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# On the classical theory of particles

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A set of classical relativistic equations of motion of an electron in an electromagnetic field is postulated. These equations are free from 'run-away' solutions, and give the same results as the Maxwell-Lorentz theory for non-relativistic motions when the external electromagnetic field does not vary too rapidly. For the scattering of light by an electron, the scattering cross-section is independent of the frequency and is a universal constant. This brings out a point of difference from the Lorentz-Dirac equations according to which the scattering cross-section varies inversely as the square of the frequency of the incident light, for large frequencies. For the motion of an electron towards a fixed proton, the equations allow a collision, unlike the Lorentz-Dirac equations according to which the electron is brought to rest before it reaches the proton.

## 1. Introduction

There has been in recent years a marked revival of interest in the classical theories of particles. After the pioneering work of Abraham and Lorentz in this field some forty or fifty years ago, comparatively little attention was paid until very recently to the development of the classical theory, the centre of research in theoretical physics being occupied by the rapidly expanding field of the quantum theory. However, the various difficulties that have now arisen in quantum electrodynamics, particularly that of infinite self-energies of point particles, have led to the view that some of these difficulties may be of classical origin, and that a promising method of approach to eliminate these difficulties would be to study and improve the classical theory first, before passing on to the quantum theory.

A substantial step in this direction was made by Dirac (1938) when he showed that the infinite self-energy, ascribed to the point electron by the Maxwell-Lorentz classical theory of electromagnetism, may be subtracted out in a Lorentz invariant way. The subtraction process led to a well-defined scheme of equations of motion of an electron in an electromagnetic field. These equations are consistent with the conservation laws and with the principle of special relativity. These equations in their non-relativistic approximation were found to be the same as the equations obtained earlier by Lorentz (1909). Lorentz regarded the electron as a small charged sphere of radius $r_0$ and obtained for the force on the electron due to its own field an expression in the form of a series in ascending powers of $r_0$. The first term of the series is $e^2\dot{\mathbf{v}}/r_0$, where $e$ is the electronic charge and $\mathbf{v}$ its velocity; dots denote differentiation with respect to the time, and units are chosen so that the velocity of light is unity. This first term describes electromagnetic inertia and may be added on to any mechanical inertia the electron may have, the two together being then represented by $m\dot{\mathbf{v}}$ in the equations of motion, where $m$ is the observed mass of the electron. The second term of the series is $\frac{2}{3}e^2\ddot{\mathbf{v}}$ and is independent of the shape of the electron. It accounts for the loss of energy by radiation at a rate $\frac{2}{3}e^2\dot{\mathbf{v}}^2$. The higher terms of the series depend on the structure of the electron, and being successively proportional to $r_0, r_0^2, \ldots$ are regarded as small since $r_0$ is small. The Lorentz equations of motion for an electron of mass $m$ and charge $e$ in an electromagnetic field described by the field vectors $\mathbf{E}$ and $\mathbf{H}$ are thus

$$m\dot{\mathbf{v}} - \tfrac{2}{3}e^2\ddot{\mathbf{v}} = e(\mathbf{E}+\mathbf{v}\times\mathbf{H}). \tag{1}$$

These equations were rederived by Dirac by using a point model for the electron and using the subtraction process; but whereas Lorentz's method of derivation makes them necessarily approximate, Dirac has suggested that according to his derivation there are hopes that these equations may be exact within the limits of the classical theory.

## 2. Self-accelerating motions. Possible ways of eliminating them

Although the difficulty of infinite self-energy is satisfactorily accounted for by Dirac's theory there arise, however, other difficulties. The equations of motion are found to have solutions which do not correspond to motions that are observed physically. Equation (1) involves $\ddot{\mathbf{v}}$ and the solution involves more arbitrary constants than are necessary to fix the actual motion. Thus in the absence of any incident field the motion is given by

$$m\dot{\mathbf{v}} - \tfrac{2}{3}e^2\ddot{\mathbf{v}} = 0, \tag{2}$$

and hence by

$$\mathbf{v} = \mathbf{A}+\mathbf{B}e^{at}, \tag{3}$$

where $a = 3m/2e^2$, and $\mathbf{A}$ and $\mathbf{B}$ are arbitrary constant vectors. If $\mathbf{B}$ is not zero the electron would rapidly build up a velocity and radiate energy at a rapid rate. Such a motion has not been observed. Similar non-physical solutions exist for motion in other electromagnetic fields.

There seem to be three possible ways of attempting to meet this difficulty:

(i) One is to suppose that out of the entire family of solutions of the equations of motion one should pick out, by the use of some extra physical condition, only those which are physically allowable. Thus for the above case of the free electron the extra condition is imposed that the acceleration should be zero, and the allowed solution corresponds to uniform motion in agreement with experiment (Dirac 1938). This condition may be suitably extended to homogeneous electric or magnetic fields (Eliezer & Mailvaganam 1945).

But there are difficulties in this method of excluding the non-physical solutions. There are some problems for which there do not seem to exist adequate physical solutions at all. For the electron in the hydrogen atom (Eliezer 1943), or an electron in the field of a thin charged plate (Eliezer 1945), all the solutions of the equations appear to have non-physical characteristics. This may not be a serious difficulty since adequate physical solutions may exist in the corresponding quantum theory, but there would then be the difficulty that the procedure of selecting some particular solution out of a family of solutions does not fit in with the general principles of quantum mechanics as at present formulated (Dirac 1946). It may not be possible to exclude quantum transitions between the physical and the non-physical motions.

(ii) Another way of eliminating this difficulty of non-physical solutions is to derive equations of motion which do not contain higher derivatives of velocity than the first derivative $\dot{\mathbf{v}}$. This is possible if the field of a moving electron is taken to be the mean of its retarded and advanced fields (Eliezer 1947). This assumption leads to the usual equations of motion without the radiation damping terms, but according to this derivation these equations are consistent with the conservation laws. A possible objection to this theory is that in leading to equations which do not contain the damping terms it does not account for the experimentally observed radiation loss of an electron. This would be a serious objection if it can be conclusively shown that the energy loss is a classical effect and that it holds for a single electron. It may be that only the many-body problem is encountered in nature and that the observed radiation damping has its origin in the complicated way in which the electrons are acted upon by the retarded and advanced fields of the other electrons. Wheeler & Feynman (1945) have attempted to build up a theory of radiation in which the force of radiation reaction on a particle is due to the field of the particles of the absorbing medium surrounding it; but it does not seem clear whether this attempt has been successful. Again, even if the observed effect applies to a single electron the radiation loss may perhaps be accountable by a quantum theory which takes over classical equations which do not contain the damping terms. Evidence on these questions seem to me to be indecisive.

(iii) A third possible approach to eliminate the difficulty of self-acceleration is to keep the extended model of the electron. The higher terms of the series for the force of radiation reaction should then be taken into account. In all applications of the Lorentz equations these higher terms have been generally assumed to be small. But for the self-accelerating motions discussed above these higher terms are not small. According to the solution (3), $\mathbf{v}$, $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$, ... are seen to increase successively by a factor $a$ so that $\mathbf{v}$, $\dot{\mathbf{v}}/a$, $\ddot{\mathbf{v}}/a^2$, ... are all of the same order ($1/a$ being the order of the radius of the electron).

In this paper I intend to study certain equations of motion with the higher terms included. These higher terms depend on the distribution of charge of the electron, and a particular distribution which would lead to simple equations of motion will be taken. These equations are expressed in relativistic form and are taken as the starting point for a classical theory of electrons.

## 3. The equations of motion

We consider the motion of an electron in an electromagnetic field. The Lorentz-Dirac equations give rise to self-accelerating motions whether we use the relativistic form of the equations or their non-relativistic approximation. Therefore in trying to eliminate these difficulties it seems the right procedure to start with the simpler non-relativistic case. We shall suppose that the electron starts with an initial motion in which the velocity, acceleration and the higher derivatives of velocity are small so that products of two or more of them may be neglected. The crucial point is to have a system of equations such that for a particle moving under no forces and with an initial motion in which the velocity and its derivatives are small, the solution of the equations of motion does not require the particle to work itself up to a rapidly increasing velocity and so contradict the conditions under which the equations are assumed to hold. If such a scheme of equations could be found, then it would provide a hopeful starting point for a consistent theory.

Page (1918) has investigated the form of the higher order terms in the equations of motion. If the electron is taken to be a charged sphere of radius $r_0$ and total charge $e$, the charge being confined to the surface of the sphere, then the equations of motion of the electron in an electric field $\mathbf{E}$ are

$$m\mathbf{f} - \tfrac{2}{3}e^2\dot{\mathbf{f}} + \frac{2}{3}\frac{e^2}{r_0}\sum_{n=2}^{\infty}(-1)^n\frac{(2r_0)^n\,\mathbf{f}^{(n)}}{(n+1)!} = e\mathbf{E}, \tag{4}$$

where $\mathbf{f}^{(n)}$ denotes the $n$th derivative of the acceleration $\mathbf{f}$ with respect to the time $t$, and $m$ is the sum of the electron's mechanical mass (if any) and its electromagnetic mass $\frac{2}{3}e^2/r_0$. This equation has some interesting features, but they cause some complications and they will be discussed in a later paper. If we consider other distributions of charge, the equations of motion will then have the form

$$m\mathbf{f} - \tfrac{2}{3}e^2\dot{\mathbf{f}} + \sum_{n=2}^{\infty}(-1)^n c_n\mathbf{f}^{(n)} = e\mathbf{E}, \tag{5}$$

where the $c$'s are positive constants which depend on the choice of the charge distribution. For a point electron the $c$'s are zero.

We take (for reasons of simplicity which will become apparent later) a model which is such that

$$c_n = m/a^n \quad (n \geqslant 2), \tag{6}$$

where $a = 3m/2e^2$ as before. The radius of the electron is of the order $1/a$. Thus we take as the equations of motion

$$m\left[\mathbf{f} - \frac{1}{a}\mathbf{f}^{(1)} + \frac{1}{a^2}\mathbf{f}^{(2)} - \ldots + (-1)^n\frac{1}{a^n}\mathbf{f}^{(n)} + \ldots\right] = e\mathbf{E}, \tag{7}$$

that is,

$$\left[1 - \frac{D}{a} + \frac{D^2}{a^2} - \ldots + (-1)^n\frac{D^n}{a^n} + \ldots\right]m\mathbf{f} = e\mathbf{E}, \tag{8}$$

where $D = d/dt$. The nature of the model corresponding to these equations is not discussed here, as such a discussion would be rather complicated, and also because we shall be looking at these equations from a different point of view later. We have to assume that the choice (6) gives a consistent model.

The series on the left-hand side of (7) is convergent only when $\mathbf{f}^{(n)}$ tends to zero as $n$ tends to infinity. It is then easy to show that the only solution of (7) satisfying this requirement is

$$m\mathbf{f} = \left(1 + \frac{D}{a}\right) e\mathbf{E}. \tag{9}$$

Equation (7) and its solution (9) are valid only when the external field $\mathbf{E}$ is such that $(D/a)^n \mathbf{E} \to \mathbf{0}$ as $n \to \infty$.

## 4. The energy equation

The energy equation may be obtained by taking the scalar product of (7) with $\mathbf{v}$, thus obtaining

$$m\left[\mathbf{v}.\mathbf{f} - \frac{1}{a}\mathbf{v}.\dot{\mathbf{f}} + \ldots + (-1)^n \frac{1}{a^n}\mathbf{v}.\mathbf{f}^{(n)} + \ldots\right] = e\mathbf{E}.\mathbf{v}. \tag{10}$$

Now

$$\mathbf{v}.\mathbf{f}^{(2n-1)} = \frac{d}{dt}[\mathbf{v}.\mathbf{f}^{(2n-2)} - \mathbf{f}.\mathbf{f}^{(2n-3)} + \ldots + (-1)^{n-1}\mathbf{f}^{(n-2)}.\mathbf{f}^{(n-1)}] + (-1)^n \mathbf{f}^{(n-1)2}, \tag{11}$$

$$\mathbf{v}.\mathbf{f}^{(2n)} = \frac{d}{dt}[\mathbf{v}.\mathbf{f}^{(2n-1)} - \mathbf{f}.\mathbf{f}^{(2n-2)} + \ldots + (-1)^{n-1}\mathbf{f}^{(n-2)}.\mathbf{f}^{(n)} + \tfrac{1}{2}(-1)^n \mathbf{f}^{(n-1)2}]. \tag{12}$$

Therefore equation (10) may be written

$$\frac{d}{dt}\left[V + \tfrac{1}{2}m\mathbf{v}^2 + m\sum_{n=1}^{\infty}\left\{-\frac{1}{a^{2n-1}}\sum_{r=0}^{n-1}(-1)^r \mathbf{v}^{(r)}.\mathbf{v}^{(2n-r-1)} + \frac{1}{a^{2n}}\left(\sum_{r=0}^{n-1}(-1)^r \mathbf{v}^{(r)}.\mathbf{v}^{(2n-r)} + \tfrac{1}{2}(-1)^n \mathbf{v}^{(n)2}\right)\right\}\right] = -m\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{a^{2n-1}}\mathbf{v}^{(n)2}, \tag{13}$$

where $V$ is the potential energy of the electron in the field $\mathbf{E}$. Writing

$$W = V + \tfrac{1}{2}m\mathbf{v}^2 + m\sum_{n=1}^{\infty}\left\{-\frac{1}{a^{2n-1}}\sum_{r=0}^{n-1}(-1)^r \mathbf{v}^{(r)}.\mathbf{v}^{(2n-r-1)} + \frac{1}{a^{2n}}\left(\sum_{r=0}^{n-1}(-1)^r \mathbf{v}^{(r)}.\mathbf{v}^{(2n-r)} + \tfrac{1}{2}(-1)^n \mathbf{v}^{(n)2}\right)\right\}, \tag{14}$$

we may interpret $W$ as the total energy of the electron. The first term gives the potential energy, the second term the usual kinetic energy, and the remaining terms correspond to energy due to the acceleration and its derivatives. If $\mathscr{R}$ denotes the rate of loss of energy, then

$$\mathscr{R} = -\frac{dW}{dt} = m\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{a^{2n-1}}\mathbf{v}^{(n)2}. \tag{15}$$

The first term of the series is $m/a\dot{\mathbf{v}}^2 = \frac{2}{3}e^2\dot{\mathbf{v}}^2$, which is the usual expression for the rate of loss of energy of a point electron. The remaining terms arise because we are using a finite model for the electron.

## 5. Some applications

### (i) *Free electron*

For an electron in the absence of any incident field, the equation of motion (9) gives $m\dot{\mathbf{v}} = 0$, and hence $\mathbf{v}$ is a constant. The solution thus corresponds to uniform motion, and there is no loss of energy by radiation.

### (ii) *Uniform electric field*

For an electron in a uniform electric field $\mathbf{E}_0$, the equation of motion (9) gives

$$m\dot{\mathbf{v}} = \left(1+\frac{D}{a}\right)e\mathbf{E}_0 = e\mathbf{E}_0.$$

The solution corresponds to the usual motion in a parabola; and according to (15) the rate of loss of energy is $\mathscr{R} = \frac{2}{3}e^4\mathbf{E}_0^2/m^2$.

### (iii) *Scattering of light*

Consider the rectilinear motion of an electron in the field of an incident beam of light of frequency $\nu$ whose electric field components are

$$E_x = \mathscr{E}_0 \cos \nu t, \quad E_y = E_z = 0. \tag{16}$$

We suppose that $\mathscr{E}_0$ is small and that the frequency $\nu$ is not larger than $a$. Then the electron's velocity and higher derivatives are small enough for products of two or more of them to be negligible, and the $n$th derivative of $\mathbf{E}$ tends to zero as $n$ tends to infinity. The equations of motion (9) are then applicable. We obtain

$$m\dot{\mathbf{v}} = \left(1+\frac{D}{a}\right)e\mathscr{E}_0\cos\nu t = e\mathscr{E}_0\left(\cos\nu t - \frac{\nu}{a}\sin\nu t\right); \tag{17}$$

that is,

$$\ddot{x} = A\cos(\nu t+\alpha), \tag{18}$$

where

$$A = \frac{e\mathscr{E}_0}{m}\left(1+\frac{\nu^2}{a^2}\right)^{\frac{1}{2}}, \quad \tan\alpha = \frac{\nu}{a} \quad (0<\alpha<\tfrac{1}{2}\pi). \tag{19}$$

Hence

$$x = -\frac{A}{\nu^2}\cos(\nu t+\alpha). \tag{20}$$

The electron performs a vibration with the same frequency as the incident wave and with a phase difference $\pi+\alpha$:

$$v^{(2n)} = \frac{A}{\nu}(-\nu^2)^n \sin(\nu t+\alpha), \quad v^{(2n+1)} = \frac{A}{\nu}(-\nu^2)^n\,\nu\cos(\nu t+\alpha).$$

The rate of loss of energy according to (15) is

$$\mathscr{R} = \frac{mA^2}{a}\sum_{n=1}^{\infty}\left\{\frac{\nu^{4n-4}}{a^{4n-4}}\cos^2(\nu t+\alpha) - \frac{\nu^{4n-2}}{a^{4n-2}}\sin^2(\nu t+\alpha)\right\}. \tag{21}$$

Hence the average rate of loss of energy is

$$\begin{aligned}\overline{\mathscr{R}} &= \frac{1}{2}\frac{mA^2}{a}\sum_{n=1}^{\infty}\left\{\frac{\nu^{4n-4}}{a^{4n-4}} - \frac{\nu^{4n-2}}{a^{4n-2}}\right\} = \frac{1}{2}\frac{mA^2}{a}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\nu^2}{a^2}\right)^{n-1} \\ &= \frac{1}{2}\frac{mA^2}{a}\frac{1}{1+\nu^2/a^2} = \frac{1}{3}\frac{e^4\mathscr{E}_0^2}{m^2}.\end{aligned} \tag{22}$$

This expression for $\overline{\mathscr{R}}$ is the same as the classical Thomson formula for the average rate of energy loss. It differs from the corresponding expression obtained when one uses the Lorentz-Dirac equations, according to which the rate of energy loss is found to be

$$\overline{\mathscr{R}} = \frac{1}{3}\frac{e^4\mathscr{E}_0^2}{m^2(1+\nu^2/a^2)}. \tag{23}$$

We have here a point of difference between the theory we are using in this paper and the Dirac classical theory of a point electron. The difference, however, is at most by a factor $\frac{1}{2}$ in the range of frequencies we are considering here, namely, $\nu \leqslant a$.

### (iv) *Harmonic oscillator*

Suppose the binding force of the oscillator is $m\omega^2 x$. The equation of motion (9) gives

$$m\ddot{x} = -\left(1+\frac{D}{a}\right)m\omega^2 x,$$

that is,

$$\ddot{x}+\frac{\omega^2}{a}\dot{x}+\omega^2 x = 0. \tag{24}$$

These equations are valid only when $\omega$ is small, and then the solution is of the form

$$x = e^{-\omega^2 t/2a}\left[C\cos\left\{\omega\left(1-\frac{\omega^2}{4a^2}\right)^{\frac{1}{2}}t\right\}+D\sin\left\{\omega\left(1-\frac{\omega^2}{4a^2}\right)^{\frac{1}{2}}t\right\}\right], \tag{25}$$

where $C$ and $D$ are arbitrary constants, showing that the motion is approximately periodic with the oscillations gradually dying out. The lifetime of the oscillator and the line breadth may be calculated in the usual way.

### (v) *Scattering of light by an oscillator*

Consider the rectilinear motion of an oscillator with binding force $m\omega^2 x$, placed in the field of an incident beam of light with electric field components given by (16). The equation of motion is

$$m\ddot{x} = \left(1+\frac{D}{a}\right)(-m\omega^2 x+e\mathscr{E}_0\cos\nu t),$$

that is,

$$\ddot{x}+\frac{\omega^2}{a}\dot{x}+\omega^2 x = A\cos(\nu t+\alpha), \tag{26}$$

where $A$ and $\alpha$ are as in (19). Ignoring the complementary function which tends to zero exponentially with time, we take as the solution

$$x = B\cos(\nu t+\alpha+\beta), \tag{27}$$

where

$$B = A\bigg/\left\{(\nu^2-\omega^2)^2+\frac{\omega^4\nu^2}{a^2}\right\}^{\frac{1}{2}}, \quad \tan\beta = \frac{\omega^2\nu}{a(\omega^2-\nu^2)}. \tag{28}$$

The average rate of energy loss is then

$$\overline{\mathscr{R}} = \frac{1}{2}\frac{mB^2\nu^4}{a(1+\nu^2/a^2)},$$

as in (22). Substituting for $B$ we obtain

$$\overline{\mathscr{R}} = \frac{e^4\mathscr{E}_0^2\nu^4}{3m^2[(\nu^2-\omega^2)^2+\omega^4\nu^2/a^2]}. \tag{29}$$

From (29) we easily derive the dispersion formula.

## 6. Higher derivatives in Dirac's theory

The higher derivatives of velocity may also be introduced into the equations of motion in Dirac's theory. I have considered in a previous paper equations of motion containing the second-order derivatives of velocity (Eliezer 1946). We shall now proceed to investigate the form of the equations of motion containing other higher derivatives. It appears that only the even higher derivatives of velocity occur in the equations of motion. This is because Dirac's theory uses a point model of the electron, and therefore the rate of energy loss is given by the usual expression for the rate of energy loss of a point charge, which in the non-relativistic limit is $\frac{2}{3}e^2\dot{\mathbf{v}}^2$. If odd derivatives of velocity appear in the equations of motion they will give rise to irreversible loss of energy of the electron in a way not consistent with conservation requirements for a point electron.

We shall first consider the relativistic equations of motion. Let $z_\mu$ ($\mu = 0, 1, 2, 3$) denote the space-time co-ordinates of the electron; dots now denote differentiation with respect to the proper time $s$, and $v_\mu = \dot{z}_\mu$. Then $v_\mu$ satisfies the condition

$$v^\mu v_\mu \equiv v^2 = 1. \tag{30}$$

The equations of motion of an electron in an electromagnetic field described by the field quantities $f_{\mu\nu}$ is

$$\tfrac{2}{3}e^2(\ddot{v}_\mu + \dot{v}^2 v_\mu) - ev^\sigma f_{\mu\sigma} = \dot{B}_\mu, \tag{31}$$

where $B_\mu$ is an arbitrary function depending on the particle's velocity and higher derivatives, and satisfying the requirements

$$(v, \dot{B}) = 0, \tag{32}$$

and

$$v_\lambda B_\mu - v_\mu B_\lambda \tag{33}$$

must be a perfect differential with respect to the proper time $s$ (Bhabha 1939). The scalar product notation

$$(a, b) \equiv a^\mu b_\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$$

is used. Let $v_\mu^{(n)}$ denote the $n$th derivative of $v_\mu$ with respect to the proper time. We solve for $B_\mu$ by assuming it to be expressible as a function of the form

$$B_\mu = B_0 v_\mu + B_1 v_\mu^{(1)} + \ldots + B_n v_\mu^{(n)} + \ldots, \tag{34}$$

where $B_0, B_1, \ldots$ are scalar quantities which are invariant functions of the velocity components and all their derivatives. The condition (32) gives

$$\dot{B}_0 + (B_1 + \dot{B}_2)(v, v^{(2)}) + \ldots + (B_{n-1} + \dot{B}_n)(v, v^{(n)}) + \ldots = 0. \tag{35}$$

We note that

$$(v, v^{(2n)}) = \frac{d}{ds}[(v, v^{(2n-1)}) - (v^{(1)} v^{(2n-2)}) + \ldots + (-1)^{n-1}(v^{(n-1)}, v^{(n)})] + \tfrac{1}{2}(-1)^n v^{(n)2}, \tag{36}$$

$$(v, v^{(2n+1)}) = \frac{d}{ds}[(v, v^{(2n)}) - (v^{(1)}, v^{(2n-1)}) + \ldots + (-1)^n (v^{(n-1)}, v^{(n+1)}) + \tfrac{1}{2}(-1)^n v^{(n)2}]. \tag{37}$$

The condition (33) requires that

$$B_1(v_\lambda v_\mu^{(1)} - v_\mu v_\lambda^{(1)}) + \ldots + B_n(v_\lambda v_\mu^{(n)} - v_\mu v_\lambda^{(n)}) + \ldots \tag{38}$$

must be a perfect differential. But

$$v_\lambda v_\mu^{(2n)} - v_\mu v_\lambda^{(2n)} = \frac{d}{ds}[\{v_\lambda v_\mu^{(2n-1)} - v_\mu v_\lambda^{(2n-1)}\} - \ldots + (-1)^n \{v_\lambda^{(n-1)} v_\mu^{(n)} - v_\mu^{(n-1)} v_\lambda^{(n)}\}], \tag{39}$$

$$v_\lambda v_\mu^{(2n+1)} - v_\mu v_\lambda^{(2n+1)} = \frac{d}{ds}[\{v_\lambda v_\mu^{(2n)} - v_\mu v_\lambda^{(2n)}\} - \ldots$$
$$+ (-1)^{n-1}\{v_\lambda^{(n-1)} v_\mu^{(n+1)} - v_\mu^{(n)} v_\lambda^{(n+1)}\}] + (-1)^n \{v_\lambda^{(n)} v_\mu^{(n+1)} - v_\mu^{(n)} v_\lambda^{(n+1)}\}. \tag{40}$$

On examining the conditions (35) and (38) we see that if we choose the $B$'s so that

$$B_1 = B_3 = \ldots = B_{2n+1} = \ldots = 0, \quad B_2, B_4, \ldots, B_{2n}, \ldots \text{ are constants,} \tag{41}$$

then (38) is automatically satisfied, and (35) gives

$$\dot{B}_0 + B_2(v, v^{(3)}) + \ldots + B_{2n}(v, v^{(2n+1)}) + \ldots = 0. \tag{42}$$

Integrating, we obtain

$$B_0 = m - \sum_{n=1}^{\infty} B_{2n}[(v, v^{(2n)}) - (v^{(1)}, v^{(2n-1)}) + \ldots + (-1)^n (v^{(n-1)}, v^{(n+1)}) + \tfrac{1}{2}(-1)^n v^{(n)^2}], \tag{43}$$

where $m$ is an arbitrary constant of integration. Therefore we take

$$B_\mu = mv_\mu + \sum_{n=1}^{\infty} B_{2n}[v_\mu^{(2n)} - \{(v, v^{(2n)}) - \ldots + (-1)^n (v^{(n-1)}, v^{(n+1)}) + \tfrac{1}{2}(-1)^n v^{(n)^2}\} v_\mu]. \tag{44}$$

This choice of $B_\mu$ leads to the equations of motion

$$m\dot{v}_\mu - \tfrac{2}{3}e^2(\ddot{v}_\mu + \dot{v}^2 v_\mu) + \sum_{n=1}^{\infty} B_{2n}[v_\mu^{(2n+1)} - (v, v^{(2n+1)})\, v_\mu - \{(v, v^{(2n)}) - \ldots$$
$$+ (-1)^n (v^{(n-1)}, v^{(n+1)}) + \tfrac{1}{2}(-1)^n v^{(n)^2}\} \dot{v}_\mu] = ev^\sigma f_{\mu\sigma}, \tag{45}$$

where the constants $B_2, B_4, \ldots$ are arbitrary. The constant $m$ which appears here as an arbitrary constant may be identified with the mass of the electron.

Suppose we now consider a motion in which the electron's velocity and all its derivatives are small, so that products of two or more of them may be neglected. If $\mathbf{v}$ now denotes $d\mathbf{z}/dt$, the equations of motion of an electron in an electric field $\mathbf{E}$ become

$$m\frac{d\mathbf{v}}{dt} - \tfrac{2}{3}e^2\frac{d^2\mathbf{v}}{dt^2} + B_2\frac{d^3\mathbf{v}}{dt^3} + \ldots + B_{2n}\frac{d^{2n+1}\mathbf{v}}{dt^{2n+1}} + \ldots = e\mathbf{E}, \tag{46}$$

that is,

$$\left[1 - \frac{D}{a} + a_2 D^2 + a_4 D^4 + \ldots + a_{2n} D^{2n} + \ldots\right] m\frac{d\mathbf{v}}{dt} = e\mathbf{E}, \tag{47}$$

where $D \equiv d/dt$, $a = 3m/2e^2$ and $a_{2n} = B_{2n}/m$.

The series

$$1 - \frac{x}{a} + a_2 x^2 + a_4 x^4 + \ldots + a_{2n} x^{2n} + \ldots \tag{48}$$

has an inverse for sufficiently small values of $x$. Suppose the inverse is

$$1 + \frac{x}{a} + c_2 x^2 + c_3 x^3 + \ldots + c_r x^r + \ldots, \tag{49}$$

where the constants $c_2$, $c_3$, ... are determined in terms of $a_2$, $a_4$, ..., that is, in terms of the arbitrary constants $B_2$, $B_4$, .... Therefore, analogous to (9) we would now have

$$m\frac{d\mathbf{v}}{dt} = \left(1+\frac{D}{a}+c_2D^2+\ldots+c_rD^r+\ldots\right)e\mathbf{E}. \tag{50}$$

In the particular case of the Lorentz-Dirac equations of motion, the constants $B_2$, $B_4$, ... are all zero, and the equations corresponding to (50) are then

$$m\frac{d\mathbf{v}}{dt} = \left(1+\frac{D}{a}+\frac{D^2}{a^2}+\ldots+\frac{D^r}{a^r}+\ldots\right)e\mathbf{E}. \tag{51}$$

If a finite model of the electron is used, and the same procedure as in the above be followed, with allowance made for the variation of the external field within the interior of the electron, then analogous to (9) the equations describing the motion of the centre of inertia of the electron may be expected to have the same form as (50). The first two terms on the right-hand side, namely $e\mathbf{E}$ and $a^{-1}e\dot{\mathbf{E}}$, are independent of the structure of the electron, while the remaining terms of the series all depend on the charge distribution. It seems therefore plausible to take the equations

$$m\frac{d\mathbf{v}}{dt} = \left(1+\frac{D}{a}\right)e\mathbf{E} \tag{9}$$

as of special physical significance.

An interesting point to note is that the familiar procedure of solving the Lorentz equations (1) approximately by first neglecting the term $\frac{2}{3}e^2\ddot{\mathbf{v}}$ altogether, and then carrying the solution to a higher approximation by replacing $\frac{2}{3}e^2\ddot{\mathbf{v}}$ by its value according to the first approximate solution leads precisely to the equations (9). For if in the first approximation $m\dot{\mathbf{v}} = e\mathbf{E}$, then in the second approximation

$$m\dot{\mathbf{v}} = e\mathbf{E}+\tfrac{2}{3}e^2\ddot{\mathbf{v}} = \left(1+\frac{D}{a}\right)e\mathbf{E}.$$

In our theory, however, we shall be taking the equations (9) as exact equations within certain limits.

## 7. Relativistic equations

The equations of motion (9) have been derived from the Maxwell-Lorentz theory. According to this derivation these equations are valid only when the velocity and all its derivatives are small so that products of two or more of them may be neglected, and the external field does not vary too rapidly. Now we shall seek a set of simple relativistic equations which under the above conditions would reduce to the equations (9), and which would be free from self-accelerating motions. Using the notation of § 6, we see that the equations

$$m\dot{v}_\mu = ev^\sigma f_{\mu\sigma}+\frac{e}{a}\left\{\frac{d}{ds}(v^\sigma f_{\mu\sigma})-\dot{v}^\sigma v^\rho f_{\rho\sigma}v_\mu\right\} \tag{52}$$

satisfy this requirement. On taking the scalar product of the equation (52) with $v^\mu$ we see that both sides give zero, owing to the condition $v^2 = 1$ and the antisymmetry of $f_{\mu\sigma}$. The last term in curly brackets in (52) gives the contribution from the radiation reaction.

We shall now postulate a classical theory in which the motion of an electron in a given electromagnetic field is described exactly by the equations (52), these equations being now looked upon as exact equations of motion applicable under all conditions within the limits of the classical theory. In taking (52) as the exact equations of motion, one should expect that for high velocities and rapidly varying fields there would be departures from the Maxwell-Lorentz theory. Maxwell's equations for empty space will be assumed to hold, that is,

$$\frac{\partial f_{\mu\sigma}}{\partial x^{\rho}}+\frac{\partial f_{\sigma\rho}}{\partial x^{\mu}}+\frac{\partial f_{\rho\mu}}{\partial x^{\sigma}}=0. \tag{53}$$

The energy equation, obtained from (52) with $\mu=0$, is

$$m\dot{v}_0-ev^{\sigma}f_{0\sigma}=\frac{e}{a}\left\{\frac{d}{ds}(v^{\sigma}f_{0\sigma})-\dot{v}^{\sigma}v^{\rho}f_{\rho\sigma}v_0\right\}. \tag{54}$$

The first term on the left-hand side gives the rate of increase of kinetic energy and the second term the rate of increase of potential energy. The right-hand side gives the rate of change of the total energy.

We shall consider some applications of the equations (52). The solution of the equations of motion contains the same number of arbitrary constants of integration as in the elementary theory which ignores radiation damping. The equations (52) do not have solutions corresponding to self-accelerating motions.

Let us investigate the cross-section for the scattering of light by an electron on the basis of the equations (52). Let us suppose that the amplitude $\mathscr{E}_0$ of the incident wave of light is so small that the velocity acquired by the electron is small enough for the non-relativistic equations to be valid. The motion is given by

$$m\dot{v}=e\mathscr{E}_0\left(\cos\nu t-\frac{\nu}{a}\sin\nu t\right),$$

which is the same as (17) except that this equation is now supposed to be valid for all values of the frequency $\nu$, whereas previously there was the restriction $\nu\leqslant a$. The rate of energy loss is found to be

$$\mathscr{R}=-\frac{1}{a}\dot{E}v=-\frac{e}{a}(-\mathscr{E}_0\nu\sin\nu t)\frac{e\mathscr{E}_0}{m}\left(\frac{\sin\nu t}{\nu}+\frac{\cos\nu t}{a}\right)$$
$$=\frac{2}{3}\frac{e^4\mathscr{E}_0^2}{m^2}\left(\sin^2\nu t+\frac{\nu}{a}\sin\nu t\cos\nu t\right). \tag{55}$$

Hence the average value of $\mathscr{R}$ over a period is

$$\overline{\mathscr{R}}=\tfrac{1}{3}e^4\mathscr{E}_0^2/m^2,$$

which agrees with the previous result. This shows that on this theory the total cross-section for the scattering of light is independent of the frequency and is a universai constant. This shows a difference from the Dirac theory, where the rate of energy loss is given by (23) and according to which the scattering cross-section varies as $\nu^{-2}$ for large frequencies.

Similar calculations may be made for the harmonic oscillator. The equations of motion are the same as (24), which for $\omega < 2a$ has solution of the form (25). If $\omega \geqslant 2a$ the auxiliary roots are real and negative, and the corresponding solution is such that the electron rapidly comes to rest without oscillating. For the scattering of light by the oscillator the average rate of energy loss $\overline{\mathscr{R}}$ is found to be given by the formula (29), now supposed to be valid for all values of $\omega$ and $\nu$. When $\omega \geqslant a\sqrt{2}$ the expression $\nu^4 \Big/ \left\{(\nu^2-\omega^2)^2+\frac{\omega^4\nu^2}{a^2}\right\}$ lies between 0 and 1 for all values of $\nu$, while if $\omega < a\sqrt{2}$ the expression has a maximum value $4a^4/\omega^2(4a^2-\omega^2)$ when $\nu = a\omega \Big/ \left(a^2-\frac{\omega^2}{2}\right)^{\frac{1}{2}}$. This shows that when $\omega$ is small $\overline{\mathscr{R}}$ has a large value for frequencies near $\omega$, giving an example of resonance fluorescence.

*The hydrogen atom.* As a further application of the relativistic equations of motion, we shall consider the rectilinear motion of an electron towards a proton which we shall take to be fixed by regarding its mass as infinitely large. If $t$, $x$ denote the time and space co-ordinates of the electron the equations of motion give

$$m\ddot{x} = -e^2\left(\frac{1}{x^2}-\frac{2\dot{x}}{ax^3}\right)\dot{t}, \quad \dot{t}^2-\dot{x}^2 = 1, \tag{56}$$

where dots denote the differentiation with respect to the proper time $s$. Writing $V = -\dot{x}$, substituting for $\dot{t}$ and transforming the independent variable to $x$, we obtain

$$V\frac{dV}{dx} = -\frac{3}{2a}\left(\frac{1}{x^2}+\frac{2V}{ax^3}\right)(1+V^2)^{\frac{1}{2}}. \tag{57}$$

If initially the electron is projected from a point $x = x_0 > 0$ towards the proton, then $V$ is positive and hence $dV/dx$ is negative. Therefore $V$ increases as $x$ decreases. That is, the electron approaches the proton with increasing speed and eventually collides with it (unlike the motion as given by the Lorentz-Dirac equations, according to which the electron is brought to rest before it could reach the proton). An approximate solution for small values of $x$ is seen to be

$$V \sim C\exp\left(\frac{3}{2a^2x^2}\right), \tag{58}$$

where $C$ is a positive constant. The solution seems to be in agreement with the usual results of mechanics.

There is another equation of motion which one may take as the starting point of a classical theory, and which is somewhat simpler than the equation (52), namely,

$$m\dot{v}_\mu = ev^\sigma\left(f_{\mu\sigma}+\frac{1}{a}\dot{f}_{\mu\sigma}\right). \tag{59}$$

For rectilinear motion in an electric field (52) and (59) are equivalent. But in other cases the equations (59) leads to some fundamental differences from the Maxwell-Lorentz theory—for example, an electron moving in a magnetic field does not lose energy by radiation, according to (59).

## 8. Discussion

The equations of motion (52) are seen to have many satisfactory features within the limits of the classical theory. They are free from self-accelerating motions. For small velocities and for fields which do not vary rapidly, these equations give the same results as the Maxwell-Lorentz theory; and in other cases they appear to give results in harmony with the usual notions of mechanics. A number of interesting points require further investigation. The extent to which Maxwell's equations need modification to be applicable when charged matter is present, the interaction of two moving electrons, the two-dimensional motion of the electron in the hydrogen atom, the passage to the quantum theory—all these questions require much detailed work, and it is hoped to deal with these in a later paper.

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