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ON THE CHROMATIC NUMBER OF GEOMETRIC HYPERGRAPHS*

SHAKHAR SMORODINSKY†

Abstract. A finite family $\mathcal{R}$ of simple Jordan regions in the plane defines a hypergraph $H = H(\mathcal{R})$ where the vertex set of H is $\mathcal{R}$ and the hyperedges are all subsets $S \subset \mathcal{R}$ for which there is a point p such that $S = \{r \in \mathcal{R} | p \in r\}$. The chromatic number of $H(\mathcal{R})$ is the minimum number of colors needed to color the members of $\mathcal{R}$ such that no hyperedge is monochromatic. In this paper we initiate the study of the chromatic number of such hypergraphs and obtain the following results: (i) Any hypergraph that is induced by a family of n simple Jordan regions such that the maximum union complexity of any k of them (for $1 \leq k \leq m$) is bounded by $\mathcal{U}(m)$ and $\frac{\mathcal{U}(m)}{m}$ is a nondecreasing function is $O(\frac{\mathcal{U}(n)}{n})$-colorable. Thus, for example, we prove that any finite family of pseudo-discs can be colored with a constant number of colors. (ii) Any hypergraph induced by a finite family of planar discs is four colorable. This bound is tight. In fact, we prove that this statement is equivalent to the four-color theorem. (iii) Any hypergraph induced by n axis-parallel rectangles is $O(\log n)$-colorable. This bound is asymptotically tight. Our proofs are constructive. Namely, we provide deterministic polynomial-time algorithms for coloring such hypergraphs with only "few" colors (that is, the number of colors used by these algorithms is upper bounded by the same bounds we obtain on the chromatic number of the given hypergraphs). As an application of (i) and (ii) we obtain simple constructive proofs for the following: (iv) Any set of n Jordan regions with near linear union complexity admits a conflict-free (CF) coloring with polylogarithmic number of colors. (v) Any set of n axis-parallel rectangles admits a CF-coloring with $O(\log^2(n))$ colors.

Key words. hypergraphs, conflict-free, coloring, wireless

AMS subject classification. combinatorics

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1. Introduction.

A *hypergraph* is a pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E} \subset 2^V$. The set V is called the *ground set* or the *vertex set* and the elements of $\mathcal{E}$ are called *hyperedges*. A k-coloring of a hypergraph $H = (V, \mathcal{E})$, for some positive integer k, is a function $\chi : V \to \{1, 2, \ldots, k\}$ such that no $S \in \mathcal{E}$ with $|S| \geq 2$ is monochromatic. Let $\chi(H)$ denote the minimum integer k for which H has a k-coloring. $\chi(H)$ is called the *chromatic number* of H.

Let $\mathcal{R}$ be a set of regions in the plane. For a point $p \in \cup_{r \in \mathcal{R}} r$, put $r(p) = \{r \in \mathcal{R} \mid p \in r\}$. Let $H(\mathcal{R})$ denote the hypergraph $(\mathcal{R}, \{r(p) \mid p \in \cup_{r \in \mathcal{R}}\})$. We say that $H(\mathcal{R})$ is the hypergraph *induced* by $\mathcal{R}$.

DEFINITION 1.1. *Let $\mathcal{R}$ be a family of n simple Jordan regions in the plane. The* union complexity *of $\mathcal{R}$ is the number of vertices (i.e., intersection of boundaries of pairs of regions in $\mathcal{R}$) that lie on the boundary $\partial \bigcup_{r \in \mathcal{R}} r$.*

In this work we initiate the study of the chromatic number of hypergraphs that are induced by simple geometric regions such as discs, pseudo-discs, axis-parallel rectangles, etc. Our main result (section 5) is a theorem correlating the chromatic number of the underlying hypergraphs with the union complexity of the regions inducing those hypergraphs. Specifically, we prove the following theorem.

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THEOREM 1.2. *Let $\mathcal{R}$ be a set of n regions and let $\mathcal{U} : \mathbb{N} \to \mathbb{N}$ be a function such that $U(m)$ is the maximum complexity of any k regions in $\mathcal{R}$ over all $k \leq m$, for $1 \leq m \leq n$. We assume that $\frac{\mathcal{U}(m)}{m}$ is a nondecreasing function. Then, $\chi(H(\mathcal{R})) = O(\frac{\mathcal{U}(n)}{n})$. Furthermore, such a coloring can be computed in polynomial time.*

In section 3 we study the chromatic number of a hypergraph that is induced by discs and prove the following theorem.

THEOREM 1.3. *Let $\mathcal{D}$ be a finite family of discs in the plane. Then the hypergraph $H(\mathcal{D})$ is four colorable.*

As one can easily see, this bound is tight by taking four pairwise touching discs. In such a case, every pair of discs needs to be colored with distinct colors. This bound is somewhat surprising in the following sense. In the restricted case where we are given a finite family $\mathcal{R}$ of discs such that every two are either touching (i.e., the boundaries of the two discs share a common point but the interiors of the discs are disjoint) or disjoint, it is easy to see that bounding the chromatic number of $H(\mathcal{R})$ by four is equivalent to bounding the chromatic number of the "kissing" graph induced by the discs (i.e., the vertex set of the graph is $\mathcal{R}$ and the edges are the touching pairs) by four. However, this graph is planar. On the other hand, a classical theorem due to Koebe [12] asserts that every planar graph can be realized as a kissing discs graph. In section 3 we show that the four-color theorem is equivalent to coloring a hypergraph induced by a finite family of discs (not necessarily interior disjoint but also discs that might have arbitrary overlaps) with at most four colors. As mentioned above, one direction of the proof easily follows from Koebe's theorem [12].

In section 4 we study the chromatic number of a hypergraph induced by n axis-parallel rectangles and prove the following theorem.

THEOREM 1.4. *Let $\mathcal{R}$ be a family of n axis-parallel rectangles. Then $\chi(H(\mathcal{R})) = O(\log n)$.*

This bound is asymptotically tight as demonstrated recently by a lower bound construction of Pach and Tardos [14].

To the best of our knowledge, these questions were not addressed previously. Beyond their purely theoretical interest, we apply our results to obtain a simple framework for tackling the problems of conflict-free colorings.

DEFINITION 1.5 (see [7, 16]). *A coloring of regions is* conflict-free *(CF) if for any covered point in the plane, there exists a region that covers it with a unique color (i.e., no other region covering that point has the same color).*

In section 6 we show how to apply our results on proper colorings of regions to obtain simple deterministic, polynomial-time algorithms for CF-colorings of those regions.

CF-coloring problems were recently introduced in [7, 16] in the context of frequency assignment in cellular networks. In addition to this practical motivation, this new coloring model has drawn much attention of researchers through its own theoretical interest and such colorings have been the focus of several recent papers [1, 4, 6, 8, 9, 10, 13].

Even et al. [7] have shown that any family of n discs in the plane admits a CF-coloring with only $O(\log n)$ colors and that this bound is tight in the worst case. Furthermore, such a coloring can be computed in deterministic polynomial time.[1] The results of Even et al. [7] were further extended by Har-Peled and Smorodinsky

[1] In [7] it is shown that finding the minimum number of colors needed to CF-color a given collection of discs is NP-hard even when all discs are congruent, and an $O(\log n)$ approximation-ratio algorithm is provided.

[9] by combining more involved probabilistic and geometric ideas. The main result of [9] is a randomized algorithm which CF-colors any set of n pseudo-discs with $O(\log n)$ colors with high probability. As an application of our main result, we obtain a simple deterministic, polynomial-time algorithm for CF-colorings regions. The performance (i.e., the number of colors used by our algorithm) depends on the union complexity of the underlying regions. For example, we obtain a simple determnistic, polynomial-time algorithm that CF-colors any set of n pseudo-discs with only $O(\log n)$ colors.

2. Preliminaries. We start with some basic definitions and lemmas.

DEFINITION 2.1. *The* Delaunay graph $G(H)$ *of a hypergraph* $H = (V, \mathcal{E})$ *is a simple graph* $G = (V, E)$*, where the edge set* E *is defined as* $E = \{(x, y) \mid \{x, y\} \in \mathcal{E}\}$ *(i.e.,* G *is the graph on the vertex set* V *whose edges consist of all hyperedges in* H *of cardinality two).*

LEMMA 2.2. *For every hypergraph* H *we have*

$$\chi(G(H)) \leq \chi(H).$$

Proof. Simply because any proper coloring of the vertices of H is also a proper coloring for $G(H)$. ☐

DEFINITION 2.3. *We say that a hypergraph* $H = (V, \mathcal{E})$ *has rank* i *for* $i \geq 2$ *if for any hyperedge* $S \in \mathcal{E}$ *with* $|S| > i$ *there exists a hyperedge* $S' \in \mathcal{E}$ *such that* $S' \subset S$ *and* $|S'| = i$*.*

LEMMA 2.4. *Let* $H = (V, \mathcal{E})$ *be a hypergraph of rank two. Then* $\chi(H) = \chi(G(H))$*.*

Proof. By Lemma 2.2 we have $\chi(G(H)) \leq \chi(H)$. It remains to prove that $\chi(H) \leq \chi(G(H))$. Let χ be a proper coloring of $G(H)$ with $k = \chi(G(H))$ colors. This coloring is also a proper coloring of H. Indeed, let $e \in \mathcal{E}$ be a hyperedge with cardinality > 1. Then, by assumption, there exists an edge $e' \subset e$ in $G(H)$ and this edge is nonmonochromatic. Then, obviously, e is nonmonochromatic. This completes the proof of the lemma. ☐

DEFINITION 2.5. *A simple graph* $G = (V, E)$ *is called* k*-degenerate for some positive integer* k*, if every (vertex-induced) subgraph of* G *has a vertex of degree at most* k*.*

LEMMA 2.6. *Let* $G = (V, E)$ *be a* k*-degenerate graph. Then* $\chi(G) \leq k + 1$*.*

Proof. Proceed by induction on $n = |V|$. Let $v \in V$ be a vertex of degree at most k. By the induction hypothesis, the graph $G \setminus v$ (obtained by removing v and all of its incident edges from G) is $k+1$ colorable. Since v has at most k neighbors there is always a free color that can be assigned to v which is distinct from the colors of its neighbors. ☐

3. Hypergraphs induced by discs. In this section we show that any hypergraph induced by a finite family of discs in the plane is four colorable.

Let H^+ denote the set of all positive halfspaces in $\mathbb{R}^3$ (i.e., those halfspaces consisting of all points that lie above some fixed plane). For a given set $P \subset \mathbb{R}^3$, put $H^+(P) = \{h \cap P | h \in H^+\}$.

A transformation to points and half-spaces. In what follows, we show that the problem of coloring n arbitrary discs in the plane reduces (but is not equivalent) to that of coloring a set of points P in $\mathbb{R}^3$ with respect to the set of ranges $H^+(P)$ (i.e., coloring the hypergraph $H = (P, H^+(P))$).

We transform a point $p = (a, b)$ in $\mathbb{R}^2$ to the plane p^* in $\mathbb{R}^3$, with the parametrization $z = -2ax - 2by + a^2 + b^2$ and transform a disc S in $\mathbb{R}^2$, with center (x, y) and radius $r \geq 0$, to the point S^* in $\mathbb{R}^3$, with coordinates $(x, y, r^2 - x^2 - y^2)$.

It is easily seen that in this transformation a point $p \in \mathbb{R}^2$ lies inside (respectively, on the boundary of, outside) a disc S, if and only if the point $S^* \in \mathbb{R}^3$ lies above (respectively, on, below) the plane p^*. Indeed, assume that point $p = (a, b)$ lies inside (respectively, on the boundary of, outside) the disc S with center (x, y) and radius r. This can be formulated by the inequality: $(a-x)^2 + (b-y)^2 < r^2$ or $-2ax - 2by + a^2 + b^2 < r^2 - x^2 - y^2$ (respectively, an equality $=$, or inequality with $>$), which is equivalent to that of the point $(x, y, r^2 - x^2 - y^2) = S^*$ lies above (respectively on, or below) the plane $z = -2ax - 2by + a^2 + b^2$ (which is the dual p^* of p), as asserted.

Given a collection $\mathcal{S} = \{S_1, \ldots, S_n\}$ of n distinct discs in the plane, one can use the above transformation to obtain a collection $\mathcal{S}^* = \{S_1^*, \ldots, S_n^*\}$ of n points in $\mathbb{R}^3$, such that any valid coloring of $\mathcal{S}^*$ with respect to $H^+(\mathcal{S}^*)$ with k colors induces a coloring of the discs of $\mathcal{S}$ with the same set of k colors.

Remark. We note that the two coloring problems are not equivalent. Indeed, the set of all planes in $\mathbb{R}^3$ that are images (under the above transformation) of points in the plane are such that they are all tangent to the paraboloid $z = -x^2 - y^2$. Since we color the points in $\mathbb{R}^3$ so that *any* positive halfspace is not monochromatic (not only positive halfspaces bounded by planes which are tangent to the paraboloid), we actually result in a coloring of a hypergraph with potentially more hyperedges than the original hypergraph.

LEMMA 3.1. *Let $P \subset \mathbb{R}^3$ be a finite set. Let H be the hypergraph induced by $H^+(P)$ (that is, $H = (P, H^+(P))$). Then $\chi(H) \leq 4$.*

Proof. Recall that $G(H)$ is the graph whose vertex set is P and whose edge set is $E = \{h \cap P \mid h \in H^+ \text{ and } |h \cap P| = 2\}$. Thus G, contains the skeleton graph of the upper convex hull of P, although G may contain additional edges. The rank of the hypergraph H is 2. Indeed, every subset $P' \in H^+(P)$ such that $|P'| > 2$ must contain an edge of E. To see this, let $h \in H^+$ be a positive halfspace containing at least three points of P. Without loss of generality, assume that the plane π bounding h is in "general position" with respect to the points of P (i.e., no line passing through two points of P is parallel to π). This can be achieved by a proper perturbation of π such that the set of points above the perturbed plane does not change. Then, we can translate π upwards keeping the translated plane π' parallel to π until the positive halfspace bounded by π' contains exactly two points of P'. By definition, these pair of points form an edge in G, so the rank of H is indeed 2. By Lemma 2.4, it is enough to color the vertices of G properly (i.e., such that no color class contains an edge). We will show that G is a planar graph and by the Four-Color Theorem (see, e.g., [2, 3]) it is four colorable. To show that G is planar, we project P onto the plane orthogonally and draw the graph G using straight line segments to represent the edges. We want to show that in this drawing there are no crossings. Assume to the contrary that there are two edges $e_1 = (p_1, q_1), e_2 = (p_2, q_2)$ whose projections cross. Let l be the line parallel to the z-axis that passes through this crossing point. Since $e_1, e_2 \in E$ belong to G, there exists a plane π_1 (respectively, π_2) such that the positive halfspace bounded by π_1 (respectively, π_2) contains only e_1 (respectively, only e_2). l must pass through a point v_1 on the line segment (in $\mathbb{R}^3$) connecting p_1, q_1 and a point v_2 on the line segment connecting p_2, q_2. Assume without loss of generality that v_1 is below v_2. See Figure 1 for an illustration. It is easy to see that π_1 intersect l in a point q that is below v_1. Indeed, since p_1 and q_1 are above π_1 (recall that p_1 and q_1 are the only points of P above π_1) then (by convexity) also the point v_1 is above π_1. Thus q is also below v_2. However, we know that both p_2 and q_2 lie below π_1 (π_1

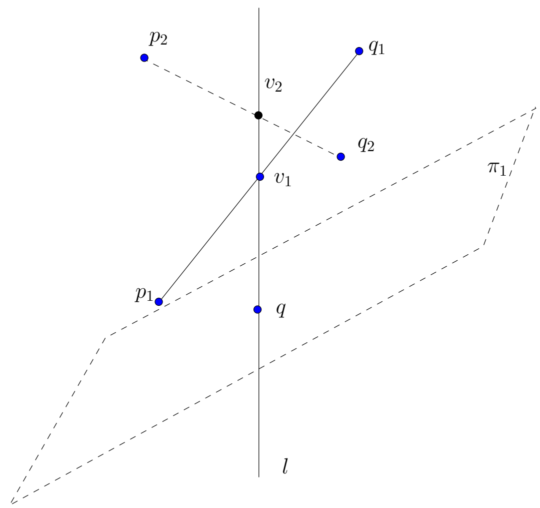

FIG. 1. *Illustration of the proof of Lemma* 3.1 *with the two edges* e_1, e_2*, whose projection cross. The plane* π_1 *must intersect* l *below* v_1 *but above* v_2*, a contradiction.*

separates p_1, q_1 from the rest of the points in P). This means that the line segment connecting p_2, q_2 is also below π_1 which means that v_2 is below q, a contradiction. Thus G is a planar graph and therefore four-colorable. This completes the proof of the lemma. ∎

Proof of Theorem 1.3. We use the above "lifting transformation" such that the discs are transformed into points in $\mathbb{R}^3$. By Lemma 3.1, there is a coloring of the transformed points with four colors, such that any positive halfspace that contains at least two of these points contains at least two points with distinct colors. We use the same coloring for the preimages of the points and obtain a valid coloring for the hypergraph $H(\mathcal{D})$. ∎

Remark. It is not clear how to obtain a different proof of Theorem 1.3 without the lifting transformation. The major problem is that $H(\mathcal{D})$ may have rank greater than two. Indeed, if a point p is contained in at least three discs of $\mathcal{D}$, it does not necessarily imply that two of those discs have a point common only to them. This is illustrated in Figure 2. In section 5 we obtain a general upper bound on the chromatic number of regions with low union complexity. Discs are an example of such regions. Therefore, section 5 provides a different way to obtain an upper bound. However, the method we develop in section 5 will only imply an upper bound of six on the chromatic number of discs.

4. Axis-parallel rectangles. In this section we deal with coloring axis-parallel rectangles. We show that any hypergraph that is induced by a family of n axis-parallel rectangles admits an $O(\log n)$ coloring. This bound is asymptotically tight.

We show that the maximum number of colors $f(n)$ needed to color n axis-parallel rectangles satisfies the recursion $f(n) \leq 8 + f(\frac{n}{2})$, and thus implies the asserted bound. We start with a lemma concerning a restricted case when all rectangles of $\mathcal{R}$ intersect some vertical line.

LEMMA 4.1. *Let* $\mathcal{R}$ *be a finite family of axis-parallel rectangles all of which intersect some vertical line* l*. Then* $\chi(H(\mathcal{R})) \leq 8$*.*

Proof. We assume that the rectangles are in general position in the sense that no

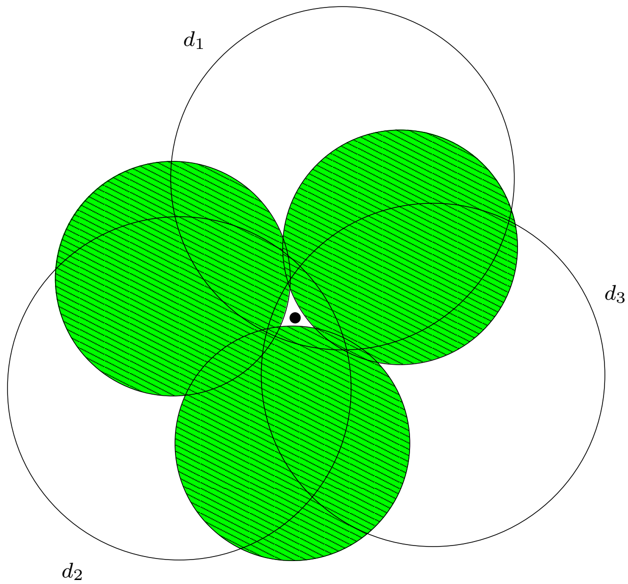

FIG. 2. *An example of a set $\mathcal{D}$ of six discs* (taken from [16]) *whose induced hypergraph $H(\mathcal{D})$ has rank three; there is a point covered only by the three discs d_1, d_2, and d_3. However, there is no point that is covered by two of those discs and no other disc. Note that not every coloring of the Delaunay graph $G(H(\mathcal{D}))$ induces a valid coloring of $H(\mathcal{D})$. Indeed in a coloring of $G(H(\mathcal{D}))$, the discs d_1, d_2, d_3 might get the same color. However, then we would have the hyperedge $\{d_1, d_2, d_3\}$ monochromatic.*

two vertical (or two horizontal) sides share a point. This can be achieved by a small perturbation which does not decrease the number of colors needed. It is easily seen that in this case the hypergraph $H = H(\mathcal{R})$ has rank two. Therefore, by Lemma 2.4, it is enough to show that $\chi(G(H)) \leq 8$. We will show that $G = G(H)$ is 7-degenerate and therefore by Lemma 2.6 is 8-colorable. It is sufficient to argue that the average degree of every (vertex-induced) subgraph of G is less than 8. Let p be a point that is covered by exactly two rectangles $r_1, r_2 \in \mathcal{R}$. Assume without loss of generality that p is to the right of l (see Figure 3). We will charge the pair r_1, r_2 to one of the horizontal sides of one of the rectangles of $\mathcal{R}$ so that each horizontal side is charged at most twice. We translate p to the right until it reaches a vertical side of one of the rectangles r_1, r_2. Assume without loss of generality that this is the side of r_1. Then we move downward until we reach a horizontal side e of some rectangle at a point p'. The important fact is that the horizontal line segment connecting the line l to the point p' is contained in $r_1 \cap r_2$ (we consider the rectangles in $\mathcal{R}$ as closed regions). We charge the pair r_1, r_2 to e. We show that to the right of l at most one charge can occur at such a side. There are two cases to consider: The side e is an upper horizontal side of some rectangle r_3 (note that it cannot be a lower horizontal side of r_3 since then p would have belonged to $r_1 \cap r_2 \cap r_3$). Indeed, assume that e is charged twice to the right of l and that the other charge can occur at a point p''. Assume without loss of generality that p'' is to the left of p'. It is easily seen that p'' belongs to $r_1 \cap r_2$ and therefore could not belong to any other rectangle of $\mathcal{R}$. The second case is when e is a lower horizontal side of either r_1 or r_2. In a similar manner it is easily seen that such a side can be charged at most once to the right of l. Altogether we charge each horizontal side of a rectangle in $\mathcal{R}$ at most once to the right of l. A symmetric argument implies that every horizontal side of a rectangle in $\mathcal{R}$ is charged at most once to the left of l. Altogether we have at most $4n$ charges. We have just shown that $G(H)$ has at most $4n$ edges. As a matter of fact, G has at most $4n - 4$ edges since the uppermost (respectively, the lowermost) horizontal side of the rectangles in $\mathcal{R}$ cannot be charged. Thus, the average degree of G is at most $8 - \frac{8}{n}$ and therefore there must exist a vertex with degree at most 7. Obviously, this charging scheme works for any subgraph of G as well. Thus G is 7-degenerate. By Lemmas 2.6 and 2.4, H is 8-colorable, as asserted. □

Proof of Theorem 1.4. Let l be a vertical line such that at most $n/2$ of the rectangles in $\mathcal{R}$ lie fully to the right of l and at most $n/2$ rectangles of $\mathcal{R}$ lie fully to its left. Let $\mathcal{R}'$ (respectively, $\mathcal{R}''$) denote the subset of rectangles that lie to the right (respectively, to the left) of l. Let $\mathcal{R}_l$ denote the subset of rectangles in $\mathcal{R}$ that intersect the line l. Let $f(n)$ denote the maximum number of colors needed to color a family of n axis-parallel rectangles in the plane. We color (separately) the rectangles in $\mathcal{R}_l$ with eight colors and recursively color the set $\mathcal{R}'$ and $\mathcal{R}''$ using the same set of colors but keeping this set disjoint from the colors used to color $\mathcal{R}_l$. Thus $f(n)$ obeys the recursive relation

$$f(n) \leq 8 + f(n/2),$$

which is easily seen to imply that $f(n) \leq 8 \log n$. This completes the proof of the theorem. □

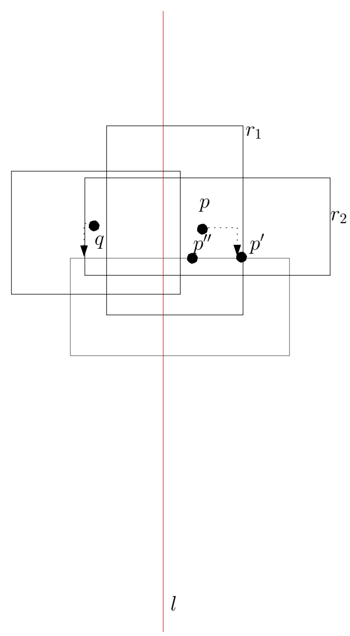

FIG. 3. *Illustration of the charging scheme in the proof of Lemma* 4.1. *Note that here the Delaunay graph of the four rectangles is the clique* K_4*, so any coloring must use at least four colors.*

5. Chromatic number of regions with low-union complexity. In this section we show a relation between the chromatic number of a hypergraph induced by a finite family of regions $\mathcal{R}$ to the union complexity of $\mathcal{R}$. For example, we show that if $\mathcal{R}$ is a family of planar simple Jordan regions such that any finite subset of $\mathcal{R}$ has linear union complexity, then there exists a constant $c = c(\mathcal{R})$ such that any hypergraph induced by a finite subset of $\mathcal{R}$ is c colorable. Thus, for example, since pseudo-discs have linear union complexity (see, e.g., [11]), there is a constant c such that any family of pseudo-discs can be colored with c colors.

LEMMA 5.1. *Let $\mathcal{R}$ be a set of n regions and let $\mathcal{U} : \mathbb{N} \to \mathbb{N}$ be a function such that $U(m)$ is the maximum complexity of any k regions in $\mathcal{R}$ over all $k \leq m$, for $1 \leq m \leq n$. Then, the Delaunay graph G of the hypergraph $H = H(\mathcal{R})$ has a vertex with degree at most $O(\frac{\mathcal{U}(n)}{n})$.*

Proof. Let $\mathcal{A}(\mathcal{R})$ denote the arrangement of the boundary curves of the regions in $\mathcal{R}$ and let F_2 denote the set of faces of $\mathcal{A}(\mathcal{R})$ that are contained in exactly two regions of $\mathcal{R}$. Obviously, the number of edges in G is bounded by $|F_2|$. We may assume that the regions of $\mathcal{R}$ are in general position, in the sense that no three distinct boundaries pass through a common point. This can be enforced by a slight perturbation of the curves, which does not decrease $|F_2|$. Let $S_{\leq 2}(\mathcal{R})$ be the set of vertices of the arrangement $\mathcal{A}(\mathcal{R})$ that lie in the interior of at most 2 regions of $\mathcal{R}$. By the analysis of Clarkson and Shor [5], we have $|S_{\leq 2}(\mathcal{R})| = O(\mathcal{U}(n))$. We charge a face $f \in F_2$ to one of the vertices on the boundary ∂f, if ∂f has vertices. Thus, the only faces unaccountable for by this charging scheme are the faces that have no vertices on their boundary. However, the number of such faces is only $O(n)$, as we can charge such a face to the region of $\mathcal{R}$ that forms its outer boundary. It is easily seen that in this charging scheme a vertex is charged at most four times, since it can belong to the boundary of at most four faces. Note also that every charged vertex is contained in at most two regions of $\mathcal{R}$ and therefore belongs to $S_{\leq 2}(\mathcal{R})$. Thus $E(G) \leq |F_2| \leq 4 \cdot S_{\leq 2}(\mathcal{R}) + n = O(\mathcal{U}(n) + n)$. Thus, the average degree of G is $O(\frac{\mathcal{U}(n)}{n} + 1)$ and therefore G must contain a vertex with degree at most $O(\frac{\mathcal{U}(n)}{n})$ as asserted. ◻

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 5.1 there exists a constant c such that the Delaunay graph G of $H(\mathcal{R})$ has a vertex with degree at most $c \cdot \frac{\mathcal{U}(n)}{n}$. We prove that $\chi(H(\mathcal{R})) \leq c \cdot \frac{\mathcal{U}(n)}{n} + 1$. The proof is by induction on n. Let $r \in \mathcal{R}$ be a region with at most $c \cdot \frac{\mathcal{U}(n)}{n}$ neighbors in G. By the induction hypothesis, the hypergraph $H(\mathcal{R} \setminus \{r\})$ is $c \cdot \frac{\mathcal{U}(n-1)}{n-1} + 1 \leq c \cdot \frac{\mathcal{U}(n)}{n} +$ 1-colorable (by our monotonicity assumption on $\frac{\mathcal{U}(n)}{n}$). We need to choose a color (out of the $c \cdot \frac{\mathcal{U}(n)}{n} + 1$ colors that are available for us) for r such that the coloring of $\mathcal{R}$ is valid. Obviously, points that are not covered by r are not affected by the coloring of r. Note also that any point $p \in r$ that is contained in at least two regions of $\mathcal{R} \setminus r$ is not affected by the color of r since by induction the set of regions in $\mathcal{R} \setminus \{r\}$ containing such points is nonmonochromatic. We thus only need to color r with a color that is different from the colors of all regions $r' \in \mathcal{R} \setminus r$ for which there is a point p that is contained only in $r \cap r'$. However, by our choice of r, there are at most $c \cdot \frac{\mathcal{U}(n)}{n}$ such regions. Thus we can assign to r a color among the $c \cdot \frac{\mathcal{U}(n)}{n} + 1$ colors available to us and keep the coloring of $\mathcal{R}$ proper. This completes the inductive step. As for the algorithmic perspective, we briefly sketch the simple ideas behind it. Here, we do not attempt to optimize the efficiency of the algorithm. Assume a model of computation as in [15] in which computing the intersection points of any pair of regions in $\mathcal{R}$, and a few similar operations, can be performed in constant time. One can compute the arrangement $\mathcal{A}(\mathcal{R})$ using standard methods as in [15]. In addition, one can compute in polynomial time, for each face f of the arrangement

$\mathcal{A}(R)$, its depth $d(f)$ which is the number of ranges in $\mathcal{R}$ containing f. Next, we compute the graph $G(H(\mathcal{R}))$. Note that the edges of $G(H(\mathcal{R}))$ consist of all pairs of regions (r_1, r_2) whose intersection contains a face of depth two. This can be done by checking for each face f of the arrangement and each region $r \in \mathcal{R}$ whether $f \subset r$. This takes time which is proportional to $\mathcal{A}(\mathcal{R}) \cdot n$. Let $r \in \mathcal{R}$ be a vertex of minimum degree in $G(H(\mathcal{R}))$. We update the depth of the faces of the arrangement $\mathcal{A}(\mathcal{R} \setminus \{r\})$ and construct $G(H(\mathcal{R} \setminus \{r\}))$ and color $\mathcal{R} \setminus \{r\}$, recursively. This can be done in total time proportional to the sum $\sum_{i=1}^{n} i f(i)$, where $f(i)$ is the maximum complexity of the arrangement of any i regions in $\mathcal{R}$. Thus if f is polynomial then the total running time is polynomial in n. See Algorithm 1 for a pseudo-code. □

Algorithm 1. Color($\mathcal{R}$): Color the hypergraph $H(\mathcal{R})$.

1: **Order the elements of $\mathcal{R}$:** Compute a permutation $\mathcal{R} = \{r_1, \ldots, r_n\}$ such that the degree of r_i in the Delaunay graph $G(H(\{r_1, \ldots, r_i\}))$ is bounded by $c \cdot \frac{\mathcal{U}(i)}{i}$, for $i = \{1, \ldots, n\}$.
2: **Color the elements:** For$(i = 2; i \leq n; i++)$ $r_i \leftarrow$ A color different from its neighbors in $G(H(\{r_1, \ldots, r_i\}))$.

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6. Application to conflict-free colorings. Among other results, Even et al. [7] proved that any set of n discs in the plane can be CF-colored with $O(\log n)$ colors and that this bound is tight in the worst case. They also provide a deterministic polynomial time algorithm for coloring a given collection of n discs with only $O(\log n)$ colors. Har-Peled and Smorodinsky [9] extended this result to any family of regions with linear union complexity. For example, they provide a randomized algorithm for CF-coloring any family of n pseudo-discs with $O(\log n)$ colors with high probability. In particular, this randomized algorithm serves as a probabilistic proof that a CF-coloring of any family of n pseudo-discs with only $O(\log n)$ colors exists. One of the open problems left in [9] is to obtain a deterministic framework for CF-colorings any family of regions with linear union complexity. As an application of Theorem 1.2 and Algorithm 1, we provide such a framework. Our algorithm outperforms the one used in [9] by being deterministic and conceptually simpler. The number of colors used in our algorithm for CF-coloring the given regions depends on their union complexity.

Algorithm 2 CF-Color($\mathcal{R}$): CF-Color the hypergraph $H(\mathcal{R})$.

1: $i \leftarrow 0$: i denotes an unused color
2: **while** $\mathcal{R} \neq \emptyset$
3: **Increment:** $i \leftarrow i + 1$
4: **Color the hypergraph $H(\mathcal{R})$:** find a coloring χ of $H(\mathcal{R})$ with "few" colors, using (in most cases) Algorithm 1
5: $\mathcal{R}' \leftarrow$ **Largest color class of** χ
6: **Color:** $f(x) \leftarrow i$ for all $x \in \mathcal{R}'$
7: **Prune:** $\mathcal{R} \leftarrow \mathcal{R} \setminus \mathcal{R}'$
8: **end while**

THEOREM 6.1. *Algorithm* 2 *outputs a valid CF-coloring of* $\mathcal{R}$.

Proof. For a point $p \in \cup_{r \in \mathcal{R}} r$, let i be the maximal index (color) for which there is a region $r \in \mathcal{R}$ that contains p and is colored with i. We claim that there is exactly

Table 1

Summary of relation between the union complexity of the underlying objects $\mathcal{R}$, the chromatic number $H(\mathcal{R})$, and its CF-chromatic number.

Type of regions	$\mathcal{U}(n)$	$\chi(H(\mathcal{R}))$	$\chi_{CF}(H(\mathcal{R}))$
pseudo-discs, etc.	$O(n)$	$O(1)$	$O(\log n)$
convex fat regions, etc.	$O(n^{1+\delta})$	$O(n^\delta)$	$O(n^\delta)$
Axis-parallel rectangles	$\Theta(n^2)$	$O(\log n)$	$O(\log^2 n)$

one such region. Indeed, assume to the contrary that there is another such region r'. Consider the $i'th$ iteration where some of the regions of $\mathcal{R}$ were colored with i (including r and r'). Since r and r' belong to an independent set, there must have been a third region r'' containing p that wasn't colored in the $i'th$ iteration. This means that the color of r'' is greater than i, a contradiction to the maximality of i. This completes the proof of the theorem. □

Remark. Algorithm 2 yields a CF-coloring of regions with "low" union complexity with only "few colors" in the following sense: If $\mathcal{R}$ has union complexity bounded by $\mathcal{U}(n)$, then by Theorem 1.2, $H(\mathcal{R})$ can be colored with $O(\frac{\mathcal{U}(n)}{n})$ colors. So the largest color class is at least $\frac{n^2}{\mathcal{U}(n)}$ by the pigeonhole principle. Thus, in the *prune* step of Algorithm 2 we discard at least this many regions so, in total, Algorithm 2 does only few iterations. This depends on the function $\frac{n^2}{\mathcal{U}(n)}$. Table 1 summarizes the relation between the union complexity of the underlying objects, the chromatic number of the induced hypergraph, and its CF-chromatic number. The bounds given on the chromatic number and the CF-chromatic number are also bounds on the numbers of colors produced by Algorithms 1 and 2, respectively.

Theorem 6.2. *Let $\mathcal{R}$ be a set of n axis-parallel rectangles. Then Algorithm* 2 *applied to $\mathcal{R}$, provides a CF-coloring of $\mathcal{R}$ with $O(\log^2 n)$ colors in polynomial-time.*

Remark. Note that the union complexity of n rectangles can be quadratic. Thus, we cannot apply Theorem 1.2 directly to $\mathcal{R}$, since we would obtain a coloring of $H(\mathcal{R})$ with potentially n colors. Thus, in the prune step of Algorithm 2 we might discard only a constant number of rectangles and the algorithm might use linear number of colors. The bound on the chromatic number of $H(\mathcal{R})$ is asymptotically tight as already mentioned. However, it is not clear that the bound $O(\log^2 n)$ on the CF-chromatic number of $\mathcal{R}$ is asymptotically tight. Maybe one can get better bounds. We leave this as an open problem.

Proof. By Theorem 1.4, we can color $H(\mathcal{R})$ with $O(\log|\mathcal{R}|)$ colors. Thus, in each prune step of Algorithm 2 we discard at least $\Omega(\frac{|\mathcal{R}|}{\log|\mathcal{R}|})$ rectangles. It is easily seen that the total number of iterations (which is the number of colors used by the algorithm) will be $O(\log^2 n)$. □

Definition 6.3 (see [11]). *A family $\mathcal{R}$ of Jordan regions in the plane is called a family of* pseudo-discs *if the boundaries of each pair of them intersect at most twice.*

Theorem 6.4. *Let $\mathcal{R}$ be a family of n pseudo-discs. Then Algorithm* 2 applied to $\mathcal{R}$ provides a CF-coloring with $O(\log n)$ colors in polynomial-time.

Proof. The complexity of the union of any m regions of $\mathcal{R}$ is $O(m)$ (see [11]). By Theorem 1.2, there is a constant c such that Algorithm 1 provides a coloring χ of $H(\mathcal{R})$ with $\leq c$ colors. Such a coloring can be computed in polynomial-time. In the prune step of Algorithm 2 we discard at least $\frac{|\mathcal{R}|}{c}$ regions. Thus, Algorithm 2 provides a CF-coloring of $\mathcal{R}$ with only $\frac{\log n}{\log \frac{c}{c-1}}$ colors in polynomial-time. □

7. Discussion and open problems. Naturally, the problems addressed in this paper have analogous versions in higher dimensions. For example, what is the minimum number of colors that always suffice to color any hypergraph induced by any set $\mathcal{B}$ of n balls in $\mathbb{R}^3$? Unfortunately, the complexity of the union of n balls could be quadratic already in $\mathbb{R}^3$, and we cannot apply the methods developed in this paper directly. Moreover, for $d \geq 4$ and any $n > 1$, there exist families of n balls in $\mathcal{R}^d$ that are pairwise touching and therefore require n distinct colors for any proper coloring of $H(\mathcal{R})$ as any two balls contain a point witnessing the fact that the two balls must be colored with distinct colors. The best upper bound known for CF-coloring any set of n balls in $\mathbb{R}^3$ is the trivial bound n. It is interesting to relax the CF-coloring requirements as follows: what is the minimum number of colors needed to color any hypergraph induced by a set $\mathcal{B}$ of n balls in $\mathbb{R}^3$ such that every hyperedge of cardinality at least 3 is nonmonochromatic. It is easily seen that this number is bounded by $O(\sqrt{n})$ since the maximum degree of any element is bounded by $O(n)$ in the 3-uniform hypergraph consisting of all hyperedges of $H(\mathcal{B})$ with cardinality 3. However, we conjecture that fewer colors are enough. This relates to the notion of 2-CF-coloring studied in [9]. Any improvement over the $O(\sqrt{n})$ bound would imply a better bound on 2-CF-coloring of balls in $\mathbb{R}^3$. Here, we omit the detailed description of this relation.

Another open problem is to bound the chromatic number of any hypergraph induced by n axis-parallel boxes in $\mathbb{R}^d$ (for $d > 2$). We conjecture that $O(\log^{d-1} n)$ colors always suffice.

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