# On the Characterizations of $f$-Biharmonic Legendre Curves in Sasakian Space Forms 

Şaban Güvençç, Cihan Özgür ${ }^{\text {a }}$<br>${ }^{a}$ Balikesir University, Faculty of Arts and Sciences, Department of Mathematics, 10145, Balikesir, Turkey


#### Abstract

We consider $f$-biharmonic Legendre curves in Sasakian space forms. We find curvature characterizations of these types of curves in four cases.


## 1. Introduction

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds and $\phi:(M, g) \rightarrow(N, h)$ a smooth map. The energy functional of $\phi$ is defined by

$$
E(\phi)=\frac{1}{2} \int_{M}|d \phi|^{2} v_{g}
$$

where $v_{g}$ is the canonical volume form in $M$. If $\phi$ is a critical points of the energy functional $E(\phi)$, then it is called harmonic [5]. $\phi$ is called a biharmonic map if it is a critical point of the bienergy functional

$$
E_{2}(\phi)=\frac{1}{2} \int_{M}|\tau(\phi)|^{2} v_{g}
$$

where $\tau(\phi)$ is the tension field of $\phi$ which is defined by $\tau(\phi)=$ trace $\nabla d \phi$. The Euler-Lagrange equation of the bienergy functional $E_{2}(\phi)$ gives the biharmonic equation

$$
\tau_{2}(\phi)=-J^{\phi}(\tau(\phi))=-\Delta^{\phi} \tau(\phi)-\operatorname{trace}^{N}(d \phi, \tau(\phi)) d \phi=0
$$

where $J^{\phi}$ is the Jacobi operator of $\phi$ and $\tau_{2}(\phi)$ is called the bitension field of $\phi$ [8].
Now, if $\phi: M \rightarrow N(c)$ is an isometric immersion from $m$-dimensional Riemannian manifold $M$ to $n$-dimensional Riemannian space form $N(c)$ of constant sectional curvature $c$, then

$$
\tau(\phi)=m H
$$

and

$$
\tau_{2}(\phi)=-m \Delta^{\phi} H+c m^{2} H
$$

[^0]Thus, $\phi$ is biharmonic if and only if

$$
\Delta^{\phi} H=c m H
$$

(see [10]). In a different setting, in [4], B.Y. Chen defined a biharmonic submanifold $M \subset \mathbb{E}^{n}$ of the Euclidean space as its mean curvature vector field $H$ satisfies $\Delta H=0$, where $\Delta$ is the Laplacian. Replacing $c=0$ in the above equation, we obtain Chen's definition.
$\phi$ is called an $f$-biharmonic map if it is a critical point of the $f$-bienergy functional

$$
E_{2, f}(\phi)=\frac{1}{2} \int_{M} f|\tau(\phi)|^{2} v_{g}
$$

The Euler-Lagrange equation of this functional gives the $f$-biharmonic equation

$$
\tau_{2, f}(\phi)=f \tau_{2}(\phi)+(\Delta f) \tau(\phi)+2 \nabla_{g r a d f}^{\phi} \tau(\phi)=0
$$

(see [9]). It is clear that any harmonic map is biharmonic and any biharmonic map is $f$-biharmonic. If the map is non-harmonic biharmonic map, then it is called proper biharmonic. Likewise, if the map is non-biharmonic $f$-biharmonic map, then it is called proper $f$-biharmonic [11].
$f$-biharmonic maps were introduced in [9]. Ye-Lin Ou studied $f$-biharmonic curves in real space forms in [11]. D. Fetcu and C. Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [6] and [7]. We studied biharmonic Legendre curves in generalized Sasakian space forms and $\mathcal{S}$-space forms in [13] and [12], respectively. In the present paper, we consider $f$-biharmonic Legendre curves in Sasakian space forms. We obtain curvature equations for this kind of curves.

The paper is organized as follows. In Section 2, we give a brief introduction about Sasakian space forms. In Section 3, we obtain our main results. We also give two examples of proper $f$-biharmonic Legendre curves in $\mathbb{R}^{7}(-3)$.

## 2. Sasakian Space Forms

Let $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ be a contact metric manifold. If the Nijenhuis tensor of $\varphi$ equals $-2 d \eta \otimes \xi$, then ( $M, \varphi, \xi, \eta, g$ ) is called Sasakian manifold [2]. For a Sasakian manifold, it is well-known that:

$$
\begin{align*}
& \left(\nabla_{X} \varphi\right) Y=g(X, Y) \xi-\eta(Y) X,  \tag{1}\\
& \nabla_{X} \xi=-\varphi X . \tag{2}
\end{align*}
$$

(see [3]).
A plane section in $T_{p} M$ is a $\varphi$-section if there exists a vector $X \in T_{p} M$ orthogonal to $\xi$ such that $\{X, \varphi X\}$ span the section. The sectional curvature of a $\varphi$-section is called $\varphi$-sectional curvature. For a Sasakian manifold of constant $\varphi$-sectional curvature (i.e. Sasakian space form), the curvature tensor $R$ of $M$ is given by

$$
\begin{gather*}
R(X, Y) Z=\frac{c+3}{4}\{g(Y, Z) X-g(X, Z) Y\}+ \\
\frac{c-1}{4}\{g(X, \varphi Z) \varphi Y-g(Y, \varphi Z) \varphi X+2 g(X, \varphi Y) \varphi Z  \tag{3}\\
+\eta(X) \eta(Z) Y-\eta(Y) \eta(Z) X+g(X, Z) \eta(Y) \xi-g(Y, Z) \eta(X) \xi\},
\end{gather*}
$$

for all $X, Y, Z \in T M[3]$.
A submanifold of a Sasakian manifold is called an integral submanifold if $\eta(X)=0$, for every tangent vector $X$. A 1-dimensional integral submanifold of a Sasakian manifold ( $M^{2 m+1}, \varphi, \xi, \eta, g$ ) is called a Legendre curve of $M$ [3]. Hence, a curve $\gamma: I \rightarrow M=\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ is called a Legendre curve if $\eta(T)=0$, where $T$ is the tangent vector field of $\gamma$.

## 3. f-Biharmonic Legendre curves in Sasakian Space Forms

Let $\gamma: I \rightarrow M$ be a curve parametrized by arc length in an $n$-dimensional Riemannian manifold $(M, g)$. If there exist orthonormal vector fields $E_{1}, E_{2}, \ldots, E_{r}$ along $\gamma$ such that

$$
\begin{align*}
E_{1} & =\gamma^{\prime}=T \\
\nabla_{T} E_{1} & =\kappa_{1} E_{2} \\
\nabla_{T} E_{2} & =-\kappa_{1} E_{1}+\kappa_{2} E_{3}  \tag{4}\\
& \cdots \\
\nabla_{T} E_{r} & =-\kappa_{r-1} E_{r-1}
\end{align*}
$$

then $\gamma$ is called a Frenet curve of osculating order $r$, where $\kappa_{1}, \ldots, \kappa_{r-1}$ are positive functions on $I$ and $1 \leq r \leq n$.
It is well-known that a Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if $\kappa_{1}$ is a non-zero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a helix of order $r$ if $\kappa_{1}, \ldots, \kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is shortly called a helix.

An arc-length parametrized curve $\gamma:(a, b) \rightarrow(M, g)$ is called an $f$-biharmonic curve with a function $f:(a, b) \rightarrow(0, \infty)$ if the following equation is satisfied [11]:

$$
\begin{equation*}
f\left(\nabla_{T} \nabla_{T} \nabla_{T} T-R\left(T, \nabla_{T} T\right) T\right)+2 f^{\prime} \nabla_{T} \nabla_{T} T+f^{\prime \prime} \nabla_{T} T=0 . \tag{5}
\end{equation*}
$$

Now let $M=\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ be a Sasakian space form and $\gamma: I \rightarrow M$ a Legendre Frenet curve of osculating order $r$. Differentiating

$$
\begin{equation*}
\eta(T)=0 \tag{6}
\end{equation*}
$$

and using (4), we get that

$$
\begin{equation*}
\eta\left(E_{2}\right)=0 \tag{7}
\end{equation*}
$$

Using (3), (4) and (7), it can be seen that

$$
\begin{aligned}
\nabla_{T} \nabla_{T} T=-\kappa_{1}^{2} E_{1}+ & \kappa_{1}^{\prime} E_{2}+\kappa_{1} \kappa_{2} E_{3}, \\
\nabla_{T} \nabla_{T} \nabla_{T} T= & -3 \kappa_{1} \kappa_{1}^{\prime} E_{1}+\left(\kappa_{1}^{\prime \prime}-\kappa_{1}^{3}-\kappa_{1} \kappa_{2}^{2}\right) E_{2} \\
& +\left(2 \kappa_{1}^{\prime} \kappa_{2}+\kappa_{1} \kappa_{2}^{\prime}\right) E_{3}+\kappa_{1} \kappa_{2} \kappa_{3} E_{4} \\
R\left(T, \nabla_{T} T\right) T= & -\kappa_{1} \frac{(c+3)}{4} E_{2}-3 \kappa_{1} \frac{(c-1)}{4} g\left(\varphi T, E_{2}\right) \varphi T
\end{aligned}
$$

(see [7]). If we denote the left-hand side of (5) with $f_{.} \tau_{3}$, we find

$$
\begin{align*}
\tau_{3}= & \nabla_{T} \nabla_{T} \nabla_{T} T-R\left(T, \nabla_{T} T\right) T+2 \frac{f^{\prime}}{f} \nabla_{T} \nabla_{T} T+\frac{f^{\prime \prime}}{f} \nabla_{T} T \\
= & \left(-3 \kappa_{1} \kappa_{1}^{\prime}-2 \kappa_{1}^{2} \frac{f^{\prime}}{f}\right) E_{1} \\
& +\left(\kappa_{1}^{\prime \prime}-\kappa_{1}^{3}-\kappa_{1} \kappa_{2}^{2}+\kappa_{1} \frac{(c+3)}{4}+2 \kappa_{1}^{\prime} \frac{f^{\prime}}{f}+\kappa_{1} \frac{f^{\prime \prime}}{f}\right) E_{2}  \tag{8}\\
& +\left(2 \kappa_{1}^{\prime} \kappa_{2}+\kappa_{1} \kappa_{2}^{\prime}+2 \kappa_{1} \kappa_{2} \frac{f^{\prime}}{f}\right) E_{3}+\kappa_{1} \kappa_{2} \kappa_{3} E_{4} \\
& +3 \kappa_{1} \frac{(c-1)}{4} g\left(\varphi T, E_{2}\right) \varphi T .
\end{align*}
$$

Let $k=\min \{r, 4\}$. From (8), the curve $\gamma$ is $f$-biharmonic if and only if $\tau_{3}=0$, that is,
(1) $c=1$ or $\varphi T \perp E_{2}$ or $\varphi T \in \operatorname{span}\left\{E_{2}, \ldots, E_{k}\right\}$; and
(2) $g\left(\tau_{3}, E_{i}\right)=0$, for all $i=\overline{1, k}$.

So we can state the following theorem:

Theorem 3.1. Let $\gamma$ be a non-geodesic Legendre Frenet curve of osculating order $r$ in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ and $k=\min \{r, 4\}$. Then $\gamma$ is $f$-biharmonic if and only if
(1) $c=1$ or $\varphi T \perp E_{2}$ or $\varphi T \in \operatorname{span}\left\{E_{2}, \ldots, E_{k}\right\}$; and
(2) the first $k$ of the following equations are satisfied (replacing $\kappa_{k}=0$ ):

$$
\begin{gathered}
3 \kappa_{1}^{\prime}+2 \kappa_{1} \frac{f^{\prime}}{f}=0 \\
\kappa_{1}^{2}+\kappa_{2}^{2}=\frac{c+3}{4}+\frac{3(c-1)}{4}\left[g\left(\varphi T, E_{2}\right)\right]^{2}+\frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}}+\frac{f^{\prime \prime}}{f}+2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \frac{f^{\prime}}{f} \\
\kappa_{2}^{\prime}+\frac{3(c-1)}{4} g\left(\varphi T, E_{2}\right) g\left(\varphi T, E_{3}\right)+2 \kappa_{2} \frac{f^{\prime}}{f}+2 \kappa_{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}=0, \\
\kappa_{2} \kappa_{3}+\frac{3(c-1)}{4} g\left(\varphi T, E_{2}\right) g\left(\varphi T, E_{4}\right)=0
\end{gathered}
$$

From Theorem 3.1, it can be easily seen that a curve $\gamma$ with constant geodesic curvature $\kappa_{1}$ is $f$-biharmonic if and only if it is biharmonic. Since Fetcu and Oniciuc studied biharmonic Legendre curves in Sasakian space forms in [7], we study curves with non-constant geodesic curvature $\kappa_{1}$ in this paper. If $\gamma$ is a non-biharmonic $f$-biharmonic curve, then we call it proper $f$-biharmonic.

Now we give the interpretations of Theorem 3.1.

Case I. $c=1$.
In this case $\gamma$ is proper $f$-biharmonic if and only if

$$
\begin{gather*}
3 \kappa_{1}^{\prime}+2 \kappa_{1} \frac{f^{\prime}}{f}=0  \tag{9}\\
\kappa_{1}^{2}+\kappa_{2}^{2}=1+\frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}}+\frac{f^{\prime \prime}}{f}+2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \frac{f^{\prime}}{f} \\
\kappa_{2}^{\prime}+2 \kappa_{2} \frac{f^{\prime}}{f}+2 \kappa_{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}=0, \\
\kappa_{2} \kappa_{3}=0
\end{gather*}
$$

Hence, we can state the following theorem:
Theorem 3.2. Let $\gamma$ be a Legendre Frenet curve in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right), c=1$ and $m>1$. Then $\gamma$ is proper $f$-biharmonic if and only if either
(i) $\gamma$ is of osculating order $r=2$ with $f=c_{1} \kappa_{1}^{-3 / 2}$ and $\kappa_{1}$ satisfies

$$
\begin{equation*}
t \pm \frac{1}{2} \arctan \left(\frac{2+c_{3} \kappa_{1}}{2 \sqrt{-\kappa_{1}^{2}-c_{3} \kappa_{1}-1}}\right)+c_{4}=0 \tag{10}
\end{equation*}
$$

where $c_{1}>0, c_{3}<-2$ and $c_{4}$ are arbitrary constants, $t$ is the arc-length parameter and

$$
\begin{equation*}
\frac{1}{2}\left(-\sqrt{c_{3}^{2}-4}-c_{3}\right)<\kappa_{1}(t)<\frac{1}{2}\left(\sqrt{c_{3}^{2}-4}-c_{3}\right) ; \text { or } \tag{11}
\end{equation*}
$$

(ii) $\gamma$ is of osculating order $r=3$ with $f=c_{1} \kappa_{1}^{-3 / 2}, \frac{\kappa_{2}}{\kappa_{1}}=c_{2}$ and $\kappa_{1}$ satisfies

$$
\begin{equation*}
t \pm \frac{1}{2} \arctan \left(\frac{2+c_{3} \kappa_{1}}{2 \sqrt{-\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-c_{3} \kappa_{1}-1}}\right)+c_{4}=0 \tag{12}
\end{equation*}
$$

where $c_{1}>0, c_{2}>0, c_{3}<-2 \sqrt{\left(1+c_{2}^{2}\right)}$ and $c_{4}$ are arbitrary constants, $t$ is the arc-length parameter and

$$
\begin{equation*}
\frac{1}{2\left(1+c_{2}^{2}\right)}\left(-\sqrt{c_{3}^{2}-4\left(1+c_{2}^{2}\right)}-c_{3}\right)<\kappa_{1}(t)<\frac{1}{2\left(1+c_{2}^{2}\right)}\left(\sqrt{c_{3}^{2}-4\left(1+c_{2}^{2}\right)}-c_{3}\right) . \tag{13}
\end{equation*}
$$

Proof. From the first equation of (9), it is easy to see that $f=c_{1} \kappa_{1}^{-3 / 2}$ for an arbitrary constant $c_{1}>0$. So, we find

$$
\begin{equation*}
\frac{f^{\prime}}{f}=\frac{-3}{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}, \frac{f^{\prime \prime}}{f}=\frac{15}{4}\left(\frac{\kappa_{1}^{\prime}}{\kappa_{1}}\right)^{2}-\frac{3}{2} \frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}} \tag{14}
\end{equation*}
$$

If $\kappa_{2}=0$, then $\gamma$ is of osculating order $r=2$ and the first two of equations (9) must be satisfied. Hence the second equation and (14) give us the ODE

$$
\begin{equation*}
3\left(\kappa_{1}^{\prime}\right)^{2}-2 \kappa_{1} \kappa_{1}^{\prime \prime}=4 \kappa_{1}^{2}\left(\kappa_{1}^{2}-1\right) \tag{15}
\end{equation*}
$$

Let $\kappa_{1}=\kappa_{1}(t)$, where $t$ denotes the arc-length parameter. If we solve (15), we find (10). Since (10) must be well-defined, $-\kappa_{1}^{2}-c_{3} \kappa_{1}-1>0$. Since $\kappa_{1}>0$, we have $c_{3}<-2$ and (11).

If $\kappa_{2}=$ constant $\neq 0$, we find $f$ is a constant. Hence $\gamma$ is not proper $f$-biharmonic in this case. Let $\kappa_{2} \neq$ constant. From the fourth equation of (9), we have $\kappa_{3}=0$. So, $\gamma$ is of osculating order $r=3$. The third equation of (9) gives us $\frac{\kappa_{2}}{\kappa_{1}}=c_{2}$, where $c_{2}>0$ is a constant. Replacing in the second equation of (9), we have the ODE

$$
3\left(\kappa_{1}^{\prime}\right)^{2}-2 \kappa_{1} \kappa_{1}^{\prime \prime}=4 \kappa_{1}^{2}\left[\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-1\right]
$$

which has the general solution (12) under the condition $c_{3}<-2 \sqrt{\left(1+c_{2}^{2}\right)}$. (13) must be also satisfied.
Remark 3.3. If $m=1$, then $M$ is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write $\kappa_{1}>0$ and $\kappa_{2}=1$. The first and the third equations of (9) give us $f$ is a constant. Hence $\gamma$ cannot be proper $f$-biharmonic.

Case II. $c \neq 1, \varphi T \perp E_{2}$.
In this case, $g\left(\varphi T, E_{2}\right)=0$. From Theorem 3.1, we obtain

$$
\begin{gather*}
3 \kappa_{1}^{\prime}+2 \kappa_{1} \frac{f^{\prime}}{f}=0,  \tag{16}\\
\kappa_{1}^{2}+\kappa_{2}^{2}=\frac{c+3}{4}+\frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}}+\frac{f^{\prime \prime}}{f}+2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \frac{f^{\prime}}{f}, \\
\kappa_{2}^{\prime}+2 \kappa_{2} \frac{f^{\prime}}{f}+2 \kappa_{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}=0, \\
\kappa_{2} \kappa_{3}=0 .
\end{gather*}
$$

Firstly, we need the following proposition from [7]:
Proposition 3.4. [7] Let $\gamma$ be a Legendre Frenet curve of osculating order 3 in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ and $\varphi T \perp E_{2}$. Then $\left\{T=E_{1}, E_{2}, E_{3}, \varphi T, \nabla_{T} \varphi T, \xi\right\}$ is linearly independent at any point of $\gamma$. Therefore $m \geq 3$.

Now we can state the following Theorem:
Theorem 3.5. Let $\gamma$ be a Legendre Frenet curve in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right), c \neq 1$ and $\varphi T \perp E_{2}$. Then $\gamma$ is proper biharmonic if and only if
(1) $\gamma$ is of osculating order $r=2$ with $f=c_{1} \kappa_{1}^{-3 / 2}, m \geq 2,\left\{T=E_{1}, E_{2}, \varphi T, \nabla_{T} \varphi T, \xi\right\}$ is linearly independent and (a) if $c>-3$, then $\kappa_{1}$ satisfies
$t \pm \frac{1}{\sqrt{c+3}} \arctan \left(\frac{c+3+2 c_{3} \kappa_{1}}{\sqrt{c+3} \sqrt{-4 \kappa_{1}^{2}-4 c_{3} \kappa_{1}-c-3}}\right)+c_{4}=0$,
(b) if $c=-3$, then $\kappa_{1}$ satisfies
$t \pm \frac{\sqrt{-\kappa_{1}\left(\kappa_{1}+c_{3}\right)}}{c_{3} \kappa_{1}}+c_{4}=0$,
(c) if $c<-3$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{1}{\sqrt{-c-3}} \ln \left(\frac{c+3+2 c_{3} \kappa_{1}-\sqrt{-c-3} \sqrt{-4 \kappa_{1}^{2}-4 c_{3} \kappa_{1}-c-3}}{(c+3) \kappa_{1}}\right)+c_{4}=0 ; \text { or }
$$

(2) $\gamma$ is of osculating order $r=3$ with $f=c_{1} \kappa_{1}^{-3 / 2}, \frac{\kappa_{2}}{\kappa_{1}}=c_{2}=$ constant $>0, m \geq 3,\left\{T=E_{1}, E_{2}, E_{3}, \varphi T, \nabla_{T} \varphi T, \xi\right\}$ is linearly independent and
(a) if $c>-3$, then $\kappa_{1}$ satisfies
$t \pm \frac{1}{\sqrt{c+3}} \arctan \left(\frac{c+3+2 c_{3} \kappa_{1}}{\sqrt{c+3} \sqrt{-4\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-4 c_{3} \kappa_{1}-c-3}}\right)+c_{4}=0$,
(b) if $c=-3$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{\sqrt{-\kappa_{1}\left[\left(1+c_{2}^{2}\right) \kappa_{1}+c_{3}\right]}}{c_{3} \kappa_{1}}+c_{4}=0
$$

(c) if $c<-3$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{1}{\sqrt{-c-3}} \ln \left(\frac{c+3+2 c_{3} \kappa_{1}-\sqrt{-c-3} \sqrt{-4\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-4 c_{3} \kappa_{1}-c-3}}{(c+3) \kappa_{1}}\right)+c_{4}=0
$$

where $c_{1}>0, c_{2}>0, c_{3}$ and $c_{4}$ are convenient arbitrary constants, $t$ is the arc-length parameter and $\kappa_{1}(t)$ is in convenient open interval.

Proof. The proof is similar to the proof of Theorem 3.2.
Case III. $c \neq 1, \varphi T \| E_{2}$.
In this case, $\varphi T= \pm E_{2}, g\left(\varphi T, E_{2}\right)= \pm 1, g\left(\varphi T, E_{3}\right)=g\left( \pm E_{2}, E_{3}\right)=0$ and $g\left(\varphi T, E_{4}\right)=g\left( \pm E_{2}, E_{4}\right)=0$. From Theorem 3.1, $\gamma$ is biharmonic if and only if

$$
\begin{gather*}
3 \kappa_{1}^{\prime}+2 \kappa_{1} \frac{f^{\prime}}{f}=0,  \tag{17}\\
\kappa_{1}^{2}+\kappa_{2}^{2}=c+\frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}}+\frac{f^{\prime \prime}}{f}+2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \frac{f^{\prime}}{f} \\
\kappa_{2}^{\prime}+2 \kappa_{2} \frac{\kappa^{\prime}}{f}+2 \kappa_{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}=0, \\
\kappa_{2} \kappa_{3}=0 .
\end{gather*}
$$

Since $\varphi T \| E_{2}$, it is easily proved that $\kappa_{2}=1$. Then, the first and the third equations of (17) give us $f$ is a constant. Thus, we give the following Theorem:
Theorem 3.6. There does not exist any proper $f$-biharmonic Legendre curve in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ with $c \neq 1$ and $\varphi T \| E_{2}$.

Case IV. $c \neq 1$ and $g\left(\varphi T, E_{2}\right)$ is not constant 0,1 or -1 .
Now, let $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$ be a Sasakian space form and $\gamma: I \rightarrow M$ a Legendre curve of osculating order $r$, where $4 \leq r \leq 2 m+1$ and $m \geq 2$. If $\gamma$ is $f$-biharmonic, then $\varphi T \in \operatorname{span}\left\{E_{2}, E_{3}, E_{4}\right\}$. Let $\theta(t)$ denote the angle function between $\varphi T$ and $E_{2}$, that is, $g\left(\varphi T, E_{2}\right)=\cos \theta(t)$. Differentiating $g\left(\varphi T, E_{2}\right)$ along $\gamma$ and using (1) and (4), we find

$$
\begin{align*}
-\theta^{\prime}(t) \sin \theta(t) & =\nabla_{T} g\left(\varphi T, E_{2}\right)=g\left(\nabla_{T} \varphi T, E_{2}\right)+g\left(\varphi T, \nabla_{T} E_{2}\right) \\
& =g\left(\xi+\kappa_{1} \varphi E_{2}, E_{2}\right)+g\left(\varphi T,-\kappa_{1} T+\kappa_{2} E_{3}\right)  \tag{18}\\
& =\kappa_{2} g\left(\varphi T, E_{3}\right) .
\end{align*}
$$

If we write $\varphi T=g\left(\varphi T, E_{2}\right) E_{2}+g\left(\varphi T, E_{3}\right) E_{3}+g\left(\varphi T, E_{4}\right) E_{4}$, Theorem 3.1 gives us

$$
\begin{align*}
& 3 \kappa_{1}^{\prime}+2 \kappa_{1} \frac{f^{\prime}}{f}=0  \tag{19}\\
& \kappa_{1}^{2}+\kappa_{2}^{2}=\frac{c+3}{4}+\frac{3(c-1)}{4} \cos ^{2} \theta+\frac{\kappa_{1}^{\prime \prime}}{\kappa_{1}}+\frac{f^{\prime \prime}}{f}+2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \frac{f^{\prime}}{f^{\prime}}  \tag{20}\\
& \kappa_{2}^{\prime}+\frac{3(c-1)}{4} \cos \theta g\left(\varphi T, E_{3}\right)+2 \kappa_{2} \frac{f^{\prime}}{f}+2 \kappa_{2} \frac{\kappa_{1}^{\prime}}{\kappa_{1}}=0  \tag{21}\\
& \kappa_{2} \kappa_{3}+\frac{3(c-1)}{4} \cos \theta g\left(\varphi T, E_{4}\right)=0 \tag{22}
\end{align*}
$$

If we put (14) in (20) and (21) respectively, we obtain

$$
\begin{align*}
& \kappa_{1}^{2}+\kappa_{2}^{2}=\frac{c+3}{4}+\frac{3(c-1)}{4} \cos ^{2} \theta-\frac{\kappa_{1}^{\prime \prime}}{2 \kappa_{1}}+\frac{3}{4}\left(\frac{\kappa_{1}^{\prime}}{\kappa_{1}}\right)^{2}  \tag{23}\\
& \kappa_{2}^{\prime}-\frac{\kappa_{1}^{\prime}}{\kappa_{1}} \kappa_{2}+\frac{3(c-1)}{4} \cos \theta g\left(\varphi T, E_{3}\right)=0 \tag{24}
\end{align*}
$$

If we multiply (24) with $2 \kappa_{2}$, using (18), we find

$$
\begin{equation*}
2 \kappa_{2} \kappa_{2}^{\prime}-2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} \kappa_{2}^{2}+\frac{3(c-1)}{4}\left(-2 \theta^{\prime} \cos \theta \sin \theta\right)=0 \tag{25}
\end{equation*}
$$

Let us denote $v(t)=\kappa_{2}^{2}(t)$, where $t$ is the arc-length parameter. Then (25) becomes

$$
\begin{equation*}
v^{\prime}-2 \frac{\kappa_{1}^{\prime}}{\kappa_{1}} v=-\frac{3(c-1)}{4}\left(-2 \theta^{\prime} \cos \theta \sin \theta\right) \tag{26}
\end{equation*}
$$

which is a linear ODE. If we solve (26), we obtain the following results:
i) If $\theta$ is a constant, then

$$
\begin{equation*}
\frac{\kappa_{2}}{\kappa_{1}}=c_{2} \tag{27}
\end{equation*}
$$

where $c_{2}>0$ is an arbitrary constant. From (18), we find $g\left(\varphi T, E_{3}\right)=0$. Since $\|\varphi T\|=1$ and $\varphi T=$ $\cos \theta E_{2}+g\left(\varphi T, E_{4}\right) E_{4}$, we get $g\left(\varphi T, E_{4}\right)= \pm \sin \theta$. By the use of (20) and (27), we find

$$
3\left(\kappa_{1}^{\prime}\right)^{2}-2 \kappa_{1} \kappa_{1}^{\prime \prime}=4 \kappa_{1}^{2}\left[\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-\frac{c+3+3(c-1) \cos ^{2} \theta}{4}\right] .
$$

ii) If $\theta=\theta(t)$ is a non-constant function, then

$$
\begin{equation*}
\kappa_{2}^{2}=-\frac{3(c-1)}{4} \cos ^{2} \theta+\lambda(t) \cdot \kappa_{1}^{2} \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda(t)=-\frac{3(c-1)}{2} \int \frac{\cos ^{2} \theta \kappa_{1}^{\prime}}{\kappa_{1}^{3}} d t \tag{29}
\end{equation*}
$$

If we write (28) in (23), we have

$$
[1+\lambda(t)] \cdot \kappa_{1}^{2}=\frac{c+3+6(c-1) \cos ^{2} \theta}{4}-\frac{\kappa_{1}^{\prime \prime}}{2 \kappa_{1}}+\frac{3}{4}\left(\frac{\kappa_{1}^{\prime}}{\kappa_{1}}\right)^{2} .
$$

Now we can state the following Theorem:

Theorem 3.7. Let $\gamma: I \rightarrow M$ be a Legendre curve of osculating order $r$ in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$, where $r \geq 4, m \geq 2, c \neq 1, g\left(\varphi T, E_{2}\right)=\cos \theta(t)$ is not constant 0,1 or -1 . Then $\gamma$ is proper $f$-biharmonic if and only if $f=c_{1} \kappa_{1}^{-3 / 2}$ and
(i) if $\theta$ is a constant,

$$
\begin{aligned}
& \frac{\kappa_{2}}{\kappa_{1}}=c_{2} \\
& 3\left(\kappa_{1}^{\prime}\right)^{2}-2 \kappa_{1} \kappa_{1}^{\prime \prime}=4 \kappa_{1}^{2}\left[\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-\frac{c+3+3(c-1) \cos ^{2} \theta}{4}\right] \\
& \kappa_{2} \kappa_{3}= \pm \frac{3(c-1) \sin 2 \theta}{8}
\end{aligned}
$$

(ii) if $\theta$ is a non-constant function,

$$
\begin{aligned}
& \kappa_{2}^{2}=-\frac{3(c-1)}{4} \cos ^{2} \theta+\lambda(t) \cdot \kappa_{1}^{2} \\
& 3\left(\kappa_{1}^{\prime}\right)^{2}-2 \kappa_{1} \kappa_{1}^{\prime \prime}=4 \kappa_{1}^{2}\left[(1+\lambda(t)) \kappa_{1}^{2}-\frac{c+3+6(c-1) \cos ^{2} \theta}{4}\right] \\
& \kappa_{2} \kappa_{3}= \pm \frac{3(c-1) \sin 2 \theta \sin w}{8}
\end{aligned}
$$

where $c_{1}$ and $c_{2}$ are positive constants, $\varphi T=\cos \theta E_{2} \pm \sin \theta \cos w E_{3} \pm \sin \theta \sin w E_{4}$, $w$ is the angle function between $E_{3}$ and the orthogonal projection of $\varphi$ T onto span $\left\{E_{3}, E_{4}\right\}$. w is related to $\theta$ by $\cos w=\frac{-\theta^{\prime}}{k_{2}}$ and $\lambda(t)$ is given by

$$
\lambda(t)=-\frac{3(c-1)}{2} \int \frac{\cos ^{2} \theta \kappa_{1}^{\prime}}{\kappa_{1}^{3}} d t
$$

We can give the following direct corollary of Theorem 3.7:
Corollary 3.8. Let $\gamma: I \rightarrow M$ be a Legendre curve of osculating order $r$ in a Sasakian space form $\left(M^{2 m+1}, \varphi, \xi, \eta, g\right)$, where $r \geq 4, m \geq 2, c \neq 1, g\left(\varphi T, E_{2}\right)=\cos \theta$ is a constant and $\theta \in(0,2 \pi) \backslash\left\{\frac{\pi}{2}, \pi, \frac{3 \pi}{2}\right\}$. Then $\gamma$ is proper $f$-biharmonic if and only if $f=c_{1} \kappa_{1}^{-3 / 2}, \frac{\kappa_{2}}{\kappa_{1}}=c_{2}=$ constant $>0$,

$$
\begin{aligned}
& \kappa_{2} \kappa_{3}= \pm \frac{3(c-1) \sin 2 \theta}{8} \\
& \kappa_{4}= \pm \frac{\eta\left(E_{5}\right)+g\left(\varphi E_{2}, E_{5}\right) \kappa_{1}}{\sin \theta}(\text { if } r>4) ; \text { and }
\end{aligned}
$$

(i) if $a>0$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{1}{2 \sqrt{a}} \arctan \left(\frac{1}{2 \sqrt{a}} \frac{2 a+c_{3} \kappa_{1}}{\sqrt{-\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-c_{3} \kappa_{1}-a}}\right)+c_{4}=0
$$

(ii) if $a=0$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{\sqrt{-\kappa_{1}\left[\left(1+c_{2}^{2}\right) \kappa_{1}+c_{3}\right]}}{c_{3} \kappa_{1}}+c_{4}=0
$$

(iii) if a $<0$, then $\kappa_{1}$ satisfies

$$
t \pm \frac{1}{2 \sqrt{-a}} \ln \left(\frac{2 a+c_{3} \kappa_{1}-2 \sqrt{-a} \sqrt{-\left(1+c_{2}^{2}\right) \kappa_{1}^{2}-c_{3} \kappa_{1}-a}}{2 a \kappa_{1}}\right)+c_{4}=0
$$

where $a=\left[c+3+3(c-1) \cos ^{2} \theta\right] / 4, \varphi T=\cos \theta E_{2} \pm \sin \theta E_{4}, c_{1}>0, c_{2}>0, c_{3}$ and $c_{4}$ are convenient arbitrary constants, $t$ is the arc-length parameter and $\kappa_{1}(t)$ is in convenient open interval.

In order to obtain explicit examples, we will first need to recall some notions about the Sasakian space form $\mathbb{R}^{2 m+1}(-3)$ [3]:

Let us consider $M=\mathbb{R}^{2 m+1}$ with the standard coordinate functions $\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{m}, z\right)$, the contact structure $\eta=\frac{1}{2}\left(d z-\sum_{i=1}^{m} y_{i} d x_{i}\right)$, the characteristic vector field $\xi=2 \frac{\partial}{\partial z}$ and the tensor field $\varphi$ given by

$$
\varphi=\left[\begin{array}{ccc}
0 & \delta_{i j} & 0 \\
-\delta_{i j} & 0 & 0 \\
0 & y_{j} & 0
\end{array}\right]
$$

The associated Riemannian metric is $g=\eta \otimes \eta+\frac{1}{4} \sum_{i=1}^{m}\left(\left(d x_{i}\right)^{2}+\left(d y_{i}\right)^{2}\right)$. Then $(M, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant $\varphi$-sectional curvature $c=-3$ and it is denoted by $\mathbb{R}^{2 m+1}(-3)$. The vector fields

$$
\begin{equation*}
X_{i}=2 \frac{\partial}{\partial y_{i}}, X_{m+i}=\varphi X_{i}=2\left(\frac{\partial}{\partial x_{i}}+y_{i} \frac{\partial}{\partial z}\right), i=\overline{1, m}, \xi=2 \frac{\partial}{\partial z} \tag{30}
\end{equation*}
$$

form a $g$-orthonormal basis and the Levi-Civita connection is calculated as

$$
\begin{aligned}
& \nabla_{X_{i}} X_{j}=\nabla_{X_{m+i}} X_{m+j}=0, \nabla_{X_{i}} X_{m+j}=\delta_{i j} \xi, \nabla_{X_{m+i}} X_{j}=-\delta_{i j} \xi, \\
& \nabla_{X_{i}} \xi=\nabla_{\xi} X_{i}=-X_{m+i}, \nabla_{X_{m+i}} \xi=\nabla_{\xi} X_{m+i}=X_{i},
\end{aligned}
$$

(see [3]).
Now, let us produce examples of proper $f$-biharmonic Legendre curves in $\mathbb{R}^{7}(-3)$ :
Let $\gamma=\left(\gamma_{1}, \ldots, \gamma_{7}\right)$ be a unit speed curve in $\mathbb{R}^{7}(-3)$. The tangent vector field of $\gamma$ is

$$
T=\frac{1}{2}\left[\gamma_{4}^{\prime} X_{1}+\gamma_{5}^{\prime} X_{2}+\gamma_{6}^{\prime} X_{3}+\gamma_{1}^{\prime} X_{4}+\gamma_{2}^{\prime} X_{5}+\gamma_{3}^{\prime} X_{6}+\left(\gamma_{7}^{\prime}-\gamma_{1}^{\prime} \gamma_{4}-\gamma_{2}^{\prime} \gamma_{5}-\gamma_{3}^{\prime} \gamma_{6}\right) \xi\right]
$$

Thus, $\gamma$ is a unit speed Legendre curve if and only if $\eta(T)=0$ and $g(T, T)=1$, that is,

$$
\gamma_{7}^{\prime}=\gamma_{1}^{\prime} \gamma_{4}+\gamma_{2}^{\prime} \gamma_{5}+\gamma_{3}^{\prime} \gamma_{6}
$$

and

$$
\left(\gamma_{1}^{\prime}\right)^{2}+\ldots+\left(\gamma_{6}^{\prime}\right)^{2}=4
$$

For a Legendre curve, we can use the Levi-Civita connection and (30) to write

$$
\begin{align*}
& \nabla_{T} T=\frac{1}{2}\left(\gamma_{4}^{\prime \prime} X_{1}+\gamma_{5}^{\prime \prime} X_{2}+\gamma_{6}^{\prime \prime} X_{3}+\gamma_{1}^{\prime \prime} X_{4}+\gamma_{2}^{\prime \prime} X_{5}+\gamma_{3}^{\prime \prime} X_{6}\right)  \tag{31}\\
& \varphi T=\frac{1}{2}\left(-\gamma_{1}^{\prime} X_{1}-\gamma_{2}^{\prime} X_{2}-\gamma_{3}^{\prime} X_{3}+\gamma_{4}^{\prime} X_{4}+\gamma_{5}^{\prime} X_{5}+\gamma_{6}^{\prime} X_{6}\right) \tag{32}
\end{align*}
$$

From (31) and (32), $\varphi T \perp E_{2}$ if and only if

$$
\gamma_{1}^{\prime \prime} \gamma_{4}^{\prime}+\gamma_{2}^{\prime \prime} \gamma_{5}^{\prime}+\gamma_{3}^{\prime \prime} \gamma_{6}^{\prime}=\gamma_{1}^{\prime} \gamma_{4}^{\prime \prime}+\gamma_{2}^{\prime} \gamma_{5}^{\prime \prime}+\gamma_{3}^{\prime} \gamma_{6}^{\prime \prime}
$$

Finally, we can give the following explicit examples:

Example 3.9. Let us take $\gamma(t)=\left(2 \sinh ^{-1}(t), \sqrt{1+t^{2}}, \sqrt{3} \sqrt{1+t^{2}}, 0,0,0,1\right)$ in $\mathbb{R}^{7}(-3)$. Using the above equations and Theorem 3.5, $\gamma$ is a proper $f$-biharmonic Legendre curve with osculating order $r=2, \kappa_{1}=\frac{1}{1+t^{2}}, f=c_{1}\left(1+t^{2}\right)^{3 / 2}$ where $c_{1}>0$ is a constant. We can easily check that the conditions of Theorem 3.5 (i.e. $c \neq 1, \varphi T \perp E_{2}$ ) are verified, where $c_{3}=-1$ and $c_{4}=0$.

Example 3.10. Let $\gamma(t)=\left(a_{1}, a_{2}, a_{3}, \sqrt{2} t, 2 \sinh ^{-1}\left(\frac{t}{\sqrt{2}}\right), \sqrt{2} \sqrt{2+t^{2}}, a_{4}\right)$ be a curve in $\mathbb{R}^{7}(-3)$, where $a_{i} \in \mathbb{R}, i=\overline{1,4}$. Then we calculate

$$
\begin{aligned}
& T=\frac{\sqrt{2}}{2} X_{1}+\frac{1}{\sqrt{2+t^{2}}} X_{2}+\frac{\sqrt{2} t}{2 \sqrt{2+t^{2}}} X_{3}, \\
& E_{2}=\frac{-t}{\sqrt{2+t^{2}}} X_{2}+\frac{\sqrt{2}}{\sqrt{2+t^{2}}} X_{3}, \\
& E_{3}=\frac{\sqrt{2}}{2} X_{1}-\frac{1}{\sqrt{2+t^{2}}} X_{2}-\frac{\sqrt{2} t}{2 \sqrt{2+t^{2}}} X_{3}, \\
& \kappa_{1}=\kappa_{2}=\frac{1}{2+t^{2}}, r=3 .
\end{aligned}
$$

From Theorem 3.5, it follows that $\gamma$ is proper $f$-biharmonic with $f=c_{1}\left(2+t^{2}\right)^{3 / 2}$, where $c_{1}>0, c_{2}=1, c_{3}=-1$ and $c_{4}=0$.

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    Communicated by Ljubica Velimirović
    Email addresses: sguvenc@balikesir.edu.tr (Şaban Güvenç), cozgur@balikesir.edu.tr (Cihan Özgür)

