# ON THE CHARACTERISTICS OF THE GENERAL QUEUEING PROCESS, WITH APPLICATIONS TO RANDOM WALK[1]

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**Summary.** The authors continue the study (initiated in [1]) of the general queueing process (arbitrary distributions of service time and time between successive arrivals, many servers) for the case ($\rho < 1$) where a limiting distribution exists. They discuss convergence with probability one of the mean waiting time, mean queue length, mean busy period, etc. Necessary and sufficient conditions for the finiteness of various moments are given. These results have consequences for the theory of random walk, some of which are pointed out.

This paper is self-contained and may be read independently of [1]; the necessary results of [1] are quoted. No previous knowledge of the theory of queues is required for reading either [1] or the present paper.

**Introduction.** We recapitulate very briefly some of the results obtained in [1] in the notation of [1] to which we shall adhere without further mention.[2]

Let $S$ be the totality of points $(x_1, x_2, \cdots, x_s)$ of Euclidean $s$-space such that $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_s$. Let $x$ and $y$ be generic points of $S$. Occasionally another letter will represent a point in $S$; it will always be clear from the context when this is so; for example, $O$ will frequently denote the origin in $s$-space.

For $i \geqq 1$, let $t_i \geqq t_0 = 0$ be the time of arrival of the $i$th person at a system of $s \geqq 1$ machines, where he waits his turn until a machine is available to serve him, say at time $t_i + w_{i1} \geqq t_i$. This machine is then occupied by him for time $R_i \geqq 0$. Let $g_i = t_i - t_{i-1}$. $\{R_i\}$ and $\{g_i\}$ are independent sequences of identically distributed and independent chance variables. An $s$-dimensional random walk $\{w_i\}$, with $w_{i1}$ its first component, is useful for the study of the theory of queues. The random walk $\{w_i\}$ is constructed as follows: $w_i = (w_{i1}, \cdots, w_{is})$. Unless the contrary is explicitly stated we have $w_1 = O$. To obtain $w_{i+1}$ from $w_i$, reorder in ascending size the quantities

$$(w_{i1} + R_i - g_{i+1})^+, \qquad (w_{i2} - g_{i+1})^+, \qquad (w_{i3} - g_{i+1})^+, \cdots, (w_{is} - g_{i+1})^+.$$

The resulting sequence is $w_{i+1}$. We have $w_{i1} \leqq w_{i2} \leqq \cdots \leqq w_{is}$ for all $i$. As usual, $a^+ = (a + |a|)/2$. The times $t_i + w_{ij}$ $(1 \leqq j \leqq s)$ are easily seen to be the earliest times after (or at) $t_i$ at which the $s$ machines have finished serving those of the first $s - 1$ arrivals which they serve.

Let $F_i(F_i^*)$ be the d.f. (distribution function) of $w_i(w_{i1})$. It was shown in [1] that $F(x) = \lim_{i\to\infty} F_i(x)$ exists and satisfies a certain integral equation (I.E.);

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[2] The definition of $\nu$ on p. 14 of [1] should be modified trivially to read $\nu = 1$ in the case $b = \infty$.

$F^*(z) = \lim_{i\to\infty} F_i^*(z)$ also exists. Assume $\rho = ER_i / sEg_i$ exists. $F$ and $F^*$ are d.f.'s if $\rho < 1$, and $F$ is then the unique d.f. solution to the I.E. Except in the trivial case where $P\{R_i = sg_i\} = 1$, if $\rho \geqq 1$ then $F \equiv 0 \equiv F^*$, and the I.E. has no d.f. solution. Always $F^*(z) = F(z, \infty, \cdots, \infty)$. Results on the limiting length of the line are also proved in [1].

Let $F_i(x \mid y)$ be the d.f. of $w_i$, given that $w_1 = y$; i.e.,

$$F_i(x \mid y) = P\{w_i \leqq x \mid w_1 = y\}.$$

It was proved in [1] that, for all $y \in S$,

$$\lim_{i\to\infty} F_i(x \mid y) = F(x).$$

Throughout this paper we shall assume that $\rho < 1$. The case $\rho \geqq 1$ has little interest and was essentially disposed of in [1]; results proved in the present paper are trivial when $\rho \geqq 1$. Throughout this paper we shall assume that $Eg_1 < \infty$. However, it can be shown, always easily and sometimes trivially, that all the results of [1] and all the queueing results of the present paper except Theorem 3 are valid also when $Eg_1 = \infty$. In order to eliminate the completely trivial we also assume, as was done in [1], that $ER_1 > 0$, $Eg_1 > 0$. Since $\rho < 1$ we have then $0 < ER_1 < \infty$, $0 < Eg_1 < \infty$.

In two or three places below we shall cite the first paragraph of Section 3 of [1]. To ease the reader's task we now quote this paragraph in full:

Let $\varphi_j(a, b, c)$, $j = 1, \cdots, s$ be the value of $w_{(i+1),j}$ when $w_i = a$, $R_i = b$, $g_{i+1} = c$. If $d$ is a point in $s$-space, we shall say that $a \leqq d$ if every coordinate of $a$ is not greater than the corresponding coordinate of $d$. If now $a \leqq d$, then obviously

$$\varphi_j(a, b, c) \leqq \varphi_j(d, b, c)$$

for $1 \leqq j \leqq s$. Applying this argument $k$ times we obtain the following result: Let $R_{i+j-1} = b_{i+j-1}$, $g_{i+j} = c_{i+j}$, $j = 1, \cdots, k$. Let $w_{i+k} = e_1$ when $w_i = a_1$, and let $w_{i+k} = e_2$ when $w_i = a_2$. Then $a_1 \leqq a_2$ implies $e_1 \leqq e_2$.

The results of [1] also imply that $F(x)$ determines a stationary and metrically transitive flow; this is the process $\{w_n^0\}$ defined in Section 1, below, where the relevant references to [1] are given.

**1. Convergence of the mean waiting time.** Let $k$ be any positive number. Define $W_n = \sum_{i=1}^{s} w_{ni}$. Since $w_{ni}$ is a nonnegative chance variable and $F_n(x) \to F(x)$, we easily have that

$$(1.1)\qquad \begin{aligned} \liminf_n \, (Ew_{ni})^k &\geqq \int (x_i)^k \, dF(x), \\ \liminf_n \, E(W_n)^k &\geqq \int (x_1 + \cdots + x_s)^k \, dF(x), \end{aligned}$$

where, of course, the right members may be infinite. From the fact (proved in

[1]) that $F_n(x)$ approaches $F(x)$ from above for every $x$, we have that

$$E(w_{ni})^k \leqq \int (x_i)^k \, dF(x).$$

Hence

$$\lim_n E(w_{ni})^k = \int (x_i)^k \, dF(x). \tag{1.2}$$

Let $F_n^W(z \mid y)$ be the d.f. of $W_n$, given that $w_1 = y(\varepsilon S)$. Hence $F_n^W(z \mid 0)$ is the d.f. of $W_n$. Then

$$F_{n+1}^W(z \mid 0) - F_n^W(z \mid 0) = \int [F_n^W(z \mid y) - F_n^W(z \mid 0)] \, dF_2(y).$$

It follows from the first paragraph of Section 3 of [1] that, if $y \varepsilon S$, the integrand in the last integral is never positive for any $z$. Hence the left member in the last equation is never positive for any $z$. Hence $F_n^W(z \mid 0)$ approaches its limit (which is a distribution function obtainable from $F(x)$ in an obvious way) from above. Consequently, as before,

$$E(W_n)^k \leqq \int \left( \sum_{i=1}^{s} x_i \right)^k dF(x).$$

From this and (1.1) we obtain

$$\lim_n E(W_n)^k = \int \left( \sum_{i=1}^{s} x_i \right)^k dF(x) = m_k' \text{ (say)}.$$

The question as to when $m_k' < \infty$ will be discussed in a later section. We define

$$m_k = \int (x_1)^k \, dF(x),$$

and

$$V_{nk} = \frac{1}{n} \sum_{i=1}^{n} (w_{i1})^k.$$

We now prove

**Theorem 1.** *We have, for any positive $k$,*

$$P\{\lim_{n \to \infty} V_{nk} = m_k\} = 1. \tag{1.3}$$

**Proof.** Let $w_1^0$ be an $s$-dimensional chance variable with the d.f. $F(x)$, and let $w_{n+1}^0$ be obtained from $w_n^0$ by using $R_n$ and $g_{n+1}$ in exactly the same manner as one obtains $w_{n+1}$ from $w_n$. Thus $w_n^0$ pertains at time $t_n$. Then the process $\{w_n^0, n = 1, 2, \cdots\}$ is easily seen to be stationary, because $F(x)$ satisfies the integral equation derived in [1]. (see Section 3 of [1] for details). It is proved in Section 8 of [1] that $F(x)$ is the only d.f. which satisfies the integral equation.

We shall show that this implies easily that there cannot be a Borel set $B$ in $s$-dimensional Euclidean space such that

$$0 < \int_B dF < 1,$$

and $w_1^0 \varepsilon B$ implies with probability one that $w_n^0 \varepsilon B$, $n \geqq 2$. For let $\bar{B}$ be the complement of $B$, and $F(x \mid B)$ and $F(x \mid \bar{B})$ be, respectively, the conditional distribution functions on $B$ and $\bar{B}$ implied by $F(x)$. Then $F(x \mid B)$ satisfies the integral equation. On a set of $w_1^0$ of probability one according to $F(x \mid \bar{B})$, $w_n^0 \varepsilon \bar{B}$ for $n \geqq 2$ with probability one, since otherwise $P\{w_n^0 \varepsilon B\}$ (when $F$ is the distribution function of $w_1^0$) would not be independent of $n$, contradicting the stationarity of $\{w_n^0\}$. Hence $F(x \mid \bar{B})$ must also satisfy the integral equation. Clearly, $F(x \mid B)$ and $F(x \mid \bar{B})$ are not identical, in contradiction to the fact that $F(x)$ is the only d.f. that satisfies the integral equation. From the fact that there is no invariant set $B$ such that $0 < \int_B dF < 1$, the fact that $w_n^0$ is a Markoff process, and Theorem 1.1, page 460 of [6] (which asserts that any set in the space of the Markoffian chance variables $w_1^0$, $w_2^0$, $\cdots$ that is invariant under a shift transformation differs from a set $B$ by a set of probability zero), we conclude that the process $w_n^0$ is metrically transitive. Hence, by the ergodic theorem,

$$P\{\lim_{n\to\infty} V_{nk}^0 = m_k\} = 1, \tag{1.4}$$

where

$$V_{nk}^0 = \frac{1}{n} \sum_{i=1}^{n} (w_{i1}^0)^k,$$

and of course $w_{i1}^0$ is the first component of the vector $w_i^0$.

From the argument in the first paragraph of Section 3 of [1], it follows that always

$$V_{nk} \leqq V_{nk}^0. \tag{1.5}$$

Hence

$$P\{\limsup_{n\to\infty} V_{nk} \leqq m_k\} = 1. \tag{1.6}$$

We shall prove that also

$$P\{\liminf_{n\to\infty} V_{nk} \geqq m_k\} = 1. \tag{1.7}$$

This will prove the theorem.

We shall now deduce (1.7) from (1.4), and for this purpose divide the argument into consideration of the four cases of Section 8 of [1]. As there defined, denote by $[a, b]$ and $[c, d]$ the smallest closed intervals for which

$$P\{a \leqq R_1 \leqq b\} = P\{c \leqq g_1 \leqq d\} = 1.$$

Of course, $b$ or $d$ or both may be $+\infty$.

CASE 1: $b > sc$. Let $t$ be so large that the point $T = (t, t, \cdots, t)$ of $S$ is such that

$$\int_{x<T} dF(x) > 0.$$

It follows from (1.4) that there exists in $S$ a point $x < T$ such that

$$P\{\lim_{n\to\infty} V^0_{nk} = m_k \mid w^0_1 = x\} = 1. \tag{1.8}$$

It is proved in [1] that there exists an integer $r$ such that $P\{w_{(sr)} > T\} > 0$, say $=\alpha$. From this it follows that

$$P\{w_n > T \quad \text{for at least one } n\} = 1. \tag{1.9}$$

Let $h$ be the smallest index $n$ for which $w_n > T$; $h < \infty$ with probability one. Obviously $R_h, R_{h+1}, \cdots$ and $g_{h+1}, g_{h+2}, \cdots$ are distributed independently of $h$ and $w_h$, and have the same distribution as $R_1, R_2, \cdots$ and $g_2, g_3, \cdots$. Consequently, if we define, for $n > h$,

$$V_{nk}(h) = \frac{(w_{h1})^k + (w_{(h+1),1})^k + \cdots + (w_{n,1})^k}{n},$$

we have, using (1.8) and the argument in the first paragraph of Section 3 of [1], that

$$P\{\liminf_{n\to\infty} V_{nk}(h) \geqq m_k\} = 1. \tag{1.10}$$

Obviously from the definition of $V_{nk}(h)$ it follows that

$$P\{\lim_{n\to\infty} (V_{nk}(h) - V_{nk}) = 0\} = 1. \tag{1.11}$$

The desired result (1.7) follows from (1.10) and (1.11).

CASE 2: $a < d$. It is proved in Section 8 of [1] that, in this case,

$$P\{w^0_n = 0 \quad \text{for some } n \geqq 1\} = 1. \tag{1.12}$$

The desired result (1.3) follows from (1.4) and (1.12) by means of an argument like that in Case 1.

CASE 3: $c = d \leqq a = b < sc$. It is proved in [1] that in this case there is a point in $S$, there called $\bar{w}$, such that

$$P\{w_n = w_{n+1} = \cdots = \bar{w} \quad \text{for some } n \geqq 1\} = 1. \tag{1.13}$$

The desired result (1.3) follows at once.

CASE 4: *$d \leqq a$, $b \leqq sc$, and either $a < b$ or $c < d$.* It is proved in [1] that, in this case, there exists an $\epsilon > 0$ such that the set

$$\Gamma^\epsilon = \{y \mid y \in S, y \leqq \bar{w}^\epsilon\},$$

where

$$\bar{w}^\epsilon = (0, u^\epsilon_{s-1}, u^\epsilon_{s-2}, \cdots, u^\epsilon_1)$$

and

$$u_j^\epsilon = \max(0, b - jc - \epsilon),$$

has the following properties:

(a) $$P\{w_n^0 \varepsilon \Gamma^\epsilon \text{ for some } n \geqq 1\} = 1.$$

(This implies at once that

(1.14) $$\int_{\Gamma^\epsilon} dF(x) > 0.)$$

(b) $$P\{w_s > \bar{w}^\epsilon\} > 0.$$

(This implies, using the argument in the first paragraph of Section 3 of [1], that

(1.15) $$P\{w_n > \bar{w}^\epsilon \text{ for at least one } n > 1\} = 1.)$$

The desired result now follows exactly as in Case 1, the place of $T$ being taken by $\bar{w}^\epsilon$.

In exactly the same manner as that employed in this section we could have proved that

(1.16) $$P\left\{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (W_i)^k = m_k'\right\} = 1$$

and similar theorems about other moments.

**2. Generalization of the lemma of Section 4 of [1].** We shall prove the following essential generalization of the fundamental lemma of Section 4 of [1] both for its use as a tool in a subsequent section and for its intrinsic interest:

LEMMA. *If, for any positive* $k > 0$,

(2.1) $$ER_1^{k+1} < \infty,$$

*then*

(2.2) $$\sup_n E(w_{ns} - w_{n1})^k < \infty;$$

*or, what is equivalent,*

(2.3) $$\sup_n E\left((s-1)w_{ns} - \sum_{j=1}^{s-1} w_{nj}\right)^k < \infty.$$

PROOF. Define $Y_i$ exactly as in (4.5) of [1], i.e.,

$$Y_i = \max[(s-1)R_i, (s-1)R_{i-1} - R_i, (s-1)R_{i-2} - R_{i-1} - R_i, \cdots, (s-1)R_1 - R_2 - \cdots - R_i].$$

Then (4.6) of [1] is

(2.4) $$L(y', n) = P\{Y_n \leqq y'\} = P\{R_1 \leqq hy', R_2 \leqq h(R_1 + y'), \cdots, R_n \leqq h(R_1 + \cdots + R_{n-1} + y')\},$$

where $h = (s-1)^{-1}$. Let $H(z)$ be the d.f. of $R_1$.

Define $L(y', 0) = 1$. Obviously $L(y', n)$ is nonincreasing in $n$ and, for $n \geqq 0$,

$$
\begin{aligned}
L(y', n) - L(y', n + 1) &= P\{Y_n \leqq y', R_{n+1} > h(R_1 + \cdots + R_n + y')\} \\
&\leqq P\{R_{n+1} > h(R_1 + \cdots + R_n + y')\} \\
&\leqq E\{1 - H(h[R_1 + \cdots + R_n + y'])\}.
\end{aligned} \tag{2.5}
$$

Hence

$$
\begin{aligned}
1 - L(y', n) &= \sum_{i=1}^{n} [(Ly', i - 1) - L(y', i)] \\
&\leqq \sum_{i=0}^{\infty} E\{1 - H(h[R_1 + \cdots + R_i + y'])\}.
\end{aligned} \tag{2.6}
$$

Let $d$ be a small positive number and define

$$
\begin{aligned}
D_i &= d \text{ when } R_i \geqq \frac{d}{h} \\
D_i &= 0 \text{ otherwise.}
\end{aligned}
$$

We choose $d$ so small that $d < 1$ and

$$p = P\{D_1 = d\} > 0.$$

(We have earlier excluded the trivial case where $R_i = 0$ with probability one.) Since $R_i \geqq D_i/h$, if we replace the former by the latter in the right member of (2.6) we do not diminish any term of this member. It is well known (e.g., [2], p. 101) from approximations to the binomial distribution that, for suitable positive $c_1$, $c_2$, we have

$$P\left\{D_1 + \cdots + D_n \leqq \frac{npd}{2}\right\} < c_1 e^{-c_2 n} \tag{2.7}$$

When $k \geqq 1$ we have, from (2.6),

$$
\begin{aligned}
E(Y_n)^k &\leqq k \sum_{j=0}^{\infty} (j + 1)^{k-1} P\{Y_n > j\} \\
&\leqq k \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (j + 1)^{k-1} E\{1 - H(h[R_1 + \cdots R_i + j])\} \\
&\leqq k \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (j + 1)^{k-1} E\{1 - H(D_1 + \cdots D_i + j)\} \\
&\leqq k \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (j + 1)^{k-1} E\{1 - H(D_1 + \cdots + D_{i+j})\} \\
&\leqq \sum_{j=0}^{\infty} (j + 2)^{k} E\{1 - H(D_1 + \cdots + D_j)\}.
\end{aligned} \tag{2.8}
$$

We have now, applying (2.7) to the right member of (2.8),

$$E(Y_n)^k \leqq c_1 \sum_{j=0}^{\infty} (j+2)^k e^{-c_2 j} + \sum_{j=0}^{\infty} (j+2)^k \left(1 - H\left(\frac{jpd}{2}\right)\right). \tag{2.9}$$

The first series on the right of (2.9) obviously converges. Now consider the second. We have

$$\begin{aligned} \sum_{j=0}^{\infty} (j+2)^k \left(1 - H\left(\frac{jpd}{2}\right)\right) &= \sum_{j=0}^{\infty} (j+2)^k P\left\{R_1 > \frac{jpd}{2}\right\} \\ &\leqq \left(\frac{2}{pd}\right)^{k+1} E(R_1+2)^{k+1} \end{aligned} \tag{2.10}$$

In [1] (relation (4.5)) it is shown that

$$\left((s-1)w_{ns} - \sum_{j=1}^{s-1} w_{nj}\right) \leqq Y_{n-1}$$

Hence (2.3) and the lemma follow for $k \geqq 1$. The proof for $0 < k < 1$ is almost the same; only a few obvious changes are needed in (2.8), (2.9), and (2.10).

**3. Finiteness of $m_k'$.** Of great interest is the question of when $m_k$ is finite. In this section we shall give a sufficient condition for $m_k'$ to be finite (and hence a fortiori for $m_k \leqq m_k'$ to be finite). We shall later see that this condition is essentially necessary for $m_k'$ to be finite.

THEOREM 2. *If* $k > 0$, *and*

$$ER_1^{k+1} < \infty, \tag{3.1}$$

*then*

$$m_k' < \infty, \tag{3.2}$$

*and*

$$m_k < \infty. \tag{3.3}$$

PROOF. We assume that there exists a number $T > 0$ such that $g_1 < T$ with probability one. When we bear in mind how $w_{n+1}$ is related to $w_n$, it follows immediately that, if Theorem 2 holds in this case, it a fortiori holds in general.

In order to carry out the proof we shall assume that $m_k' = \infty$ and obtain a contradiction. Let $A$ be the set $\{x \mid x_1 < T\}$. Then from (2.2) we obtain that

$$\sup_n \int_A (x_s)^k \, dF_n(x) < \infty, \tag{3.4}$$

and hence

$$\sup_n \int_A (x_1 + \cdots x_s)^k \, dF_n(x) < \infty. \tag{3.5}$$

From the manner in which we obtain $w_{n+1}$ from $w_n$ we have that

$$W_{n+1} = W_n + R_n - sg_{n+1} \tag{3.6}$$

if $w_{n1} \geqq T$, and always we have

$$W_{n+1} \leqq W_n + R_n . \tag{3.7}$$

We now note the inequality (2.15.1) on page 39 of [7], which states that $r > 1$, $x \geqq 0$, $y \geqq 0$ imply that

$$x^r - y^r \leqq rx^{r-1}(x - y). \tag{3.8}$$

Putting $r = k + 1$, $x = W_{n+1}$, $y = W_n$, we have, from (3.6),

$$\begin{aligned} W_{n+1}^{k+1} - W_n^{k+1} &\leqq (k + 1)(W_n + R_n - sg_{n+1})^k(R_n - sg_{n+1}) \\ &= (k + 1)W_n^k \left\{ \left(1 + \frac{R_n - sg_{n+1}}{W_n}\right)^k (R_n - sg_{n+1}) \right\}. \end{aligned} \tag{3.9}$$

Consider the expression in brackets in the last expression of (3.9). By (3.1), the boundedness of $g_{n+1}$, and the independence of $W_n$ from $g_{n+1}$ and $R_n$, the conditional expected value of this bracketed expression, given $W_n$, tends to $E(R_n - sg_{n+1}) < 0$ as $W_n \to \infty$. Hence, if $EW_n^k \to \infty (= m_k')$ as $n \to \infty$, (3.9) implies that

$$\lim_{n\to\infty} E\{W_{n+1}^{k+1} - W_n^{k+1} \mid w_{n1} \geqq T\} = -\infty. \tag{3.10}$$

Similarly, putting $x = W_n + R_n$, $y = W_n$, and noting that $(a + b)^k \leqq 2^k(a^k + b^k)$ if $a, b, k \geqq 0$, (3.7) yields

$$\begin{aligned} W_{n+1}^{k+1} - W_n^{k+1} \leqq (W_n + R_n)^{k+1} - W_n^{k+1} &\leqq (k + 1)(W_n + R_n)^k R_n \\ &\leqq (k + 1)2^k(W_n^k + R_n^k)R_n . \end{aligned} \tag{3.11}$$

From (3.1), (3.5) and the independence of $R_n$ and $W_n$, we conclude that there is a number $c < \infty$ such that

$$\sup_n E\{W_{n+1}^{k+1} - W_n^{k+1} \mid w_{n1} < T\} < c. \tag{3.12}$$

From (3.10), (3.12), and the fact that (3.5) and $m_k' = \infty$ imply that $\bar{A}$ has probability $>\epsilon > 0$ according to $F_n$ for all sufficiently large $n$, we conclude that there is an integer $N_0$ such that $EW_{n+1}^{k+1} \leqq EW_n^{k+1}$ for $n \geqq N_0$. Since, for $n \leqq N_0$, $EW_n^{k+1} \leqq E(R_1 + \cdots R_{N_0})^{k+1} < \infty$, we conclude that $\sup_n EW_n^{k+1} < \infty$, contradicting the assumption that $m_k' = \infty$. This completes the proof.

**4. Necessity of the condition (3.1).** The present section is devoted to the proof of

THEOREM 3. *If, for any positive* $k$,

$$ER_1^{k+1} = \infty \tag{4.1}$$

*and* $Eg_1 < \infty$, *then*

$$m_k' = \infty . \tag{4.2}$$

It will easily be seen from our proof that Theorem 3 is a fortiori true if $\rho \geqq 1$. Only the case $\rho < 1$ requires proof and this is the case we shall consider.

PROOF. Let $m$ be so large that

$$\int_M dF(x) = \alpha > 0$$

where $M$ is the set of all points $(x_1, x_2, \cdots, x_s)$ in $S$ such that $x_s \leqq m$. We have already remarked in Section 1 that the process $\{w_n^0\}$ there defined is stationary and metrically transitive. Let $\nu_1^0, \nu_2^0, \cdots$ be the indices $n$ for which $w_n^0 \in M$, and define

$$\mu_i^0 = \nu_{i+1}^0 - \nu_i^0 .$$

It follows from the ergodic theorem that

$$E\mu_i^0 = \frac{1}{\alpha} < \infty.$$

Let $\{w_n'\}$ be the process obtained from $\{w_n\}$ as follows: $w_1' = w_1 = 0$. Thereafter $w_n' = w_n$ until the first index $n$, say $\nu_1'$, such that $w_{\nu_1'} \in M$; define $w_{\nu_1}' = 0$. We now obtain each successive $w_{n+1}'$ from its predecessor $w_n'$ by using $R_n$ and $g_{n+1}$ in exactly the same manner as $w_{n+1}$ is obtained from $w_n$, until the next index, say $\nu_2'$, for which $w_{\nu_2}'$ would be in $M$; instead set $w_{\nu_2}' = 0$. Continue in this manner to define $\{w_n'\}$. Define $\mu_i' = \nu_{i+1}' - \nu_i'$. Then $\mu_1', \mu_2', \cdots$ are independent, identically distributed chance variables. It follows from the construction of the process $\{w_n'\}$ and the first paragraph of Section 3 of [1] that $E\mu_i' \leqq E\mu_i^0$. Hence $E\mu_i'$ is finite. It follows from the strong law of large numbers that

$$P\left\{\lim_{n\to\infty} \frac{\nu_n'}{n} = E\mu_1'\right\} = 1. \tag{4.3}$$

We shall later show that

$$P\left\{\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} (w_{is}')^k = \infty\right\} = 1. \tag{4.4}$$

Since $w_n' \leqq w_n$ it follows at once that

$$P\left\{\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} (w_{is})^k = \infty\right\} = 1. \tag{4.5}$$

Hence

$$P\left\{\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} (W_i)^k = \infty\right\} = 1. \tag{4.6}$$

The desired result (4.2) follows from (1.16) and (4.6).

It remains to prove (4.4). Let $j(n)$ be defined for all integral $n$ by

$$\nu_{j(n)}' \leqq n < \nu_{j(n)+1}' .$$

We shall later prove that

$$(4.7)\qquad E\{(w'_{1s})^k + (w'_{2s})^k + \cdots + (w'_{\mu'_1,s})^k\} = \infty.$$

From this and the strong law of large numbers it follows that

$$(4.8)\qquad P\left\{\lim_{n\to\infty} (j(n))^{-1} \sum_{i=1}^{\nu'_{j(n)}} (w'_{is})^k = \infty\right\} = 1.$$

From (4.3) and (4.8) we obtain that

$$(4.9)\qquad P\left\{\lim_{n\to\infty} (\nu'_{j(n)})^{-1} \sum_{i=1}^{\nu'_{j(n)}} (w'_{is})^k = \infty\right\} = 1.$$

From (4.9) we have at once that

$$(4.10)\qquad P\left\{\lim_{n\to\infty} (\nu'_{j(n)})^{-1} \sum_{i=1}^{n} (w'_{is})^k = \infty\right\} = 1.$$

Also

$$(4.11)\qquad P\left\{\lim_{n\to\infty} \frac{n}{\nu'_{j(n)}} = 1\right\} = 1.$$

From (4.10) and (4.11) we have the desired result (4.4).

It remains to prove (4.7). Let $N$ be an integer so large that

$$(4.12)\qquad P\left\{\sum_{i=1}^{n} g_i < 2nEg_1 \text{ for all } n \geqq N\right\} > \tau > 0$$

The existence of such an $N$ follows from the strong law of large numbers. We may also assume $N$ so large that $2NEg_1 > m$. Let $T = 4NEg_1$. Suppose that $t \geqq T$ and the largest integer contained in $(t/4Eg_1)$ is $t'$. Then $t' \geqq N$, and (4.12) implies that the conditional probability of the event $A_1$,

$$(4.13)\qquad A_1 = \{\mu'_1 > t' \text{ and } w'_{ns} > 2t'Eg_1 \text{ for } 2 \leqq n \leqq t'\},$$

given that $w'_{2s} = t$, is greater than $\tau$. ($\mu'_1 > t'$ is implied by the other events in (4.13).) When the event $A_1$ occurs, we have

$$(4.14)\qquad \sum_{n=1}^{\mu'_1} (w'_{ns})^k \geqq \sum_{n=1}^{t'} (w'_{ns})^k > t'(2t'Eg_1)^k \geqq ct^{k+1}$$

with $c > 0$. From (4.1) and the construction of the process $\{w'_n\}$ we have (by considering $((R_1 - g_2)^+)^{k+1}$ on the set where $g_2 < c$ where $c < \infty$ is chosen so that $P\{g_2 < c\} > 0$) that

$$(4.15)\qquad E(w'_{2s})^{k+1} = \infty.$$

The desired result (4.7) follows from (4.14) and (4.15). This completes the proof of Theorem 3.

The following theorem can be proved in essentially the same manner as Theorem 3:

THEOREM 4. *If, for a positive integer $N$, an integer $j$ ($1 \leqq j \leqq s$), and a positive $k$*

$$E(w_{Nj})^{k+1} = \infty, \tag{4.16}$$

*then*

$$\int (x_j)^k \, dF(x) = \infty. \tag{4.17}$$

Theorem 3 is a special case of Theorem 4 for the case $N = 2, j = s$. For then (4.1) implies (4.16), and (4.17) implies (4.2). Let $M_i$ denote the $i$th smallest of $R_1, \cdots, R_s$, and suppose

$$E(M_i)^{k+1} = \infty. \tag{4.18}$$

Then (4.16) holds with $N = s$, $j = i$. This also implies Theorem 3, for (4.1) implies (4.18) for $i = s$. Finally we remark that (4.18) with $i = 1$ implies

$$m_k = \infty. \tag{4.19}$$

**5. Implications for the one-dimensional random walk.** The results of the preceding sections imply not only results on the behavior of queues in general, but also results on the random walk in $s$-dimensional space. We shall content ourselves with pointing out two of these implications for the one-dimensional random walk, although the results for the $s$-dimensional walk obtained in earlier sections are more general and usually more difficult to prove. Without further remark all problems treated in this section are to be assumed to be one-dimensional.

THEOREM 5. *Let $u_1, u_2, \cdots$ be independent, identically distributed chance variables. Let $S_n = \sum_{i=1}^{n} u_i$, and define*

$$v = \sup(0, S_1, S_2, S_3, \cdots).$$

*If*

$$-\infty \leqq Eu_1 < 0, \tag{5.1}$$

*and, for $k > 0$,*

$$E(u_1^+)^{k+1} < \infty, \tag{5.2}$$

*then*

$$Ev^k < \infty. \tag{5.3}$$

THEOREM 6. *With the definitions of Theorem* 5, *if*

(5.4) $$-\infty < Eu_1 < 0,$$

*and, for* $k > 0$,

(5.5) $$E(u_1^+)^{k+1} = \infty,$$

*then*

(5.6) $$Ev^k = \infty.$$

PROOF. Consider the process: $w_1^* = u_1^+$, $w_{n+1}^* = (w_n^* + u_{n+1})^+$, $n \geqq 1$. Let $F_n^*(z)$ be the d.f. of $w_n^*$, and let

$$F^*(z) = \lim_{n\to\infty} F_n^*(z)$$

when the latter exists. It was shown in [3] and follows from the results of [1] for the case $s = 1$ that, when $u_n = R_n - g_{n+1}$, $F^*(z)$ exists, is a distribution function, and equals the limiting d.f. $F(z)$ of $w_n$. It was also shown in [3] that the distribution function of $v$ is then $F^*(z)$. An examination of the proofs of these statements shows that they are valid for the process $\{w_n^*\}$ even when $u_n$ is not of the form $R_n - g_{n+1}$, provided only that (5.1) is satisfied. An examination of the proofs of Section 1 and Theorem 3 of the present paper shows that they too hold even if $u_n$ is not of the form $R_n - g_{n+1}$. But then Theorem 6 is simply a restatement of Theorem 3.

It is sufficient to prove Theorem 5 for chance variables $\{u_n^*\}$, where $u_n^* = \max(u_n, -T)$ and $T > 0$ is so large that $Eu_n^* < 0$. But $u_n^* = (u_n^* + T) - T$ and is therefore of the form $R_n - g_{n+1}$, with $R_n = (u_n^* + T)$, $g_{n+1} \equiv T$. Theorem 5 is then simply a restatement of Theorem 2.

While the results of the present paper on the queueing process and the corresponding $s$-dimensional random walk are new, Theorems 5 and 6 on the one-dimensional random walk were also obtained by Darling, Erdös, and Kakutani, to whom the problem was communicated by us. These writers also obtained other related results, and they have informed us that many of these results are implicit in [4]. In the course of the present work we have had interesting discussions with Professor Shizuo Kakutani.

**6. The mean queue length.** As in [1], Section 9, let $Q_i$ be the number of individuals in the queue waiting to be served, just before the service of the $i$th individual begins. To avoid trivial circumlocutions we assume $G(0) = 0$ ($G(x)$ is the d.f. of $g_i$). In [1] the limit $D(x)$ of $D_n(x)$, the d.f. of $Q_n$, is shown to exist and $D(x)$ is explicitly given. We shall now be concerned with

$$\bar{Q}_n = n^{-1} \sum_{i=1}^{n} Q_i .$$

Let $\{w_n^0\}$ be the process defined in Section 1. We now construct a process $\{w_n^0, Q_n^0\}$, where $Q_1^0, Q_2^0, \cdots$ remain to be defined. Let $t_n = \sum_{i=1}^{n} g_i$. We define

$Q_n^0$ to be equal to the number of indices $i$ which satisfy

(6.1) $$t_n < t_i \leqq t_n + w_{n1}^0 .$$

It follows that the process $\{w_n^0, Q_n^0\}$ is stationary and metrically transitive, so that, by the ergodic theorem, $\bar{Q}_n^0 = n^{-1} \sum_{i=1}^n Q_i^0$ approaches a constant limit $c$,

$$c = \int x \, dD(x),$$

with probability one. (It is easy to prove that $c$ is contained between $Ew_{n1}^0/Eg_1 - 1$ and $Ew_{n1}^0/Eg_1$.) Since $w_{n1} \leqq w_{n1}^0$ it follows from (6.1) that $Q_n \leqq Q_n^0$. Hence

(6.2) $$P\{\limsup_n \bar{Q}_n \leqq c\} = 1.$$

Just as in Section 1, one proves that

(6.3) $$P\{\liminf_n \bar{Q}_n \geqq c\} = 1.$$

Hence

(6.4) $$P\{\lim_n \bar{Q}_n = c\} = 1.$$

**7. The duration of busy periods.** A busy period is a closed time interval, say $t' \leqq t \leqq t''$, such that all $s$ servers are occupied throughout this interval, $t'' - t' > 0$, and the interval is maximal, i.e., if $\tau' \leqq t' < t'' \leqq \tau''$, $\tau'' - \tau' > t'' - t'$, then all $s$ servers are not occupied for some time point in the interval $(\tau', \tau'')$. The length of the busy period is $t'' - t'$, $t'$ is its beginning, and $t''$ is its end. Let $B_i$ be the sum of the lengths of all busy periods at or before $t_i$; if $t_i$ is in the interior of a busy period, we count into $B_i$ the length of the interval from the beginning of the period until $t_i$.

It is easy to verify that whether or not any time point $t$ with $t_i < t < t_{i+1}$ is in a busy interval depends only on $w_i$, $R_i$, and $g_{i+1}$. Since the value of $B_n$ is unaffected by removing from busy periods any of the points $t_i$ $(1 \leqq i \leqq n)$ contained in them, it follows that the process

$$\{B_n, w_n\}, n = 1, 2, \cdots$$

is Markoffian.

Let $\{w_n^0\}$ be the process defined in Section 1. Define $B_1^0 = 0$. Define $B_n^0$, $n \geqq 2$, to be the same function of the process $\{w_n^0\}$ as $B_n$ is of the process $\{w_n\}$. Since $w_n^0 \geqq w_n$ with probability one, it follows that $B_n^0 \geqq B_n$ with probability one.

Since the process $\{w_n^0\}$ is stationary and metrically transitive, so is the process

$$\{B_{n+1}^0 - B_n^0\}, n = 1, 2, \cdots .$$

Hence

$$P\left\{\lim \frac{B_n^0}{n} = E(B_2^0)\right\} = 1.$$

In essentially the same manner as in Section 1 one proves easily that

$$P\left\{\lim \frac{B_n}{n} = E(B_2^0)\right\} = 1.$$

From this we obtain immediately that

$$P\left\{\lim \frac{B_n}{t_n} = \frac{E(B_2^0)}{Eg_1}\right\} = 1.$$

This gives the long-term average time spent in busy periods.

The limiting distribution of the length of a busy period can be obtained in a very tedious but straightforward manner from the marginal distributions of the process $\{w_n^0\}$.

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