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On the Characteristic Properties of Travel-Time Curves

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(Presented by B. M. Naimark)

Summary

The uniqueness of the determination of a velocity cross-section from the travel-time curves for surface and deep sources was investigated by Gerver & Markushevich (1966, 1967). The Earth was assumed spherically symmetrical with a finite number of waveguides.

The present report states the conditions when a solution of this inverse problem exists.

1. Statement of the problem

The formulation of the problem investigated by Gerver & Markushevich (1966, 1967) is the following (see Fig. 1). The wave propagates from a surface point A with velocity $v(r)$ inside the circle $r \leqslant R$ according to the laws of geometric optics. The travel time is known at every point where the pulse arrives at the surface.

Let us denote by $\tilde{\theta}$ the angular epicentral distance and by α the angle between a ray and radius CA at the point A. Then on the plane θ, t we shall have the travel-time curve $\tilde{\Gamma}_0\{\tilde{\theta}=\tilde{\theta}(\alpha),\ t=t(\alpha)\}$, $\alpha \in (0, \frac{1}{2}\pi)$. The psoblem is to determine $v(r)$ knowing $\tilde{\Gamma}_0$.

The ray may perform some number of revolutions around the centre of circle. Assuming this number k is known, we can consider instead of $\tilde{\Gamma}_0$ the curve

$$\Gamma_0\{\theta=\theta(\alpha),\ t=t(\alpha)\}, \quad \alpha \in (0, \tfrac{1}{2}\pi),$$

where

$$\theta(\alpha) = \tilde{\theta}(\alpha)+2\pi k.$$

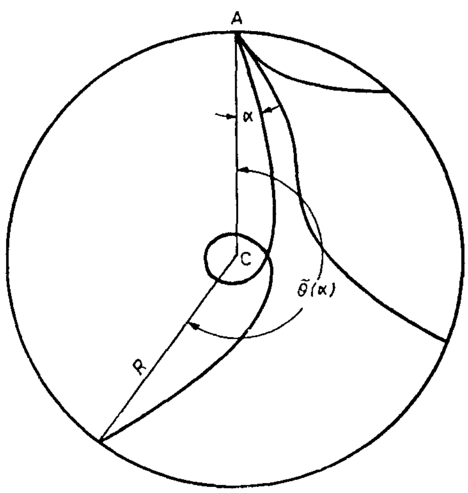

FIG. 1

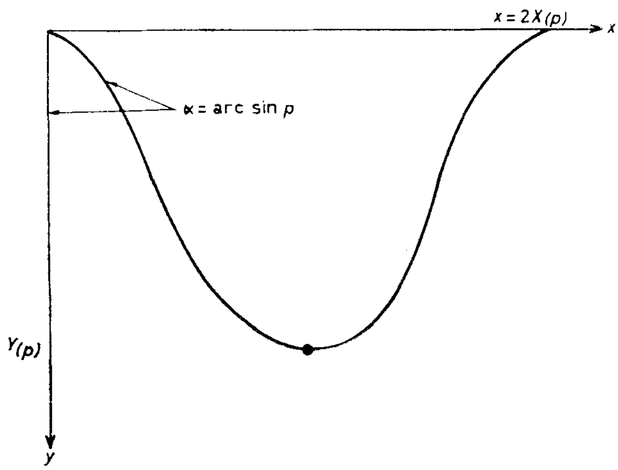

FIG. 2

The transformation

$$x=\frac{R}{v(R)}\theta,\quad y=\frac{R}{v(R)}\ln\frac{R}{r},\quad u(y)=\frac{v(Re^{-\{[v(R)/R]\,y\}})}{v(R)e^{-\{[v(R)/R]\,y\}}} \tag{1.1}$$

reduces our problem to the more simple one for a half-plane $-\infty<x<\infty$, $y\geqslant 0$, where travel-time curves

$$\tilde{\Gamma}\{x=2\tilde{X}(p),\ t=2T(p)\}\ \text{and}\ \Gamma\{x=2X(p),\ t=2T(p)\},\ p\in(0,1)$$

are given, and the velocity $u(y)$ (given $u(0)=1$) is to be determined.

Here $p=\sin\alpha$ is a ray parameter,

$X(p)$ is an abscissa of the deepest point on the ray with parameter p,

$T(p)$ is a travel time along the ray from source to the deepest point,

$$\tilde{X}(p)\equiv X(p)\left[\mathrm{mod}\,\frac{\pi R}{v(R)}\right],\quad 0\leqslant\tilde{X}(p)<\frac{\pi R}{v(R)}\ \text{(see Fig. 2).}$$

2. The case of the surface source

The functions $X(p)$ and $T(p)$ can be determined from Γ (in some cases additional information is needed) (Gerver & Markushevich 1967).

According to Gerver & Markushevich (1966),

$$X(p)=\int_0^{Y(p)}\frac{pu(y)\,dy}{\sqrt{[1-p^2u^2(y)]}},\quad T(p)=\int_0^{Y(p)}\frac{dy}{u(y)\sqrt{[1-p^2u^2(y)]}},\quad p\in(0,1), \tag{2.1}$$

where $Y(p)=\inf\{y, pu(y)\geqslant 1\}$ is the ordinate of the deepest point of the ray with parameter p.

We assume that $u(y)$ is a positive piecewise smooth function which does not go to infinity in any finite interval on the positive semi-axis, and $u(y)\to\infty$ if $y\to\infty$.

We assume also that $u(y)$ forms only a finite number of waveguides. To be definite we assume that the first waveguide does not begin at the surface. Fig. 3 shows $u(y)$ with two waveguides.

If $X(p)$ and $T(p)$ are known, we can regard (2.1) as a system of equations with $u(y)$ unknown. We have already found out (Gerver & Markushevich 1966) that the system has no unique solution. But it may also have no solution at all.

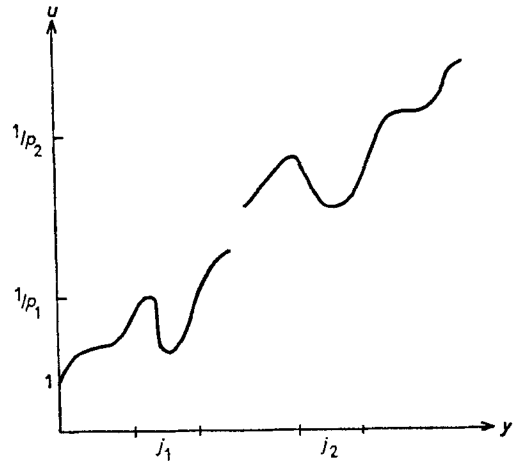

FIG. 3

The system (2.1) has a solution $u(y)$ which satisfies the above limitations if some restrictions are imposed on $X(p)$ and $T(p)$. These restrictions are given by the following theorem.

Theorem 1. The following conditions are necessary and sufficient for the curve $\Gamma\{2X(p), 2T(p)\}$, $p\in(0, 1)$, to be a travel-time curve from a surface source in a half-plane with the velocity $u(y)$, if $u(y)$ satisfies the limitations described above.

A. The functions $X(p)$ and $T(p)$:

(1) are positive,

(2) are differentiable almost everywhere,

(3) $T'(p)-pX'(p)=0$ almost everywhere in (0, 1),

(4) for all points p, where $X(p)$ and $T(p)$ are not differentiable, we have $X(p\pm0)=X(p)=T(p\pm0)=T(p)=\infty$ (except maybe for a finite number of them).

B. The function $\tau(p)=T(p)-pX(p)$:

(1) monotonically decreases,

(2) $\tau(1-0)=0$,

(3) is continuous everywhere, except at the points p_i, $p_1>p_2>p_3>\ldots>p_n$, where it has jumps $\sigma_i=\tau(p_i-0)-\tau(p_i+0)$.

C. The function $\phi(q)=\dfrac{2}{\pi}\displaystyle\int_q^1\frac{X(p)\,dp}{\sqrt{(p^2-q^2)}}$

(1) does not increase for $q\in(0, 1)$,

(2) strictly decreases for $q\in(0, p_1)$,

(3) $\phi(+0)=+\infty$,

(4) there exists a $C>0$, for which $\phi'(q)<-Cq/\sqrt{(p_i^2-q^2)}$ at any $q\in(p_{i+1}, p_i)$, where $\phi'(q)$ is finite, $1\leqslant i\leqslant n$.

(5) function $g(y)$, the inverse function for $\phi(q)$, is a piecewise doubly smooth one.

D. The function $\tau(p)+\displaystyle\int_p^1\sqrt{(z^2-p^2)}\,d\phi(z)$ has a continuous derivative for $p\neq p_i$, $i=1, 2, 3, \ldots, n$.

Some specific features of the conditions A, B, C and D are to be noted.

The function $\tau(p)=T(p)-pX(p)$ is determined only on the set of p where $X(p)$ and $T(p)$ are finite. But this set is dense in (0, 1), and $\tau(p)$ is continuous everywhere, except at a finite number of points, at which it has jumps. Hence $\tau(p)$ is given everywhere in (0, 1). The number of jumps is equal to the number of waveguides.

If a waveguide begins at the surface, then $p_1=1$. In this case condition B.2 is replaced by

B.2′. $$\tau(1-0)=\sigma_1>0$$

and condition C.1 becomes useless.

It is difficult to verify the condition D; let us introduce instead of it the condition D′.

D′. $X(p)=\infty$ on not more than a numerable set of $p\in(0, 1)$. This condition is not the necessary one, but conditions A, B, C and D′ together are sufficient for Γ to be a travel-time curve.

The condition C implies a corollary:

$$\overline{\lim_{p\to p^0-0}} X(p)\geqslant \underline{\lim_{p\to p^0+0}} X(p) \text{ at any } p^0\in(0, 1).$$

Therefore the curve in Fig. 4 is not a travel-time curve.

3. The case of a deep source

Now we shall consider the travel-time curve from a deep source. Using the transformation (1.1) we can again reduce the problem to that of the half-plane. Let the depth of source be $y=d$.

Let

$$f(y)=\left(\sup\{u(y^0),\ 0\leqslant y^0\leqslant y\}\right)^{-1},\quad f(d-0)=P,\quad f(d+0)=Q.$$

It is clear that $P\geqslant Q$. The rays which go up from the source give the part of travel-time curve $\Gamma_1\{X_1(p),\ T_1(p)\}$, where

$$X_1(p)=\int_0^d\frac{pu(y)\,dy}{\sqrt{[1-p^2u^2(y)]}},\quad T_1(p)=\int_0^d\frac{dy}{u(y)\sqrt{[1-p^2u^2(y)]}},\quad p\in[0, P). \tag{3.1}$$

If $X_1(Q)<\infty$, then Γ_1 is an arc $0I$ for $Q=P$ or an arc $0J$ for $Q<P$ (see Fig. 5).

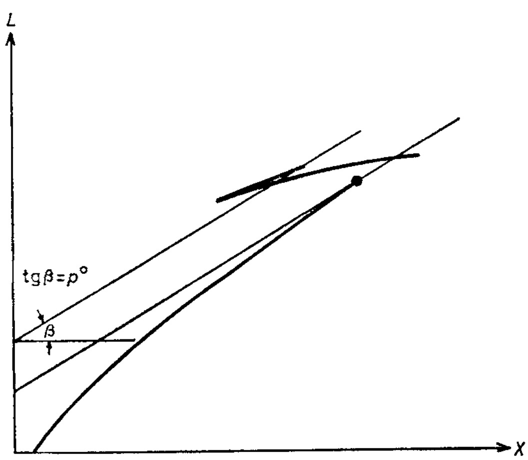

FIG. 4

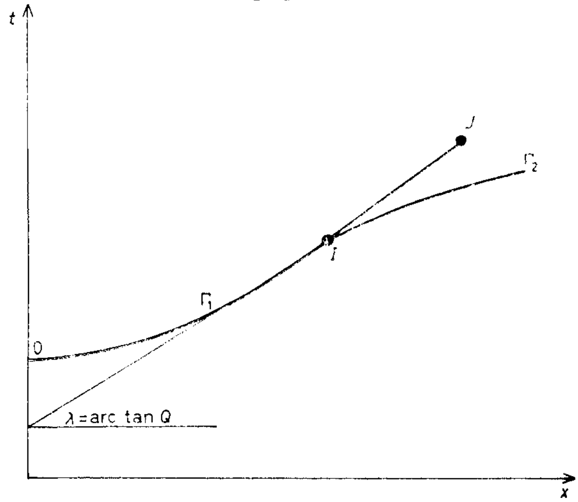

FIG. 5

Introducing the function $H(r)=mes\{y,\ y\leqslant d,\ u(y)\leqslant r\}$ we have:

$$X_1(p)=\int_0^{p^{-1}}\frac{pr\,dH(r)}{\sqrt{(1-p^2r^2)}},\quad T_1(p)=\int_0^{p^{-1}}\frac{dH(r)}{r\sqrt{(1-p^2r^2)}},\quad p\in[0,\ P). \tag{3.2}$$

The equations (3.2) can be treated as a system with unknown $H(r)$. It is evident that $H(r)$ is a monotonically non-decreasing function and $H(0)=0$. The uniqueness of solution of (3.2) is proved by Gerver & Markushevich (1967). But this solution will be non-decreasing if some restrictions are imposed on $X_1(p)$ and $T_1(p)$. Our purpose is to find these restrictions. Evidently, it is equivalent to determining such properties of Γ_1 which are necessary for Γ_1 to be a part of travel-time curve from a deep source. The restrictions on $u(y)$ in this case will be much less severe than in Section 2: the function $u(y)$, $y\in[0,\ D]\supset[0,\ d]$ is assumed to be positive, bounded and measurable.

Let us introduce the values $\beta_i=\int_0^1 v^i\,T_1(vP)\,dv,\ i=1,2,\ldots$.

Further C_i, $i=1,2,\ldots$, are to be determined from the triangular system of equations:

$$\beta_{2k+1}=\frac{k!}{(2k+1)!}\sum_{i=1}^{k}\frac{(2k-i+1)!}{(k-i+1)!}C_i,\quad k=0,\ 1,\ldots\ .$$

Let $C_0=\tfrac{1}{2}T_1(0)$.

Lemma. The equation

$$T_1(p)=\int_0^{P^{-1}}\frac{dH(i)}{2\sqrt{(1-p^2i^2)}},\quad p\in[0,\ P)$$

with nondecreasing $H(v)$ is satisfied if and only if the values C_i are moments of the function

$$H_1(z)=\int_1^z\frac{Pt}{4\sqrt{(t-1)}}\,dH\left(\frac{2\sqrt{(t-1)}}{Pt}\right),\quad 1\leqslant z\leqslant 2$$

that is if

$$C_i=\int_1^2 z^i dH_1(z)\,. \tag{3.3}$$

Theorem 2. The curve $\Gamma_1\{X_1(p), T_1(p)\}$, $p \in [0, P)$ is part of the travel-time curve from a deep source if the following necessary and sufficient conditions are satisfied:

A. The quadratic forms

$$\sum_0^m C_{i+j} x_i x_j; \quad \sum_0^m (3C_{i+j+1} - 2C_{i+j} - C_{i+j+2}) x_i x_j$$

are not negative for any m.

B. The functions $X_1(p)$ and $T_1(p)$ are differentiable at $p \in [0, P)$.

C. $T_1'(p) - pX_1'(p) = 0$,

D. $X_1(0) = 0$.

The condition A is equivalent to condition A′ of non-negativity of the forms

$$\sum_0^m (C_{i+j+1} - C_{i+j}) x_i x_j; \quad \sum_0^m (2C_{i+j} - C_{i+j+1}) x_i x_j$$

for any m. It means (Krein 1951) that C_i are the moments (3.3).

The above two theorems also give us necessary and sufficient conditions

(a) for a curve on the x, t plane to be a travel-time curve of a wave reflected from a deep interface,

(b) for a velocity cross-section to correspond to the part Γ_2 of a travel-time curve from a deep source.

It is easy to see this from the following. A travel-time curve of reflected waves is a twice magnified curve Γ_1 from the source placed at the same depth as the reflected boundary. As to the curve $\Gamma_2\{X_2(p), T_2(p)\}$, $p \in (0, P)$, it is shown by Gerver & Markushevich (1967) that functions

$$X_d(p) = \frac{X_2(p) - X_1(p)}{2} \quad \text{and} \quad T_d(p) = \frac{T_2(p) - T_1(p)}{2}$$

are analogous to $X(p)$ and $T(p)$, if the surface is moved to the depth $y = d$. Consequently, they must satisfy the conditions of Theorem 1 with some modifications because $p \in (0, P)$, but not $(0, 1)$.

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