

ON THE CHOICE OF A MODEL TO FIT DATA FROM AN EXPONENTIAL FAMILY

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Let X_1, \dots, X_n be iid observations coming from an exponential family. The problem of interest is this: Given a finite number of models m_j (smoothly curved manifolds in \mathbb{R}^k), choose the best model to fit the observations, with some penalty for choosing models with dimensions which are too large. A result of Schwarz is made more specific and is extended to the case where the models are curved manifolds. If $S(Y, n, j)$ is—up to a constant $C(n)$ independent of the model—the log of the posterior probability of the j th model, where the sample mean $Y_n = (1/n)\sum_{i=1}^n X_i$ has been replaced by Y , Schwarz suggested an asymptotic expansion of $S(Y, n, j)$ whose leading terms are $\gamma(Y, n, j) = n \sup_{\psi \in m_j \cap \Theta} (Y\psi - b(\psi)) - \frac{1}{2}k_j \log n$, in the case where the models are affine subspaces of \mathbb{R}^k . We establish a similar asymptotic expansion, including the next term, with uniform bounds for Y in a compact neighborhood of $\nabla b(\theta)$, where θ is the true value of the parameter. We suggest a criterion for the choice of the best model that consists of maximizing the three leading terms in the expansion $S(Y, n, j)$. We show that the criterion gives the correct model with probabilities $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$.

0. Introduction. This article is concerned with the problem of choosing, among a finite number of possibly curved models (manifolds in \mathbb{R}^k), the “best” model to fit iid observations X_i , $i = 1, 2, \dots$, whose law belongs to an exponential family.

In Section 1 (Proposition 1.2) we show that maximizing the quantities

$$\gamma(n, j) := \log M_j(X_1, \dots, X_n) - \frac{1}{2}k_j \log n$$

leads to a correct choice of a model with probabilities $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, where P_θ is the true law of the observations in the exponential family; here $M_j(X_1, \dots, X_n)$ is the maximum of the likelihood function of the n first observations on the j th model and k_j is the dimension of the j th model. It follows in particular from Proposition 1.2 that this procedure is consistent [see, for example, Woodroffe (1982)].

The main conclusions of this paper, which completes and extends a result of Schwarz (1978), are as follows:

1. For any model m_j where the true parameter θ is in $\text{int}(m_j \cap \Theta)$, the quantities $S(n, j)$, where $S(n, j)$ is the log of the posterior probability of the j th model plus a constant $C(n)$ independent of the model, have an asymptotic expansion whose leading three terms $\Gamma(n, j)$ are given by Theorem 2.3.

Received March 1984; revised May 1986.

AMS 1980 subject classifications. Primary 62F12; secondary 62H99.

Key words and phrases. Curved models, Schwarz's criterion, exponential families.

2. When choosing between two models m_1 and m_2 , if θ belongs to m_1 and not to m_2 , then the procedure that consists of maximizing the $\Gamma(n, j)$, $j = 1, 2$, will lead to the correct choice of a model (i.e., m_1 rather than m_2) with $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, even if $k_1 \geq k_2$ (Proposition 1.2 and Remark 2.3).
3. When choosing between two models m_1 and m_2 where the true value θ is in $m_1 \cap m_2$ and $k_1 \neq k_2$, with $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, the procedure that consists of maximizing the $\Gamma(n, j)$, $j = 1, 2$, will lead to the choice of the model with the smallest dimension and in this case, for any suitably smooth prior, a Bayes procedure will also lead to the same choice with $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$ (Proposition 1.2, Remark 2.3 and Corollary 2.3).
4. When choosing between two models m_1 and m_2 with $k_1 = k_2$ and where the true value θ of the parameter belongs to $m_1 \cap m_2$, then by assumption ($**$) of Section 1 we really have a choice of three models, m_1 , m_2 and $m_1 \cap m_2$: say $m_1 \cap m_2 = m_{i_0}$ for some index i_0 ; then, the procedure based on the $\Gamma(n, j)$, $j = 1, 2$, i_0 will pick $m_1 \cap m_2$ and coincide with the Bayes procedure with $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$.

Section 2.1 deals with the problem of existence and unicity of MLE's on a curved model [note that Amari (1982) gives a geometrical interpretation of maximum likelihood estimation but is not concerned with the existence problem]. Sections 2.2 and 2.3 are devoted to obtaining an asymptotic expansion for the log of the posterior probability of the j th model; this posterior probability is a function $P(Y_n)$ of the sample mean Y_n of the n first observations. We first obtain an asymptotic expansion for $P(Y)$ as a function of Y (Proposition 2.2) *which is uniform in Y , for Y in some compact set*. The expansion has a precision of $n^{-(k_j+1)/2}$. In Section 2.3 we apply Proposition 2.2 to the case where Y is replaced by Y_n —this is where we need the uniformity in Y in Proposition 2.2—and obtain the desired asymptotic expansion for the log of the posterior probability of the j th model with a precision of $n^{-1/2}$ in probability.

Section 3 is concerned with the choice of a degree for a polynomial regression. One major issue is that the observations are not iid. We show that, by assuming that *both variables* in the regression are random variables, we can still apply the results of Sections 1 and 2. We describe the different models, which are curved.

To conclude this section, we note that the quantities $\gamma(n, j)$ introduced by Schwarz (1978) arise very naturally as the leading terms of the asymptotic expansion in Section 2.3 and that maximizing the $\gamma(n, j)$ [commonly called the Bayesian information criterion (BIC)] is a consistent procedure. Another well-known procedure is the Akaike information criterion (AIC) [Akaike (1974)]. It has been shown that the AIC is not consistent as $n \rightarrow +\infty$ [e.g., Woodroffe (1982) and Hannan (1980)]. The point has been made, though, that inconsistency may not be of great consequence from the point of view of prediction [Geisser and Eddy (1979)]. The AIC seems to have optimality properties in cases such as the selection of the order of the model for estimating parameters of a linear process, the key assumption being that the dimension of the models is allowed to increase with sample size [Shibata (1980, 1981)].

1. Criterion for the correct choice of a model when the models are C^∞ manifolds in \mathbb{R}^k .

1.1. *The Bayes procedure. Statement of the problem.* Let $X_i, i = 1, 2, \dots$, be iid observations from an exponential family in standard form with densities $f(X, \phi) = \exp(X\phi - b(\phi))$, with respect to a finite measure on \mathbb{R}^k , with $\phi \in \Theta$ the natural parameter space [Lehmann (1959), page 51]. We assume that we have a finite number of competing models $m_j \cap \Theta$ where m_j is a C^∞ k_j -dimensional connected manifold embedded in \mathbb{R}^k [for terminology and basic facts in differential geometry we refer to Spivak (1979)]. An important special case is the case where m_j is a k_j -dimensional affine space in \mathbb{R}^k [Schwarz (1978)]. We assume that

- (*) for each $i \neq j$, if a point in the closure of m_i is in $m_j \cap \text{int } \Theta$, then it is in m_i .

We will also assume that

- (**) if $k_i = k_j$ for some pair $(i, j), i \neq j$, and if $m_i \cap m_j \neq \emptyset$, then $m_i \cap m_j$ is also an available model m_r and is of lower dimension.

On each of these m_j a natural analogue λ_j of Lebesgue measure is defined. If m_j is a k_j -dimensional affine space in \mathbb{R}^k , the defined measure will reduce to the Lebesgue measure. If m_j is a one-dimensional curve (or a two-dimensional surface) in \mathbb{R}^k , this measure will be the usual arc length (or the usual surface element). Note that the standard inner product of \mathbb{R}^k induces a natural C^∞ Riemannian structure on m_j . Let g_{ij} be the coefficients of this Riemannian metric on a coordinate neighborhood U of a point p of m_j .

We define *the standard volume element* dV on the Riemannian manifold m_j by

$$dV = (\det g_{ij})^{1/2} |dX^1(p) \wedge \cdots \wedge dX^{k_j}(p)|$$

in a coordinate system (X, U) about the point p [see Spivak (1979), pages 417–418 for details]. This definition does not depend on the coordinate system [Spivak (1979), page 281]. The set function $\lambda_j(A) = \int_A dV$ is then defined for any Borel subset A of m_j and is countably additive.

We will assume that the conditional prior distribution μ_j of the parameter ϕ given the j th model has a density f_j with respect to λ_j which is a nowhere zero C^∞ function on $m_j \cap \Theta$ (this assumption will also hold for measures obtained from volume elements of Riemannian metrics smoothly related to the original one). Let α_j be the prior probability of the j th model. The prior distribution of ϕ is then $\mu = \sum \alpha_j \mu_j$. Note that μ is concentrated on $\cup m_j$ and that the μ_j are mutually orthogonal, clearly if two m_j are of different dimensions, and by assumption (**) if they have the same dimension.

Let $P(n, j)$ be the posterior probability of the j th model given the prior and the n first observations X_1, \dots, X_n . We have

$$P(n, j) = \alpha_j \int_{m_j \cap \Theta} \exp\left(\phi \sum_{i=1}^n X_i - nb(\phi)\right) d\mu_j(\phi) \Big/ \int_{\Theta} \exp\left(\phi \sum_{i=1}^n X_i - nb(\phi)\right) d\mu(\phi).$$

Then a Bayes choice of a model is a choice that maximizes $P(n, j)$.

Let $Y_n = (1/n)\sum_{i=1}^n X_i$ and

$$\begin{aligned} S(n, j) &= \log \alpha_j + \log \int_{m_j \cap \Theta} \exp n(\phi Y_n - b(\phi)) d\mu_j(\phi) \\ &= \log P(n, j) + C(n) \end{aligned}$$

for some $C(n)$. Let θ be the true value of the parameter. We assume throughout that $\theta \in m_j$ for some j (this is no loss of generality since we can always adjoin \mathbb{R}^k as a model).

DEFINITION. The correct choice between models is the model of lowest dimension which contains θ .

1.2. *Criterion for the correct choice of a model.* Using the notation of Section 1.1, let

$$\gamma(n, j) = n \sup_{\phi \in m_j \cap \Theta} (Y_n \phi - b(\phi)) - \frac{1}{2} k_j \log n$$

[Schwarz (1978)]. The following proposition proves the consistency of Schwarz's criterion.

PROPOSITION 1.2. *Assume $\theta \in \text{int } \Theta$ and let m_1 and m_2 be two different models. If $\theta \in m_2 \setminus m_1$, or if $\theta \in m_1 \cap m_2$ with $k_2 < k_1$, then*

$$\lim_{n \rightarrow +\infty} P_\theta^n(\gamma(n, 1) < \gamma(n, 2)) = 1.$$

PROOF. Let $f(\phi) = \nabla b(\theta)\phi - b(\phi)$ for $\phi \in \Theta$. The function f attains its unique maximum at θ [Barndorff-Nielsen (1978), Theorems 9.13 and 9.1 and (1), page 141]. Let $\theta \in m_2 \setminus m_1$. Since $\theta \notin \bar{m}_1$ by assumption, let us pick $\varepsilon > 0$ and a neighborhood N of θ such that

$$N \cap m_1 = \emptyset$$

and, for $\phi \notin N$,

$$\nabla b(\theta)\phi - b(\phi) + \varepsilon \leq \nabla b(\theta)\theta - b(\theta).$$

We have

$$(*) \quad \sup_{\phi \in m_1 \cap \Theta} \nabla b(\theta)\phi - b(\phi) + \varepsilon \leq \nabla b(\theta)\theta - b(\theta).$$

Since, by the strong law of large numbers, $Y_n \rightarrow \nabla b(\theta)$ with $P_\theta^\infty = 1$ as $n \rightarrow +\infty$

(note that $E_\theta X_1 = \nabla b(\theta)$ [Barndorff-Nielsen (1978), page 114]),

$$\sup_{\phi \in m_i \cap \Theta} (Y_n \phi - b(\phi)) \rightarrow \sup_{\phi \in m_i \cap \Theta} \nabla b(\theta) \phi - b(\phi) \quad \text{with } P_\theta^\infty = 1$$

as $n \rightarrow +\infty$, by continuity of the function $Y \rightarrow \sup_{\phi \in m_i \cap \Theta} Y\phi - b(\phi)$ which follows from its convexity, $i = 1, 2$. (Note that

$$\nabla b(\theta) \in \text{int} \left\{ Y \in \mathbb{R}^k \mid \sup_{\phi \in \Theta} Y\phi - b(\phi) < +\infty \right\}$$

[Barndorff-Nielsen (1978), page 151].) So with probabilities $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, we have

$$(*) * \quad \left| \sup_{\phi \in m_i \cap \Theta} (Y_n \phi - b(\phi)) - \sup_{\phi \in m_i \cap \Theta} (\nabla b(\theta) \phi - b(\phi)) \right| < \varepsilon/4,$$

$i = 1, 2$.

Using (*) and (**), with $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, we have

$$\sup_{\phi \in m_1 \cap \Theta} (Y_n \phi - b(\phi)) + \varepsilon/2 < \sup_{\phi \in m_2 \cap \Theta} (Y_n \phi - b(\phi)),$$

which completes the proof of the first part of the proposition.

Let $\theta \in m_1 \cap m_2$ and $k_2 < k_1$. We put $S_{n,i} = \sup_{\phi \in m_i \cap \Theta} Y_n \phi - b(\phi)$, $i = 1, 2$. To prove the proposition, it is enough to show that $|S_{n,1} - S_{n,2}| = O_p(1/n)$. Since $\nabla b(\text{int } \Theta)$ is open, with probabilities $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, there exists a unique MLE $\hat{\theta}_n$ that satisfies

$$\sup_{\phi \in \Theta} Y_n \phi - b(\phi) = Y_n \hat{\theta}_n - b(\hat{\theta}_n)$$

and $Y_n = \nabla b(\hat{\theta}_n)$ [Barndorff-Nielsen (1978), Theorem 9.13, page 151]. Let $U_n = Y_n \theta - b(\theta)$ and $\hat{U}_n = Y_n \hat{\theta}_n - b(\hat{\theta}_n)$. Since $\theta \in m_1 \cap m_2$, $0 \leq S_{n,i} - U_n \leq \hat{U}_n - U_n$, $i = 1, 2$. Now $\hat{U}_n - U_n = Y_n(\hat{\theta}_n - \theta) + b(\theta) - b(\hat{\theta}_n)$ and $b(\hat{\theta}_n) - b(\theta) = (\hat{\theta}_n - \theta) \nabla b(\theta) + O_p(1/n)$ as obtained by writing a Taylor formula for b about θ and from the efficiency of the MLE $\hat{\theta}_n$ [Huber (1967)]. So $\hat{U}_n - U_n = (Y_n - \nabla b(\theta))(\hat{\theta}_n - \theta) + O_p(1/n)$. By the CLT, $Y_n - \nabla b(\theta) = O_p(1/\sqrt{n})$, and by efficiency of $\hat{\theta}_n$, $\|\hat{\theta}_n - \theta\| = O_p(1/\sqrt{n})$, so $\hat{U}_n - U_n = O_p(1/n)$. \square

REMARK 1.2. Proposition 1.2 still holds for any sequence a_n of positive real numbers in place of $\log n$ such that $a_n/n \rightarrow 0$ as $n \rightarrow +\infty$ and $a_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

The aim of the following section is to establish an asymptotic expansion for the $S(n, j)$ and show the role of $\log n$.

2. Asymptotic expansion of the $S(n, j)$.

2.1. *Study of the map $\phi \rightarrow Y\phi - b(\phi)$ on $m_j \cap \Theta$ when Y is in a neighborhood of $\nabla b(0)$ and $0 \in m_j \cap \text{int } \Theta$.* We note that, by a translation of the parameter, we can assume in this section that $\theta = 0$.

PROPOSITION 2.1. *There exists a neighborhood W of $\nabla b(0)$ in \mathbb{R}^k such that, if $Y \in W$, the map $\phi \rightarrow Y\phi - b(\phi)$ attains its maximum on $m_j \cap \Theta$ at a unique point $\hat{\theta}_Y$.*

The idea of the proof is to write the function $\phi \rightarrow Y\phi - b(\phi)$ in local coordinates near 0 and apply the implicit function theorem.

PROOF. We consider for $\varepsilon > 0$ the following neighborhoods of 0 in Θ :

$$N_{\varepsilon, Y} = \{\phi \in \Theta / Y\phi - b(\phi) > -b(0) - \varepsilon\}, \quad M_\varepsilon = N_{\varepsilon, \nabla b(0)}.$$

Then

$$\sup_{\phi \in \bar{N}_{\varepsilon, Y} \cap m_j \cap \Theta} (Y\phi - b(\phi)) = \sup_{\phi \in m_j \cap \Theta} (Y\phi - b(\phi)).$$

The function $\phi \rightarrow \nabla b(0)\phi - b(\phi)$ attains its unique maximum on $\text{int } \Theta$ at 0, and if $Y \in \nabla b(\text{int } \Theta)$, $\phi \rightarrow Y\phi - b(\phi)$ attains its unique maximum at some $\hat{\theta}$ in $\text{int } \Theta$ [Barndorff-Nielsen (1978), Theorem 9.13].

Lemma 2.1.1 follows easily from remarks on level sets [Barndorff-Nielsen (1978), page 150].

LEMMA 2.1.1. *There exist a compact set $K \subset \mathbb{R}^k$ and a constant C such that, if $\|Y - \nabla b(0)\| \leq C$, $M_\varepsilon \subset K$ and $N_{\varepsilon, Y} \subset K$ for $0 < \varepsilon \leq 1$.*

We put $M = m_j$ and $m = k_j = \dim m_j$, and we choose coordinates in \mathbb{R}^k and a coordinate neighborhood $M \cap V$ of 0 on M such that $M \cap V = \{X, y_{m+1}(X), \dots, y_k(X); X = (x_1, \dots, x_m) \in U\}$ for some neighborhood U of 0 in R^m with $|y_{m+l}(X)| \leq D\|X\|^2, \forall X \in U, l = 1, \dots, k - m$ for some D [cf. Guillemin and Pollack (1974), page 19]. It is easy to prove that we can pick $\varepsilon \leq 1$ small enough so that $M \cap M_{2\varepsilon} \subset M \cap V$ and $\delta_\varepsilon > 0$ such that if $\|Y - \nabla b(0)\| < \delta_\varepsilon$, $N_{\varepsilon, Y} \subset M_{2\varepsilon}$ (the existence of such a δ_ε follows easily from Lemma 2.1.1). In our choice of coordinates near 0, 0 has coordinates 0 in R^m and any $\phi \in M_{2\varepsilon} \cap M$, thus any $\phi \in N_{\varepsilon, Y} \cap M$ with $\|Y - \nabla b(0)\| < \delta_\varepsilon$ can be written $\phi(X) = X + O(\|X\|^2)$, as $X \rightarrow 0$, where $X = (x_1, \dots, x_m, 0, \dots, 0)$.

We would like to evaluate the function $F(X) = Y\phi(X) - b(\phi(X))$ in a neighborhood of 0. We will need the following lemma, which follows from a Taylor formula with integral remainder.

LEMMA 2.1.2. *Let k be a positive integer and $f(X) = O(\|X\|^k)$ denote a C^∞ function of X in a neighborhood U of 0 in R^d such that $f(X)/\|X\|^k$ is bounded in $U \setminus \{0\}$. Then*

$$\frac{\partial f}{\partial x_l}(X) = O(\|X\|^{k-1}), \quad l = 1, \dots, d, \text{ as } X \rightarrow 0.$$

Put

$$Q_{ij} = \frac{1}{2} \frac{\partial^2 b}{\partial \theta_i \partial \theta_j}(0)$$

and

$$Q(\phi) = \sum_{i,j=1}^k Q_{ij}\phi_i\phi_j.$$

Then

$$F(X) = YX + YO(\|X\|^2) - b(0) - \nabla b(0)(X + O(\|X\|^2)) - Q(X + O(\|X\|^2)) + O(\|X + O(\|X\|^2)\|^3)$$

near $X = 0$.

Using Lemma 2.1.2 we get, for $i = 1, \dots, m$,

$$\frac{\partial F}{\partial x_i}(X) = -2 \sum_{j=1}^m Q_{ij}x_j + (Y - \nabla b(0))O^i(\|X\|) + \left(Y_i - \frac{\partial b}{\partial \theta_i}(0) \right) + O(\|X\|^2),$$

where $O^i(\|X\|) = \partial/\partial x_i(O(\|X\|^2))$ and is an $O(\|X\|)$ by Lemma 2.1.2. Therefore,

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(0) = -2Q_{ij} + (Y - \nabla b(0))O^j(1) + O(\|X\|).$$

An application of the implicit function theorem [e.g., Dieudonné (1972), page 277] then shows that, for Y in a neighborhood of $\nabla b(0)$, the equation $\nabla F(X) = 0$ has a unique C^∞ solution ξ_Y in a neighborhood of 0 (note that Q is positive definite). Note also that $\sup_{\phi \in \bar{N}_{\epsilon, Y} \cap M \cap \Theta} Y\phi - b(\phi)$ is attained at a point $\bar{\theta}_Y$ in $N_{\epsilon, Y} \cap M \cap \Theta$. So for ϵ small enough, $\bar{\theta}_Y$ must satisfy $\nabla F(\phi^{-1}(\bar{\theta}_Y)) = 0$, so by the preceding, for ϵ small enough and $\|Y - \nabla b(0)\| < \delta_\epsilon$, it is unique. This completes the proof of Proposition 2.1. \square

2.2. *Asymptotic behavior of the integrals $J_n = \int_{m_j \cap \Theta} \exp(n(Y\phi - b(\phi))) d\mu_j(\phi)$, as $n \rightarrow +\infty$ uniformly in Y for $\|Y - \nabla b(0)\| \leq \sigma$, for some $\sigma > 0$.* This calculation will be an example of ‘‘Laplace’s method’’ for multidimensional integrals [see Hsu (1948, 1951) and Skinner (1980)]. We will use the notations of Section 2.1. As in Section 2.1 we assume that $\theta = 0$. We have the following proposition:

PROPOSITION 2.2. *Assume that $0 \in m_j \cap \text{int } \Theta$ and that the density f_j of μ_j on $m_j \cap \Theta$ is C^∞ and nowhere vanishing on $m_j \cap \Theta$. Then there exists a positive number σ such that on the compact set $\{\|Y - \nabla b(0)\| \leq \sigma\}$, uniformly in Y ,*

$$J_n = e^{n(Y\bar{\theta}_Y - b(\bar{\theta}_Y))} \left\{ \left(\frac{2\pi}{n} \right)^{k_j/2} \frac{f_j(\bar{\theta}_Y)(\det g_{ij}(\xi_Y))^{1/2}}{\{\det(-\partial^2 F/\partial x_i \partial x_j(\xi_Y))\}^{1/2}} + O(n^{-(k_j+1)/2}) \right\}.$$

The idea of the proof is to write the integral J_n in terms of the μ_j -measure of a neighborhood $N'_{\epsilon, Y}$ of $\bar{\theta}_Y$, and then to estimate $\mu_j(N'_{\epsilon, Y})$ by noticing that $N'_{\epsilon, Y}$ lies between two ellipsoids, and estimating the volume of these ellipsoids.

PROOF. By using Proposition 2.1 and its proof we choose $\sigma' > 0$ small enough so that $\bar{\theta}_Y$ exists and equals $\phi(\xi_Y)$ (see Section 2.1) for $\|Y - \nabla b(0)\| \leq \sigma'$. We put $f(\phi) = \exp(Y\phi - b(\phi) - (Y\bar{\theta}_Y - b(\bar{\theta}_Y)))$ and $I_n = \int_{m_j \cap \Theta} f^n d\mu_j(\phi) = E_{\mu_j} f^n$. Note that $0 < f \leq 1$. Let $g = 1 - f$; then $0 \leq g < 1$. If $G(t) = \mu_j(g \leq t)$ is the distribution function of g , then, integrating by parts,

$$E f^n = \int_0^1 (1-t)^n dG(t) = n \int_0^1 (1-t)^{n-1} G(t) dt$$

and $G(t) = \mu_j(f \geq 1-t) = \mu_j(N'_{\varepsilon, Y})$ where $\varepsilon = -\log(1-t)$ and $N'_{\varepsilon, Y} = \{\phi \mid Y\phi - b(\phi) > Y\bar{\theta}_Y - b(\bar{\theta}_Y) - \varepsilon\}$. It is easy to check that for ε small enough, say $\varepsilon \leq \varepsilon_0$, there exists $\theta_\varepsilon > 0$ such that if $\|Y - \nabla b(0)\| \leq \theta_\varepsilon$, $N'_{\varepsilon, Y} \subset M_{4\varepsilon}$ and $M_{4\varepsilon}$ is included in a coordinate neighborhood on m_j near 0 as in Section 2.1.

We wish to estimate the $\mu_j(N'_{\alpha, Y})$ for α small enough and $\|Y - \nabla b(0)\| \leq \min(\theta_\alpha, \sigma')$. By the preceding and Section 2.1, $\phi \in N'_{\alpha, Y}$ can be written $\phi(X) = X + O(\|X\|^2)$. As in Section 2.1, we define $F(X) = F(X, Y) = Y\phi(X) - b(\phi(X))$. Then

$$\mu_j(N'_{\alpha, Y}) = \int_{F(X) - F(\xi_Y) \geq -\alpha} f_j(\phi(X)) (\det g_{ij}(X))^{1/2} dX,$$

where f_j is the density of μ_j on $m_j \cap \Theta$ and g_{ij} are the coefficients of the Riemannian structure induced on m_j by the Euclidean structure of R^k , expressed in the chosen coordinate system on m_j . If A is the quadratic form defined by

$$A(V) = -\frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 F}{\partial x_i \partial x_j}(\xi_Y) V_i V_j,$$

where $m = \dim m_j$, we have $F(X) - F(\xi_Y) = -A(X - \xi_Y) + R(Y, X)$, where $R(Y, X)$ denotes the integral remainder in a Taylor expansion for F about ξ_Y . Note that A is positive definite for $\|Y - \nabla b(0)\| \leq \eta^*$ for some η^* . Let α_i be the positive eigenvalues of A , $i = 1, \dots, m$. Then $\min \alpha_i$ and $\max \alpha_i$ are continuous functions of Y since $\max \alpha_i = \|A\|$ and $\min \alpha_i = \|A^{-1}\|^{-1}$ where $\|A\| = \sup_{\|X\| \leq 1} \|AX\|$.

We define

$$(1) \quad \rho = \inf_{\|Y - \nabla b(0)\| \leq \eta^*} (\min \alpha_i).$$

Note that $\rho > 0$. We will use the following lemma, which is easily proved.

LEMMA 2.2.1. *There exists a constant K independent of Y such that $|R(Y, X)| \leq K \|X - \xi_Y\|^3$ for $X \in \phi^{-1}(M_{4\varepsilon_0})$ and $\|Y - \nabla b(0)\| \leq \eta^*$.*

Note that if Δ_ε is the diameter of $\phi^{-1}(M_{4\varepsilon})$, $\Delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, we can pick $0 < \varepsilon_1 < \varepsilon_0$ small enough so that $K \Delta_{\varepsilon_1} < \rho/2$, hence $\rho - K \Delta_{\varepsilon_1} > \rho/2$, where K is as in Lemma 2.2.1. We take $\alpha \leq \varepsilon_1$ and $\|Y - \nabla b(0)\| < \min(\theta_{\varepsilon_1}, \eta^*, \sigma')$; then it is straightforward to show that $X \in \phi^{-1}(N'_{\alpha, Y}) \Rightarrow \|X - \xi_Y\| \leq \sqrt{2\alpha/\rho}$.

Clearly

$$\mu_j(N'_{\alpha, Y}) = \int_{A(X - \xi_Y) - R(Y, X) \leq \alpha} H_j(X) dX,$$

where

$$H_j(X) = f_j(\phi(X))(\det g_{ij}(X))^{1/2}.$$

With a few elementary arguments it can be shown that

$$\begin{aligned} (2) \quad \int_{A(X - \xi_Y) + \partial_\alpha^* \|X - \xi_Y\|^2 \leq \alpha} H_j(X) dX &\leq \int_{A(X - \xi_Y) - R(Y, X) \leq \alpha} H_j(X) dX \\ &\leq \int_{A(X - \xi_Y) - \partial_\alpha^* \|X - \xi_Y\|^2 \leq \alpha} H_j(X) dX, \end{aligned}$$

where $\partial_\alpha^* = K\sqrt{2\alpha/\rho}$.

Let $E = \{X | A(X - \xi_Y) + \partial_\alpha^* \|X - \xi_Y\|^2 \leq \alpha\}$. We wish to estimate $\int_E H_j(X) dX$. For this we will need to estimate the volume of the ellipsoid E . To do this we will make the change of variable $X - \xi_Y = P(Z - \xi_Y)$ in R^m , where P is an orthogonal matrix such that $P^t A P$ is diagonal and A also denotes the matrix of the positive definite form A . The Jacobian of the transformation is $|dX/dZ| = \det P = \pm 1$. We have $A(X - \xi_Y) = \sum_{i=1}^m \alpha_i (Z - \xi_Y)_i^2$ and $\|Z - \xi_Y\|^2 = \|X - \xi_Y\|^2$ since P is orthogonal. Then

$$\int_E dX = \int_{\sum_{i=1}^m (\alpha_i + \partial_\alpha^*) (Z - \xi_Y)_i^2 \leq \alpha} dZ.$$

Therefore,

$$\int_E dX = (\alpha\pi)^{m/2} / \prod_{i=1}^m (\alpha_i + \partial_\alpha^*)^{1/2} \Gamma\left(\frac{m}{2} + 1\right).$$

We note here that

$$\prod_{i=1}^m \alpha_i^{-1/2} = \left\{ \det \left(-\frac{1}{2} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} (\xi_Y) \right) \right) \right\}^{-1/2} = 2^{m/2} \left\{ \det \left(-\frac{\partial^2 F}{\partial x_i \partial x_j} (\xi_Y) \right) \right\}^{-1/2}.$$

We expand the function $H_j(X)$ about ξ_Y : $H_j(X) = H_j(\xi_Y) + h_j(X - \xi_Y)$ with $|h_j(X - \xi_Y)| \leq M \|X - \xi_Y\|$, for some M independent of Y , $X \in \phi^{-1}(M_{4\epsilon_0})$ and $\|Y - \nabla b(0)\| \leq \min(\theta_{\epsilon_1}, \eta^*, \sigma')$. We have

$$\int_E H_j(X) dX = H_j(\xi_Y) \lambda(E) + \int_E h_j(X - \xi_Y) dX;$$

where λ is the Lebesgue measure on R^m , and $|\int_E h_j(X - \xi_Y) dX| \leq M \int_E \|X - \xi_Y\| dX$. It is easy to show that if $A(X - \xi_Y) + \partial_\alpha^* \|X - \xi_Y\|^2 \leq \alpha$, then

$$\|X - \xi_Y\| \leq \frac{\sqrt{\alpha}}{\sqrt{\rho + \partial_\alpha^*}};$$

it follows that

$$\int_E H_j(X) dX = H_j(\xi_Y)(\alpha\pi)^{m/2} \left/ \left(\Gamma\left(\frac{m}{2} + 1\right) \prod_{i=1}^m (\alpha_i + \partial_\alpha^*)^{1/2} \right) \right. + H(\alpha),$$

with $|H(\alpha)| \leq M\pi^{m/2}\alpha^{(m+1)/2}/(\Gamma(m/2 + 1)(\rho + \partial_\alpha^*)^{(m+1)/2})$. By the same reasoning we get

$$\begin{aligned} & \int_{A(X - \xi_Y) - \partial_\alpha^* \|X - \xi_Y\|^2 \leq \alpha} H_j(X) dX \\ &= H_j(\xi_Y)(\alpha\pi)^{m/2} \left/ \Gamma\left(\frac{m}{2} + 1\right) \prod_{i=1}^m (\alpha_i - \partial_\alpha^*)^{1/2} \right. + H^1(\alpha), \end{aligned}$$

where

$$|H^1(\alpha)| \leq M\pi^{m/2}\alpha^{(m+1)/2} \left/ \left(\Gamma\left(\frac{m}{2} + 1\right) (\rho - \partial_\alpha^*)^{(m+1)/2} \right) \right.$$

for α small enough so that $\partial_\alpha^* < \rho$, say for $\alpha \leq \alpha_0$, with $\alpha_0 < \varepsilon_1$. Now

$$\prod_{i=1}^m (\alpha_i + \partial_\alpha^*)^{-1/2} = \prod_{i=1}^m \alpha_i^{-1/2} (1 + O(\sqrt{\alpha})) \quad \text{as } \alpha \rightarrow 0,$$

uniformly in Y . Also $H(\alpha) = O(\alpha^{(m+1)/2})$ and $H^1(\alpha) = O(\alpha^{(m+1)/2})$ uniformly in Y for $\|Y - \nabla b(0)\| \leq \min(\theta_{\alpha_0}, \eta^*, \sigma')$. Using these estimates, and inequality (2), we have

$$\begin{aligned} & \int_{A(X - \xi_Y) - R(Y, X) \leq \alpha} H_j(X) dX \\ &= H_j(\xi_Y)(\alpha\pi)^{m/2} \left/ \left(\Gamma\left(\frac{m}{2} + 1\right) \prod_{i=1}^m \alpha_i^{1/2} \right) \right. + O(\alpha^{(m+1)/2}) \end{aligned}$$

uniformly in Y for $\|Y - \nabla b(0)\| \leq \min(\theta_{\alpha_0}, \eta^*, \sigma')$. We have now shown the following lemma.

LEMMA 2.2.2. *There exist positive numbers σ and α_0 such that if $\|Y - \nabla b(0)\| \leq \sigma$ and $\alpha \leq \alpha_0$, then $\mu_j(N'_{\alpha, Y}) = C_m(Y)\alpha^{m/2} + \beta(\alpha)$ with*

$$\begin{aligned} & C_m(Y) \\ &= f_j(\phi(\xi_Y))(\det g_{ij}(\xi_Y))^{1/2} \left\{ \det \left(-\frac{\partial^2 F}{\partial x_i \partial x_j}(\xi_Y) \right) \right\}^{-1/2} (2\pi)^{m/2} \left/ \Gamma\left(\frac{m}{2} + 1\right) \right. \end{aligned}$$

and $|\beta(\alpha)| \leq \alpha^{(m+1)/2}\beta^1(\alpha)$ where $\beta^1(\alpha)$ is bounded and independent of Y .

Proposition 2.2 now follows easily from Lemma 2.2.2 applied to $G(t) = \mu_j(N'_{\varepsilon, Y})$ with $\varepsilon = -\log(1 - t)$, and known facts about Euler's beta and gamma functions [see Haughton (1983) for details]. \square

2.3. *Asymptotic expansion of the $S(n, j)$.* The following theorem will show the special role of $\alpha_n = \log n$ in Schwarz's criterion. We will use the notation and assumptions of the previous sections.

THEOREM 2.3. *Let the true $\theta \in m_j \cap \text{int } \Theta$. If*

$$S(n, j) = \log \alpha_j + \log \int_{m_j \cap \Theta} \exp(Y_n \phi - b(\phi)) d\mu_j(\phi)$$

with $Y_n = n^{-1} \sum_{i=1}^n X_i$, if $\bar{\theta}_n^j$ is the unique point on $m_j \cap \Theta$ where the function $Y_n \phi - b(\phi)$ attains its maximum, defined with probabilities converging to 1 as $n \rightarrow \infty$, then

$$\begin{aligned} S(n, j) &= n \sup_{\phi \in m_j \cap \Theta} (Y_n \phi - b(\phi)) - \frac{1}{2} k_j \log \left(\frac{n}{2\pi} \right) + \log \alpha_j + \log f_j(\bar{\theta}_n^j) \\ &\quad - \frac{1}{2} \log \det \left(\frac{\partial^2 b}{\partial \phi_r \partial \phi_s}(\bar{\theta}_n^j) \right) + O_p(n^{-1/2}). \end{aligned}$$

PROOF. We will need a few lemmas. Let $M = m_j$. We can assume that $\theta = 0$.

LEMMA 2.3.1. *If g_{ij} are the coefficients of the Riemannian structure induced on M by the Euclidean structure of R^k , corresponding to the coordinate neighborhood $M \cap V$ of Section 2.1, then $g_{ij}(0) = \delta_{ij}$.*

PROOF. An easy calculation shows that

$$g_{ij}(X) = \delta_{ij} + \sum_{l=1}^{k-m} \frac{\partial y_{m+l}}{\partial x_i}(X) \frac{\partial y_{m+l}}{\partial x_j}(X). \quad \square$$

LEMMA 2.3.2. *Let f be a C^∞ function on a convex neighborhood U of 0 in R^k . Then there exists $\sigma > 0$ such that, if $\|Y - \nabla b(0)\| \leq \sigma$, we have $f(\xi_Y) = f(0) + R(Y)$, where $|R(Y)| \leq C \|\xi_Y\|$ for some constant C independent of Y (where ξ_Y is as defined in Section 2.1).*

PROOF. The proof is similar to the proof of Lemma 2.2.1 and is omitted.

LEMMA 2.3.3. *If $Y_n = n^{-1} \sum_{i=1}^n X_i$, then $\|\xi_{Y_n}\| = O_p(n^{-1/2})$.*

PROOF. The lemma follows from the central limit theorem.

Theorem 2.3 now follows easily from Proposition 2.2 and Lemmas 2.3.1 and 2.3.2. \square

We now give a proposition which will show that when choosing between models m_i and m_j such that $k_i \neq k_j$ and $\theta \in m_i \cap m_j \cap \text{int } \Theta$, with probabili-

ties $P_\theta^n \rightarrow 1$ as $n \rightarrow +\infty$, the Bayes choice and the choice based on the quantities

$$\Gamma(n, j) = n \sup_{\phi \in m_j \cap \Theta} Y_n \phi - b(\phi) - \frac{1}{2} k_j \log \left(\frac{n}{2\pi} \right) + \log \alpha_j + \log f_j(\bar{\theta}_n^j) - \frac{1}{2} \log \left(\det \left(\frac{\partial^2 b}{\partial \phi_r \partial \phi_s} \right) \right)$$

coincide.

COROLLARY 2.3. *If $\theta \in m_{j_1} \cap m_{j_2} \cap \text{int } \Theta$ and $k_{j_1} \neq k_{j_2}$, then $P_\theta^n(S(n, j_1) > S(n, j_2))$ and $\Gamma(n, j_1) \leq \Gamma(n, j_2) \rightarrow 0$ as $n \rightarrow +\infty$.*

The proof is straightforward and is left to the reader.

REMARK 2.3. Note that $\Gamma(n, j) = \gamma(n, j) + O_p(1) + O_p(n^{-1/2})$ so $\Gamma(n, j) = \gamma(n, j) + o_p(\log n)$, where $\gamma(n, j)$ was defined in Section 1.2. Proposition 1.2 therefore holds with $\gamma(n, j)$ replaced by $\Gamma(n, j)$.

3. Choice of degree in a polynomial regression. Let (x_i, y_i) be a set of data in \mathbb{R}^2 where $y_i = \sum_{j=0}^d a_j x_i^j + \varepsilon_i$ and the ε_i are iid $N(0, \varepsilon^2)$. If we consider x_i not as a random variable but as an ‘‘incidental parameter,’’ given the ‘‘structural parameters’’ $(a_0, a_1, \dots, a_d, d, \varepsilon)$, the law of the y_i is $N(m_i, \varepsilon^2)$ with $m_i = \sum_{j=0}^d a_j x_i^j$. Schwarz’s criterion does not apply to the observations y_i since they are not iid. We will show that we can still apply a Schwarz criterion to this problem by considering x_i as a random variable. This will also show the necessity of considering ‘‘curved models.’’ We will assume therefore that the ε_i are iid $N(0, \varepsilon^2)$, the x_i are iid $N(m, \tau^2)$ and that all the ε_i are independent of all the x_i . Let $\eta = (d, a_0, \dots, a_d, \varepsilon^2, m, \tau^2)$. We assume that η has a prior law of the form $\sum \alpha_k \mu_k$ where α_k is the prior probability that $d = k$.

Now let $z_i = (x_i, y_i)$. The z_i are iid given η and their density is

$$(3) \quad f(x, y) = \frac{1}{2\pi\tau\varepsilon} \exp \left[-\frac{(x - m)^2}{2\tau^2} - \frac{\left(y - \sum_{j=0}^d a_j x^j \right)^2}{2\varepsilon^2} \right],$$

so $f(x, y) = \exp[\sum_{j=1}^{3d+2} \theta_j T_j(x, y) - b(\theta)]$, where the $T_i(x, y)$ are defined by

$$(4) \quad \begin{aligned} T_1(x, y) &= -x^2, & T_2(x, y) &= -y^2, & T_3(x, y) &= y, \dots, \\ T_{3+j}(x, y) &= x^j y, & j &= 0, 1, \dots, d, & T_{4+d}(x, y) &= -x, \\ T_{5+d}(x, y) &= -x^3, & T_{6+d}(x, y) &= -x^4, \dots, & T_{3d+2}(x, y) &= -x^{2d}, \end{aligned}$$

the θ_i are defined by

$$\begin{aligned}
 \theta_1 &= 1/2\tau^2 + (2a_0a_2 + a_1^2)/2\varepsilon^2, & \theta_2 &= 1/2\varepsilon^2, \\
 \theta_3 &= a_0/\varepsilon^2, & \theta_4 &= a_1/\varepsilon^2, \dots, \theta_{3+j} = a_j/\varepsilon^2, & j &= 0, \dots, d, \\
 \theta_{4+d} &= a_0a_1/\varepsilon^2 - m/\tau^2, & \theta_{5+d} &= \sum_{j=0}^3 a_j a_{3-j}/2\varepsilon^2, \\
 (5) \quad \theta_{6+d} &= \sum_{j=0}^4 a_j a_{4-j}/2\varepsilon^2, \dots, \theta_{2d+2} = \sum_{j=0}^d a_j a_{d-j}/2\varepsilon^2, \\
 \theta_{2d+3} &= \sum_{j=1}^d a_j a_{d+1-j}/2\varepsilon^2, \dots, \theta_{3d+1} = \sum_{j=d-1}^d a_j a_{2d-1-j}/2\varepsilon^2, \\
 \theta_{3d+2} &= a_d^2/2\varepsilon^2,
 \end{aligned}$$

and where $b(\theta)$ is defined by normalization. The family of distributions with densities $f(x, y)$ with respect to the Lebesgue measure on \mathbb{R}^2 is an exponential family with natural parameter space

$$\Theta = \left\{ \theta \in \mathbb{R}^{3d+2} \mid \int_{\mathbb{R}^2} \exp \left[\sum_{j=1}^{3d+2} \theta_j T_j(x, y) \right] dx dy < \infty \right\}.$$

Let us describe the different models: We define the *maximum model* m_d to be the set of $\theta \in \mathbb{R}^{3d+2}$ defined parametrically by (5) with $(a_0, \dots, a_d) \in \mathbb{R}^{d+1}$, $\varepsilon^2 > 0$, $\tau^2 > 0$. Then m_d is a closed manifold in \mathbb{R}^{3d+2} of dimension $3d + 2 - (2d - 2) = d + 4$ [Spivak (1979), page 65, Proposition 12]. The other models, corresponding to $a_d = 0, a_d = a_{d-1} = 0, \dots, a_d = a_{d-1} = \dots = a_1 = 0$ will be denoted by $m_{d-1}, m_{d-2}, \dots, m_0$. Note that $m_j = \{\theta \in m_d / \theta_{3+j+1} = \dots = \theta_{3+d} = 0\}$, so m_j is a closed submanifold of m_d of dimension $j + 4$ in m_d . In particular, $\dim m_1 = 5$ and $\dim m_0 = 4$, as submanifolds of m_d . Note that all the m_j are curved, and that, assuming the appropriate regularity for the density of μ_k with respect to the volume element on m_k , the results of Section 2.3 apply.

Acknowledgment. I would like to thank Richard M. Dudley for help and guidance in preparing this paper.

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