On the Classical d-Orthogonal Polynomials Defined by Certain Generating Functions, II

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Abstract

This paper is a direct sequel to [5]. The present part deals with the problem of finding all d-orthogonal polynomial sets generated by $G(x,t) = e^t\Psi(xt)$. The resulting polynomials reduce to Laguerre polynomials for d=1 and to two-orthogonal polynomials associated with MacDonald functions for d=2, recently considered by the authors [6] and by Van Assche and Yakubovich [36]. Various properties for the obtained polynomials are singled out.

1 Introduction and preliminaries

During the two past decades, there has been increased interest in an extension of the notion of orthogonal polynomials known as multiple orthogonal polynomials (see, for instance, [3,10]). This notion, which is closely related to simultaneous Padé approximation, has many applications in various fields of mathematics as the number theory and the special functions theory. However, only recently examples of multiple orthogonal polynomials appeared in the literature. A convenient framework to discuss such examples consists of considering a subclass of multiple orthogonal polynomials known as d-orthogonal polynomials (see, for instance, [5,6,12-16, 27,36]). Our purpose in this work is to investigate some d-orthogonal polynomials defined by specified generating functions. The resulting polynomials are natural extensions of

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certain classical orthogonal polynomials. Our previous paper [6] dealt with polynomials generated by $G[(d+1)xt-t^{d+1}]$. The present one is devoted to the polynomials generated by $G(x,t)=e^t\Psi(xt)$ and defined by (1.7)-(1.8).
Next, we present some basic definitions which we need below.
Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u,f\rangle$ the effect of the functional $u\in\mathcal{P}'$ on the polynomial $f\in\mathcal{P}$. Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials in $\mathcal{P}$ such that $\deg P_n(x)=n$ for all n. The corresponding monic polynomial sequence $\{\widehat{P}_n\}_{n\geq 0}$ is given by $P_n=\lambda_n\widehat{P}_n$, $n\geq 0$, where λ_n is the normalization coefficient and its dual sequence $\{u_n\}_{n\geq 0}$ is defined by $\langle u_n,\widehat{P}_m\rangle=\delta_{n,m}$, $n,m\geq 0$.
Definition 1.1: Let d be an arbitrary positive integer. The polynomial sequence $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial sequence (d-OPS) with respect to the d-dimensional functional $\mathcal{U}={}^t(u_0,\cdots,u_{d-1})$ if it fulfils [27,37]

$$\begin{cases}\langle u_k,P_mP_n\rangle=0, & m>dn+k\ ,n\geq 0,\\ \langle u_k,P_nP_{dn+k}\rangle\neq 0, & n\geq 0,\end{cases}\tag{1.1}$$

for each integer k belonging to $\{0,1,\ldots,d-1\}$.
The orthogonality conditions (1.1) are equivalent to the fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a $(d+1)$-order recurrence relation [37] which we write in the *monic* form as

$$\widehat{P}_{m+d+1}(x)=(x-\beta_{m+d})\widehat{P}_{m+d}(x)-\sum_{\nu=0}^{d-1}\gamma_{m+d-\nu}^{d-1-\nu}\widehat{P}_{m+d-1-\nu}(x)\,,\ m\geq 0,\tag{1.2}$$

with the initial conditions

$$\begin{cases}\widehat{P}_0(x)=1\quad ,\quad \widehat{P}_1(x)=x-\beta_0\quad\text{and}\quad\text{if}\quad d\geq 2:\\ \widehat{P}_n(x)=(x-\beta_{n-1})\widehat{P}_{n-1}(x)-\sum_{\nu=0}^{n-2}\gamma_{n-1-\nu}^{d-1-\nu}\widehat{P}_{n-2-\nu}(x)\,,\ 2\leq n\leq d,\end{cases}\tag{1.3}$$

and the regularity conditions

$$\gamma_{n+1}^0\neq 0\,,\quad n\geq 0.$$

When $d=1$, the recurrence (1.2) with (1.3) is the well-known second-order recurrence relation

$$\begin{cases}\widehat{P}_{n+2}(x)=(x-\beta_{n+1})\widehat{P}_{n+1}(x)-\gamma_{n+1}\widehat{P}_n(x),\ n\geq 0,\\ \widehat{P}_0(x)=1,\ \widehat{P}_1(x)=x-\beta_0.\end{cases}$$

Let $\{Q_n\}_{n\geq 0}$ be the sequence of the derivatives defined by $Q_n(x)=D_xP_{n+1}(x)$, $n\geq 0$, where D_x is the derivative operator d/dx. According to Hahn's property [20], if the sequence $\{Q_n\}_{n\geq 0}$ is also d-orthogonal, the sequence $\{P_n\}_{n\geq 0}$ is called "*classical*" d-OPS.
The generalized hypergeometric functions are defined by (see, for instance, [26] p.136):

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};\ z\right)=\sum_{m=0}^{+\infty}\frac{(a_1)_m\cdots(a_p)_m}{(b_1)_m\cdots(b_q)_m}\frac{z^m}{m!},$$

where

- p and q are positive integers or zero (interpreting an empty product as 1),
- z is a complex variable,
- $(a)_m$ is Pochhammer's symbol given by:

$$(a)_m = \begin{cases} 1 & \text{if } m = 0, \\ a(a+1)\cdots(a+m-1) & \text{if } m = 1, 2, 3, \ldots, \end{cases}$$

- the numerator parameters $a_1, \ldots, a_p$ and the denominator parameters $b_1, \ldots, b_q$ take on complex values provides that $b_j \neq 0, -1, -2, \ldots; \; j = 1, \ldots, q$.

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates and we are led to a generalized hypergeometric polynomial of the type

$$\begin{aligned} P_n\left(x; a_1, \ldots, a_p; \alpha_1, \ldots, \alpha_q\right) &= {}_{p+1}F_q\left(\begin{matrix} -n, a_1, \ldots, a_p \\ \alpha_1+1, \ldots, \alpha_q+1 \end{matrix}; x\right) \\ &= \sum_{m=0}^{n} \frac{(-n)_m (a_1)_m \cdots (a_p)_m}{(\alpha_1+1)_m \cdots (\alpha_q+1)_m} \frac{x^m}{m!}, \end{aligned} \tag{1.4}$$

where $\alpha_j \neq -1, -2, \ldots; \; j = 1, \ldots, q$. The polynomial P_n is of degree n. These polynomials are generated by (cf.[7] p.947):

$$e^t {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ \alpha_1+1, \ldots, \alpha_q+1 \end{matrix}; -xt\right) = \sum_{n=0}^{\infty} P_n\left(x; a_1, \ldots, a_p; \alpha_1, \ldots, \alpha_q\right) \frac{t^n}{n!}, \tag{1.5}$$

and satisfy the differential equation (cf. [29] p.75):

$$\left(x(\theta - n)\prod_{i=1}^{p}(\theta + a_i) - \theta\prod_{j=1}^{q}(\theta + \alpha_j)\right)y = 0, \tag{1.6}$$

where $\theta = x\frac{d}{dx}$.

Now, let us consider the following Problem **P**: *Find all d-orthogonal polynomial sequences* $\{P_n\}_{n\geq 0}$ *generated by*

$$e^t\,\Psi(xt) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n, \tag{1.7}$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n \neq 0. \tag{1.8}$$

Such characterization takes into account the fact that polynomial sets which are obtainable from one another by a linear change of the variable are considered equivalent.

Before proceeding to a discussion of the case $d = 1$, let's recall some properties satisfied by polynomials generated by (1.7):

Lemma 1.1 (cf. [6]): *The following statements are equivalent:*

1) The polynomials P_n*;* $n \geq 0$*; are generated by* (1.7)*.*

2) The polynomials P_n*;* $n \geq 0$*; are generated by*

$$(1-t)^{-\lambda} F\left(\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} P_n(x) t^n, \tag{1.9}$$

where

$$F(z) \;=\; \sum_{n=0}^{\infty} (\lambda)_n\, c_n z^n, \quad c_n \neq 0. \tag{1.10}$$

3) The sequence of polynomials $R_n(x) = x^n P_n\left(\frac{1}{x}\right)$ *is an Appell one, namely, generated by* $\Psi(t)e^{xt}$.
4) The polynomials P_n*;* $n \geq 1$*; satisfy the differential-recurrence relation*

$$x\, P_n'(x) \;=\; nP_n(x) - nP_{n-1}(x). \tag{1.11}$$

5) The polynomials P_n*;* $n \geq 0$*; possess a multiplication formula of the form*

$$P_n(xy) \;=\; \sum_{k=0}^{n} \binom{n}{k}\, y^k\, (1-y)^{n-k}\, P_k(x). \tag{1.12}$$

Some other interesting properties of the polynomials P_n were given by Rainville (cf. [28,29]).
Problem **P**, for $d = 1$, was set and treated under different aspects by many authors who took as starting point for their characterizations one of the properties given by Lemma 1.1. Indeed:
1) Feldheim [18] proved that the only orthogonal polynomials which satisfy the multiplication formula (1.12) are those of Laguerre.
2) Toscano [33] proved that $\{P_n(x)\}_{n\geq 0}$ is an orthogonal polynomial set and $\left\{x^n P_n\left(\frac{1}{x}\right)\right\}_{n\geq 0}$ is an Appell if and only if $\{P_n(x)\}_{n\geq 0}$ is the Laguerre polynomial set.
3) Abdul-Halim and Al-Salam [1] proved that the only orthogonal polynomials of the form (1.4) where the a's and α's are independent of x and n, are the Laguerre polynomials ($p = 0$, $q = 1$).
Also, this result may be deduced from Al-Salam-Chihara's characterization of classical orthogonal polynomials (cf.[2]) and the identity (1.11).
We now state our main result:

Theorem 1.2: *The polynomial sequence* $\{\ell_n^{\vec{\alpha}_d}\}_{n\geq 0}$ *defined by*

$$\ell_n^{\vec{\alpha}_d}(x) := \ell_n^{(\alpha_1,\ldots,\alpha_d)}(x) = {}_1F_d\left(\begin{matrix} -n \\ \alpha_1+1,\ldots,\alpha_d+1 \end{matrix}\,;\, x\right), \quad \alpha_j \neq -1,-2,\ldots\;;\; j = 1,\ldots,d. \tag{1.13}$$

is the only d-OPS generated by (1.7)*. Moreover, it is classical in Hahn's sense.*
At first, we prove the above theorem. Then, we establish the connection of the polynomials $\ell_n^{\vec{\alpha}_d}; n \geq 0$; with other polynomial sets in the literature. After that, we state some of their properties which generalize in a natural way the Laguerre polynomials ones. These properties are: a differential equation of order $d+1$, generating functions defined by means of the hyper-Bessel functions, some differentiation formulas and a Koshlyakov type formula involving the Meijer G-function.

2 Proof of the main result

Starting point for the proof of Theorem 1.2 is the following fundamental lemma:
Lemma 2.1: *Let P_n; $n \geq 0$; be the polynomials generated by* (1.7). *The following statements are equivalent:*
1) $\{P_n\}_{n\geq 0}$ is a d-OPS.
2) Each polynomial P_n ; $n = 0, 1, \cdots$, satisfies a differential equation of the type

$$\Big(x(\theta - n) - S(\theta)\Big)\, y \;=\; 0, \quad \theta = x\frac{d}{dx}, \tag{2.1}$$

where S is a polynomial of degree $d+1$ in θ given by

$$S(\theta) \;=\; \sum_{k=0}^{d+1} s_{n,k}\,(\theta - n)_k, \qquad s_{n,0} \neq 0 \text{ and } s_{n,d+1} \neq 0. \tag{2.2}$$

Proof: The iteration of (1.11) leads to the relation (cf.[28] p.242):

$$\frac{P_{n-k}(x)}{(n-k)!} \;=\; (-1)^k\,(\theta - n)_k\,\frac{P_n(x)}{n!} \tag{2.3}$$

where $(T)_k$ denotes

$$(T)_k \;=\; (T)(T+1)\cdots(T+k-1), \qquad (T)_0 \;=\; 1.$$

This identity allows us to replace any condition, given by a recurrence relation linking P_n and P_{n-j}; $0 \leq j \leq r$, by another expressed by a differential equation satisfied by P_n where the differential operator is defined in terms of $(\theta - n)_j$; $0 \leq j \leq r$, and vice versa. *For the sake of simplicity we illustrate this method for the case $d = 2$, see below at the end of this section.*
Thus, the combination of (2.3) and (1.2) provides the equivalence: 1) $\Longleftrightarrow$ 2).
Now, we return to our problem and prove the main result.
Proof of Theorem 1.2. Without loss of generality (use a change of variable if necessary), the polynomial $S(\theta)$ may be written as:

$$S(\theta) = \theta \prod_{j=1}^{d} (\theta + \alpha_j), \quad \alpha_j \in \mathbb{C}\ ,\ j = 1, \ldots, d. \tag{2.4}$$

By induction, we verify that the expression of the coefficients $s_{n,0}$ and $s_{n,d+1}$ in terms of the roots $-\alpha_j$; $j = 1, \ldots, d$; are given by

$$s_{n,0} \;=\; n \prod_{j=1}^{d} (n + \alpha_j) \;\text{ and }\; s_{n,d+1} \;=\; 1.$$

So, the conditions (2.2) are ensured if and only if $\alpha_j \neq -1, -2 \ldots$. In this case, the polynomials P_n, $n = 0, 1, \cdots$, are solutions of the generalized hypergeometric equation:

$$\Big(x(\theta - n) - \theta \prod_{j=1}^{d} (\theta + \alpha_j)\Big) y = 0 \tag{2.5}$$

which, according to (1.6), means that P_n; $n \geq 0$; are the polynomials $\ell_n^{\vec{\alpha}_d}$; $n \geq 0$; defined by (1.13).
Thus, if we consider as equivalent two polynomial sets which are obtainable from one another by a linear change of the variable, we state that the polynomials $\ell_n^{\vec{\alpha}_d}$; $n \geq 0$, are the only d-OPS generated by (1.7).
Finally, in order to verify that these polynomials are classical in Hahn's sense, let us recall the identity (see [29] p.107) :

$$D_x \, {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; x\right) = \frac{\prod_{i=1}^p a_i}{\prod_{j=1}^q b_j} \, {}_pF_q\left(\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_q+1 \end{matrix} ; x\right). \tag{2.6}$$

From (2.6) and (1.13), we deduce

$$\prod_{j=1}^d (\alpha_j+1) \, D_x \ell_{n+1}^{\vec{\alpha}_d}(x) = -(n+1)\ell_n^{\vec{\alpha}_d+1}(x), \; n \geq 0, \tag{2.7}$$

where $\vec{\alpha}_d+1 = (\alpha_1+1, \dots, \alpha_d+1)$. So the sequence $\left\{D_x \ell_{n+1}^{\vec{\alpha}_d}\right\}_{n\geq 0}$ is also d-orthogonal, which is the Hahn property [20].
On the other hand, from (1.13), it may be seen that $\ell_n^{\vec{\alpha}_d}$ has the explicit formula

$$\ell_n^{\vec{\alpha}_d}(x) = \sum_{k=0}^n \binom{n}{k} \xi_k x^k, \; n \geq 0, \tag{2.8}$$

where

$$\xi_m := \xi_m(\vec{\alpha}_d) = \frac{(-1)^m}{\prod_{j=1}^d (\alpha_j+1)_m}, \; m \geq 0. \tag{2.9}$$

Also, if we denote by $\{\hat{\ell}_n^{\vec{\alpha}_d}\}_{n\geq 0}$, the monic polynomials corresponding to $\{\ell_n^{\vec{\alpha}_d}\}_{n\geq 0}$, it is easily seen that $\ell_n^{\vec{\alpha}_d} = \xi_n \hat{\ell}_n^{\vec{\alpha}_d}$. Thus (2.7) gives $D_x \hat{\ell}_{n+1}^{\vec{\alpha}_d} = (n+1)\hat{\ell}_n^{\vec{\alpha}_d+1}, n \geq 0$.
Now, we shall illustrate the possibility of deriving a $(d+1)$-order recurrence relation satisfied by the polynomials $\ell_n^{\vec{\alpha}_d}$ from the differential equation (2.5) and the identity (2.3) for $d = 2$:
The polynomials

$$\ell_n^{(\alpha,\beta)}(x) = {}_1F_2\left(\begin{matrix} -n \\ \alpha+1, \beta+1 \end{matrix} ; x\right), \; n \geq 0$$

satisfy the differential equation:

$$\Big(x(\theta-n) - \theta(\theta+\alpha)(\theta+\beta)\Big) \, \ell_n^{(\alpha,\beta)}(x) = 0.$$

Use the following relations, which one can easily verify,

$$\begin{cases} \theta &= (\theta-n)+n, \\ \theta^2 &= (\theta-n)_2 + (2n-1)(\theta-n) + n^2, \\ \theta^3 &= (\theta-n)_3 + 3(n-1)(\theta-n)_2 + (3n^2-3n+1)(\theta-n) + n^3, \end{cases}$$

to rewrite the last equation under the form

$$\Big(n(\alpha+n)(\beta+n)-\Big[3n^2-3n+1+(2n-1)(\alpha+\beta)+\alpha\beta-x\Big](\theta-n) \\ +(3n-3+\alpha+\beta)(\theta-n)_2+(\theta-n)_3\Big)\ell_n^{(\alpha,\beta)}(x) \;=\; 0,$$

which, combined with (2.3), leads to the recurrence relation:

$$(\alpha+n)(\beta+n)\ell_n^{(\alpha,\beta)}(x)-\Big[3n^2-3n+1+(2n-1)(\alpha+\beta)+\alpha\beta-x\Big]\ell_{n-1}^{(\alpha,\beta)}(x) \\ +(n-1)(3n-3+\alpha+\beta)\ell_{n-2}^{(\alpha,\beta)}(x)-(n-1)(n-2)\ell_{n-3}^{(\alpha,\beta)}(x)=0.$$

Notice that, this recurrence relation was also obtained by Rainville (cf. [29] p.243) using Sister Celine's technique.

Next, we give some properties of the polynomials $\ell_n^{\vec{\alpha}_d}$, $n\geq 0$.

3 Polynomials related to $\ell_n^{\vec{\alpha}_d}$

From (1.13), we deduce that the polynomial sequence $\left\{\ell_n^{\vec{\alpha}_d}\right\}_{n\geq 0}$ is a special case of the Brafman polynomials (cf. [8] p.186). Three particular cases are worthy of note:
1) For $d=1$, with $\vec{\alpha}_1=(\alpha)$, $\alpha>-1$, we obtain

$$\ell_n^{(\alpha)}(x) \;=\; \frac{n!}{(\alpha+1)_n}\, L_n^{(\alpha)}(x), \tag{3.1}$$

where $L_n^{(\alpha)}$, $n\geq 0$, are the classical Laguerre polynomials.
2) For $d=2$, $\vec{\alpha}_2=(\alpha,\beta)$, $\alpha,\beta\neq -1,-2,\ldots$, we have

$$\ell_n^{(\alpha,\beta)}(x) \;=\; n!\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+n+\beta)}\, x^{-\frac{\alpha}{2}}\, J_n^{(\alpha,\beta-\frac{\alpha}{2})}\left(x^{\frac{1}{2}}\right),$$

where $J_n^{(\nu,\sigma)}$ are the Bateman functions (cf.[4] p.575).
These polynomials have been studied recently by the authors [6] and by Van Assche and Yakubovich [36] in order to solve an open problem, formulated by Prudnikov in [35], which consists of constructing orthogonal polynomials associated with the weight function

$$\rho_\nu(x)=2x^{\frac{\nu}{2}}K_\nu(2\sqrt{x}), \qquad x>0,$$

where K_ν is the MacDonald function (modified Bessel function).
3) For $\vec{\alpha}_d(\alpha) \;=\; \left(\frac{\alpha+1}{d}-1,\ldots,\frac{\alpha+d}{d}-1\right)$, $\alpha>-1$, we have

$$\ell_n^{\vec{\alpha}_d(\alpha)}(x) \;=\; \frac{n!}{(\alpha+1)_{dn}}\, Z_n^{\alpha}\left(dx^{\frac{1}{d}},d\right),$$

where $Z_n^{\alpha}(x,k); n\geq 0$; are known in the literature as Konhauser polynomials, earlier introduced by Toscano as follows (cf. [32] or [23]):

$$Z_n^{\alpha}(x,k) \;=\; \frac{(\alpha+1)_{kn}}{n!}\;{}_1F_k\left(\begin{matrix}-n\\ \frac{\alpha+1}{k},\ldots,\frac{\alpha+k}{k}\end{matrix};\left(\frac{x}{k}\right)^k\right).$$

In particular, $Z_n^{\alpha}(x,1)=L_n^{(\alpha)}(x)$. The case $k=2$ was encountered by Spencer and Fano [30] in certain calculations involving the penetration of gamma rays through matter.

4 Generating functions

From (1.5), we deduce a generating function for the polynomials $\ell_n^{\vec{\alpha}_d}$, $n \geq 0$, expressed by the hyper-Bessel function $\mathcal{J}_{\vec{\alpha}_d}$ defined by (cf.[11] or [22]):

$$\mathcal{J}_{(\alpha_1,\alpha_2,\ldots,\alpha_d)}(z) \;=\; {}_0F_d\left(\begin{matrix} - \\ \alpha_1+1,\alpha_2+1,\ldots,\alpha_d+1 \end{matrix}\;;\; -\left(\frac{z}{d+1}\right)^{d+1}\right).$$

In fact, from (1.5) we have

$$e^t\mathcal{J}_{(\alpha_1,\alpha_2,\ldots,\alpha_d)}\left((d+1)(xt)^{\frac{1}{d+1}}\right) \;=\; \sum_{n=0}^{\infty}\ell_n^{\vec{\alpha}_d}(x)\frac{t^n}{n!}. \tag{4.1}$$

When $d = 1$, we have the well-known generating function for Laguerre polynomials (cf. [17] p.189)

$$e^tJ_\alpha\left(2\sqrt{xt}\right) \;=\; \sum_{n=0}^{\infty}\frac{1}{(\alpha+1)_n}L_n^{(\alpha)}(x)t^n,$$

where J_ν is the Bessel function of the first kind of order ν.
Another generating function for the polynomials $\ell_n^{\vec{\alpha}_d}$; $n \geq 0$; may be derived from the Chaundy identity (cf. [9] p.62) or from (1.9), that is,

$$(1-t)^{-\lambda}\,{}_1F_d\left(\begin{matrix}\lambda \\ \alpha_1+1,\ldots,\alpha_d+1\end{matrix};\; -\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty}(\lambda)_n\,\ell_n^{\vec{\alpha}_d}(x)\frac{t^n}{n!},\quad |\,t\,|< 1. \tag{4.2}$$

If $\lambda = \alpha_i + 1$, this identity takes the form

$$\begin{aligned}(1-t)^{-(\alpha_i+1)}{}_0F_{d-1}&\left(\begin{matrix} - \\ \alpha_1+1,\ldots,\alpha_{i-1}+1,\alpha_{i+1}+1,\ldots,\alpha_d+1\end{matrix};-\frac{xt}{1-t}\right)\\ &= \sum_{n=0}^{\infty}(\alpha_i+1)_n\ell_n^{\vec{\alpha}_d}(x)\frac{t^n}{n!},\ |\,t\,|< 1.\end{aligned}$$

When $d = 1$, we have the well-known generating function for Laguerre polynomials (cf. [17] p.189)

$$\frac{1}{(1-t)^{\alpha+1}}\exp\left(-\frac{xt}{1-t}\right) \;=\; \sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n,\quad |\,t\,|< 1.$$

5 Connection between the polynomials $\ell_n^{\vec{\alpha}_d}$ and the d-OPS of Hermite type

Recall that the d-OPS of Hermite type are at the same time d-orthogonal and Appell polynomials [12]. The Gould-Hopper polynomials $g_n^{d+1}(x,h)$; $n \geq 0$; are (apart from a linear transform) d-OPS of Hermite type. They are generated by the following relation (cf.[19] p.58):

$$\exp(xt+ht^{d+1}) \;=\; \sum_{n=0}^{\infty}g_n^{d+1}(x,h)\frac{t^n}{n!} \tag{5.1}$$

and generalize the Hermite polynomials H_n ; $n \geq 0$. In fact, for $d = 1$, we have

$$g_n^2(2x, -1) \;=\; H_n(x), \quad n \geq 0. \tag{5.2}$$

The explicit form of these polynomials is (cf. [19] p.58):

$$g_n^{d+1}(x, h) \;=\; \sum_{s=0}^{[\frac{n}{d+1}]} \frac{n!}{s!(n-(d+1)s)!}\, h^s\, x^{n-(d+1)s}, \tag{5.3}$$

which means that the sequence $\left\{g_n^{d+1}\right\}_{n\in\mathbb{N}}$ is d-symmetric, namely,

$$g_n^{d+1}(\omega_{d+1}x, h) = \omega_{d+1}^n g_n^{d+1}(x, h),$$

where $\omega_{d+1} = \exp(2i\pi/(d+1))$.
Let f be a complex function. Put

$$\Pi_{[d+1,k]}(f)(x) = \frac{1}{d+1}\sum_{\ell=0}^{d} \omega_{d+1}^{-k\ell} f\left(\omega_{d+1}^{\ell} x\right),$$

where k is an integer such that $0 \leq k \leq d$. Apply this operator to the two members of the identity

$$\exp(-t^{d+1})\, \exp\left(-(d+1)xt\right) \;=\; \sum_{n=0}^{\infty} g_n^{d+1}\Big((d+1)x, -1\Big)\, \frac{t^n}{n!} \tag{5.4}$$

considered as functions of the variable x.
Using the fact that the sequence $\left\{g_n^{d+1}\Big((d+1)x, -1\Big)\right\}_{n\in\mathbb{N}}$ is d-symmetric and the identities

$$\sum_{\ell=0}^{d} \omega_{d+1}^{m\ell} = \begin{cases} d+1 & \text{if } m \equiv 0 \bmod (d+1), \\ 0 & \text{otherwise,} \end{cases}$$
$$((d+1)n+k)! = k!(d+1)^{(d+1)n} \prod_{j=0}^{d} \left(\tfrac{k+1+j}{d+1}\right)_n,$$

we obtain

$$\exp(-t^{d+1})\, \frac{((d+1)xt)^k}{k!}\, {}_0F_d\left(\begin{matrix} - \\ \Delta^*(d+1, k+1) \end{matrix}; (xt)^{d+1}\right)$$
$$= \sum_{n=0}^{\infty} g_{(d+1)n+k}^{d+1}\Big((d+1)x, -1\Big)\, \frac{t^{(d+1)n+k}}{((d+1)n+k)!} \tag{5.5}$$

where $\Delta^*(d+1, k+1)$ abbreviates the set of the d parameters

$$\left\{\frac{k+1+j}{d+1};\; j = 0, 1, \ldots, d,\; j \neq d-k\right\}. \tag{5.6}$$

Now, combining (4.1) and (5.5) to obtain

$$g_{(d+1)n+k}^{d+1}\Big((d+1)x, -1\Big) \;=\; (-1)^n \frac{((d+1)n+k)!}{n!\, k!}\, ((d+1)x)^k\, \ell_n^{\vec{\alpha}_{d,k}}(x^{d+1}), \quad \text{for all } n \in \mathbb{N} \tag{5.7}$$

where the components of the vector $\vec{\alpha}_{d,k} + 1$ are given by the set (5.6).

Notice that for $d=1$, the identity (5.7) is reduced to the well-known relation between Hermite and Laguerre polynomials (cf.[31] p.102):

$$\begin{cases} H_{2n}(x) &= (-1)^n\, 2^{2n}\, n!\, L_n^{(-\frac{1}{2})}(x^2), \\ H_{2n+1}(x) &= (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(\frac{1}{2})}(x^2). \end{cases}$$

For $d=2$, see the analogous result given in [6] Section 5.

6 Differentiation formulas and recurrence relations

Notations : If $\vec{\alpha}_d=(\alpha_1,\dots,\alpha_i,\dots,\alpha_d)$, we put $\vec{\alpha}_d(i-)=(\alpha_1,\dots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},\dots,\alpha_d)$ and $\mid\vec{\alpha}_d\mid=\sum_{i=1}^d \alpha_i$.

Theorem 6.1 : *The polynomials* $\ell_n^{\vec{\alpha}_d}$, $n=0,1,\cdots$, *satisfy the following relations :*

$$\prod_{i=1}^d(\alpha_i+1)\,D_x\ell_n^{\vec{\alpha}_d}(x) = -n\,\ell_{n-1}^{\vec{\alpha}_d+1}(x), \tag{6.1}$$

$$\prod_{i=1}^d(\alpha_i+1)_k\;D_x^k\ell_n^{\vec{\alpha}_d}(x) = (-1)^k\frac{n!}{(n-k)!}\,\ell_{n-k}^{\vec{\alpha}_d+k}(x), \tag{6.2}$$

$$x\,D_x\ell_n^{\vec{\alpha}_d}(x) = n\,\ell_n^{\vec{\alpha}_d}(x)-n\,\ell_{n-1}^{\vec{\alpha}_d}(x), \tag{6.3}$$

$$x\,\ell_{n-1}^{\vec{\alpha}_d+1}(x) = \prod_{i=1}^d(\alpha_i+1)\left(-\ell_n^{\vec{\alpha}_d}(x)+\ell_{n-1}^{\vec{\alpha}_d}(x)\right), \tag{6.4}$$

$$x\,D_x\,\ell_n^{\vec{\alpha}_d}(x) = \alpha_i\left(-\,\ell_n^{\vec{\alpha}_d}(x)+\ell_n^{\vec{\alpha}_d(i-)}(x)\right), \tag{6.5}$$

$$\{dxD_x+\mid\vec{\alpha}_d\mid\}\;\ell_n^{\vec{\alpha}_d}(x) = \sum_{i=1}^d\alpha_i\ell_n^{\vec{\alpha}_d(i-)}(x), \tag{6.6}$$

$$(n+\alpha_i)\,\ell_n^{\vec{\alpha}_d}(x) = n\,\ell_{n-1}^{\vec{\alpha}_d}(x)+(\alpha_i)\ell_n^{\vec{\alpha}_d(i-)}(x), \tag{6.7}$$

$$[dn+\mid\vec{\alpha}_d\mid]\;\ell_n^{\vec{\alpha}_d}(x) = d\,n\,\ell_{n-1}^{\vec{\alpha}_d}(x)+\sum_{i=1}^d\alpha_i\,\ell_n^{\vec{\alpha}_d(i-)}(x), \tag{6.8}$$

$$-\frac{d\,n\,x}{\prod_{i=1}^d(\alpha_i+1)}\ell_{n-1}^{\vec{\alpha}_d+1}(x)+(\mid\vec{\alpha}_d\mid)\,\ell_n^{\vec{\alpha}_d}(x) = \sum_{i=1}^d(\alpha_i-1)\ell_n^{\vec{\alpha}_d(i-)}(x), \tag{6.9}$$

$$\left(-D_x\prod_{j=1}^d(\theta+\alpha_j)\right)\ell_n^{\vec{\alpha}_d}(x) = n\,\ell_{n-1}^{\vec{\alpha}_d}(x), \qquad n\geq 1. \tag{6.10}$$

Proof : 1) (6.1) is the relation (2.7) obtained in Section 2.
2) The iteration of (6.1) leads to (6.2).
3) (6.3) may be deduced from (5.2) and Lemma 1.1.
4) If we combine (6.1) and (6.3), we obtain (6.4).
5) (6.5) may be deduced from (6.3) and the identity (13) p.82 in [29].
6) We derive (6.6) from (6.5).
7) (6.7) may be deduced from (1.13) and the identity (15) p.82 in [29]. Also, (6.7) results from (6.3) and (6.5) by eliminating their common term $xD_x\ell_n^{\vec{\alpha}_d}(x)$.
8) We derive (6.8) from (6.7).
9) We derive (6.9) from (6.1) and (6.6) by eliminating the derivative term.
10) If we combine (6.3) and (2.5), we obtain (6.10).

7 A hierarchy of polynomials $\ell_n^{\vec{\alpha}_d}$

We now obtain a representation of $\ell_n^{\vec{\alpha}_d}$ in terms of integrals containing polynomials $\ell_n^{\vec{\alpha}_{d'}}$ of least order ($d' < d$).
Recall that the inverse Laplace transform of a ${}_pF_q$ is given by (cf., for instance, [26] p.60):

$$\omega^{\beta-1}{}_pF_{q+1}\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta,\beta_1,\dots,\beta_q\end{matrix};\ \omega z\right)=\frac{\Gamma(\beta)}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\omega t}t^{-\beta}{}_pF_q\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix};\ \frac{z}{t}\right)dt,\quad(7.1)$$

where ω is real, $\omega \neq 0$, $\Re(\beta) > 0$, $c > 0$ and $p \leq q$. From this, if we put $p = 1$, $\alpha_1 = -n$, $q = d-1$, $\beta_j = \alpha_j + 1$; $1 \leq j \leq d-1$, $\beta = \alpha_d + 1$ and $\omega = 1$, and we use the definition (1.13), we obtain a representation of $\ell_n^{\vec{\alpha}_d}$ in terms of an integral containing the polynomials $\ell_n^{\vec{\alpha}_{d-1}}$, that is

$$\ell_n^{\vec{\alpha}_d}(z) \;=\; \frac{\Gamma(\alpha_d+1)}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{t}t^{-\alpha_d-1}\ell_n^{\vec{\alpha}_{d-1}}\left(\frac{z}{t}\right)dt.\qquad(7.2)$$

The iteration of this identity to the polynomial $\ell_n^{\vec{\alpha}_{d-1}}$ leads to a representation of $\ell_n^{\vec{\alpha}_d}$ in terms of $(d-1)$-fold integral containing the Laguerre polynomials.

8 Koshlyakov formula

Recall that the Koshlyakov formula links two Laguerre polynomials of different parameters, that is (cf. [24] or [25]):

$$L_n^{(\alpha+\beta)}(x)=\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta)\Gamma(n+\alpha+1)}\int_0^1 t^{\alpha}(1-t)^{\beta-1}L_n^{(\alpha)}(xt)dt,\qquad \alpha>-1,\qquad \beta>0.\qquad(8.1)$$

This identity may be used to define transmutation operators between two differential operators of the same order.
Next, we generalize this identity to the polynomials $\ell_n^{\vec{\alpha}_d}$.
Let us recall (cf. [17] p.200)

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ \nu+\mu,b_2,\dots,b_q\end{matrix};\ x\right)\;=\;\frac{\Gamma(\mu)}{B(\mu,\nu)}\,I_{1,1}^{\nu-1,\mu}\left({}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ \nu,b_2,\dots,b_q\end{matrix};\ x\right)\right),\qquad(8.2)$$

where

$$I_{1,1}^{\nu,\mu}\,f(z)\;=\;\frac{1}{\Gamma(\mu)}\int_0^1 x^{\nu}(1-x)^{\mu-1}f(xz)dx.$$

The iteration of (8.2) leads to the following identity:

$${}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ \nu_1+\mu_1,\dots,\nu_q+\mu_q\end{matrix};\ x\right)=\prod_{i=1}^{q}\left(\frac{\Gamma(\mu_i)}{B(\mu_i,\nu_i)}\right)\left(\prod_{i=1}^{q}I_{1,1}^{\nu_i-1,\mu_i}\right)\left({}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ \nu_1,\dots,\nu_q\end{matrix};\ x\right)\right).\qquad(8.3)$$

The repeated integral operator possesses a single integral representation (cf. [21]):

$$I_{1,q}^{\vec{\nu}_q-1,\vec{\mu}_q}(f)(x)=\begin{cases}\int_0^1 G_{qq}^{q0}\left(t\left|\begin{matrix}\mu_1+\nu_1-1,\dots,\mu_q+\nu_q-1\\ \nu_1-1,\dots,\nu_q-1\end{matrix}\right.\right)f(xt)dt, & \text{for}\quad \sum_{i=1}^{q}\mu_i>0;\\ f(x), & \text{for}\quad \mu_1=\mu_2=\dots=\mu_q=0,\end{cases}\qquad(8.4)$$

where

$$G_{pq}^{mn}\left(x\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\right)$$

is Meijer's G-function (see, for instance, [17] p.207). In particular, we have

$$G_{11}^{10}\left(x\,\middle|\,\begin{matrix}\alpha+\beta\\ \alpha\end{matrix}\right) = \frac{1}{\Gamma(\beta)}(1-x)^{\beta-1}x^{\alpha}, \tag{8.5}$$

$$G_{22}^{20}\left(t\,\middle|\,\begin{matrix}\gamma_1+\delta_1,\gamma_2+\delta_2\\ \gamma_1,\gamma_2\end{matrix}\right) = \begin{cases}\frac{t^{\gamma_2}(1-t)^{\delta_1+\delta_2-1}}{\Gamma(\gamma_1+\gamma_2)}{}_2F_1\left(\begin{matrix}\gamma_2+\delta_2-\gamma_1,\delta_1\\ \delta_1+\delta_2\end{matrix};\,1-t\right), & \text{for } t<1;\\ 0 & \text{, for } t>1.\end{cases} \tag{8.6}$$

Thus the identity (8.3) may be rewritten under the form

$$\begin{aligned}{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ \alpha_1+\beta_1+1,\ldots,\alpha_q+\beta_q+1\end{matrix};\,x\right) &= \prod_{i=1}^{q}\left(\frac{\Gamma(\alpha_i+\beta_i+1)}{\Gamma(\alpha_i+1)}\right)\\ &\times\int_0^1 G_{qq}^{q0}\left(t\,\middle|\,\begin{matrix}\alpha_1+\beta_1,\ldots,\alpha_q+\beta_q\\ \alpha_1,\ldots,\alpha_q\end{matrix}\right){}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ \alpha_1+1,\ldots,\alpha_q+1\end{matrix};\,xt\right)dt\end{aligned} \tag{8.7}$$

from which, using (1.13), we deduce an integral relation connecting $\ell_n^{\vec{\alpha}_d+\vec{\beta}_d}$ and $\ell_n^{\vec{\alpha}_d}$ if $\sum_{i=0}^{d}\beta_i>0$, that is

$$\ell_n^{\vec{\alpha}_d+\vec{\beta}_d}(x)=\prod_{i=1}^{d}\frac{\Gamma(\alpha_i+\beta_i+1)}{\Gamma(\alpha_i+1)}\int_0^1 G_{dd}^{d0}\left(t\,\middle|\,\begin{matrix}\alpha_1+\beta_1,\ldots,\alpha_d+\beta_d\\ \alpha_1,\ldots,\alpha_d\end{matrix}\right)\ell_n^{\vec{\alpha}_d}(xt)dt. \tag{8.8}$$

For $d=1$, this identity is reduced to Koshlyakov's formula (8.1) by virtue of (8.5) and for $d=2$, it's equivalent to

$$\begin{aligned}\ell_n^{(\alpha_1+\beta_1,\alpha_2+\beta_2)}(x) &= \frac{\Gamma(\alpha_1+\beta_1+1)\Gamma(\alpha_2+\beta_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\beta_1+\beta_2)}\\ &\times\int_0^1 t^{\alpha_2}(1-t)^{\beta_1+\beta_2-1}{}_2F_1\left(\begin{matrix}\alpha_2+\beta_2-\alpha_1,\beta_1\\ \beta_1+\beta_2\end{matrix};\,1-t\right)\ell_n^{(\alpha_1,\alpha_2)}(xt)dt\end{aligned} \tag{8.9}$$

by virtue of (8.6) if $\beta_1+\beta_2>0$. Next, we obtain an integral expression for the polynomials $\ell_n^{\vec{\alpha}_d}$ in terms of Gould-Hopper polynomials $g_{(d+1)n}^{d+1}$, $n\geq 0$. Put

$$\vec{\alpha}_{d,0}-1 = \left(\frac{-1}{d+1},\frac{-2}{d+1},\ldots,\frac{-d}{d+1}\right).$$

Replace in the integral relation (8.8), $\vec{\alpha}_d$ by $\vec{\alpha}_{d,0}-1$ and $\vec{\beta}_d$ by $\vec{\alpha}_d-(\vec{\alpha}_{d,0}-1)$, to obtain

$$\begin{aligned}\ell_n^{\vec{\alpha}_d}(x) &= \frac{(-1)^n\prod_{i=1}^{d}\Gamma(\alpha_i+1)}{((d+1)n)!\sqrt{\frac{(2\pi)^d}{d+1}}}\\ &\times\int_0^1 G_{dd}^{d0}\left(t\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_d\\ \frac{-1}{d+1},\ldots,\frac{-d}{d+1}\end{matrix}\right)g_{(d+1)n}^{d+1}\left((d+1)(xt)^{\frac{1}{d+1}},-1\right)dt,\end{aligned} \tag{8.10}$$

since we have (cf., for instance, [26] p.12)

$$\prod_{j=1}^{d}\Gamma\left(\frac{j}{d+1}\right) = \sqrt{\frac{(2\pi)^d}{d+1}}.$$

For $d=1$, we have, after a change of variable, the Uspensky representation (cf. [34] p.604):

$$L_n^{(\alpha)}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\sqrt{\pi}(2n)!\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi} H_{2n}\left(\sqrt{x}\cos\varphi\right)\sin^{2\alpha}\varphi d\varphi.$$

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