ON THE CHARACTERISTIC OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK AND OF THEIR INTEGRALS

BY

W. K. HAYMAN

Imperial College, London

1. Introduction

Suppose that $F(z)$ is meromorphic in $|z|<1$ and satisfies $F(0)=0$ there, and that $f(z)=F'(z)$. We define as usual

$$m(r, F)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|F(re^{i\theta})|\,d\theta,$$

$n(r, F)$ as the number of poles in $|z|\leqslant r$ and

$$N(r, F)=\int_0^r\frac{n(t, F)\,dt}{t}.$$

Then

$$T(r, F)=m(r, F)+N(r, F)$$

is called the Nevanlinna characteristic function of $F(z)$. The function $T(r, F)$ is a convex increasing function of $\log r$, so that

$$T(1, F)=\lim_{r\to 1}T(r, F)$$

always exists as a finite or infinite limit. If $T(1, F)$ is finite we say that $F(z)$ has bounded characteristic in $|z|<1$.

Examples show that $F(z)$ may have bounded characteristic in $|z|<1$, even if $f(z)$ does not.(1) We may take for instance $f(z)$ to be a regular function

$$f(z)=\sum_{n=1}^{\infty}\lambda_n^{\alpha}z^{\lambda_n-1},$$

(1) The first such example is due to Frostman [3].

where $0<\alpha<1$, $\lambda_n=K^n$ and K is a large positive integer depending on α. Then for $|z|\leqslant 1$

$$|F(z)|\leqslant \sum \lambda_n^{\alpha-1}=\sum_{n=1}^{\infty} K^{n(\alpha-1)}=\frac{1}{K^{1-\alpha}-1}<1,$$

so that $|F(z)|<1$ in $|z|<1$, $T(1,F)=0$. For $|z|=e^{-1/\lambda_n}$, we have $\lambda_n^{\alpha}|z|^{\lambda_n}=K^{n\alpha}/e$, while

$$\frac{1}{\lambda_n^{\alpha}}\sum_{m=n+1}^{\infty}\lambda_m^{\alpha}|z|^{\lambda_m}\leqslant\sum K^{\alpha(m-n)}e^{-K^{m-n}}<\sum\frac{K^{\alpha(m-n)}}{K^{m-n}}=\sum_{t=1}^{\infty}K^{(1-\alpha)t}<\frac{1}{K^{1-\alpha}-1},$$

and
$$\frac{1}{\lambda_n^{\alpha}}\sum_{m=1}^{n-1}\lambda_m^{\alpha}|z|^{\lambda_m}\leqslant\frac{1}{K^{n\alpha}}\sum_{m=1}^{n-1}K^{m\alpha}<\frac{1}{K^{\alpha}-1}.$$

Thus if K is so large that $K^{\alpha}>10$, $K^{1-\alpha}>10$, then

$$\sum_{m\neq n}\lambda_m^{\alpha}|z|^{\lambda_m}<\frac{2}{9}\lambda_n^{\alpha}|z|^{\lambda_n},$$

$$|z|\,|f(z)|>\left(\frac{1}{e}-\frac{2}{9}\right)\lambda_n^{\alpha}>\frac{1}{9}K^{n\alpha}.$$

Thus for $r=e^{-1/\lambda_n}$, and so for $e^{-1/\lambda_n}<r<e^{-1/\lambda_{n+1}}$ we have

$$T(r,f)>\log\left(\frac{K^{n\alpha}}{9}\right)=\alpha\log\lambda_n+O(1)=\alpha\log\frac{1}{1-r}+O(1).$$

$$\varliminf_{r\to\infty}\frac{T(r,f)}{\log\dfrac{1}{1-r}}\geqslant\alpha. \tag{1.1}$$

On the other hand, if $F(z)$ is bounded

$$f(z)=\frac{O(1)}{1-r},\quad\text{and}\quad \log|f(z)|\leqslant\log\frac{1}{1-r}+O(1).$$

Thus
$$\varlimsup_{r\to 1}\frac{T(r,f)}{\log\dfrac{1}{1-r}}\leqslant 1. \tag{1.2}$$

This result remains true for functions for bounded characteristic.

For the sharpest results on the bounds for $T(r,f)$ if $T(1,F)$ is finite see Kennedy [8], where more refined examples of the above type are constructed and sharper positive theorems are proved. Since the minimum modulus of $f(z)$ is unbounded in the above examples while the maximum modulus of $F(z)$ remains bounded, all means of $F(z)$ and no means of $f(z)$ on $|z|=r$ remain bounded as $r\to 1$.

It is natural to ask whether conversely $f(z)$ can have bounded characteristic, while $F(z)$ has bounded characteristic. This problem was raised during a recent Conference at Cornell University.(1)

At first sight the evidence appears to be in the opposite direction. Let us write

$$f_1(re^{i\theta}) = \sup_{0 \leqslant t \leqslant r} |f(te^{i\theta})|. \tag{1.3}$$

Then clearly
$$|F(z)| \leqslant r f_1(z), \quad |z| = r < 1, \tag{1.4}$$

If we write
$$I_\lambda(r, f) = \left\{\frac{1}{2\pi}\int |f(re^{i\theta})|^\lambda \, d\theta\right\}, \quad 0 < \lambda < \infty,$$

$$I_\infty(r, f) = \lim_{\lambda \to +\infty} I_\lambda(r, f) = \sup_{|z|=r} |f(z)|,$$

then Hardy and Littlewood [4] proved that if $f(z)$, $F(z)$ are regular, then

$$I_\lambda(r, F) \leqslant I_\lambda(r, f_1) \leqslant A(\lambda)\, I_\lambda(r, f), \quad 0 < r < 1, \; 0 < \lambda < \infty,$$

where $A(\lambda)$ depends only on λ, and also the stronger inequality [5]

$$I_\mu(r, F) \leqslant A(\lambda)\, I_\lambda(r, f), \quad 0 < \lambda \leqslant 1,$$

where $\mu = \lambda/(1-\lambda)$, and in particular $\mu = +\infty$, if $\lambda = 1$.

If $f(z)$ is regular then $\log^+|f(z)|$ is subharmonic. Hence it follows from the Hardy–Littlewood maximum theorem [4], that for $\lambda > 1$

$$I_\lambda(r, \log^+ F(z)) \leqslant I_\lambda(r, \log^+ |f_1(z)|) \leqslant A(\lambda)\, I_\lambda(r, \log^+ f(z)), \quad 0 < r < 1.$$

The result we require, would follow at least for regular functions $F(z)$ if the above inequality were to remain true for $\lambda = 1$. In fact such an extension is not possible.

2. Statement of results

We shall prove the following theorems, using the notation introduced above.

THEOREM 1. *Suppose that $f(z)$ is meromorphic and of bounded characteristic in $|z| < R$, where $0 < R < \infty$, and that $f(0) \neq \infty$. Then we have for $0 < r < R$*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ f_1(re^{i\theta})\, d\theta \leqslant T(R, f) + \frac{1}{\pi}\log\frac{R+r}{R-r}\, m(R, f) + \psi\left(\frac{r}{R}\right) N(R, f)$$
$$\leqslant \left[1 + \psi\left(\frac{r}{R}\right)\right] T(R, f), \tag{2.1}$$

(1) [9, Problem 6]. See also [7, p. 349].

where
$$\psi(t)=\frac{(1-t)\log\left(1+\frac{2\pi\sqrt{t}}{1-t}\right)}{\pi\sqrt{t}\log\frac{1}{t}}. \tag{2.2}$$

The first inequality of (2.1) *is sharp if* $f(z)$ *is regular, so that*

$$T(R,f)=m(R,f), \qquad N(R,f)=0.$$

For meromorphic functions the inequality (2.1) is no longer sharp. However, we note that
$$\psi(t)\to 0 \text{ as } t\to 0 \tag{2.3}$$

and
$$\psi(t)=\frac{1}{\pi}\log\frac{1}{1-t}+O(1), \quad \text{as } t\to 1. \tag{2.4}$$

Thus the bound in (2.1) is asymptotic to the correct bound as $r\to 0$ and as $r\to R$. We shall also prove an analogue of Theorem 1 in which $-\log|f|$ is replaced by a subharmonic function (Theorem 4).

We deduce immediately

THEOREM 2. *Suppose that* $F(z)$ *is meromorphic in* $|z|<R$, *that* $F(0)=0$, *and that* $f(z)=F'(z)$ *has bounded characteristic in* $|z|<R$. *Then we have for* $0<r<R$

$$m(r,F)\leqslant T(R,f)+\frac{1}{\pi}\log\frac{R+r}{R-r}m(R,f)+\psi\left(\frac{r}{R}\right)N(R,f)+\log^+ r$$
$$\leqslant\left[1+\psi\left(\frac{r}{R}\right)\right]T(R,f)+\log^+ r. \tag{2.5}$$

Hence we have
$$T(r,F)\leqslant\left[2+\psi\left(\frac{r}{R}\right)\right]T(R,f)+\log^+ r, \quad 0<r<R \tag{2.6}$$

and if $F(z)$ *is regular the sharper inequality*
$$T(r,F)\leqslant\left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)T(R,f). \tag{2.7}$$

A result of the same general type as (2.6) but with $A/(1-t)\log[1/(1-t)]$ instead of $\psi(t)$ was proved by Chuang [2]. A version of (2.7) with the correct order of magnitude but a less precise form is due to Biernacki [1, Lemma 1, p. 103].

In the opposite direction we can show by examples that the orders of magnitude of the bounds of Theorem 2 are correct as $r \to R$. We have in fact

THEOREM 3. *Given $C>0$ there exists $f(z)$ regular and satisfying $|f(z)|>1$ in $|z|<1$ and*

$$T(1, f) = \log |f(0)| = C, \tag{2.8}$$

while at the same time

$$T(r, F) > .12\, C \log \frac{1}{1-r}, \quad r_0 < r < 1. \tag{2.9}$$

Here $$F(z) = \int_0^z f(\zeta)\, d\zeta, \quad r_0 = 1 - [\min(\tfrac{1}{2}, C)]^A$$

and A is a positive absolute constant.

It is interesting to compare the results of Theorems 2 and 3 with the corresponding inequalities in the opposite direction. If $T(1, F)$ is finite, then (1.1) and (1.2) show that

$$\varlimsup_{r\to 1} \frac{T(r, f)}{\log \dfrac{1}{1-r}} \leqslant 1,$$

and that equality is possible even if $T(1, F) = 0$. Thus the restriction on the order of magnitude of $T(r, f)$ when $T(1, F)$ is finite is similar to that on $T(r, F)$ when $T(1, f)$ is finite. However, in the first case the constant multiplying $\log [1/(1-r)]$ is bounded by one, while in the second case it is bounded by a fixed multiple of $T(1, f)$ and so can be as large as we please.

We shall prove Theorems 1 to 3 in turn. We reserve for a late paper the applications of these results to integral functions and functions meromorphic in the plane.

3. Some preliminary results

In order to prove Theorem 1, we need some preliminary estimates. We suppose that $0 < r < R$, $0 < |\phi| < \pi$ and write

$$P(R, r, \phi) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\phi + r^2}, \tag{3.1}$$

$$p(R, r, \phi) = \sup_{0 \leqslant t \leqslant r} P(R, t, \phi). \tag{3.2}$$

We also suppose that $0<x<R$ and write

$$G(R, r, x, \phi)=\log\left|\frac{R^2-xre^{i\phi}}{R(re^{i\phi}-x)}\right|, \tag{3.3}$$

and

$$g(R, r, x, \phi)=\sup_{0\leq t\leq x} G(R, r, t, \phi). \tag{3.4}$$

LEMMA 1. *We have with the above notation*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p(R, r, \phi)\,d\phi = 1+\frac{1}{\pi}\log\frac{R+r}{R-r}, \tag{3.5}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(R, r, R, \phi)\,d\phi < \log\frac{R}{r}+\frac{R^2-r^2}{2\pi rR}\log\left(1+\frac{2\pi rR}{R^2-r^2}\right). \tag{3.6}$$

We note that P, p, G and g are homogeneous functions in R, r and x and so we may suppose without loss in generality that $R=1$.

3.1. We proceed to prove (3.5). We note that for $R=1$ and $\frac{1}{2}\pi\leq|\phi|<\pi$ $P(1, r, \phi)$ is a decreasing function of r. For $0<|\phi|<\frac{1}{2}\pi$ the function $P(1, r, \phi)$ increases from 0 to $|\operatorname{cosec}\phi|$ as r increases from 0 to $\cos\phi/(1+|\sin\phi|)$ and then decreases again. If ϕ_0 is the number in the range $0<\phi_0<\frac{1}{2}\pi$ given by

$$\frac{\cos\phi_0}{1+\sin\phi_0}=r, \quad \text{or} \quad \tan\frac{\phi_0}{2}=\frac{1-r}{1+r},$$

then

$$p(1, r, \phi)=\sup_{0\leq t\leq r} P(1, t, \phi)=\begin{cases} P(1, r, \phi), & 0<|\phi|<\phi_0 \\ |\operatorname{cosec}\phi|, & \phi_0<|\phi|<\frac{1}{2}\pi \\ 1, & \frac{1}{2}\pi\leq|\phi|\leq\pi. \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} p(1, r, \phi)\,d\phi=\pi+2\int_0^{\phi_0} P(1, r, \phi)\,d\phi+2\int_{\phi_0}^{\frac{1}{2}\pi}\operatorname{cosec}\phi\,d\phi.$$

On setting $t=\tan(\frac{1}{2}\phi)$, $t_0=\tan(\frac{1}{2}\phi_0)=(1-r)/(1+r)$, this becomes

$$\begin{aligned}\pi+4\int_0^{t_0}\frac{(1-r^2)\,dt}{(1-r)^2+(1+r)^2t^2}+2\int_{t_0}^{1}\frac{dt}{t} &= \pi+4\tan^{-1}\left(\frac{1+r}{1-r}t_0\right)+\log\cot\frac{\phi_0}{2} \\ &= 2\pi+2\log\left(\frac{1+r}{1-r}\right).\end{aligned}$$

This gives (3.5) when $R=1$ and so generally.

3.2. We proceed to prove (3.6). Suppose that $R=1$, $0<|\phi|<\frac{1}{2}\pi$ and set

$$G(1, r, x, \phi)=\log K.$$

We obtain

$$\frac{(1-r^2)(1-x^2)}{r^2+x^2-2rx\cos\phi}=K^2-1, \quad (K^2-r^2)x^2-2(K^2-1)rx\cos\phi+K^2r^2-1=0.$$

For fixed K, r and ϕ this is a quadratic in x, and the maximum value of K occurs when this quadratic has equal roots, i.e. when

$$(K^2-r^2)(K^2r^2-1)=r^2\cos^2\phi\,(K^2-1)^2.$$

This may be written as

$$(K^2+1)^2-K^2\left(\frac{1+r^4-2r^2\cos 2\phi}{r^2\sin^2\phi}\right)=0.$$

Since the maximum value of K is greater than one, we deduce that

$$K^2-K\sqrt{b}+1=0, \quad \text{where } b=\frac{1+r^4-2r^2\cos 2\phi}{r^2\sin^2\phi},$$

or

$$K=\tfrac{1}{2}[\sqrt{b}+\sqrt{b-4}]=c+\sqrt{c^2+1},$$

where

$$c=\tfrac{1}{2}\sqrt{b-4}=\frac{1-r^2}{2r|\sin\phi|}. \tag{3.7}$$

Again if $\frac{1}{2}\pi<|\phi|<\pi$, it is evident that K decreases with increasing x for $0<x<1$, so that K attains its maximum value when $x=0$. Thus

$$g(1, r, 1, \phi)=\begin{cases}\log[c+\sqrt{c^2+1}], & 0<|\phi|<\frac{1}{2}\pi. \\ \log\dfrac{1}{r}, & \frac{1}{2}\pi\leqslant|\phi|<\pi;\end{cases}$$

where c is given by (3.7).

Hence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}g(1, r, 1, \phi)\,d\phi&=\frac{1}{\pi}\int_0^{\pi}g(1, r, 1, \phi)\,d\phi\\ &=\frac{1}{2}\log\frac{1}{r}+\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log[c+\sqrt{1+c^2}]\,d\phi\\ &=\frac{1}{2}\log\frac{1}{r}+\left[\frac{\phi}{\pi}\log[c+\sqrt{1+c^2}]\right]_0^{\frac{1}{2}\pi}+\frac{1}{\pi}\int_{\phi=\frac{1}{2}\pi}^{\phi=0}\frac{\phi\,dc}{\sqrt{1+c^2}}.\end{aligned} \tag{3.8}$$

Now
$$c=\frac{1-r^2}{2r\sin\phi}=\frac{c_0}{\sin\phi},\quad \text{where } c_0=\frac{1-r^2}{2r}.$$

Hence
$$dc=\frac{-c_0\cos\phi\,d\phi}{\sin^2\phi}$$

and
$$\frac{1}{\pi}\int_{\phi=\frac{1}{2}\pi}^{0}\frac{\phi\,dc}{\sqrt{1+c^2}}=\frac{c_0}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\phi\cos\phi\,d\phi}{\sin\phi\sqrt{\sin^2\phi+c_0^2}}. \tag{3.9}$$

We now note that
$$\frac{\sin^2\phi}{\cos\phi}>\phi^2,\quad 0<\phi<\tfrac{1}{2}\pi.$$

In fact we have for $0<\phi<2$

$$\frac{\sin^2\phi}{\phi^2}-\cos\phi=\frac{1-\cos 2\phi}{2\phi^2}-\cos\phi$$
$$>\frac{1}{2\phi^2}\left\{\frac{(2\phi)^2}{2!}-\frac{(2\phi)^4}{4!}\right\}-\left(1-\frac{\phi^2}{2}+\frac{\phi^4}{4!}\right)$$
$$=\frac{\phi^2}{6}-\frac{\phi^4}{24}>0.$$

Thus we have for $0<\phi<\frac{1}{2}\pi$.

$$\frac{\sin\phi\sqrt{\sin^2\phi+c_0^2}}{\cos\phi}=\sqrt{\frac{\sin^4\phi}{\cos^2\phi}+c_0^2\tan^2\phi}>\sqrt{\phi^4+c_0^2\phi^2}.$$

Hence
$$\frac{c_0}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\phi\cos\phi\,d\phi}{\sin\phi\sqrt{\sin^2\phi+c_0^2}}<\frac{c_0}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{c_0^2+\phi^2}}$$
$$=\frac{c_0}{\pi}\left[\log\{\phi+\sqrt{c_0^2+\phi^2}\}\right]_0^{\frac{1}{2}\pi}=\frac{c_0}{\pi}\log\left[\frac{\pi}{2c_0}+\sqrt{1+\frac{\pi^2}{4c_0^2}}\right]$$
$$<\frac{c_0}{\pi}\log\left(1+\frac{\pi}{c_0}\right)=\frac{1-r^2}{2\pi r}\log\left(1+\frac{2\pi r}{1-r^2}\right).$$

Thus we obtain from (3.8) and (3.9)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}g(1,r,\phi)\,d\phi<\frac{1}{2}\log\frac{1}{r}+\frac{1}{2}\log[c_0+\sqrt{1+c_0^2}]+\frac{1-r^2}{2\pi r}\log\left(1+\frac{2\pi r}{1-r^2}\right)$$
$$=\log\frac{1}{r}+\frac{1-r^2}{2\pi r}\log\left(1+\frac{2\pi r}{1-r^2}\right)$$

as required. This completes the proof of (3.6) and so of Lemma 1.

4. Proof of Theorem 1, when $f(z)$ is regular

We can now prove

LEMMA 2. *Suppose that $f(z)$ is regular and of bounded characteristic for $|z|<R$. Then, if $f_1(z)$ is given by* (1.3)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+ f_1(re^{i\theta})\,d\theta \leqslant \left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)T(R,f), \quad 0<r<R \tag{4.1}$$

and this inequality is sharp.

To see that (4.1) is sharp, we set

$$f(z)=\exp\left\{c\,\frac{R+z}{R-z}\right\},$$

where c is a positive constant. Then since $\log|f(z)|$ is positive and harmonic

$$T(R,f)=m(R,f)=\lim_{\varrho\to R-} m(\varrho,f)=\log|f(0)|=c.$$

Also
$$\log^+ f_1(re^{i\phi})=cp(R,r,\phi),$$

and now Lemma 1, (3.5) shows that equality holds in (4.1).

To prove (4.1) in general we may suppose without loss in generality that $f(z)$ is regular in $|z|\leqslant R$, since the general case can be deduced from this by a limit argument. Now the Poisson–Jensen formula shows that for $0<r<R$, $0\leqslant\theta\leqslant 2\pi$

$$\log^+|f(re^{i\theta})|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\phi})|\,P(R,r,\theta-\phi)\,d\phi.$$

Hence
$$\log^+ f_1(re^{i\theta})\leqslant\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\phi})|\,p(R,r,\theta-\phi)\,d\phi.$$

Thus
$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log^+ f_1(re^{i\theta})\,d\theta &\leqslant \frac{1}{2\pi}\int_0^{2\pi}d\theta\,\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\phi})|\,p(R,r,\theta-\phi)\,d\phi\\ &=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\phi})|\,d\phi\left(\frac{1}{2\pi}\int_0^{2\pi}p(R,r,\theta-\phi)\,d\theta\right)\\ &=\left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)m(R,f),\end{aligned}$$

by (3.5). This completes the proof of Lemma 2.

5. Completion of proofs of Theorems 1 and 2

We suppose now that $f(z)$ is meromorphic in $|z| \leqslant R$, $f(0) \neq \infty$, and that $b_\nu = |b_\nu| e^{i\phi_\nu}$, $\nu = 1$ to N are the poles of $f(z)$ in $|z| \leqslant R$ with due count of multiplicity. Then the Poisson–Jensen formula yields for $z = re^{i\theta}$, using (3.1), (3.3),

$$\log^+ |f(z)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| P(R, r, \theta - \phi)\, d\phi + \sum_{\nu=1}^{N} G(R, |b_\nu|, r, \phi_\nu - \theta).$$

Thus in view of (3.2) and (3.4) we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ f_1(re^{i\theta})\, d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| p(R, r, \theta - \phi)\, d\phi + \sum_{\nu=1}^{N} \frac{1}{2\pi} \int_0^{2\pi} g(R, |b_\nu|, r, \phi - \theta)\, d\theta. \quad (5.1)$$

In view of Lemma 1, (3.5) we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ f(Re^{i\phi})\, d\phi \frac{1}{2\pi} \int_0^{2\pi} p(R, r, \theta - \phi)\, d\theta = m(R, f) \left(1 + \frac{1}{\pi} \log \frac{R+r}{R-r}\right). \quad (5.2)$$

We next set $R_1 = (rR)^{\frac{1}{2}}$ and suppose first that $R_1 < |b_\nu| < R$. Then $G(R, |b_\nu|, r, \phi - \theta)$ is a positive harmonic function of $z = re^{i\theta}$ for $0 \leqslant r \leqslant R_1$. Hence we may apply Lemma 2 with R_1 instead of R and $G(R, b_\nu, r, \phi - \theta)$ instead of $\log^+ |f(re^{i\theta})|$.

This yields for $R_1 < |b_\nu| < R$

$$\frac{1}{2\pi} \int_0^{2\pi} g(R, |b_\nu|, r, \phi - \theta)\, d\theta \leqslant \left(1 + \frac{1}{\pi} \log \frac{R_1 + r}{R_1 - r}\right) \frac{1}{2\pi} \int_0^{2\pi} G(R, |b_\nu|, r, \phi - \theta)\, d\theta$$
$$= \left(1 + \frac{1}{\pi} \log \frac{R_1 + r}{R_1 - r}\right) G(R, |b_\nu|, 0, 0) = \left(1 + \frac{1}{\pi} \log \frac{R_1 + r}{R_1 - r}\right) \log \frac{R}{|b_\nu|}. \quad (5.3)$$

Next if $0 < |b_\nu| \leqslant R_1$ we have from Lemma 1, (3.6)

$$\frac{1}{2\pi} \int_0^{2\pi} g(R, |b_\nu|, r, \phi - \theta)\, d\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} g(R, |b_\nu|, R, \phi - \theta)\, d\theta$$
$$\leqslant \log \frac{R}{|b_\nu|} + \frac{R^2 - |b_\nu|^2}{2\pi |b_\nu| R} \log \left(1 + \frac{2\pi |b_\nu| R}{R^2 - |b_\nu|^2}\right). \quad (5.4)$$

It remains to estimate the right-hand sides of (5.3) and (5.4). To do this we set

$$h(t)=\frac{1-t^2}{2\pi t\log 1/t}\log\left(1+\frac{2\pi t}{1-t^2}\right),\quad 0<t<1,$$

and note that $h(t)$ is an increasing function of t for $0<t<1$. In fact

$$H(t)=\frac{t(1-t^2)}{(1+t^2)}\frac{d}{dt}\log h(t)=\frac{(1-t^2)}{(1+t^2)\log 1/t}+\frac{1}{\left(1+\frac{1-t^2}{2\pi t}\right)\log\left(1+\frac{2\pi t}{1-t^2}\right)}-1.$$

We apply the elementary inequality

$$2\log x<x-x^{-1},\quad x>1$$

in turn with $x=t^{-\frac{1}{2}}$ and $x=1+2\pi t/(1-t^2)$, and set $t=y^2$. This gives

$$H(t)>\frac{y(1+y^2)}{1+y^4}+\frac{(1-y^4)}{1-y^4+\pi y^2}-1$$

$$=\frac{y(1-y)\left[(1+y^2)(1+y+y^2+y^3)-\pi y(1-y)(1+y+y^2)\right]}{(1+y^4)(1-y^4+\pi y^2)}$$

$$>\frac{y(1-y)(1+y+y^2)\left[1+y^2-\pi y(1-y)\right]}{(1+y^4)(1-y^4+\pi y^2)}>0,\quad 0<y<1.$$

Thus $h(t)$ is an increasing function of t for $0<t<1$.

We deduce that for $0<t<R_1/R=(r/R)^{\frac{1}{2}}$, we have

$$h(t)<h[(r/R)^{\frac{1}{2}}]=\psi\left(\frac{r}{R}\right).$$

Thus (5.4) gives for $|b_\nu|\leqslant R_1$

$$\frac{1}{2\pi}\int_0^{2\pi}g(R,|b_\nu|,r,\phi-\theta)\,d\theta\leqslant\log\frac{R}{|b_\nu|}\left[1+h\left(\frac{|b_\nu|}{R}\right)\right]\leqslant\log\frac{R}{|b_\nu|}\left[1+\psi\left(\frac{r}{R}\right)\right].\tag{5.5}$$

Next we note that

$$L(t)=\frac{1-t^2}{2t\log 1/t}$$

decreases with increasing t for $0<t<1$. To see this set

$$\frac{1}{t}=\frac{1+y}{1-y},\quad\text{so that}\quad\frac{1-t^2}{2t}=\frac{2y}{1-y^2}$$

and
$$\frac{1}{L(t)}=\frac{(1-y^2)}{2y}\log\left(\frac{1+y}{1-y}\right)=\frac{1-y^2}{y}(y+\tfrac{1}{3}y^3+\dots)$$
$$=1-(1-\tfrac{1}{3})y^2-(\tfrac{1}{3}-\tfrac{1}{5})y^4-\dots$$

which is clearly a decreasing function of y. In particular we see that
$$L(t)>1. \tag{5.6}$$

Thus
$$\frac{1}{\pi}\log\frac{1+t}{1-t}=\frac{1}{\pi}\log\left(1+\frac{2t(1+t)}{1-t^2}\right)<\frac{1}{\pi}\log\left(1+\frac{2\pi t}{1-t^2}\right)=\frac{h(t)}{L(t)}<h(t).$$

Thus (5.3) yields for $R_1<|b_\nu|<R$
$$\frac{1}{2\pi}\int_0^{2\pi}g(R,|b_\nu|,r,\phi-\theta)\,d\theta\leqslant\left[1+h\left(\frac{r}{R_1}\right)\right]\left(\log\frac{R}{|b_\nu|}\right)=\left[1+\psi\left(\frac{r}{R}\right)\right]\log\frac{R}{|b_\nu|}. \tag{5.7}$$

On combining (5.1), (5.2), (5.5) and (5.7) we obtain
$$\frac{1}{2\pi}\int_0^{2\pi}\log^+ f_1(re^{i\theta})\,d\theta\leqslant m(R,f)\left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)+\sum\log\frac{R}{|b_\nu|}\left(1+\psi\left(\frac{r}{R}\right)\right),$$

which is the first inequality of (2.1). The second inequality follows at once, since by (2.2), (5.6) we have, setting $t=r/R$,
$$\psi(t)>\frac{1}{\pi}\log\left(1+\frac{2\pi\sqrt{t}}{1-t}\right)>\frac{1}{\pi}\log\left(1+\frac{2t}{1-t}\right)=\frac{1}{\pi}\log\frac{R+r}{R-r}. \tag{5.8}$$

This completes the proof of Theorem 1.

In view of (1.4) we have
$$\log^+|F(re^{i\theta})|\leqslant\log^+ f_1(re^{i\theta})+\log^+ r,$$
so that
$$m(r,F)\leqslant\frac{1}{2\pi}\int_0^{2\pi}\log^+ f_1(re^{i\theta})\,d\theta+\log^+ r,$$

and now (2.5) and (2.7) follow from (2.1). Also since the poles of $F(z)$ occur at the same points as the poles of $f(z)$ and have smaller multiplicity, we deduce that
$$N(r,F)\leqslant N(r,f)\leqslant T(r,f)\leqslant T(R,f),$$
so that
$$T(r,F)=m(r,F)+N(r,F)\leqslant\left[2+\psi\left(\frac{r}{R}\right)\right]T(R,f)+\log^+ r$$

by (2.5). This proves (2.6) and completes the proof of Theorem 2.

6. An extension to subharmonic functions

Suppose that $u(z)$ is subharmonic in $|z| \leqslant R$. In view of Riesz' decomposition theorem [10] there exists a positive mass distribution $d\mu\, e(\zeta)$ in $|\zeta| < R$, such that for $0 < r \leqslant R$

$$n(r) = \int_{|\zeta| \leqslant r} d\mu\, e(\zeta)$$

is finite and

$$u(z) - \int_{|\zeta| \leqslant r} \log |z - \zeta|\, d\mu\, e(\zeta)$$

remains harmonic for $|z| < r$. We also have the Poisson–Jensen formula, [6, p. 473] which asserts that for $z = re^{i\theta}$, $0 < r < R$,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\phi})\, P(R, r, \phi - \theta)\, d\theta + \int_{|\zeta| < R} \log \left| \frac{R(z-\zeta)}{R^2 - \bar{z}\zeta} \right| d\mu\, e(\zeta). \tag{6.1}$$

We set

$$u^+(z) = \max(u(z), 0), \quad u^-(z) = -\min(u(z), 0).$$

$$T(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\theta})\, d\theta,$$

$$m(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u^-(re^{i\theta})\, d\theta,$$

$$N(r, u) = \int_0^r \frac{n(t)\, dt}{t} = \int_{|\zeta| < r} \log \frac{r}{|\zeta|}\, d\mu\, e(\zeta).$$

Then if we put $z = 0$ in (6.1) we obtain the analogue of Nevanlinna's first fundamental theorem, namely

$$T(R, u) = m(R, u) + N(R, u) + u(0). \tag{6.2}$$

Suppose now that $u(0)$ is finite, so that $N(r, u)$ is finite for $0 < r \leqslant R$, and set

$$u_1(re^{i\theta}) = \sup_{0 \leqslant t \leqslant r} u^-(te^{i\theta}).$$

We then prove the following analogue of Theorem 1.

THEOREM 4. *We have with the above notation*

$$\frac{1}{2\pi} \int_0^{2\pi} u_1(re^{i\theta})\, d\theta \leqslant \left(1 + \frac{1}{\pi} \log \frac{R+r}{R-r}\right) m(R, u) + \left[1 + \psi\left(\frac{r}{R}\right)\right] N(R, u)$$

$$\leqslant \left[1 + \psi\left(\frac{r}{R}\right)\right] (T(R, u) - u(0)),$$

where $\psi(t)$ is given by (2.2).

To put the above result in its setting we recall that $u(z) = -\infty$ is possible for a dense set of z in $|z| < R$, so that $u_1(re^{i\theta}) = +\infty$ may hold for a dense set of θ. To prove Theorem 4, we deduce from (6.1) and (3.1), (3.3) that

$$u^-(re^{i\theta}) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u^-(Re^{i\phi})\,P(R, r, \phi-\theta)\,d\theta + \int_{0<t<R}\int_{0\leqslant\phi<2\pi} G(R, t, r, \phi-\theta)\,d\mu\,e(te^{i\phi}).$$

Using (3.2), (3.4) we deduce at once that

$$u_1(re^{i\theta}) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u^-(Re^{i\phi})\,p(R, r, \phi-\theta)\,d\phi + \int_{0<t<R}\int_{0\leqslant\phi<2\pi} g(R, t, r, \phi-\theta)\,d\mu\,e(te^{i\phi}). \tag{6.3}$$

We now integrate both sides with respect to θ and invert the order of integration, which is justified since all integrands are positive. In view of (5.5) and (5.7) we have

$$\frac{1}{2\pi}\int_0^{2\pi} g(R, t, r, \phi-\theta)\,d\theta < \log\frac{R}{t}\left[1+\psi\left(\frac{r}{R}\right)\right], \quad 0<t<R,\ 0<r<R. \tag{6.4}$$

Thus we deduce from (6.3), using (3.5) and (6.4)

$$\frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})\,d\theta \leqslant \left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)\frac{1}{2\pi}\int_0^{2\pi} u^-(Re^{i\theta})\,d\theta + \left[1+\psi\left(\frac{r}{R}\right)\right]\int_{|\zeta|<R}\log\frac{R}{|\zeta|}\,d\mu\,e(\zeta).$$

This is the first inequality of Theorem 4. In view of (5.8) we deduce

$$\left(1+\frac{1}{\pi}\log\frac{R+r}{R-r}\right)m(r, u) + \left[1+\psi\left(\frac{r}{R}\right)\right]N(r, u) \leqslant \left[1+\psi\left(\frac{r}{R}\right)\right][m(r, u)+N(r, u)] = \left[1+\psi\left(\frac{r}{R}\right)\right][T(r, u)-u(0)],$$

by (6.2). This completes the proof of Theorem 4.

We note that if $u(z)$ is non-positive in $|z| \leqslant R$ so that $T(r, u) = 0$, we have

$$u(0) \geqslant \frac{1}{2\pi}\int_0^{2\pi} -u_1(re^{i\theta})\,d\theta \geqslant \left[1+\psi\left(\frac{r}{R}\right)\right]u(0), \tag{6.5}$$

where $\psi(t) \to 0$ as $t \to 0$. In fact the left-hand inequality of (6.5) is trivial, since

$$-u_1(re^{i\theta}) = \inf_{0 \leqslant t \leqslant r} u(te^{i\theta}) \leqslant u(0)$$

in this case. The right-hand inequality follows from Theorem 4. The inequality (6.5) shows that on most radial segments, going outward from the origin and having length r, $u(z)$ is not much smaller than $u(0)$, provided that r is small compared with R.

7. Outline of proof of Theorem 3

We proceed to construct the counter examples whose existence is asserted in Theorem 3. To do this we define(1) a function $\alpha(t)$ in the interval $[0, 1]$, to satisfy the following conditions

(i) $\alpha(t)$ is increasing for $0 \leqslant t \leqslant 1$, and $\alpha(0) = 0$, $\alpha(1) = 1$.

(ii) Suppose that $\alpha(t)$ has already been defined when t is of the form $p\,10^{-N}$, where p is an integer, such that $0 \leqslant p \leqslant 10^N$. Then we define

$$\alpha\left(\frac{p + \frac{1}{10}}{10^N}\right) = \alpha\left(\frac{p + \frac{9}{10}}{10^N}\right) = \frac{1}{2}\left[\alpha\left(\frac{p}{10^N}\right) + \alpha\left(\frac{p+1}{10^N}\right)\right], \quad 0 \leqslant p < 10^N.$$

It follows from (i) and (ii) that $\alpha(t)$ is constant for $(p + \frac{1}{10})\,10^{-N} \leqslant t \leqslant (p + \frac{9}{10})\,10^{-N}$. Thus $\alpha(t)$ is defined at all points of the form $p\,10^{-N}$, where p, N are positive integers and $p \leqslant 10^N$. Clearly $\alpha[(p+1)\,10^{-N}] - \alpha[p\,10^{-N}] = 0$ or 2^{-N}. Thus $\alpha(t)$ is uniformly continuous on the points $p\,10^{-N}$ and so there is a unique continuous extension of $\alpha(t)$ to all real numbers t, such that $0 \leqslant t \leqslant 1$. This extension is the unique function $\alpha(t)$ in $[0, 1]$, which satisfies (i) and (ii).

We set
$$f(z) = \exp\left\{C \int_0^1 \frac{e^{it} + z}{e^{it} - z}\, d\alpha(t)\right\},$$

where C is a positive constant and

$$F(z) = \int_0^z f(\zeta)\, d\zeta.$$

We shall then prove that $F(z)$ satisfies (2.9). It is trivial that (2.8) holds, since for $0 \leqslant r < 1$, $0 < \theta \leqslant 2\pi$,

(1) We could probably improve our estimates somewhat by replacing 10 by a smaller integer, e.g. 5 or 6 in this definition, but at the cost of considerably more delicate analysis.

$$\log|f(re^{i\theta})| = C\int_0^1 P(1, r, t-\theta)\,d\alpha(t) > 0,$$

so that $\log|f(z)|$ is positive and harmonic in $|z|<1$, and

$$T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta = \log|f(0)| = C, \quad 0<r<1.$$

The idea of our proof is as follows. We set $\theta_0 = (p+\frac{1}{2})\,10^{-N}$, $h = \frac{1}{2}\,10^{-N}$, $\alpha_0 = 2^{-N}$, and suppose that $\alpha(\theta_0+h) - \alpha(\theta_0-h) = 2^{-N}$. Then when $z = re^{i\theta_0}$, and r is near to $1-h$,

$$|f(z)| \simeq \exp\frac{C\alpha_0}{2}\left\{\frac{1+re^{ih}}{1-re^{ih}} + \frac{1+re^{-ih}}{1-re^{-ih}}\right\} \simeq \exp\frac{C\alpha_0(1-r^2)}{(1-r)^2+h^2}.$$

Thus if δ is a small absolute constant and N is large

$$|F(e^{i\theta_0})| \simeq \int_0^1 \exp\frac{C\alpha_0(1-r^2)}{(1-r)^2+h^2}\,dr > \exp\frac{C(1-\delta)\,\alpha_0}{h}.$$

Also if $\theta_0 - h(\frac{4}{5}-\delta) < \arg z < \theta_0 + h(\frac{4}{5}-\delta)$, $|z| > 1-\frac{1}{8}\delta^2 h$, then $|f(z)|$ is much smaller than $F(e^{i\theta_0})$, and so

$$|F(z)| = \left|F(e^{i\theta_0}) + \int_{e^{i\theta_0}}^{z} f(\zeta)\,d\zeta\right| > |\tfrac{1}{2}F(e^{i\theta_0})|.$$

In particular

$$\frac{1}{2\pi}\int_{\theta_0-h(\frac{4}{5}-\delta)}^{\theta_0+h(\frac{4}{5}-\delta)}\log^+|F(re^{i\theta})|\,d\theta > \frac{C\alpha_0(1-\delta)\,2h(\frac{4}{5}-\delta)}{2\pi h} \simeq \frac{4C\alpha_0}{5\pi}.$$

The argument is applicable for fixed N and 2^N distinct values of θ_0, provided that

$$(1-r) < \frac{\delta^2 h}{8} = \frac{\delta^2}{8}\,10^{-N}.$$

For each N the total contribution to $m(r,f)$ from all the 2^N intervals of length $\frac{4}{5}\,10^{-N}$, in which $\alpha(t)$ is constant is about $2^N 4C\alpha_0/(5\pi)$ i.e. $4C/(5\pi)$.

For N we have

$$10^N \leq \frac{\delta^2}{8(1-r)}, \quad N \leq \frac{1}{\log 10}\log\frac{1}{1-r} + A.$$

Thus the total contribution of all the intervals for varying N to $m(r,F)$ is at least

$$\frac{4C}{5\pi \log 10}\left[\log \frac{1}{1-r} + A\right] > \frac{C}{10}\log \frac{1}{1-r},$$

when r is sufficiently near 1, which is the type of result we require.

8. The saddle points

Unfortunately a good deal of rather delicate analysis is required for the actual lower bound for $|F(e^{i\theta_0})|$ and to this we now turn, using a saddle point technique.

LEMMA 3. *With the above notation set* $1 - ze^{-i\theta_0} = \zeta = \xi + i\eta$, *and write*

$$\frac{1}{C}\log f(z) = \int_0^1 \frac{e^{it}+z}{e^{it}-z}\, d\alpha(t) = g(\zeta) = u + iv.$$

Then if N is sufficiently large, $g(\zeta)$ has the following properties:

(a) $\mathcal{R}\, g''(\zeta) < \frac{-.7\alpha_0}{h^3}$, *and* $|g''(\zeta)| < \frac{5\alpha_0}{h^3}$, *for* $\frac{4h}{5} < \xi < h$, $|\eta| < \frac{h}{20}$.

(b) *There exists* $\zeta_0 = \xi_0 + i\eta_0$ *such that* $.84h < \xi_0 < .96h$, $|\eta_0| < \frac{h}{630}$, *and* $g'(\zeta_0) = 0$.

(c) *We have for* $|\zeta - \zeta_0| < \frac{h}{120}$, $|\arg(-g''(\zeta))| \leqslant \frac{\pi}{4}$.

We divide the interval $[0, 1]$ up as follows. We denote the interval $\theta_0 - h \leqslant t \leqslant \theta_0 + h$, i.e. $p\,10^{-N} \leqslant t \leqslant (p+1)\,10^{-N}$, by J_0. We define intervals J_ν, J'_ν, $0 \leqslant \nu \leqslant N-1$, as follows. Suppose that J_ν is an interval of the form $p_\nu 10^{\nu-N} \leqslant t \leqslant (p_\nu+1)\,10^{\nu-N}$, where p_ν is an integer, and that $\alpha(t)$ increases by $2^{\nu-N}$ in J_ν. This is true for $\nu = 0$ with $p_0 = p$. Then by (ii) we must have $p_\nu \equiv 0$ or $9 \pmod{10}$. If $p_\nu \equiv 0 \pmod{10}$, we define $p_{\nu+1} = p_\nu/10$, $p'_\nu = p_\nu + 9$. If $p_\nu \equiv 9 \pmod{10}$, we define $p'_\nu = p_\nu - 9$, and $p_{\nu+1} = p'_\nu/10$. Then if J'_ν is the interval $p'_\nu 10^{\nu-N} \leqslant t \leqslant (p'_\nu + 1)\,10^{\nu-N}$, and

$$J_{\nu+1} \text{ the interval } p_{\nu+1}10^{\nu+1-N} \leqslant t \leqslant (p_{\nu+1}+1)\,10^{\nu+1-N},$$

it is evident that $\alpha(t)$ increases by $2^{\nu-N}$ in J'_ν and so by $2^{\nu+1-N}$ in $J_{\nu+1}$. Hence our inductive hypothesis is satisfied also for $J_{\nu+1}$. We note that J_ν contains $J_{\nu-1}$, for $1 \leqslant \nu \leqslant N-1$. Also if $\phi(t)$ is continuous in $[0, 1]$, then

$$\int_{J_{\nu+1}} \phi(t)\, d\alpha(t) = \int_J \phi(t)\, d\alpha(t) + \int_{J'_\nu} \phi(t)\, d\alpha(t), \quad 0 \leqslant \nu \leqslant N-2.$$

Thus
$$\int_0^1 \phi(t)\,d\alpha(t) = \left(\int_{J'_{N-1}} + \int_{J_{N-1}}\right) \phi(t)\,d\alpha(t)$$
$$= \int_{J'_{N-1}} + \int_{J'_{N-2}} + \int_{J_{N-2}} \phi(t)\,d\alpha(t) = \dots$$
$$= \int_{J_0} \phi(t)\,d\alpha(t) + \sum_{\nu=0}^{N-1} \int_{J'_\nu} \phi(t)\,d\alpha(t). \tag{8.1}$$

In order to estimate the integrals occurring in this identity we note that if t, t' lie in J_ν, J'_ν respectively, then
$$|t-t'| \geqslant 8.\,10^{\nu-N} = 16h\,10^\nu. \tag{8.2}$$

In particular this inequality holds if t lies in J_0 and t' in J'_ν.

8.1. We now set $1 - ze^{-i\theta_0} = \zeta = \xi + i\eta$, and suppose that
$$\frac{4h}{5} < \xi < h, \quad |\eta| < \frac{h}{20}. \tag{8.3}$$

We write
$$g(\zeta) = \int_0^1 \frac{e^{it} + e^{i\theta_0}(1-\zeta)}{e^{it} - e^{i\theta_0}(1-\zeta)}\,d\alpha(t) = g_0(\zeta) + g_1(\zeta),$$

where
$$g_0(\zeta) = \int_{J_0} \frac{e^{it} + e^{i\theta_0}(1-\zeta)}{e^{it} - e^{i\theta_0}(1-\zeta)}\,d\alpha(t),$$
$$g_1(\zeta) = \sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{e^{it} + e^{i\theta_0}(1-\zeta)}{e^{it} - e^{i\theta_0}(1-\zeta)}\,d\alpha(t).$$

Thus
$$g_0''(\zeta) = \int_{J_0} \frac{4e^{i(t+2\theta_0)}\,d\alpha(t)}{[e^{it} - e^{i\theta_0}(1-\zeta)]^3}.$$

We set $t = \theta_0 + \tau$, and note that
$$\frac{4e^{i(t+2\theta_0)}}{[e^{it} - e^{i\theta_0}(1-\zeta)]^3} = \frac{-4e^{i\tau}}{[1-e^{i\tau}-\zeta]^3} = \frac{-4[(1-\cos\tau-\xi) + i(\sin\tau+\eta)]^3\,e^{i\tau}}{|1-e^{i\tau}-\zeta|^6}$$

Thus
$$g_0''(\zeta) = \int_{J_0} \frac{\{4[\xi - i(\tau+\eta)]^3 + O(h^4)\}\,d\alpha(t)}{|\xi + i(\tau+\eta)|^6 + O(h^7)},$$
$$\mathfrak{R}\,g_0''(\zeta) = \int_{J_0} \frac{4\xi[\xi^2 - 3(\eta+\tau)^2]}{[\xi^2 + (\eta+\tau)^2]^3}\,d\alpha(t) + O\left(\frac{\alpha_0}{h^2}\right).$$

We note that $d\alpha(t) > 0$, only for $\frac{4}{5}h \leqslant |\tau| \leqslant h$. Since ξ, η satisfy (8.3) we have $.8h \leqslant \xi \leqslant h$, and $.75h \leqslant |\tau + \eta| \leqslant 1.05h$. Consider now

$$\phi(a, b) = \frac{4a(3b^2 - a^2)}{[a^2 + b^2]^3}$$

in the range $.8h \leqslant a \leqslant h$, $.75h \leqslant b \leqslant 1.05h$. Then

$$\frac{\partial \phi(a, b)}{\partial a} = \frac{12(a^4 + b^4 - 6a^2b^2)}{(a^2 + b^2)^4} < 0$$

in the range. Thus, for fixed b, $\phi(a, b)$ is smallest when $a = h$. Also

$$\frac{\partial \phi(a, b)}{\partial b} = \frac{48ab(a^2 - b^2)}{(a^2 + b^2)^4},$$

so that for fixed a, $\phi(a, b)$ first increases to a maximum at $b = a$ and then decreases. Thus

$$\begin{aligned} \phi(a, b) &\geqslant \min \{\phi(h, 1.05h), \phi(h, .75h)\} \\ &= \min \left\{ \frac{4(2.3075)h^{-3}}{(2.1025)^3}, \frac{(11/4)h^{-3}}{[25/16]^3} \right\} \\ &\doteq \frac{(11)(2^{16})}{10^6 h^3} > \frac{.72}{h^3}. \end{aligned}$$

Thus it follows that

$$\mathcal{R}\, g_0''(\zeta) < -\frac{.72}{h^3} \int_{J_0} d\alpha(t) + O\left(\frac{\alpha_0}{h^2}\right) = -\frac{.72\alpha_0}{h^3} + O\left(\frac{\alpha_0}{h^2}\right). \tag{8.4}$$

Also we have in the range (8.3)

$$|g_0''(\zeta)| < 4 \int_{J_0} \frac{d\alpha(t)}{|\zeta + i\tau|^3} + O\left(\frac{\alpha_0}{h^2}\right) < \frac{4\alpha_0}{h^3[(\frac{3}{4})^2 + (\frac{4}{5})^2]^{\frac{3}{2}}} + O\left(\frac{\alpha_0}{h^2}\right) < \frac{4\alpha_0}{h^3} + O\left(\frac{\alpha_0}{h^2}\right). \tag{8.5}$$

Again in view of (8.2) we have in J_ν', $|\tau| \geqslant 16h\,10^\nu$, and $|\eta| \leqslant h/20$, and

$$|1 - e^{i\tau} - \zeta| \geqslant |\sin \tau| - |\eta|.$$

Also $|\tau| \leqslant 1$, so that $|\sin \tau| \geqslant |\tau| \sin 1 > \frac{5}{6}|\tau|$. Thus in the range (8.3) and for t in J_ν'

$$|1 - e^{i\tau} - \zeta| > \frac{5}{6} 16h\,10^\nu - \frac{h}{20} > 13h\,10^\nu. \tag{8.6}$$

Thus $$|g_1''(\zeta)| \leqslant \sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{4}{|1-e^{i\tau}-\zeta|^3}\, d\alpha(t)$$

$$\leqslant \sum_{\nu=0}^{N-1} \frac{4.2^\nu \alpha_0}{(13h\, 10^\nu)^3} = \frac{4\alpha_0}{2197\, h^3}\left(1+\frac{2}{1000}+\left(\frac{2}{1000}\right)^2+\dots\right) < \frac{\alpha_0}{500\, h^3}. \tag{8.7}$$

Thus we deduce from this and (8.4) that

$$\mathcal{R}\, g''(\zeta) < \frac{\alpha_0}{h^3}[-.72+.002+O(h)] < -\frac{.7\,\alpha_0}{h^3}$$

if h is sufficiently small, i.e. if N is sufficiently large. Further by (8.5) and (8.7) we have

$$|g''(\zeta)| \leqslant |g_0''(\zeta)| + |g_1''(\zeta)| < \frac{\alpha_0}{h^3}\left[4+\frac{1}{500}+O(h)\right] < \frac{5\alpha_0}{h^3}.$$

This completes the proof of Lemma 3 (a).

8.2. We proceed to prove (b). To do this we suppose now that $\eta=0$, $0<\xi<h$. Then

$$g_1'(\zeta) = \sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{-2e^{i\tau}\, d\alpha(t)}{(e^{i\tau}-1+\zeta)^2}.$$

We note that (8.6) still holds on J'_ν. Thus

$$\left|\sum_{\nu=1}^{N} \int_{J'_\nu} \frac{-2e^{i\tau}\, d\alpha(t)}{(e^{i\tau}-1+\zeta)^2}\right| \leqslant \sum_{\nu=1}^{N} \frac{2}{(13h\, 10^\nu)^2} \int_{J'_\nu} d\alpha(t)$$

$$= \sum_{\nu=1}^{N} \frac{2^{\nu+1}\alpha_0}{(13h\, 10^\nu)^2} = \frac{4\alpha_0}{130^2 h^2}\left(1+\frac{1}{50}+\left(\frac{1}{50}\right)^2+\dots\right) < \frac{\alpha_0}{4000\, h^2}. \tag{8.8}$$

Again we have in J'_0 $$17h \leqslant |\tau| = t-\theta_0 \leqslant 19h, \tag{8.9}$$

so that $$e^{i\tau}-1+\zeta = \xi + i\tau + O(h^2).$$

Thus $$\int_{J'_0} \frac{-2e^{i\tau}\, d\alpha(t)}{(e^{i\tau}-1+\zeta)^2} = -2\int_{J'_0} \frac{[1+O(h)]\, d\alpha(t)}{[\xi+i\tau+O(h^2)]^2}$$

$$= \int_{J'_0} \frac{2(\tau^2-\xi^2)\, d\alpha(t)}{(\xi^2+\tau^2)^2} + \int_{J'_0} \frac{4i\xi\tau\, d\alpha(t)}{(\xi^2+\tau^2)^2} + O\left(\frac{\alpha_0}{h}\right)$$

$$= I + iI' + O\left(\frac{\alpha_0}{h}\right) \text{ say.}$$

In view of (8.9) and $0<\xi<h$, we deduce that

$$\frac{2\alpha_0}{(17h)^2}>I>\frac{2(17^2-1)}{(19^2+1)^2}\frac{\alpha_0}{h^2} \tag{8.10}$$

and

$$|I'|\leqslant\frac{4\xi\alpha_0}{(17h)^3}. \tag{8.11}$$

Again

$$g_0'(\xi)=\int_{J_0}\frac{-2e^{i\tau}\,d\alpha(t)}{(e^{i\tau}-1+\xi)^2}=4\int_{\frac{4}{5}h}^{h}\frac{(\tau^2-\xi^2)\,d\alpha(t)}{(\xi^2+\tau^2)^2}+O\left(\frac{\alpha_0}{h}\right),$$

since the mass $d\alpha(t)$ is symmetrical about the centre of J_0 and is zero on the interval $(-\frac{4}{5}h, \frac{4}{5}h)$.

The function $\alpha(t)$ increases by $\alpha_0/4$ in each of the intervals $(.8h, .82h)$ and $(.98h, h)$ and is constant in the interval $(.82h, .98h)$. Thus if $\xi=.95h$ then

$$\int_{\frac{4}{5}h}^{h}\frac{(\xi^2-\tau^2)\,d\alpha(t)}{[\xi^2+\tau^2]^2}>\frac{\alpha_0}{4h^2}\left\{\frac{(.95)^2-1}{[(.95)^2+(.98)^2]^2}+\frac{(.95)^2-(.82)^2}{[(.95)^2+(.82)^2]^2}\right\}$$

$$>\frac{\alpha_0}{4h^2}\frac{(.13)(1.77)-(.05)(1.95)}{4}>\frac{.13\alpha_0}{16h^2}>\frac{\alpha_0}{125h^2}.$$

Thus for this value of ξ, we have for large N in view of (8.8) and (8.10)

$$\mathcal{R}g'(\xi)<\frac{-4\alpha_0}{125h^2}+\frac{2\alpha_0}{289h^2}+\frac{\alpha_0}{4000h^2}+\frac{O(\alpha_0)}{h}<0.$$

Again, for $\xi=.85h$,

$$\int_{\frac{4}{5}h}^{h}\frac{[\xi^2-\tau^2]\,d\alpha(t)}{[\xi^2+\tau^2]^2}\leqslant\frac{\alpha_0}{4h^2}\left[\frac{(.85)^2-(.98)^2}{[(.85)^2+1^2]^2}+\frac{(.85)^2-(.8)^2}{[(.85)^2+(.8)^2]^2}\right]$$

$$<\frac{\alpha_0}{4h^2}\left[\frac{-(1.83)(.13)}{(1.73)^2}+\frac{(.05)(1.65)}{(1.36)^2}\right]<-\frac{\alpha_0}{200h^2}.$$

Thus for $\xi=.85$ we have, using (8.8), (8.10),

$$\mathcal{R}g'(\zeta)>\frac{\alpha_0}{50h^2}-\frac{\alpha_0}{4000h^2}+O\left(\frac{\alpha_0}{h}\right)>0.$$

Thus there is a value ξ_1, such that

$$.85h<\xi_1<.95h \tag{8.12}$$

and

$$\mathcal{R}g'(\xi_1)=0.$$

Also for this value ξ_1 we have by (8.8) and (8.11)

$$|\mathfrak{I}\, g'(\xi_1)| < |I'| + \frac{\alpha_0}{4000 h^2} + O\left(\frac{\alpha_0}{h}\right)$$

$$\leqslant \frac{\alpha_0}{h^2}\left[\frac{4}{4913} + \frac{1}{4000} + O(h)\right] < \frac{\alpha_0}{900 h^2},$$

when N is large. Thus

$$|g'(\xi_1)| < \frac{\alpha_0}{900 h^2}. \tag{8.13}$$

We now set $\zeta = \xi + i\eta = re^{i\phi} + \xi_1$. We suppose $r < h/20$, so that the estimates of Lemma 3 (a) hold. Also

$$g'(\zeta) = g'(\xi_1) + \int_{\xi_1}^{\zeta} g''(z)\, dz = g'(\xi_1) + \int_0^r g''(\xi_1 + te^{i\phi})\, e^{i\phi}\, dt.$$

Thus by Lemma 3 (a)

$$\mathfrak{R}\, e^{-i\phi}[g'(\zeta) - g'(\xi_1)] < -\frac{7\alpha_0 r}{10 h^3}.$$

In particular

$$|g'(\zeta) - g'(\xi_1)| \geqslant \frac{7\alpha_0 r}{10 h^3}, \quad |\zeta - \xi_1| = r.$$

We choose $r = h/630$, so that in view of (8.13)

$$\frac{7\alpha_0 r}{10 h^3} = \frac{\alpha_0}{900 h^2} > |g'(\xi_1)|.$$

Thus by Rouché's Theorem $g'(\zeta) - g'(\xi_1)$ and $g'(\zeta) = g'(\zeta) - g'(\xi_1) + g'(\xi_1)$ have equally many zeros in $|\zeta - \xi_1| < r$, i.e. at least one. We set such a zero equal to $\zeta_0 = \xi_0 + i\eta_0$ and note that

$$|\xi_1 - \zeta_0| < \frac{h}{630}, \tag{8.14}$$

where ξ_1 satisfies (8.12). This gives (b).

8.3. It remains to prove (c). We note that in view of Lemma 3 (a) and (b)

$$\mathfrak{R}\, g_0''(\zeta) < 0 \text{ for } |\zeta - \xi_1| < \frac{h}{20} = r_0 \text{ say.}$$

Also if

$$g_0''(\zeta) = a_0 + \sum_1^{\infty} a_n (\zeta - \xi_1)^n, \quad |\zeta - \xi_1| < r_0,$$

then a_0 is real and negative, and by the Borel-inequalities $|a_n| \leqslant 2|a_0|/r_0^n$. Thus for $|z-\xi_1| \leqslant r_0/5$, we have

$$|g_0''(z) - a_0| \leqslant \frac{2|a_0|/5}{1-\frac{1}{5}} = \frac{|a_0|}{2}.$$

Again in the same disk $|z-\xi_1| \leqslant r_0/5$, we have also by (8.7)

$$|g''(z) - g_0''(z)| = |g_1''(z)| < \frac{\alpha_0}{500h^3}, \quad \text{and} \quad |g''(z)| \geqslant \frac{.7\alpha_0}{h^3}.$$

In particular
$$|a_0| = |g_0''(\xi_1)| \geqslant \frac{.7\alpha_0}{h^3} - \frac{\alpha_0}{500h^3} > \frac{\alpha_0}{2h^3}.$$

We deduce further for $|z-\xi_1| \leqslant r_0/5$, that

$$|g''(z) - a_0| \leqslant |g_0''(z) - a_0| + |g_1''(z)| \leqslant \frac{|a_0|}{2} + \frac{|a_0|}{200} < \frac{|a_0|}{\sqrt{2}}.$$

Since a_0 is real and negative we deduce that

$$|\arg[-g''(z)]| \leqslant \frac{\pi}{4}, \quad |z-\xi_1| \leqslant \frac{r_0}{5} = \frac{h}{100},$$

and so in particular for $|z-\zeta_0| < h/120 < h/100 - h/630$. This completes the proof of Lemma 3.

8.4. We also need a global estimate for the growth of $u(\xi)$.

LEMMA 4. *With the notation of Lemma 3 the function $u(\xi)$ assumes its maximum value for $0 < \xi < 1$ at $\xi = \xi_1$, where $|\xi_1 - \zeta_0| < h/630$. Also $u(\xi)$ increases in the interval $[0, \xi_1]$ and decreases in the interval $[\xi_1, 3h]$ and $u(\xi) < u(3h)$ for $\xi > 3h$.*

We have seen that $u(\xi)$ has a local maximum at $\xi = \xi_1$, where by (8.14) and (8.12)

$$|\xi_1 - \zeta_0| < \frac{h}{630}, \quad \text{and} \quad .85h < \xi_1 < .95h.$$

It also follows from Lemma 3 (a) that

$$\frac{d^2u}{d\xi^2} < 0, \quad .8h < \xi < h,$$

so that $u(\xi)$ increases in the interval $[.8h, \xi_1]$ and $u(\xi)$ decreases in the interval $[\xi_1, h]$.

Suppose now that $\xi < .8h$. Then as we saw in section 8.2

$$\frac{du}{d\xi} = \mathcal{R} g_0'(\xi) + \mathcal{R} g_1'(\xi) > 4\int_{\frac{4}{5}h}^{h} \frac{(\tau^2 - \xi^2)}{(\xi^2 + \tau^2)^2}\, d\alpha(t) + I - \frac{\alpha_0}{4000 h^2} + O\left(\frac{\alpha_0}{h}\right) > 0,$$

in view of (8.10).

Thus u increases in the range $[0, .8h]$ and so in $[0, \xi_1]$. Again for $\xi > h$

$$\frac{du}{d\xi} < 4\int_{\frac{4}{5}h}^{h} \frac{(\tau^2 - \xi^2)}{(\tau^2 + \xi^2)^2}\, d\alpha(t) + 2\int_{J_0'} \frac{(\tau^2 - \xi^2)\, d\alpha(t)}{(\tau^2 + \xi^2)^2} + \frac{\alpha_0}{4000 h^2} + O\left(\frac{\alpha_0}{h}\right).$$

Since $|\tau| \geqslant 17h$, we see again that

$$2\int_{J_0'} \frac{(\tau^2 - \xi^2)\, d\alpha(t)}{(\tau^2 + \xi^2)^2} < 2\int_{J_0'} \frac{d\alpha(t)}{\tau^2} < \frac{2\alpha_0}{(17h)^2}.$$

Suppose first that $h < \xi < 2h$. Then

$$4\int_{\frac{4}{5}h}^{h} \frac{(\tau^2 - \xi^2)\, d\alpha(t)}{(\tau^2 + \xi^2)^2} < 4\int_{.8h}^{.82h} \frac{(\tau^2 - \xi^2)\, d\alpha(t)}{(\tau^2 + \xi^2)^2} < \frac{-[\xi^2 - (.82h)^2]\,\alpha_0}{[\xi^2 + (.82h)^2]^2}.$$

The right-hand side is less than

$$-\alpha_0 \frac{.32 h^2}{5^2 h^4}.$$

Thus in this range

$$\frac{du}{d\xi} < \frac{\alpha_0}{h^2}\left[-.0128 + \frac{2}{17^2} + \frac{1}{4000}\right] + O\left(\frac{\alpha_0}{h}\right) < 0.$$

Next if $\xi > 2h$

$$4\int_{\frac{4}{5}h}^{h} \frac{(\tau^2 - \xi^2)\, d\alpha(t)}{(\tau^2 + \xi^2)^2} < -4\int_{\frac{4}{5}h}^{h} \frac{\frac{3}{4}\xi^2\, d\alpha(t)}{(\frac{5}{4}\xi^2)^2} = \frac{-24\alpha_0}{25\xi^2}.$$

Thus

$$\frac{du}{d\xi} < \frac{\alpha_0}{h^2}\left[\left(\frac{2}{17}\right)^2 + \frac{1}{4000}\right] - \frac{24\alpha_0}{25\xi^2} + O\left(\frac{\alpha_0}{h}\right) < 0 \text{ if } \xi \leqslant 10h.$$

Thus u decreases also in the range $[h, 10h]$ and so in $[\xi_1, 10h]$.

Finally if $\xi > 10h$

$$u < \int_{J_0} \frac{2\, d\alpha(t)}{|e^{i\tau} - (1 - \xi)|} + \sum_{\nu=0}^{N-1} \int_{J_\nu'} \frac{2\, d\alpha(t)}{|e^{i\tau} - 1 + \xi|}.$$

In J_0

$$|e^{i\tau} - (1 - \xi)| > \xi + O(h^2) > 10h + O(h^2),$$

so that

$$\int_{J_0} \frac{2\, d\alpha(t)}{|e^{i\tau} - (1 - \xi)|} < \frac{\alpha_0}{5h} + O(\alpha_0).$$

Also in J'_ν we have by (8.6) for $0 < \xi < 1$,

$$|1 - e^{i\tau} - \xi| < 13h\, 10^\nu,$$

so that
$$\sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{2d\alpha(t)}{|e^{i\tau} - 1 + \xi|} < \sum_{\nu=0}^{N-1} \frac{2\alpha_0 2^\nu}{(13h)\, 10^\nu} < \frac{2\alpha_0}{13h(1-\frac{1}{5})} = \frac{5\alpha_0}{26h}.$$

Thus in this range we have

$$u < \frac{\alpha_0}{5h} + \frac{5\alpha_0}{26h} + O(\alpha_0) < \frac{2\alpha_0}{5h}.$$

On the other hand, we have for $\xi = 3h$

$$u(\xi) > \mathcal{R} \int_{J_0} \frac{[e^{i\tau} + (1-\xi)]\, d\alpha(t)}{e^{i\tau} - (1-\xi)} = \int_{J_0} \frac{[1-(1-\xi)^2]\, d\alpha(t)}{1 + (1-\xi)^2 - 2(1-\xi) \cos \tau}$$

$$= \int_{J_0} \frac{2\xi + O(h^2)}{\xi^2 + \tau^2}\, d\alpha(t) > \frac{2\alpha_0 \xi}{\xi^2 + h^2} + O(\alpha_0) = \frac{3\alpha_0}{5h} + O(\alpha_0).$$

Thus $u(\xi) < u(3h)$ for $\xi > 10h$, and hence also for $\xi > 3h$, since $u(\xi)$ decreases in the interval $[3h, 10h]$. This completes the proof of Lemma 4.

9. Construction of the path of integration

We shall need

LEMMA 5. *Suppose that $g(z) = u + iv$ is regular and not constant in the disk $|z - z_0| \leqslant r$ and satisfies $|\arg g''(z)| \leqslant \frac{1}{4}\pi$ there and $g'(z_0) = 0$. Then there exists an analytic Jordan arc γ with the following properties*

(a) *γ is a cross cut in $|z - z_0| < r$ with end points on $|z - z_0| = r$ and passing through the point z_0.*

(b) *If z describes γ in a suitable sense v is constant on γ and z_0 divides γ into two arcs γ_2, γ_1, such that u decreases on γ_2 and increases on γ_1.*

(c) *On γ_1 we have $|\arg (z - z_0)| \leqslant \frac{1}{8}\pi$, and on γ_2 we have $|\arg (z - z_0) - \pi| \leqslant \frac{1}{8}\pi$.*

(d) *If z_1, z_2 are points on γ_1, γ_2 respectively and $|\phi_1| \leqslant \frac{1}{8}\pi$, $|\phi_2 - \pi| \leqslant \frac{1}{8}\pi$ then for $j = 1, 2$, $u(z_j + te^{i\phi_j})$ increases while t increases through positive values as long as $z_j + te^{i\phi_j}$ remains in $|z - z_0| \leqslant r$.*

It follows from our hypotheses that $\mathcal{R} g''(z) \geqslant 0$ in $|z - z_0| < r$. Here strict inequality holds unless $g''(z) \equiv i\beta$, which conflicts with our hypotheses, unless $\beta = 0$.

In this case $g'(z) = \text{constant} = g'(z_0) = 0$, so that $g(z)$ is constant contrary to hypothesis. Thus

$$\mathfrak{R}\, g''(z) > 0, \quad |z - z_0| < r.$$

If we set $g''(z_0) = 2a_2$, it follows that $a_2 \neq 0$ and hence by classical theorems the set $v = v(z_0)$ consists near $z = z_0$ of two Jordan arcs which intersect at right angles at z_0. We have near $z = z_0$

$$g(z) - g(z_0) \sim a_2 (z - z_0)^2, \quad \text{as } z \to z_0,$$

and hence

$$\arg a_2 (z - z_0)^2 \to 0 \text{ or } \pi, \quad \text{as } z \to z_0,$$

so that $v(z) = \text{constant}$. We choose for γ that arc for which

$$\arg [(z - z_0)^2] \to -\arg a_2 = \varepsilon$$

say, where by hypothesis we may suppose $|\varepsilon| \leq \frac{1}{4}\pi$. Thus

$$\arg (z - z_0) \to \frac{\varepsilon}{2} \text{ or } \frac{\varepsilon}{2} \mp \pi \text{ as } z \to z_0 \text{ on } \gamma.$$

As z describes γ, $dg(z) = g'(z)\, dz$ is purely real. We have

$$g'(z_0 + \varrho e^{i\theta}) = \int_0^{\varrho} g''(z_0 + te^{i\theta})\, e^{i\theta} dt, \quad 0 < \varrho < r.$$

From this and our hypothesis that $|\arg g''(z)| \leq \frac{1}{4}\pi$ it follows that

$$\theta - \frac{\pi}{4} \leq \arg g'(z_0 + \varrho e^{i\theta}) \leq \theta + \frac{\pi}{4} \tag{9.1}$$

and that

$$g'(z) \neq 0, \quad 0 < |z - z_0| < r. \tag{9.2}$$

If $z = z_0 + \varrho e^{i\theta}$ is a point on γ, then by (9.1)

$$\arg dz = -\arg g'(z) \leq \frac{\pi}{8}, \text{ if } \theta = \frac{\pi}{8},$$

$$\arg dz \geq -\frac{\pi}{8}, \qquad \text{if } \theta = -\frac{\pi}{8}.$$

If we denote by γ_1, γ_2 the arcs of γ on which $\arg (z - z_0)$ approaches $\frac{1}{2}\varepsilon$ and $\frac{1}{2}\varepsilon + \pi$ respectively, then it follows that γ_1 remains in the sector

$$|\arg (z - z_0)| \leq \frac{\pi}{8}. \tag{9.3}$$

Similarly γ_2 remains in the sector

$$|\arg (z-z_0)-\pi| \leqslant \frac{\pi}{8}.$$

We have for z on γ, if s denotes arc length on γ

$$\frac{\partial u}{\partial s} = \mp |g'(z)|, \quad \frac{\partial v}{\partial s} = 0.$$

If z describes γ_1 away from z_0, then it is clear that initially

$$u(z)-u(z_0) \sim a_2 (z-z_0)^2 \sim |a_2|\,|z-z_0|^2 > 0.$$

Thus

$$\frac{\partial u}{\partial s} = |g'(z)|$$

as γ_1 is described in this direction. It follows from (9.1) that γ_1 can have no double points and continues as a Jordan arc to the boundary circle $|z-z_0|=r$. Also

$$\frac{\partial u}{\partial s} = |g'(z)| > 0$$

on the whole of γ_1. Similarly if γ_2 is described away from z_0

$$\frac{\partial u}{\partial s} = |g'(z)| > 0 \text{ on } \gamma_2$$

and γ_2 continues to the boundary circle $|z-z_0|=r$. Since γ_1, γ_2 lie in different sections of the plane they do not meet and so we have proved (a), (b) and (c).

It remains to prove (d). If $z_1 = z_0 + \varrho e^{i\theta}$ is on γ_1, and $\arg g'(z_1) = \theta_1$, then on γ_1 we have by (9.1) and (9.3)

$$|\arg dz| = |-\theta_1| \leqslant |\theta| + \frac{\pi}{4} \leqslant \frac{3\pi}{8}. \tag{9.4}$$

Hence if $|\phi_1| < \frac{1}{8}\pi$ we have $|\phi_1 + \theta_1| < \frac{1}{2}\pi$, so that

$$\frac{\partial}{\partial t} u(z_1 + te^{i\theta_1}) = |g'(z_1)| \cos(\phi_1+\theta_1) > 0, \text{ at } t=0.$$

Also for $t \geqslant 0$

$$\frac{\partial^2}{\partial t^2} u(z_1 + te^{i\phi_1}) = \mathcal{R}\frac{\partial}{\partial t^2} g(z_1+te^{i\phi_1}) = \mathcal{R}\, e^{2i\phi_1} g''(z_1+te^{i\phi_1}) > 0,$$

since by our hypotheses

$$|\arg\{e^{2i\phi_1}g''(z_1+te^{i\phi_1})\}| \leqslant \frac{\pi}{4}+2\phi_1 < \frac{\pi}{2}.$$

Thus if $u(z_1+te^{i\phi})=u_1(t)$ then

$$u_1'(t)=u_1'(0)+\int_0^t u_1''(\tau)\,d\tau \geqslant u_1'(0)>0,$$

provided that $(z_1+te^{i\phi})$ lies in $|z-z_0|<r$, so that $u_1(t)$ increases with t as required.

Similarly $u(z_2+te^{i\phi_2})$ increases with t, when z_2 lies on γ_2 and $|\phi_2-\pi|\leqslant \frac{1}{8}\pi$. This completes the proof of Lemma 5.

10. The estimate for $F(e^{i\theta_0})$

We can now prove

LEMMA 6. *We have with the notation of Lemma 3*

$$\left|\int_0^1 e^{Cg(\zeta)}\,d\zeta\right| \geqslant e^{Cu(\zeta_0)}\left(\frac{A_1h^3}{C\alpha_0}\right)^{\frac{1}{2}},$$

provided that $N\geqslant A_2(1+\log^+ 1/C)$ *where* A_1, A_2 *are positive absolute constants.*

Let ζ_0 be the zero whose existence is asserted in Lemma 3. We apply Lemma 5 with $z_0=\zeta_0=\xi_0+i\eta_0$, $r=10^{-3}h$ and $-g(\zeta)$ instead of $g(z)$. Let γ be the corresponding cross cut with end points ζ_2, ζ_3, where $|\zeta_2|<|\zeta_0|<|\zeta_3|$. Let ζ_1, ζ_4 be the points

$$\zeta_1=\xi_0-\frac{h}{130},\quad \zeta_4=\xi_0+\frac{h}{130} \tag{10.1}$$

and let Γ be the contour $0\,\zeta_1\,\zeta_2\,\zeta_3\,\zeta_4\,1$, taken along straight line segments from 0 to ζ_1, ζ_1 to ζ_2, ζ_3 to ζ_4 and from ζ_4 to 1 and along γ from ζ_2 to ζ_3. We proceed to estimate

$$I=\int_\Gamma e^{Cg(\zeta)}\,d\zeta$$

by considering the integrals along each of these arcs in turn.

Set $$\zeta=\zeta_0+re^{i\phi(r)},\quad -10^{-3}h\leqslant r\leqslant 10^{-3}h,$$

on γ, where $r=-10^{-3}h$, $10^{-3}h$ correspond to ζ_2, ζ_3 respectively, and

$$g(\zeta)=U(r)+iv,$$

on γ, where v is constant. Also by (9.4) we have on γ

$$|\arg d\zeta| = |\arg g'(\zeta)| \leqslant \frac{3\pi}{8}.$$

$$\left|\int_\gamma e^{Cg(\zeta)}\,d\zeta\right| \geqslant \mathcal{R}\int_\gamma e^{C(g(\zeta)-iv)}\,d\zeta \geqslant \int_{-10^{-3}h}^{10^{-3}h} e^{CU(r)}\cos\frac{3\pi}{8}\,dr.$$

Now since $g'(\zeta_0)=0$, we have

$$g(\zeta) = g(\zeta_0) + \int_{\zeta_0}^{\zeta} (\zeta - z)\, g''(z)\,dz.$$

We integrate along a straight line segment from ζ_0 to $\zeta_0 + re^{i\phi(r)}$, and note that in this segment we have by Lemma 3 (a)

$$|g''(z)| < \frac{5\alpha_0}{h^3}, \quad |z-\zeta_0| < r.$$

Thus for $|z - z_0| = r$ we have

$$|u(z) - u(z_0)| \leqslant |g(z) - g(z_0)| \leqslant \frac{5\alpha_0 r^2}{h^3}.$$

Thus
$$\left|\int_\gamma e^{Cg(\zeta)}\,d\zeta\right| \geqslant \cos\frac{3\pi}{8}\, e^{u(z_0)} \int_{-10^{-3}h}^{10^{-3}h} \exp\left(\frac{-C\,5\alpha_0 r^2}{h^3}\right) dr$$

$$\geqslant \cos\frac{3\pi}{8}\, e^{u(z_0)} \left(\frac{h^3}{5C\alpha_0}\right)^{\frac{1}{2}} \int_{-t_0}^{t_0} e^{-t^2}\,dt,$$

where
$$t_0 = 10^{-3}\left(\frac{5C\alpha_0}{h}\right)^{\frac{1}{2}} = C^{\frac{1}{2}}\,10^{-2.5}\,5^{\frac{1}{2}N}.$$

Hence $t_0 \to \infty$ as $N \to \infty$, and we deduce that

$$\left|\int_\gamma e^{Cg(\zeta)}\,d\zeta\right| \geqslant A_3\, e^{u(\zeta_0)} \left(\frac{h^3}{C\alpha_0}\right)^{\frac{1}{2}}, \text{ if } N \geqslant A_4 \log\frac{A_5}{C}, \tag{10.2}$$

where A_3, A_4, A_5 are absolute constants.

Again we have for $z = \zeta_0 + re^{i\phi}$, $0 < r \leqslant 10^{-3}h$,

$$e^{-i\phi} g'(z) = e^{-i\phi}\int_{z_0}^{z} g''(\zeta)\,d\zeta = \int_0^r g''(\zeta_0 + te^{i\phi})\,dt.$$

Thus we have in view of Lemma 3 (a)

$$|g'(z)| \geqslant \left| \int_0^r \mathcal{R} g''(\zeta_0 + te^{i\phi})\,dt \right| \geqslant \frac{.7\alpha_0 r}{h^3}.$$

Also on γ
$$g(z) - g(\zeta_0) = u(z) - u(z_0) = \int_\gamma \frac{\partial u}{ds}\,ds = -\int_\gamma |g'(z)|\,ds$$
$$\leqslant -\int_0^r \frac{.7\alpha_0}{h^3} t\,dt \leqslant -\frac{.35\alpha_0 r^2}{h^3}.$$

In particular we have if $\zeta = \zeta_2$ or ζ_3, so that $r = 10^{-3}h$,

$$|e^{Cg(\zeta)}| \leqslant e^{Cu(\zeta_0)} \exp\left(\frac{-A_6 C\alpha_0}{h}\right). \tag{10.3}$$

Next we have $\quad |\tan\arg(\zeta_2 - \zeta_1)| < \frac{2}{5}$, so that $|\arg(\zeta_2 - \zeta_1)| < \frac{\pi}{8}$,

since $\quad \mathcal{R}(\zeta_2 - \zeta_1) \geqslant \frac{h}{130} - \frac{h}{1000} > .0065h, \quad |\mathcal{J}(\zeta_2 - \zeta_1)| < \frac{h}{630} + 10^{-3}h < .0026h.$

Also
$$|\zeta_2 - \zeta_0| \leqslant \sqrt{\left(\frac{h}{630}\right)^2 + \left(\frac{h}{130}\right)^2} < \frac{h}{120}.$$

Thus by applying Lemma 5 (d) and using Lemma 3 (c) we see that $|e^{Cg(\zeta)}|$ increases as ζ describes the segment $\zeta_1\zeta_2$, so that (10.3) holds on this segment also. Similarly (10.3) holds on the segment $\zeta_3\zeta_4$.

Finally by Lemma 4 and (10.1)

$u(\zeta) \leqslant u(\zeta_4)$ on the segment $[\zeta_4, 1]$ and $u(\zeta) \leqslant u(\zeta_1)$ on the segment $[0, \zeta_1]$

so that (10.3) holds on these segments also.

Thus (10.3) holds on all of Γ except γ. Since the total length of the four segments which make up this part of Γ is at most 2, we deduce from (10.2) and (10.3) that for $N \geqslant A_4 \log(A_5/C)$ we have

$$\left| \int_\Gamma f(\zeta)\,d\zeta \right| \geqslant A_3 e^{Cu(\zeta_0)} \left(\frac{h^3}{C\alpha_0}\right)^{\frac{1}{2}} - 2e^{Cu(\zeta_0)} \exp\left(\frac{-A_6 C\alpha_0}{h}\right) \geqslant \frac{1}{2} A_3 e^{Cu(\zeta_0)} \left(\frac{h^3}{C\alpha_0}\right)^{\frac{1}{2}},$$

provided that
$$2\exp\left(\frac{-A_6 C\alpha_0}{h}\right) < \frac{1}{2} A_3 \left(\frac{h^3}{C\alpha_0}\right)^{\frac{1}{2}},$$

i.e. if
$$A_6 C 5^N > \frac{1}{2}\log C + \log\frac{4}{A_3} + \frac{3}{2}\left[\log 2 + N \log\frac{10}{2^{\frac{3}{2}}}\right],$$

which is true if $N > A_7 \log(A_8/C)$. This completes the proof of Lemma 6.

11. Proof of Theorem 3

To complete the proof of Theorem 3, we need to estimate $|f(z)|$ from above in the neighbourhood of $e^{i\theta_0}$ and to estimate the quantity $u(\zeta_0)$ which occurs in Lemma 6 from below. The result is contained in

LEMMA 7. *We have for all sufficiently large N*

$$u(\zeta_0) > \frac{1.1\,\alpha_0}{h}. \tag{11.1}$$

Also if $z = re^{i\theta}$ and ζ are related as in Lemma 3 and

$$U(z) = u[\zeta(z)] = \mathcal{R}\int_0^1 \frac{e^{it}+z}{e^{it}-z}\,d\alpha(t),$$

then if $0 < \delta < \frac{4}{5}$, $|\theta - \theta_0| < (\frac{4}{5} - \delta)h$ and $1 - \frac{1}{8}\delta^2 h < r < 1$, we have

$$U(z) < \tfrac{1}{2}u(\zeta_0). \tag{11.2}$$

It follows from the arguments leading to the proof of Lemma 6 that $u(\zeta)$ assumes its maximum value on the path Γ at the point ζ_0. Also Γ contains the interval $[0, \zeta_1]$ of the real axis and

$$\zeta_1 = \xi_0 - \frac{h}{130} > .83h,$$

by Lemma 3 (b). Again when $\zeta = .83h$, $z = re^{i\theta_0}$, where $r = 1 - .83h$

$$U(z) = \int_0^1 \frac{(1-r^2)\,d\alpha(t)}{1 - 2r\cos(\theta_0 - t) + r^2} \geq \int_{J_0} \frac{(1-r^2)\,d\alpha(t)}{1 - 2r\cos(\theta_0 - t) + r^2}$$

$$= \int_{\frac{4}{5}h}^{h} \left[\frac{4(1-r)}{(1-r)^2 + \tau^2} + O(1)\right] d\alpha(\theta_0 + \tau).$$

The function $\alpha(\theta_0 + \tau)$ increases by $\alpha_0/4$ in each of the intervals $[.8h, .82h]$ and $[.98h, h]$. Thus

$$U(z) \geq (1-r)\,\alpha_0 \left[\frac{1}{(1-r)^2 + (.82)^2h^2} + \frac{1}{(1-r)^2 + h^2}\right] + O(\alpha_0)$$

$$= \frac{.83\,\alpha_0}{h}\left[\frac{1}{(.83)^2 + (.82)^2} + \frac{1}{1 + (.83)^2}\right] + O(\alpha_0) > \frac{1.101\,\alpha_0}{h}[1 + o(1)].$$

Since $u(\zeta_0) > u(\zeta) = U(z)$, we deduce (11.1) for small h, i.e. large N.

Next suppose that $z = re^{i\theta}$, where $r > 1 - \delta^2 h/8$ and $|\theta - \theta_0| < \frac{4}{5}h(1-\delta)$. Then we have by (8.1)

$$U(z) = \int_{J_0} \frac{(1-r^2)\,d\alpha(t)}{1 - 2r\cos(\theta - t) + r^2} + \sum_{\nu=1}^{N-1} \int_{J'_\nu} \frac{(1-r^2)\,d\alpha(t)}{1-2r\cos(\theta-t)+r^2}.$$

We have
$$1 - 2r\cos(\theta - t) + r^2 \geq \sin^2(\theta - t) \geq \frac{4(\theta - t)^2}{\pi^2}, \tag{11.3}$$

since $|\theta - t| \leq 1 < \frac{1}{2}\pi$, and also since θ lies in J_0 and so in J_ν we have for t in J'_ν

$$|\theta - t| \geq 16h\,10^\nu$$

by (8.2). Thus

$$\sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{(1-r^2)\,d\alpha(t)}{1-2r\cos(\theta-t)+r^2} \leq \sum_{\nu=0}^{N-1} \int_{J'_\nu} \frac{2(1-r)\,d\alpha(t)}{\left[\frac{32}{\pi}h\,10^\nu\right]^2}$$

$$= \frac{2\pi^2(1-r)\,\alpha_0}{32^2h^2} \sum_{\nu=0}^{N-1} (50)^{-\nu} < \frac{(1-r)\,\alpha_0}{45h^2} < \frac{\alpha_0}{360h}.$$

Again in J_0 we have $|t - \theta_0| \geq \frac{4}{5}h$ in the intervals in which $\alpha(t)$ is not constant and $|\theta - \theta_0| < \frac{4}{5}h(1-\delta)$. Thus by (11.3)

$$1 - 2r\cos(\theta - t) + r^2 \geq \frac{4}{\pi^2}(\tfrac{4}{5}\delta h)^2,$$

so that
$$\int_{J_0} \frac{(1-r^2)\,d\alpha(t)}{1-2r\cos(\theta-t)+r^2} < \frac{2\alpha_0(1-r)\,25\pi^2}{64\delta^2h^2} < \frac{\alpha_0}{2h},$$

if $(1-r) < 32\delta^2 h/(25\pi^2)$ i.e. certainly if $r > 1 - \delta^2 h/8$. Thus in this case

$$U(z) < \frac{\alpha_0}{2h} + \frac{\alpha_0}{360h} < \frac{1}{2}u(\zeta_0)$$

by (11.1). This completes the proof of Lemma 7.

11.1. It remains to put our results together. If we set

$$F(z) = \int_0^z f(z)\,dz,$$

then with the notation of Lemma 3

$$|F(e^{i\theta_0})| = \left|\int_0^{e^{i\theta_0}} f(z)\,dz\right| = \left|\int_0^1 e^{C\phi(\zeta)}\,e^{i\theta_0}\,d\zeta\right| > e^{Cu(\zeta_0)}\left(\frac{A_1h^3}{C\alpha_0}\right)^{\frac{1}{2}}$$

by Lemma 6, provided that $N \geq A(1+\log^+ 1/C)$. Suppose next that $z_0 = re^{i\theta}$, with

$$r > 1 - \frac{\delta^2 h}{8}, \quad \text{and} \quad |\theta - \theta_0| < \frac{4h}{5}(1-\delta).$$

We integrate $f(z)$ from $e^{i\theta_0}$ so z_0, first along a radius from $e^{i\theta_0}$ to $re^{i\theta_0}$, and then along the smaller arc of $|z| = r$ from $re^{i\theta_0}$ to z_0. On this path we have by (11.2)

$$|f(z)| = e^{CU(z)} < e^{\frac{1}{2}Cu(\zeta_0)}.$$

Also the length of the path is less than 2. Thus

$$|F(z_0)| \geq |F(e^{i\theta_0})| - \left|\int_{z_0}^{z} f(z)\,dz\right| \geq e^{Cu(\zeta_0)}\left[\left(\frac{A_1 h^3}{C\alpha_0}\right)^{\frac{1}{2}} - 2e^{-\frac{1}{2}Cu(\zeta_0)}\right] > \exp\left[(1.1-\delta)\,\alpha_0 C/h\right],$$

by (11.1), provided that $N > A(\delta)(1+\log^+ 1/C)$, where $A(\delta)$ depends only on δ. Thus

$$\int_{\theta_0 - \frac{4}{5}h(1-\delta)}^{\theta_0 + \frac{4}{5}h(1-\delta)} \log^+ |F(re^{i\theta})|\,d\theta > \frac{8h(1-\delta)}{5}(1.1-\delta)\frac{\alpha_0 C}{h}.$$

There are just $2^N = 1/\alpha_0$ different values of θ_0 for fixed N, and their total contribution is thus at least $1.6(1-\delta)(1.1-\delta)C$. For N we have the inequalities

$$A(\delta)(1+\log^+ 1/C) < N \quad \text{and} \quad h = \frac{1}{2}10^{-N} > \frac{8}{\delta^2}(1-r),$$

so that

$$N \log 10 < \log \frac{\delta^2}{16(1-r)}.$$

The number N_0 of distinct values of N satisfying these inequalities itself satisfies

$$N_0 > \frac{1}{\log 10}\left\{\log\frac{1}{1-r} - \log\frac{16}{\delta^2}\right\} - A(\delta)\left(1 + \log^+\frac{1}{C}\right) - 1,$$

$$> \frac{(1-\delta)}{\log 10}\left\{\log\frac{1}{1-r}\right\},$$

if

$$\log\frac{1}{1-r} > A_1(\delta)\left(1+\log^+\frac{1}{C}\right), \tag{11.4}$$

where $A_1(\delta)$ also depends only on δ. In this case

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(re^{i\theta})|\,d\theta > \frac{N_0}{2\pi}1.6(1-\delta)(1.1-\delta)C > \frac{1.6(1-\delta)^2(1.1-\delta)C}{2\pi \log 10}\log\frac{1}{1-r}.$$

We note that $(1.6)(1.1)/(2\pi \log 10) = .121 \ldots$. Thus if δ is a sufficiently small absolute constant and (11.4) holds we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(re^{i\theta})|\, d\theta > .12\, C \log \frac{1}{1-r}.$$

This gives Theorem 3.

References

[1]. Biernacki, M., Sur la caractéristique $T(f)$ des fonctions méromorphes dans un cercle. *Ann. Univ. Mariae Curie-Sklodowska, Sect. A*, 9 (1955), 99–125.

[2]. Chuang, C. T., Sur la comparaison de la croissance d'une fonction méromorphe et de celle de sa dérivée. *Bull. Sci. Math.* (2), 75 (1951), 171–190.

[3]. Frostman, O., Sur les produits de Blaschke. *Kungl. Fysiogr. Sällsk. i Lund Förh.*, 12 no. 15 (1942), 169–182.

[4]. Hardy, G. H., & Littlewood, J. E., A maximal theorem with function-theoretic applications. *Acta Math.*, 54 (1930), 81–116.

[5]. ——, Some properties of fractional integrals. II. *Math. Z.*, 34 (1931), 403–439.

[6]. Hayman, W. K., The minimum modulus of large integral functions. *Proc. London Math. Soc.* (3), 2 (1952), 469–512.

[7]. ——, On Nevanlinna's second theorem and extensions. *Rend. Circ. Mat. Palermo* (2), 2 (1953), 346–392.

[8]. Kennedy, P. B., On the derivative of a function of bounded characteristic. *Quart. J. Math. Oxford*, to be published.

[9]. Classical Function Theory Problems. *Bull. Amer. Math. Soc.*, 68 (1962), 21–24.

[10]. Riesz, F., Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel. I, *Acta Math.*, 48 (1926), 329–343, and II, *Acta Math.*, 54 (1930), 321–360.

Received April 16, 1964