, we may write $V_1 = \bigoplus_i L_i$ as a direct sum of one dimensional k_1-subspaces invariant by G, and we may find homomorphisms $\phi: G \longrightarrow k_1^\times$, such that, if $v \in L_i$, then $g(v) = \phi_i(g)v$. Since G acts irreducibly on V, the ϕ_i must all be distinct, and the ϕ_i and the L_i must be permuted transitively by $\mathrm{Gal}(k_1/k)$, the Galois group of k_1 over k, acting on V_1.

Since G preserves a symplectic form, V, and hence V_1, must be self-contragredient as a G-module. Thus for each i, there must be j such that $\phi_j = \phi_i^{-1}$. Hence ϕ_i and ϕ_i^{-1} are conjugate by $\mathrm{Gal}(k_1/k)$. Thus, there is $\tau \in \mathrm{Gal}(k_1/k)$ such that $\tau(\phi_i(g)) = \phi_i(g)^{-1}$. Identifying the field extension generated by $\phi_i(G)$ with k'', as we may, it follows that τ is a Galois automorphism of k'' over k, of order 2. Let k' be the fixed field of τ. Then k'' is a quadratic extension of k', and the relation $\tau(\phi_i(g)) = \phi_i(g)^{-1}$ shows $\phi_i(G)$ is a subgroup of the norm-units of k'' over k'.

Now, in k'', $\ker \mathrm{tr}(k''/k')$ is a complement to k', is a k'-subspace, and consists of elements x satisfying an equation $x^2 = d$, d being a nonsquare in k'. Consider the form $\gamma(x, y) = \mathrm{tr}(k''/k)(c(\tau(x)y - \tau(y)x))$. Clearly γ is antisymmetric, and a brief calculation shows it is invariant under multiplication by the norm units of k'' over k'. In particular, it is invariant by G. Writing $\tau(x)y - \tau(y)x = \tau(x\tau(y)) - x\tau(y)$, we see that, if $c \in k'$, then $c(\tau(x)y - \tau(y)x) \in \ker(\mathrm{tr}(k''/k')) \subseteq \ker(\mathrm{tr}(k''/k))$. However, if $c \in \ker \mathrm{tr}(k''/k')$, then for fixed $x \neq 0$, $y \longrightarrow c(\tau(x)y - \tau(y)x)$ maps k'' onto k', so $\mathrm{tr}(k''/k)(c)\tau(x)y - \tau(y)x))$ is in this case nondegenerate, k' being separable over k, and k not having characteristic 2.

Thus γ and $\langle\ ,\ \rangle$ are two symplectic forms on k'', invariant by G. Since G is irreducible, we must have $\langle\ ,\ \rangle = \gamma(zx, y)$ for some $z \in k''$. Since $\langle\ ,\ \rangle$ is antisymmetric, we need $\gamma(zx, y) = \gamma(x, zy)$ also. We compute $\gamma(zx, v) =$ $\mathrm{tr}(k''/k)(c(\tau(zx)y - \tau(y)zx)) = \mathrm{tr}(k''/k)(c\tau(z)(\tau(x)y - \tau(y)x)) + \mathrm{tr}(k''/k)(c(\tau(z) - z)\tau(y)x)$. Similarly $\gamma(x, zy) = \mathrm{tr}(k''/k)(c\tau(z)(\tau(x)y - \tau(y)x)) - \mathrm{tr}(k''/k)(c(\tau(z) - z)\tau(x)y)$. Thus $\mathrm{tr}(k''/k)(c(\tau(z) - z)(\tau(x)y + \tau(y)x)) = 0$ for $\langle\ ,\ \rangle$ to be antisymmetric. Since $\tau(x)y + \tau(y)x$ is arbitrary in k', and $c(\tau(z) - z) \in k'$, we must have $\tau(z) - z = 0$, or

$z \in k'$. This shows $\langle\ ,\ \rangle = \gamma$, for suitable c, and finishes the lemma and concludes the paper.

REFERENCES

1. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups* (preprint).

2. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962. MR 26 #6267.

3. H. Jacquet and R. P. Langlands, *Automorphic forms on* Gl_2, Lecture Notes in Math., no. 114, Springer-Verlag, Berlin and New York, 1970.

4. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk 17 (1962), no. 4 (106), 57–110 = Russian Math. Surveys 17 (1962), no. 4, 53–104. MR 25 #5396.

5. C. C. Moore, *Group extensions of p-adic and adelic linear groups*, Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 157–222. MR 39 #5575.

6. J. Shalika, *Representation theory of the* 2×2 *unimodular group over local fields*, Notes, Institute for Advanced Study, Princeton, N. J., 1966.

7. A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143–211. MR 29 #2324.

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