On the Classification of Differential Equations Having Orthogonal Polynomial Solutions (*).

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Summary. – *Suppose $\varphi_m(x)$ is a polynomial of degree m that satisfies the differential equation*

$$(1) \qquad \sum_{i=1}^{2n} b_i(x) y^{(i)}(x) = \lambda_m y(x)\,, \qquad m = 0, 1, 2, \ldots$$

where n is some fixed integer ≥ 1. We show that, under certain conditions, there exists an orthogonalizing weight distribution for $\{\varphi_m(x)\}$ that simultaneously satisfies n distributional differential equations of orders 1, 3, 5, ... $(2n-1)$. In particular, this weight Λ must satisfy

$$n b_{2n}(x)\Lambda' + \big(n b'_{2n}(x) - b_{2n-1}(x)\big)\Lambda = 0$$

in the distributional sense. As a corollary to this result, we get part of H. L. Krall's 1938 classification theorem which gives necessary and sufficient conditions on the existence of an OPS of solutions to (1) in terms of the moments and the coefficients of $b_i(x)$. To illustrate the theory, we consider all of the known OPS's to (1). In particular, new light is shed upon the problem of finding a real weight distribution for the Bessel polynomials.

1. – Introduction.

A problem that has attracted much interest over the past fifty years has been that of classifying all differential equations of the form

$$(1.1) \qquad \sum_{i=1}^{2n} \sum_{j=0}^{i} l_{ij} x^j y^{(i)}(x) = \lambda_m y(x)\,, \qquad n \geq 1$$

having a sequence $\{\varphi_m(x)\}_{m=0}^{\infty}$ of orthogonal polynomial solutions. This problem has become increasingly important to applied mathematicians who are looking for examples to fit the general Weyl-Titchmarsh theory of higher order differential equations. For example, see the recent papers of KRALL [10] and LITTLEJOHN and KRALL [20].

The first major result was obtained by BOCHNER [3] in 1929 when he solved the problem for $n = 1$. Indeed, he found that, up to a linear change of variable, only the classical polynomials of Jacobi, Laguerre and Hermite and the Bessel

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polynomials satisfied a second order differential equation of the form (1.1). In 1938, H. L. KRALL published necessary and sufficient conditions in order for (1.1) to have an orthogonal polynomial sequence (OPS) of solutions [13]. Using this remarkable theorem, he subsequently classified all fourth order equations having an OPS. He discovered three new differential equations having an OPS, all of which were different from the four sets mentioned above. In 1978, A. M. KRALL studied these nonclassical polynomials, naming them the Legendre type, Laguerre type and Jacobi type polynomials [11].

Little work was done on the problem for the period 1940 1978. It became clear that more efficient techniques were required to solve the problem in general. The methods of Bochner and Krall are too tedious to apply for large values of n. How ever, a pattern became apparent. Consider the following table:

Order of Differential Equation	OPS	Real Weight	Interval of Orthogonality
2	Jacobi	$(1-x)^{\alpha}(1+x)^{\beta}$	$[-1, 1]$
2	Laguerre	$x^{\alpha}\exp[-x]$	$[0, \infty)$
2	Hermite	$\exp[-x^2]$	$(-\infty, \infty)$
4	Jacobi type	$(1-x)^{\alpha}+\frac{1}{M}\delta(x)$	$[0, 1]$
4	Legendre type	$\frac{\alpha}{2}+\frac{1}{2}\delta(x-1)+\frac{1}{2}\delta(x+1)$	$[-1, 1]$
4	Laguerre type	$\exp[-x]+\frac{1}{R}\delta(x)$	$[0, \infty)$

Notice that the weight for each OPS in the fourth order case can be found by adding discrete mass(es) to the weight in the associated second order OPS. Also, observe that the weight for the Legendre type OPS has equal jumps at ± 1. It is natural to ask whether an OPS exists when unequal jumps at ± 1 are considered. In 1981, LITTLEJOHN found that such an OPS does exist and called them the Krall polynomials. They satisfy a sixth order differential equation and their various properties are extensively considered in [16]. Recently, KOORNWINDER found a new OPS that generalizes the Krall polynomials [10]. The set that he has found are orthogonal on $[-1, 1]$ with respect to the weight $A\delta(x+1)+B\delta(x-1)+$ $+C(1-x)^{\alpha}(1+x)^{\beta}$. The details have not yet been worked out to determine if his OPS satisfies a differential equation of the form (1.1). It should be pointed out that NEVAI [22] has considered adding finitely many masspoints to a weight distribution which has an associated OPS. In fact, he has produced some remarkable formulas for the resulting new OPS. However, the applications of his theory to differential equations are unclear at the moment.

In this paper, we derive some new results regarding the general classification problem. If we are to solve this problem, we must be able to classify all the weight distributions that make the polynomials orthogonal. This is the approach that we take: we show that the weight distributions fit nicely into a pattern. More specifically, *if* $\{\varphi_m(x)\}$ *is an OPS and* $\varphi_m(x)$ *satisfies* (1.1) *for* $m = 0, 1, 2, \ldots$, *we show, under certain conditions, how to construct an orthogonalizing weight distribution* Λ *for* $\{\varphi_m(x)\}$. *Under these conditions* Λ *will simultaneously satisfy* n *distributional differential equations of orders* $1, 3, 5, \ldots, (2n-1)$. We also show how part of H. L. KRALL's 1938 classification theorem follows from our results.

In § 2, we lay the foundation for our results. We assume the reader is familiar with the paper of KRALL and MORTON [12] even though we review some of its main points in § 2. In § 3, we introduce the ideas of our article by re-examining the second order case. The reader will notice how quickly the moments are generated in this case. Also in § 3, new light is shed upon the problem of finding an orthogonalizing Stieltjes function of bounded variation for the Bessel polynomials. Because of the intimate connection between self adjoint differential operators and orthogonal polynomials, we develop, in § 4, necessary and sufficient conditions for even order differential expressions to be formally self adjoint. § 5 deals with the general theory that we develop and, lastly, we follow this in § 6 by considering the examples in the fourth and sixth order cases.

2. - Background and preliminary assumptions.

We list some fundamental hypotheses and well known results that we shall assume and use throughout the remainder of this paper.

(*a*) Let $\{\varphi_m(x)\}$ be an OPS with respect to a real Stieltjes weight $\mu(x)$, where $\mu(x)$ is of bounded variation on $-\infty < x < \infty$. That is:

i) $\varphi_m(x)$ is a polynomial, with real coefficients, of degree exactly m, $m = 0, 1, \ldots$ *and*

ii) $\int_{-\infty}^{\infty} \varphi_p(x)\varphi_q(x)\, d\mu(x) = K_p \delta_{pq}$, $K_p \neq 0$ and δ_{pq} denotes the Kronecker δ-function, $p, q = 0, 1, \ldots$.

For general properties of an OPS, the reader is referred to the excellent text of CHIHARA [6].

(*b*) Assume $\varphi_m(x)$, $m = 0, 1, \ldots$ is a solution of the $2n$ th order real differential equation

$$L_{2n}(y) = \sum_{i=1}^{2n} b_i(x) y^{(i)}(x) = \lambda_m y(x)$$

where λ_m is a parameter depending only on m and where n is some fixed integer ≥ 1.

Necessarily then, $b_i(x)$ is a polynomial of degree $\leq i$, $i = 1, 2, \ldots, 2n$, so we hence forth assume that $b_i(x) = \sum_{j=0}^{i} l_{ij} x^j$, $l_{ij} \in \boldsymbol{R}$. It follows then that $\lambda_m = \sum_{j=1}^{2n} P(m, j) l_{jj}$ where $P(m, j) = m(m-1) \ldots (m-j+1)$ and $P(m, j) = 0$ for $j > m$. The reader is encouraged to consult [13] for a general account of this discussion.

(*c*) Let $\mu_m = \int_{-\infty}^{\infty} x^m \, d\mu(x)$, $m = 0, 1, \ldots$; μ_m is called the m th moment. We shall assume:

i) $\mu_0 = 1$;

ii) $|\mu_m| \leq cP^m m!$, $m = 0, 1, \ldots$ where c and P are arbitrary, but fixed, constants. (This is in accordance with the assumptions given by Krall and Morton.)

iii) The n recurrence relations given by H. L. KRALL (see Theorem 5.5) have a unique solution. This is a reasonable hypothesis; all of the known OPS's have the property that once μ_0 is known, all the other moments are uniquely determined. Note that this assumption does not imply the uniqueness of $d\mu(x)$: there exist many signed measures $d\mu(x)$ such that

$$\int_{-\infty}^{\infty} x^n \, d\mu(x) = 0 , \qquad n = 0, 1, \ldots .$$

For example, $\int_{-\infty}^{\infty} x^n g(x) \, dx = 0$, $n = 0, 1, 2, \ldots$ where

$$g(x) = \begin{cases} \exp[-x^{\frac{1}{4}}] \sin(x^{\frac{1}{4}}) , & x \geqq 0 \\ 0 , & x < 0 . \end{cases}$$

From (*a*), it follows that

$$\Delta_m = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_m \\ \mu_1 & \mu_2 & \cdots & \mu_{m+1} \\ \vdots & & & \\ \mu_m & \mu_{m+1} & \cdots & \mu_{2m} \end{vmatrix} \neq 0 , \qquad m = 1, 2 \ldots$$

(see [6]).

(*d*) Assume there exists a function $f(x)$ having $(2n-1)$ piecewise continuous derivatives on $\boldsymbol{R}$ and, on the intervals where $f(x)$ has $(2n-1)$ continuous derivatives, assume $f(x) L_{2n}(y)$ is formally self adjoint. This is also a reasonable assumption. The differential equations for the classical orthogonal polynomials, the Bessel polynomials and the nonclassical polynomials found by KRALL [11] and LITTLEJOHN [16] can all be made formally self adjoint. Indeed, for these classical and nonclassical polynomials, $f(x) = a'(x)$ where $a(x)$ is the absolutely continuous part of $\mu(x)$.

(*e*) KRALL and MORTON [12] found what seems to be the appropriate setting for the weight distribution $d\mu(x)$. More specifically, they showed that the distribution

$$w(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \mu_m \delta^{(m)}(x)}{m!}$$

acts as an orthogonalizing weight for $\{\varphi_m(x)\}$:

$$\int_{-\infty}^{\infty} \varphi_p(x)\varphi_q(x)w(x)\,dx = K_p\delta_{pq}\,, \qquad K_p \neq 0$$

They showed, that under certain conditions, $w(x) \in P'$, the dual of the vector space P which consists of all infinitely differentiable real valued functions $\psi(x)$, $-\infty < x < \infty$, such that for all $\alpha > 0$, $q \geqq 0$,

$$\lim_{|x|\to\infty} \exp\left[-\alpha|x|\right]\psi^{(q)}(x) = 0\,.$$

The reader is referred to [12] for the discussion of the topology given to P. Notice that P contains the set of all polynomials.

(*f*) For any distribution Γ in P', we will adopt the usual notation to show how Γ acts on P: for $\psi \in P$, $\langle\Gamma, \psi\rangle$ denotes this action. In particular, if $w \in P'$ where $d\mu(x) = w(x)\,dx$, then the m th moment of μ is $\langle w, x^m\rangle$. Since we will be freely using the calculus of distributions, the reader is referred to [9], [23], or [25] for an excellent review.

3. – Examples: the orthogonal polynomials in the second order case.

To illustrate the general theory developed in § 4 and § 5, we show how the weights, differential equations and moments for the Hermite, Jacobi, Laguerre and Bessel polynomials can be found

As stated in § 2, we shall assume:

(i) $\{\varphi_m(x)\}$ is an OPS with respect to $d\mu(x)$;

(ii) there exists a function $f(x)$ which is piecewise continuously differentiable on $(-\infty, \infty)$;

(iii) $\varphi_m(x)$ is a solution of the real second order differential equation;

(3 1) $$L_2(y) = b_2y'' + b_1y' = \lambda_m y\,, \qquad m = 0, 1, 2, \quad ;$$

(iv) $f(x)L_2(y)$ is formally self adjoint on the intervals where f is C^1.

The most general formally self adjoint second order differential expression is given by:

$$a_2 y'' + a_2' y' + a_0 y .$$

From (3.1) and iv), it follows that $f(x)$ satisfies the first order differential equation:

$$b_2(x) f'(x) + \big(b_2'(x) - b_1(x)\big) f(x) = 0 . \tag{3.2}$$

Separation of variables easily yields the classical solution

$$f(x) = \frac{\exp\left[\int \frac{b_1(x)\,dx}{b_2(x)}\right]}{b_2(x)}$$

However, (3.2) might be a singular equation so its general solution *in* P' might include a distributional part. For example, the general solution in P' to $x^2 f'(x) = 0$ is $f(x) = c_1 + c_2\delta(x)$ where c_1 and c_2 are arbitrary constants. *We shall make the additional assumption that there is a distributional solution* $\Lambda \in P'$ *to* (3.2) *with* $\langle \Lambda, 1\rangle = 1$. For $\psi \in P$, we have

$$0 = -\langle b_2\Lambda' + (b_2' - b_1)\Lambda, \psi\rangle = \langle \Lambda, b_2\psi' + b_1\psi\rangle . \tag{3.3}$$

Letting $b_2(x) = l_{22}x^2 + l_{21}x + l_{20}$, $b_1(x) = l_{11}x + l_{10}$, $\psi(x) = x^m$ and using the notation of § 2(g), equation (3.3) reads:

$$(ml_{22} + l_{11})\mu_{m+1} + (ml_{21} + l_{10})\mu_m + ml_{20}\mu_{m-1} = 0 , \qquad m \geq 0 . \tag{3.4}$$

This is exactly the recurrence relation given by H. L. KRALL in [13]. Before continuing, there is an important remark to be made: the true interval of orthogonality is not immediately available once we find Λ. For example, the weight for the Legendre polynomials is well-known to be

$$f(x) = \begin{cases} 1 & \text{for } -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} . \end{cases}$$

However, equation (3.2) yields just $f(x) = 1$. This turns out to be not a serious problem. We outline a method below that allows us to determine the true interval of orthogonality.

A) The Hermite Polynomials.

In this case, $b_2(x) = 1$ and $b_1(x) = -2x$. Then, (3.2) becomes $f'(x) + 2xf(x) = 0$. The general solution in P' is clearly the classical solution $f(x) = \exp[-x^2]$. (3.4) yields the recurrence relation $-2\mu_{m+1}+m\mu_{m-1}=0$. This is readily solved to yield $\mu_{2m+1}=0$ and $\mu_{2m}=\big((2m)!\mu_0\big)/(4^m m!)$.

We now show how Lagrange's identity [5] can be used to find the Hermite differential equation and weight function one $b_2(x)$ is known. Indeed, suppose $b_2(x)=1$ in (3.1). From (3.2), it follows that

$$f(x)=\exp\left[\frac{l_{11}x^2}{2}\right]+l_{10}x\,.$$

By considering a linear change of variable, we may as well assume $l_{10}=0$. Lagrange's identity yields the equation:

$$z(x)f(x)L_2(y)-y(x)f(x)L_2(z)=\frac{d[f(x)w_{01}(x)]}{dx}$$

where $w_{01}=zy'-yz'$. Suppose the interval of orthogonality is (a,b) where $-\infty\le a<$ $<b\le\infty$. Then, for $z=\varphi_p$, $y=\varphi_r$, $p\neq r$, we have

$$0=\int_a^b[zL_2(y)-yL_2(z)]f(x)\,dx=\int_a^b\frac{d[f(x)w_{01}(x)]}{dx}\,dx=f(b)w_{01}(b)-f(a)w_{01}(a)\,.$$

It follows, then, that it is necessary and sufficient that $f(a)=f(b)=0$. Indeed, by choosing $z=\varphi_0$, $y=\varphi_1$, we see that $f(a)=f(b)$. By letting $z=\varphi_0$, $y=\varphi_2$, it follows that $f(a)=f(b)=0$. Of course, this implies $l_{11}<0$ and $a=-\infty$, $b=\infty$. Again by considering a linear change of variable, we can assume $l_{11}=-2$. It follows that $f(x)=\exp[-x^2]$, $L_2(y)=y''-2xy'$, and thus, we get the Hermite polynomials.

B) The Jacobi Polynomials and the Laguerre Polynomials.

The Jacobi differential equation

$$(1-x^2)y''(x)+[\beta-\alpha-(\alpha+\beta+2)x]y'(x)+m(m+\alpha+\beta+1)y(x)=0$$

can be found in a similar way to that of the above Hermite equation by considering the case of $b_2(x)$ being a polynomial of degree 2 with two unequal real roots. We will not do this however; instead we find the weight. In this case, (3.2) becomes:

$$(1-x^2)f'(x)+\big(\beta-\alpha-(\alpha+\beta)x\big)f(x)=0\,.$$

The general solution in P' is again just the classical solution

$$f(x)=(1-x)^\alpha(1+x)^\beta\,.$$

For the Laguerre polynomials, equation (3.2) reads

$$xf'(x) + (x - \alpha)f(x) = 0 .$$

The general solution in P' to this equation is $f(x) = x^\alpha [\exp [-x]$. Using (3.4), the moments for these polynomial sets can easily be calculated.

C) The Bessel Polynomials.

The Bessel polynomials satisfy the second order differential equation $x^2y'' + 2(x+1)y' - n(n+1)y = 0$. A Stieltjes weight distribution $d\mu(x)$, where $\mu(x)$ is of bounded variation on $-\infty < x < \infty$, has never been found, even though Boas' theorem [2] guarantees its existence. Attempt after attempt has been made to find $d\mu(x)$ but each method has failed to recover μ. For an excellent treatise on the Bessel polynomials, the reader is encouraged to consult GROSSWALD's text [7].

The first order equation (3.2) is given by

$$x^2f'(x) - 2f(x) = 0 . \tag{3.5}$$

Note that $k \exp [-2/x]$ satisfies this equation (k any constant) but $k \exp [-2/x]$ is not the weight function for the Bessel polynomials. In fact, $k \exp [-2/x]$ is not in P' unless $k = 0$. KRALL and MORTON [12] found that

$$w(x) = \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}$$

makes the Bessel polynomials orthogonal on $(-\infty, \infty)$. We can show:

THEOREM 3.1. - $w(x) = \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}$ satisfies (3.5).

PROOF. - Let φ be an appropriate test function (for example, a polynomial). Note that $(x^2\varphi(x))^{(n+1)} = x^2\varphi^{(n+1)}(x) + 2(n+1)x\varphi^{(n)}(x) + n(n+1)\varphi^{(n-1)}(x)$. Hence

$$\begin{aligned}
\langle x^2w' - 2w, \varphi\rangle &= \left\langle x^2 \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n+1)}(x)}{n!(n+1)!} - 2\sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}, \varphi\right\rangle = \\
&= \sum_{n=0}^{\infty} \frac{2^{n+1}}{n!(n+1)!} \langle \delta^{n+1}(x), x^2\varphi(x)\rangle - \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!(n+1)!} \langle \delta^{(n)}(x), \varphi(x)\rangle \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^{n+1}}{n!(n+1)!} \langle \delta(x), (x^2\varphi(x))^{n+1}\rangle + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^{n+2}}{n!(n+1)!} \langle \delta(x), \varphi^n(x)\rangle \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}\varphi^{(n)}(0)}{n!(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^{n+2}\varphi^{(n)}(0)}{n!(n+1)!} = 0. \quad \square
\end{aligned}$$

Krall and Morton further showed that the inverse Fourier transform of $w(x)$ is *not* $\exp [-2/x]$. Thus (3.5) does have a distributional solution other than $\exp [-2/x]$.

This author has applied various techniques to try and find the general solution to (3.5) but to no avail. The standard methods that are used to find the distributional solutions to differential equations do not seemt o work in this case. Let Λ denote this general distributional solution to (3.5) (it may be that $\Lambda = c_1 w(x) + c_2 \exp[-2/x]$). It is not clear if this general distributional solution is in P'. However, the formal arithmetic that we carry out below indicates that the directions that we have taken in this paper are right even for the Bessel polynomials. Equation (3.4) yields the recurrence relation $2\mu_m + (m+2)\mu_{m+1} = 0$. This is easily solved to yield $\mu_m = ((-1)^m 2^m \mu_0)/(m+1)!$. Of course this is in complete agreement with other methods that compute these moments.

We now show the orthogonality of the Bessel polynomials with respect to Λ. For an appropriate test function φ, we have:

$$0 = \langle x^2\Lambda' - 2\Lambda, \varphi\rangle = \langle \Lambda, x^2\varphi' + 2(x+1)\varphi\rangle . \tag{3.6}$$

If y_n and y_m denote the n-th and m-th Bessel polynomials respectively, we see that

$$(n-m)(n+m+1)y_n y_m = x^2(y_m'' y_n - y_n'' y_m) + 2(x+1)(y_m' y^n - y_m y_n') .$$

Let $z = y_m' y_n - y_m y_n'$ so $z' = y_m'' y_n - y_n'' y_m$.

Hence,

$$(n-m)(n+m+1)\langle \Lambda, y_n y_m\rangle = \langle \Lambda, x^2 z' + 2(x+1)z\rangle = 0 \quad \text{by (3.6)} .$$

This shows the orthogonality of the Bessel polynomials with respect to Λ.

4. – A criterion for formal self adjointness of higher order differential equations.

In [17], this author investigated the problem of when an even order real differential equation can be made formally self adjoint. The following theorem is based upon a remarkably simple formula of H. L. KRALL [15]. Let

$$S_{2n}(y) = \sum_{k=0}^{2n} a_k(x) y^{(k)}(x) ,$$

where $a_k(x)$ is real valued, $a_k \in C^k(I)$, $a_{2n}(x) \neq 0$ for $x \in I$, and where I is some compact interval of the real line and n is some positive integer.

THEOREM 4.1. – $f(x)S_{2n}(y)$ is formally self adjoint if and only if $f(x)$ simultaneously satisfies the n homogeneous differential equations:

$$\sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} a_{2s}^{(2s-2k+1-j)}(x) f^{(j)}(x) - a_{2k-1}(x)f(x) = 0 , \quad k = 1, 2, \ldots, n . \tag{4.1}$$

where B_{2i} is the Bernoulli number defined by:

$$\frac{x}{\exp[x]-1} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}x^{2i}}{(2i)!}\,. \qquad \square$$

The n differential equations given by (4.1) are of orders $1, 3, 5, \ldots, (2n-1)$. The first order equation can easily be solved to give:

$$f(x) = \frac{\exp\left[\dfrac{1}{n}\displaystyle\int \frac{a_{2n-1}(x)\,dx}{a_{2n}(x)}\right]}{a_{2n}(x)}\,.$$

Notice that for $n = 1$, (4.1) is just equation (3.2).

5. – General theory.

The reader is reminded of the hypotheses stated in § 2 that we shall assume. By Theorem 4.1, $f(x)$ necessarily satisfies the n differential equations:

$$\sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} b_{2s}^{(2s-2k+1-j)}(x) f^{(j)}(x) - \\ - b_{2k-1}(x) f(x) = 0\,, \qquad k = 1, 2, \ldots, n\,. \tag{5.1}$$

We now make another assumption: *assume system* (5.1) *has a general solution* $\Lambda \in P'$ *with* $\langle \Lambda, 1\rangle = 1$.

Then, for $\varphi \in P$, we have:

$$0 = \Big\langle \sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \cdot \\ \cdot \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} b_{2s}^{(2s-2k+1-j)}(x) \Lambda^{(j)} - b_{2k-1}(x)\Lambda, \varphi \Big\rangle\,. \tag{5.2}$$

We will show that Λ *is an orthogonalizing weight distribution for* $\{\varphi_m(x)\}$.

It is well-known that if Λ is a distribution and $L(y)$ is a differential expression, then

$$\langle L(\Lambda), \varphi\rangle = \langle \Lambda, L^{+}(\varphi)\rangle\,, \qquad \text{(see [24])} \tag{5.3}$$

where L^{+} is the Lagrange adjoint of L. Our first aim will be to simplify (5.2) using (5.3). The following lemma is easy to prove.

LEMMA 5.1. – If m is a positive integer, then

$$\sum_{j=0}^{m-1} \frac{2(-1)^{j-1}}{(2m-j)!\,j!} = \frac{(-1)^m}{(m!)^2}.$$

LEMMA 5.2. –

$$\sum_{j=0}^{2s-2k+1} \sum_{i=0}^{2s-2k+1-j} (-1)^{j-1} \binom{2s-2k+1}{j} \binom{2s-2k+1-j}{i} \cdot b_{2s}^{(i+j)}(x)\,\varphi^{(2s-2k+1-j-i)}(x) = -b_{2s}(x)\,\varphi^{(2s-2k+1)}(x).$$

PROOF. – Let

$$a_{ij} = (-1)^{j-1} \binom{2s-2k+1}{j} \binom{2s-2k+1-j}{i} b_{2s}^{(i+j)}(x)\,\varphi^{(2s-2k+1-j-i)}(x).$$

Since

$$\binom{2s-2k+1}{j} \binom{2s-2k+1-j}{i} = \binom{2s-2k+1}{i} \binom{2s-2k+1-i}{j},$$

it follows that

i) $a_{ij} = a_{ji}$ if and only if $i-j$ is even, and;

ii) $a_{ij} = -a_{ji}$ if and only if $i-j$ is odd.

By considering the matrix of a_{ij}'s, it is clear that it suffices to show $\sum_{j=0}^{i} a_{i-j,j} = 0$, $i = 0, 1, \ldots, 2s-2k+1-j$, $j = 0, 1, \ldots, 2s-2k+1$.

Suppose, then, that $i = 2m-1$.

Then

$$\sum_{j=0}^{2m-1} a_{2m-1-j,j} = a_{2m-1,0} + a_{2m-2,1} + a_{2m-3,2} + \ldots + a_{2,2m-3} + a_{1,2m-2} + a_{0,2m-1} = 0$$

by ii) above.

If $i = 2m$, then $\sum_{j=0}^{2n} a_{2m-j,j} = 2\sum_{j=0}^{m-1} a_{2m-j,j} + a_{m,m}$ by i) above.

$$= 2\sum_{j=0}^{m-1} (-1)^{j-1} \binom{2s-2k+1}{j} \binom{2s-2k+1-j}{2m-j} b_{2s}^{(2m)}(x)\,\varphi^{(2s-2k+1-2m)}(x) + \\ + (-1)^{m-1} \binom{2s-2k+1}{m} \binom{2s-2k+1-m}{m} b_{2s}^{(2m)}(x)\,\varphi^{(2s-2k+1-2m)}(x).$$

Now

$$\binom{2s-2k+1}{j} \binom{2s-2k+1-j}{2m-j} = \frac{(2s-2k+1)!}{(2s-2k-2m+1)!(2m-j)!\,j!}$$

and
$$\binom{2s-2k+1}{m}\binom{2s-2k+1-m}{m}=\frac{(2s-2k+1)!}{(2s-2k-2m+1)!\,(m!)^2}.$$

Thus,

$$\sum_{j=0}^{2m} a_{2m-j,j}=\frac{(2s-2k+1)!}{(2s-2k-2m+1)!}\,b_{2s}^{(2m)}(x)\varphi^{(2s-2k+1-m)}(x)\cdot\left\{\sum_{j=0}^{m}\frac{2(-1)^{j-1}}{(2m-j)!\,j!}+\frac{(-1)^{m-1}}{(m!)^2}\right\}=0$$

by Lemma 5.1.

Hence
$$\sum_{j=0}^{2s-2k+1}\ \sum_{i=0}^{2s-2k+1-j} a_{ij}=a_{00}=-b_{2s}(x)\varphi^{(2s-2k+1)}(x)\,.\qquad\square$$

Returning to (5.2), we have:

$$\begin{aligned}0&=\left\langle\sum_{s=k}^{n}\ \sum_{j=0}^{2s-2k+1}\binom{2s}{2k-1}\binom{2s-2k+1}{j}\frac{2^{2s-2k+2}-1}{s-k+1}\cdot\right.\\&\quad\left.\cdot B_{2s-2k+2}b_{2s}^{(2s-2k+1-j)}(x)\Lambda^{j}-b_{2k-1}(x)\Lambda,\varphi\right\rangle=\\&=\left\langle\sum_{s=k}^{n}\ \sum_{j=0}^{2s-2k+1}\binom{2s}{2k-1}\binom{2s-2k+1}{j}\frac{2^{2s-2k+2}-1}{s-k+1}\cdot\right.\\&\quad\left.\cdot B_{2s-2k+2}b_{2s}^{(j)}(x)\Lambda^{(2s-2k+1-j)}-b_{2k-1}(x)\Lambda,\varphi\right\rangle=\\&=\sum_{s=k}^{n}\binom{2s}{2k-1}\frac{2^{2s-2k+2}-1}{s-k+1}B_{2s-2k+2}\left\langle\sum_{j=0}^{2s-2k+1}\binom{2s-2k+1}{j}b_{2s}^{(j)}(x)\Lambda^{(2s-2k+1-j)},\varphi(x)\right\rangle-\\&\quad-\langle\Lambda,b_{2k-1}(x)\varphi(x)\rangle\,.\end{aligned}$$

Now

$$\begin{aligned}&\sum_{j=0}^{2s-2k+1}\binom{2s-2k+1}{j}\langle\Lambda^{(2s-2k+1-j)},b_{2s}^{(j)}\varphi\rangle=\\&=\sum_{j=0}^{2s-2k+1}\binom{2s-2k+1}{j}(-1)^{j-1}\langle\Lambda,(b_{2s}^{(j)}\varphi)^{(2s-2k+1-j)}\rangle=\\&=\sum_{j=0}^{2s-2k+1}\ \sum_{i=0}^{2s-2k+1-j}(-1)^{j-1}\binom{2s-2k+1}{j}\binom{2s-2k+1-j}{i}\langle\Lambda,b_{2s}^{(i+j)}\varphi^{(2s-2k+1-j-i)}\rangle=\\&=\langle\Lambda,-b_{2s}(x)\varphi^{(2s-2k+1)}(x)\rangle\qquad\text{by Lemma 5.2}\,.\end{aligned}$$

Hence, (5.2) may be rewritten as:

$$0=\left\langle\Lambda,\sum_{s=k}^{n}\binom{2s}{2k-1}\frac{2^{2s-2k+2}-1}{s-k+1}B_{2s-2k+2}b_{2s}(x)\varphi^{(2s-2k+1)}(x)+b_{2k-1}(x)\varphi(x)\right\rangle,$$
$$k=1,2,\ldots,n\,.$$

We summarize our results so far in the following theorem.

THEOREM 5.3. – The distribution $\Lambda \in P'$ satisfies the n distributional differential equations: for $\varphi \in P$,

$$\left\langle \sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1}\binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} b_{2s}^{(2s-2k+1-j)} \Lambda^{(j)} - b_{2k-1}\Lambda, \varphi \right\rangle = 0\,, \qquad k = 1, 2, \ldots, n\,.$$

Equivalently, for all $\varphi \in P$,

$$\left\langle \Lambda, \sum_{s=k}^{n} \binom{2s}{2k-1} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2} b_{2s} \varphi^{(2s-2k+1)} + b_{2k-1}\varphi \right\rangle = 0\,, \qquad k = 1, 2, \ldots, n\,. \tag{5.3}$$

If we let $\varphi(x) = x^m$ and write $b_i(x) = \sum_{j=0}^{i} l_{ij}x^j$, (5.3) gives us n recurrence relations involving the moments of Λ. Indeed, we have:

THEOREM 5.4. – The moments $\{\mu_m\}_{m=0}^{\infty}$ of the distribution Λ simultaneously satisfy the n recurrence relations:

$$T_k(m) = \sum_{s=k}^{n} \sum_{j=0}^{2s} \binom{2s}{2k-1} P(m-2k+1,\, 2s-2k+1) \frac{2^{2s-2k+2}-1}{s-k+1} \cdot$$
$$\cdot B_{2s-2k+2} l_{2s,j} \varphi_{m-2s+j} + \sum_{j=0}^{2k-1} l_{2k-1,j}\mu_{m-2k+1+j} = 0\,, \qquad m \geq 2k-1,\; k = 1, 2, \ldots, n\,.$$

PROOF. – This follows immediately upon substituting $\varphi = x^m$ into (5.3) and replacing m by $m - 2k + 1$. □

At this point, let us recall H. L. KRALL's 1938 classification theorem:

THEOREM 5.5 (KRALL). – In order that there exist an OPS $\{\varphi_m(x)\}$ satisfying the differential equation

$$\sum_{i=1}^{2n} \sum_{j=0}^{i} l_{ij} x^j y^{(i)}(x) = \lambda_m y(x)$$

it is necessary and sufficient that:

(i) the moments $\{\mu_m\}_{m=0}^{\infty}$ satisfy the n recurrence relations

$$S_k(m) = \sum_{i=2k+1}^{2n} \sum_{u=0}^{i} \binom{i-k-1}{k} P(m-2k-1,\, i-2k-1)\, l_{i,i-u}\mu_{m-u} = 0\,, \qquad k = 0, 1, \ldots, (n-1),\; m \geq 2k+1 \tag{5.4}$$

and

(ii) $\Delta_k \neq 0$, $k = 1, 2, \ldots$, where Δ_k is defined in § 2(c). □

It is natural to ask: what is the connection between $S_k(m)$ and $T_k(m)$? Elementary calculations reveal, for example, that

$$T_n(m) = S_{n-1}(m)$$

$$T_{n-1}(m) = S_{n-2}(m) - \frac{n(n-1)(m-2n+3)(m-2n+2)}{2} S_{n-1}(m)$$

$$T_{n-2}(m) = S_{n-3}(m) - \frac{(n-1)(n-2)(m-2n+5)(m-2n+4)}{2} S_{n-2}(m) +$$

$$+ \frac{n(n-1)(5n-7)(m-2n+5)(m-2n+4)(m-2n+3)(m-2n+2)}{24} S_{n-1}(m) .$$

In fact,

$$T_{n-k}(m) = \sum_{i=1}^{k+1} A_j S_{n-k+j-2}(m) , \qquad k = 0, 1, \ldots, (n-1) , \tag{5.5}$$

where:

$$A_1 = 1 \quad \text{ and } \quad \sum_{i=1}^{j+1} A_i \binom{n-k+2j-i}{n-k+i-2} P(m-2n+2k-2i-3, 2j-2i+2) = 0 ,$$
$$j = 1, 2, \ldots, k .$$

We leave out the verification of (5.5) because of the tedious computations involved. From (5.5), it is clear that $S_k(m) = 0$, $k = 0, 1, \ldots, (n-1)$, if and only if $T_k(m) = 0$, $k = 1, 2, \ldots, n$.

Hence, from § 2 (c), Λ and w generate the same moments where w is defined in § 2. In view of these facts and Theorem 5.5, we have the following.

THEOREM 5.6. – Suppose $\{\varphi_m(x)\}$ is an OPS with moment sequence $\{\mu_m\}$. Assume the hypotheses stated in § 2. In addition, suppose the following are satisfied:

i) $\varphi_m(x)$ satisfies the real differential equation

$$\sum_{i=1}^{2n} b_i(x) y^{(i)}(x) = \lambda_m y(x) , \qquad m = 0, 1, 2, \ldots \tag{5.6}$$

where $b_i(x) = \sum_{j=0}^{i} l_{ij} x^j$, $i = 1, 2, \ldots, 2n$;

ii) The system of distributional differential equations

$$\sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1} \binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} \cdot B_{2s-2k+1} b_{2s}^{(2s-2k+1-j)}(x) \Lambda^{(j)} - b_{2k-1}(x)\Lambda = 0 , \qquad k = 1, 2, \ldots, n \tag{5.7}$$

has a nontrivial solution $\Lambda \in P'$ with $\langle \Lambda, 1\rangle = 1$, where B_i is the i-th Bernoulli number. Then Λ is an orthogonalizing weight distribution for $\{\varphi_m(x)\}$ with $\mu_m = \langle \Lambda, x^m \rangle$. In particular, Λ satisfies the first order equation $nb_{2n}(x)\Lambda' + (nb'_{2n}(x) - b_{2n-1}(x))\Lambda = 0$.

NOTE. – Compare this last statement to Theorem 2 in [1]. This theorem says that if a weight function $w(x)$ satisfies a first order equation of the form $a(x)y'(x) + b(x)y(x) = 0$ where $a(x)$ and $b(x)$ are polynomials then the OPS associated with $w(x)$ satisfies a differential equation of the form

$$A(x, n)y''(x) + B(x, n)y'(x) + C(x, n)y(x) = 0 .$$

We now point out a rather interesting corollary.

THEOREM 5.7. – Under the assumptions of Theorem 5.6, $\langle \Lambda, b_{2k-1} \rangle = 0$, $k = 1, 2, \ldots, n$.

PROOF. – Set $\varphi(x) = 1$ in (5.3).

For example, the Hermite differential equation is $y'' - 2xy' + 2ny = 0$.

Note that $\langle \exp[-x^2], -2x \rangle = \int_{-\infty}^{\infty} -2x \exp[-x^2]\,dx = 0$. Similarly, the Laguerre equation is $xy'' + (1 + \alpha - x)y' + ny = 0$. Observe that

$$\langle x^\alpha \exp[-x], 1 + \alpha - x \rangle = \int_0^\infty (1 + \alpha - x)x^\alpha \exp[-x]\,dx = 0 .$$

6. – More examples.

In this section, we show how the theory applies to the nonclassical orthogonal polynomials found by Krall and Littlejohn.

A) The Legendre type Polynomials.

In this case, the fourth order equation is:

$$(x^2 - 1)^2 y^{(4)} + 8x(x^2 - 1)y^{(3)} + (4\alpha + 12)(x^2 - 1)y'' + 8\alpha x y' = \lambda_m y .$$

From (5.6), we know that the weight Λ simultaneously satisfies

(*a*) $(x^2 - 1)^2 \Lambda' = 0$;

(*b*) $(x^2 - 1)^2 \Lambda^{(3)} + 12x(x^2 - 1)\Lambda'' + [(24 - 4\alpha)x^2 + 4\alpha]\Lambda' = 0$.

The general solution to (*a*) is $\Lambda(x) = c_1 + c_2\delta(x - 1) + c_3\delta(x + 1)$.

Substitution of Λ into (b) and using the equivalent equation from (5.3) yields

$$0 = \langle (x^2-1)^2\Lambda^{(3)} + 12x(x^2-1)\Lambda'' + [(24-4\alpha)x^2+4\alpha]\Lambda', \varphi\rangle =$$
$$= \langle \Lambda, (x^2-1)^2\varphi^{(3)} - (4\alpha+12)(x^2-1)\varphi' - 8\alpha x\varphi\rangle =$$
$$= -8\alpha\varphi(1)c_2 + 8\alpha\varphi(-1)c_3 + c_1\int_{-1}^{1}[(x^2-1)^2\varphi^{(3)} - (4\alpha+12)(x^2-1)\varphi' - 8\alpha x\varphi]\,dx =$$
$$= (-8\alpha c_2 + 8c_1)\varphi(1) + (8\alpha c_3 - 8c_1)\varphi(-1)$$

where we have used integration by parts to simplify the integral. This forces $c_1 = \alpha c_2 = \alpha c_3$ so $c_2 = c_3$. If we let $c_2 = \frac{1}{2}$, we get $\Lambda(x) = \alpha/2 + \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$. This, of course, is in complete agreement with the method used by Krall and Morton who used the Fourier transform. Our method does seem easier. Notice that the moments can be immediately calculated without using the recurrence relations in Theorem 5.4. We find that $\mu_{2m} = (\alpha + 2m + 1)/(2m+1)$ and $\mu_{2m+1} = 0$.

B) The Laguerre type Polynomials.

The fourth order differential equation for these polynomials is:

$$x^2y^{(4)} - (2x^2-4x)y^{(3)} + \big(x^2-(2R+6)x\big)y'' + \big([2R+2]x - 2R\big)y' = \lambda_m y\,.$$

The weight distribution satisfies the two equations:

a) $x^2\Lambda' + x^2\Lambda = 0$;

b) $x^2\Lambda^{(3)} + 6x\Lambda'' + [-x^2 + (2R+6)x + 6]\Lambda' + (2Rx+6)\Lambda = 0$.

The general solution to *a)* is $\Lambda(x) = c_1\delta(x) + c_2\exp[-x]$. Substitution into *b)* reveals $\Lambda(x) = (1/R)\delta(x) + \exp[-x]$. The moments can now be easily calculated. They are $\mu_0 = (R+1)/R$ and $\mu_m = m!$, $m \geq 1$.

C) The Jacobi type Polynomials.

These polynomials satisfy

$$(x^2-x)^2y^{(4)} + 2x(x-1)\big([\alpha+4]x-2\big)y^{(3)} + x\big([\alpha^2+9\alpha+14+2M]x -$$
$$-[6\alpha+12+2M]\big)y'' + \big([\alpha+2][2\alpha+2+2M]x - 2M\big)y' = \lambda_m y\,.$$

In this case, the weight satisfies:

a) $x^2(x-1)^2\Lambda' + \alpha x^2(1-x)\Lambda = 0$, and

b) $x^2(x-1)^2\Lambda^{(3)} + (12x^3 - 18x^2 + 6x)\Lambda'' +$
$+ [(-\alpha^2 - 9\alpha + 22 - 2M)x^2 + (6\alpha - 24 + 2M)x + 6]\Lambda' +$
$+ [(2M\alpha - 12\alpha)x + 6\alpha]\Lambda = 0$.

The general solution to a) is $A(x) = c_1(1-x)^\alpha + c_2\delta(x)$.

Similar to the previous two examples, substitution of this A into (b) and integration by parts reveals $Mc_2 = c_1$: Setting $c_1 = 1$, we find that $A(x) = (1-x)^\alpha + 1/M\,\delta(x)$. Again, the moments can easily be calculated. We find in this case that $\mu_0 = (\alpha+1+M)/M(\alpha+1)$ and, for $m \geq 1$, $\mu_m = m!/(\alpha+1)_{m+1}$.

D) The Krall Polynomials.

This OPS satisfies the sixth order differential equation

$$\begin{aligned}(x^2-1)^3y^{(6)} &+ 18x(x^2-1)^2y^{(5)} + \{[3AC+3BC+96]x^4 - [6AC+6BC+132]x^2 +\\ &+ [3AC+3BC+36]\}y^{(4)} + \{[24AC+24BC+168]x^3 -\\ &- [24AC+24BC+168]x\}y^{(3)} + \{[12ABC^2+42AC+42BC+72]x^2 +\\ &+ [12BC-12AC]x - [12ABC^2+30AC+30BC+72]\}y'' +\\ &+ \{[24ABC^2+12AC+12BC]x + [12BC-12AC]\}y' = \lambda_m y\,.\end{aligned}$$

From (5.2), the weight distribution A satisfies the three differential equations:

a) $(x^2-1)^3A' = 0$;

b) $5(x^2-1)^3A^{(3)} + 90x(x^2-1)^2A'' + (x^2-1)[(-6AC-6BC+258)x^2 + (6AC+6BC-18)]A' = 0$;

c) $(x^2-1)^3A^{(5)} + 30x(x^2-1)^2A^{(4)} + (x^2-1)\{(-AC-BC+268)x^2 + (AC+BC-48)\}A^{(3)} + \{(-12AC-12BC+816)x^3 + (12AC+12BC-456)x\}A'' + \{(4ABC^2-22AC-22BC+672)x^2 + (4BC-4AC)x + (-4ABC^2+2AC+2BC-120)\}A' = 0.$

The general solution to a) is

$$A(x) = c_1 + c_2\delta(x-1) + c_3\delta(x+1) + c_4\delta'(x-1) + c_5\delta'(x+1)\,.$$

Sustitution of A into b) reveals $c_4 = c_5 = 0$. Similarly, substitution of A into c) leads to $c_1 = BCc_2 = ACc_3$. If we let $c_1 = C$, we get $A(x) = C + (1/B)\delta(x-1) + (1/A)\delta(x+1)$. For this OPS, the moments are given by $\mu_{2m-1} = (A-B)/AB$ and $\mu_{2m} = 1/A + 1/B + 2C/(2m+1)$.

7. – Conclusion.

In summary, we have shown, under certain conditions, how to construct an orthogonalizing weight distribution for an OPS $\{\varphi_m(x)\}$ satisfying a real differential equation of the form

$$\sum_{i=1}^{2n}\sum_{j=0}^{i} l_{ij}x^j y^{(i)}(x) = \lambda_m y(x) .$$

Much work needs to be done if we are to classify all differential equations of this form having an OPS for a solution set. In particular, it is surprising how very little is known on the general distributional solution to first order equations of the form $a(x)y'(x) + b(x)y(x) = 0$, where $a(x)$ and $b(x)$ are polynomials. If we can successfully learn how to solve these first order equations, perhaps we can solve the above classification problem.

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