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On the Classification of Quantum Poincaré Groups

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Abstract: Using the general theory of [10], quantum Poincaré groups (without dilatations) are described and investigated. The description contains a set of numerical parameters which satisfy certain polynomial equations. For most cases we solve them and give the classification of quantum Poincaré groups. Each of them corresponds to exactly one quantum Minkowski space. The Poincaré series of these objects are the same as in the classical case. We also classify possible R-matrices for the fundamental representation of the group.

0. Introduction

The Minkowski space with the Poincaré group acting on it is the area of the quantum field theory. However, it is not known yet what is the area of a deeper theory, which would involve also the gravitational effects. It was suggested by many authors that it would be a quantum space. It means that instead of functions on spacetime we would have elements of some noncommutative algebra, called "the algebra of functions on the quantum space." On the other hand, such a quantum space should be in some sense similar to the ordinary Minkowski space. The simplest models of such a situation can be obtained by choosing some properties of Minkowski space endowed with the action of the Poincaré group and classifying all quantum groups and spaces which satisfy those properties. There are many examples of quantum Poincaré groups, the corresponding Minkowski spaces and R-matrices (cf. e.g. [4, 2, 11, 6, 5, 1, 15] and remarks in [10] concerning these papers) but such classification still doesn't exist. Our aim is to provide it. In Sect. 1 we define a quantum Poincaré group as a quantum group which is built from any quantum Lorentz group [14] and translations and satisfies some natural properties. The corresponding

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commutation relations are inhomogeneous and contain a set of parameters H_{ABCD}, T_{ABCD}. Our scheme contains the examples provided in [4, 1], but doesn't contain the examples of [2, 11, 5] (see however Remark 3.9 of [10]) because we consider quantum Poincaré groups without dilatations. Also the example [6] (formulated in the language of universal enveloping algebras) has no corresponding object in our scheme (for $q \neq \pm 1$).

It turns out that there are many quantum Lorentz groups which can be used in our construction. However all of them correspond to $q = \pm 1$. For each such quantum Lorentz group (except the classical one and one more for $q = -1$ which are considered in Remark 1.8) we classify all quantum Poincaré groups. We also provide the corresponding quantum Minkowski spaces and R-matrices for the fundamental representation of the quantum Poincaré group (for one family of considered quantum Poincaré groups there is no nontrivial R-matrix). The Poincaré series of the corresponding objects are the same as in the classical case. The proofs of our results (using [10]) are contained in Sect. 2. In particular, the question of finding all quantum Poincaré groups is reduced to a set of polynomial equations for H_{ABCD}, T_{ABCD} which we solve (in the indicated cases) using the computer MATHEMATICA program. Some results of the present paper were presented in [9]. In [16] a similar classification is provided in the case of Poisson manifolds and Poisson–Lie groups.

We use the terminology and results of [10]. The letter S means that we make a reference to [10], e.g. Theorem S3.1 denotes Theorem 3.1 of [10], (S1.2) denotes Eq. (1.2) of [10]. The small Latin indices $a, b, c, d, \ldots,$ belong to $\mathscr{I} = \{0, 1, 2, 3\}$ and the capital Latin indices $A, B, C, D, \ldots,$ belong to $\{1, 2\}$. We sum over repeated indices which are not taken in brackets (Einstein's convention). The number of elements in a set B is $\#B$ or $|B|$. The unit matrix with dimension N is denoted by $\mathbf{1}_N, \mathbf{1} = \mathbf{1}_2$. The Pauli matrices are given by

$$\sigma_0 = \mathbf{1}_2, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If V, W are vector spaces then $\tau_{VW} : V \otimes W \to W \otimes V$ is given by $\tau_{VW}(x \otimes y) = y \otimes x,\ x \in V, y \in W$. We often write τ instead of τ_{VW}. We denote $\mathbf{C}_* = \mathbf{C}\backslash\{0\}$, $\mathbf{R}_* = \mathbf{R}\backslash\{0\}$.

1. Quantum Poincaré Groups

In this section we define and (in almost all cases) classify quantum Poincaré groups as objects having the properties of the usual (spinorial) Poincaré group. The proofs of the results are shifted to Sect. 2.

The (connected component of) vectorial Poincaré group

$$\tilde{P} = SO_0(1,3) \rtimes \mathbf{R}^4 = \{(M, a) : M \in SO_0(1,3), a \in \mathbf{R}^4\}$$

has the multiplication $(M, a) \cdot (M', a') = (MM', a + Ma')$. By the Poincaré group we mean the spinorial Poincaré group (which is more important in quantum field theory than $\tilde{P}$)

$$P = SL(2, \mathbf{C}) \rtimes \mathbf{R}^4 = \{(g, a) : g \in SL(2, \mathbf{C}), a \in \mathbf{R}^4\}$$

with multiplication $(g,a) \cdot (g',a') = (gg', a + \lambda_g(a'))$, where the double covering $SL(2,\mathbf{C}) \ni g \to \lambda_g \in SO_0(1,3)$ is given by $\lambda_g(x)_i\sigma_i = g(x_j\sigma_j)g^+, g \in SL(2,\mathbf{C})$, $x \in \mathbf{R}^4$. The group homomorphism $\pi : P \ni (g,a) \to (\lambda_g, a) \in \tilde{P}$ is also a double covering. In particular, $(-\mathbf{1}_2, 0) \in P$ can be treated as rotation about 2π which is trivial in $\tilde{P}$ but nontrivial in P (it changes the sign of wave functions for fermions). Both P and $\tilde{P}$ act on Minkowski space $M = \mathbf{R}^4$ as follows $(g,a)x = (\lambda_g, a)x = \lambda_g x + a, g \in SL(2,\mathbf{C}),\ a,x \in \mathbf{R}^4$, and give affine maps preserving the scalar product in M (in a more abstract setting we would treat M as an affine space without distinguished 0). Let us consider continuous functions w_{AB}, p_i on P defined by

$$w_{AB}(g,a) = g_{AB}, \qquad p_i(g,a) = a_i\ .$$

We introduce the Hopf $*$-algebra $\mathrm{Poly}(P) = (\mathscr{B}, \Delta)$ of polynomials on the Poincaré group P as the $*$-algebra $\mathscr{B}$ with identity I generated by w_{AB} and p_i, $A,B = 1,2, i \in \mathscr{I}$ (according to the Introduction, $\mathscr{I} = \{0,1,2,3\}$ in this section) endowed with the comultiplication Δ given by $(\Delta f)(x,y) = f(x \cdot y), f \in \mathscr{B},\ x, y \in P(f^*(x) = \overline{f(x)})$. In particular,

$$\Delta w_{CD} = w_{CF} \otimes w_{FD}, \qquad \Delta p_i = p_i \otimes I + \Lambda_{ij} \otimes p_j\ , \tag{1.1}$$

$p_i^* = p_i$, where

$$\Lambda = V^{-1}(w \textcircled{\scriptsize T} \bar{w})V, \qquad V = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{1.2}$$

In order to prove (1.1) we notice that

$$\begin{aligned}(\Delta w_{CD})((g,a),(g',a')) &= w_{CD}(gg', a + \lambda_g(a')) = (gg')_{CD} = g_{CF}g'_{FD} \\ &= w_{CF}(g,a)w_{FD}(g',a') = (w_{CF} \otimes w_{FD})((g,a),(g',a'))\ , \\ (\Delta p_i)((g,a),(g',a')) &= p_i(gg', a + \lambda_g(a')) = a_i + \lambda_g(a')_i = a_i + (\lambda_g)_{ij}a'_j \\ &= p_i(g,a) + \Lambda_{ij}(g,a)p_j(g',a') = (p_i \otimes I + \Lambda_{ij} \otimes p_j)((g,a),(g',a'))\ ,\end{aligned}$$

where we used the formulae $(\sigma_i)_{CD} = V_{CD,i}$ and

$$\begin{aligned}V_{CD,i}(\lambda_g)_{ij} &= (\lambda_g)_{ij}(\sigma_i)_{CD} = (g\sigma_j g^+)_{CD} = g_{CE}(\sigma_j)_{EF}(g^+)_{FD} \\ &= w_{CE}(g,a)V_{EF,j}w_{DF}{}^*(g,a) = (w_{CE}\bar{w}_{DF}V_{EF,j})(g,a) = V_{CD,i}\Lambda_{ij}(g,a)\ .\end{aligned}$$

Since $\tau_{CD,EF} = \delta_{CF}\delta_{DE}$, we get

$$\bar{V} = \tau V \tag{1.3}$$

and $\bar{\Lambda} = \Lambda$. We put $p = (p_i)_{i \in \mathscr{I}}$. One can treat w_{CD} as continuous functions on the Lorentz group $L = SL(2,\mathbf{C})$ ($w_{CD}(g) = g_{CD},\ g \in L$). We define the Hopf $*$-algebra $\mathrm{Poly}(L) = (\mathscr{A}, \Delta)$ of polynomials on L as $*$-algebra with I generated by all w_{CD} endowed with Δ obtained by restriction of Δ for $\mathscr{B}$ to $\mathscr{A}$. Clearly w and Λ are representations of L. It is easy to check that

1. $\mathscr{B}$ is generated as an algebra by $\mathscr{A}$ and the elements $p_i,\ i \in \mathscr{I}$.
2. $\mathscr{A}$ is a Hopf $*$-subalgebra of $\mathscr{B}$.
3. $\mathscr{P} = \begin{pmatrix} \Lambda & p \\ 0 & I \end{pmatrix}$ is a representation where Λ is given by (1.2).

4. There exists $i \in \mathscr{I}$ such that $p_i \notin \mathscr{A}$.

5. $\Gamma\mathscr{A} \subset \Gamma$, where $\Gamma = \mathscr{A}X + \mathscr{A}$, $X = \operatorname{span}\{p_i : i \in \mathscr{I}\}$.

6. The left $\mathscr{A}$-module $\mathscr{A} \cdot \operatorname{span}\{p_i p_j, p_i, I : i,j \in \mathscr{I}\}$ has a free basis consisting of $10 + 4 + 1$ elements.

(5. and 6. follow from the relations $p_i a = a p_i$, $p_i p_j = p_j p_i$, $a \in \mathscr{A}$, and elementary computations, a free basis is given by $\{p_i p_j, p_i, I : i \leqq j, i,j \in \mathscr{I}\}$). According to [14], Poly(L) satisfies:

i. $(\mathscr{A}, \Delta)$ is a Hopf $*$-algebra such that $\mathscr{A}$ is generated (as a $*$-algebra) by matrix elements of a two-dimensional representation w;

ii. $w \mathbin{\text{Ⓣ}} w \cong I \oplus w^1$, where w^1 is a representation;

iii. the representation $w \mathbin{\text{Ⓣ}} \bar{w} \simeq \bar{w} \mathbin{\text{Ⓣ}} w$ is irreducible;

iv. if $\mathscr{A}', \Delta', w'$ satisfy i.–iii. and there exists a Hopf $*$-algebra epimorphism $\rho : \mathscr{A}' \to \mathscr{A}$ such that $\rho(w') = w$ then ρ is an isomorphism (the universality condition).

We say [14] that H is a quantum Lorentz group if Poly$(H) = (\mathscr{A}, \Delta)$ satisfies i.–iv.

Definition 1.1. *We say that G is a quantum Poincaré group if the Hopf $*$-algebra* Poly$(G) = (\mathscr{B}, \Delta)$ *satisfies the conditions* 1.–6. *for some quantum Lorentz group H with* Poly$(H) = (\mathscr{A}, \Delta)$ *and a representation w of H.*

Remark. 1.2. The condition 5. follows from $\mathscr{P} \mathbin{\text{Ⓣ}} w \simeq w \mathbin{\text{Ⓣ}} \mathscr{P}$, $\mathscr{P} \mathbin{\text{Ⓣ}} \bar{w} \simeq \bar{w} \mathbin{\text{Ⓣ}} \mathscr{P}$, while 6. is suggested by the requirement $W(\mathscr{P} \mathbin{\text{Ⓣ}} \mathscr{P}) = (\mathscr{P} \mathbin{\text{Ⓣ}} \mathscr{P})W$ for a "τ-like" matrix W (cf. Theorem 1.13). Moreover, the condition 4. is superfluous (it follows from the condition 6. and Proposition S0.1).

Remark. 1.3. Different choices of (H, w) can give a $*$-isomorphic $\mathscr{B}$.

Theorem 1.4. *Let G be a quantum Poincaré group,* Poly$(G) = (\mathscr{B}, \Delta)$. *Then $\mathscr{A}$ is linearly generated by matrix elements of irreducible representations of G, so $\mathscr{A}$ is uniquely determined. Moreover, we can choose w in such a way that $\mathscr{A}$ is the universal $*$-algebra generated by w_{AB}, $A, B = 1, 2$, satisfying*

$$(w \mathbin{\text{Ⓣ}} w)E = E\,, \tag{1.4}$$

$$E'(w \mathbin{\text{Ⓣ}} w) = E'\,, \tag{1.5}$$

$$X(w \mathbin{\text{Ⓣ}} \bar{w}) = (\bar{w} \mathbin{\text{Ⓣ}} w)X\,, \tag{1.6}$$

where $X = \tau Q'$ and

1) $E = e_1 \otimes e_2 - e_2 \otimes e_1$, $E' = -e^1 \otimes e^2 + e^2 \otimes e^1$,

$$Q' = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}, \quad 0 < t \leqq 1, \quad \textit{or}$$

2)

$$E, E' \textit{ as above}, \ Q' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textit{or}$$

3) $E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1,\ E' = -e^1 \otimes e^2 + e^2 \otimes e^1 + e^2 \otimes e^2,$

$$Q' = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad r \geqq 0, \quad \textit{or}$$

4)

$$E, E' \textit{ as above, } Q' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textit{or}$$

5) $E = e_1 \otimes e_2 + e_2 \otimes e_1,\ E' = e^1 \otimes e^2 + e^2 \otimes e^1,$

$$Q' = i \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}, \quad 0 < t \leqq 1, \quad \textit{or}$$

6)

$$E, E' \textit{ as above, } Q' = i \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textit{or}$$

7)

$$E, E' \textit{ as above, } Q' = i \begin{bmatrix} r & 0 & 0 & s \\ 0 & -r & s & 0 \\ 0 & s & -r & 0 \\ s & 0 & 0 & r \end{bmatrix},$$

$$r = (t + t^{-1})/2, \quad s = (t - t^{-1})/2, \quad 0 < t < 1\,,$$

$e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$, $e^1 = (1 \quad 0)$, $e^2 = (0 \quad 1)$. *Moreover, all the above triples* (E, E', Q') *give nonisomorphic* $(\mathscr{A}, \Delta)$. *We can (and will) choose* p_i *in such a way that* $p_i^* = p_i$.

In the following we assume that G is a quantum Poincaré group, $\mathrm{Poly}(G) = (\mathscr{B}, \Delta)$ and w, p are as in Theorem 1.4. We set $q = q^{1/2} = 1$ in the cases 1)–4), $q = -1$, $q^{1/2} = i$ in the cases 5)–7), $s = \pm 1$, $L = sq^{1/2}(\mathbf{1}^{\otimes 2} + q^{-1}EE')$, $\tilde{L} = q\tau L\tau$, $G = (V^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes X)(L \otimes \mathbf{1})(\mathbf{1} \otimes V)$, $\tilde{G} = (V^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{L})(X^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes V)$, $R = (V^{-1} \otimes V^{-1})(\mathbf{1} \otimes X \otimes \mathbf{1})(L \otimes \tilde{L})(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})(V \otimes V)$.

Theorem 1.5. *$\mathscr{B}$ is the universal $*$-algebra with I generated by w_{AB} and p_i satisfying* (1.4), (1.5), (1.6) *and*

$$p_i a = (a * f_{ij}) p_j + a * \eta_i - \Lambda_{ij}(\eta_j * a), \quad a \in \mathscr{A}\,, \tag{1.7}$$

$$(R - \mathbf{1}^{\otimes 2})_{kl,ij}(p_i p_j - \eta_i(\Lambda_{js}) p_s + T_{ij} - \Lambda_{im}\Lambda_{jn}T_{mn}) = 0\,, \tag{1.8}$$

$$p_i^* = p_i\,, \tag{1.9}$$

where $f=(f_{ij})_{i,j\in\mathscr{I}}, \eta=(\eta_i)_{i\in\mathscr{I}}$ *and* $T=(T_{ij})_{i,j\in\mathscr{I}}$ *are uniquely determined by* $s=\pm1$, $H_{EFCD}, T_{EFCD}\in\mathbf{C}$ *and the following properties*:

a) $\mathscr{A}\ni a\to\rho(a)=\begin{pmatrix} f(a) & \eta(a)\\ 0 & \varepsilon(a)\end{pmatrix}\in M_5(\mathbf{C})$ *is a unital homomorphism*,

b) $\rho(a^*)=\overline{\rho(S(a))},\ a\in\mathscr{A}$,

c) $f_{ij}(w_{CD})=G_{iC,Dj},\ \eta_i(w_{CD})=V^{-1}_{i,EF}H_{EFCD},\ T_{ij}=(V^{-1}\otimes V^{-1})_{ij,EFCD}T_{EFCD}$.

The $*$*-Hopf structure in* Poly(G) *is determined by*:

$$\begin{aligned}
&\Delta w=w\oplus w, \qquad \Delta\bar{w}=\bar{w}\oplus\bar{w}, \qquad \Delta p=p\oplus I+\Lambda\oplus p\,,\\
&\varepsilon(w)=\mathbf{1}, \qquad \varepsilon(\bar{w})=\mathbf{1}, \qquad \varepsilon(p)=0\,,\\
&S(w)=w^{-1}, \qquad S(\bar{w})=\bar{w}^{-1}, \qquad S(p)=-\Lambda^{-1}p\,.
\end{aligned}$$

Quantum Poincaré groups corresponding to different s are nonisomorphic.

Theorem 1.6. *For each case in Theorem* 1.4 *and each s* (*except the case* 1), $s=1,\ t=1$ *and the case* 5), $s=\pm1,\ t=1$) *we list H and T giving* (*via formulae in Theorem* 1.5) *all nonisomorphic quantum Poincaré groups G*:

1) $s=-1,\ t=1$:

$$\left.\begin{aligned} H_{EFCD}&=0\,,\\ T_{EFCD}&=V_{EF,i}V_{CD,j}T_{ij}\,,\end{aligned}\right\}, \tag{1.10}$$

where

a) $T_{03}=-T_{30}=ia,\ T_{12}=-T_{21}=ib$, *other* T_{ij} *equal* 0, $a=\cos\phi,\ b=\sin\phi$(*one parameter family for* $0\leqq\phi\leqq\pi/2$) *or*

b) $T_{02}=T_{12}=i,\ T_{20}=T_{21}=-i$, *other* T_{ij} *equal* 0, *or*

c) *all* T_{ij} *equal* 0.

1) $s=\pm1, 0<t<1$:

$$\left.\begin{aligned} &T_{1122}=ia, \qquad T_{1221}=b\,,\\ &T_{2112}=-b, \qquad T_{2211}=-ia\,,\\ &\textit{all } H_{EFCD} \textit{ and other } T_{EFCD} \textit{ equal } 0 \textit{ and}\end{aligned}\right\} \tag{1.11}$$

a) $a=\cos\phi,\ b=\sin\phi$ (*one parameter family for* $0\leqq\phi<\pi$) *or*

b) $a=b=0$.

2) $s=1$:

the first case:

$$\left.\begin{aligned} &H_{1111}=-(a+bi), \qquad H_{1122}=a+bi, \qquad H_{2112}=-2bi\,,\\ &T_{2111}=c-di, \qquad T_{1211}=-c-di\,,\\ &T_{1121}=-c+di, \qquad T_{1112}=c+di\,,\\ &\textit{other } H_{EFCD} \textit{ and } T_{EFCD} \textit{ equal } 0 \textit{ and}\end{aligned}\right\} \tag{1.12}$$

a) $a=1,\ c=d=0$ (*one parameter family for* $b\in\mathbf{R}$) *or*

b) $a=0,\ b=1,\ d=0$ (*one parameter family for* $c\geqq0$);

the second case:

$$\left.\begin{aligned} H_{1212} = a+bi, \quad T_{2112} = (a^2+b^2)/2, \quad T_{2111} = c-di\,, \\ T_{1221} = -(a^2+b^2)/2, \quad T_{1211} = -c-di, \quad T_{1121} = -c+di\,, \\ T_{1112} = c+di, \quad T_{1111} = -(a^2+b^2)/2\,, \\ \textit{other } H_{EFCD} \textit{ and } T_{EFCD} \textit{ equal } 0 \textit{ and} \end{aligned}\right\} \qquad (1.13)$$

a) $a = 1,\ b = 0,\ c = r\cos\phi,\ d = r\sin\phi$ (*two parameter family for* $r > 0$, $0 \leqq \phi < \pi/2$ *or* $r = \phi = 0$) *or*
b) $a = b = 0,\ c = 1,\ d = 0$, *or*
c) $a = b = c = d = 0$.

2) $s = -1$, (1.12) *and*
a) $a = b = 0,\ c = 1,\ d = 0$, *or*
b) $a = b = c = d = 0$.

3) $s = \pm 1,\ r \geqq 0$, *all* H_{EFCD} *and* T_{EFCD} *equal* 0.

4) $s = 1$,

$$\left.\begin{aligned} H_{2212} = -2bi, \quad H_{2122} = -bi, \quad H_{2112} = a-bi\,, \\ H_{2111} = bi, \quad H_{1222} = bi, \quad H_{1212} = a, \quad H_{1211} = -bi\,, \\ H_{1121} = -2bi, \quad H_{1112} = 3bi/4, \quad H_{1111} = -4bi\,, \\ T_{1112} = 9b^2/8 + 3abi/2, \quad T_{1121} = -9b^2/8 + 3abi/2\,, \\ T_{1211} = -9b^2/8 - 3abi/2, \quad T_{1221} = 3b^2/2\,, \\ T_{2111} = 9b^2/8 - 3abi/2, \quad T_{2112} = -3b^2/2\,, \\ \textit{other } H_{EFCD} \textit{ and } T_{EFCD} \textit{ equal } 0 \textit{ and} \end{aligned}\right\} \qquad (1.14)$$

a) $a = \cos\phi,\ b = \sin\phi$ (*one parameter family for* $0 \leqq \phi < \pi$) *or*
b) $a = b = 0$.

4) $s = -1$, *all* H_{EFCD} *and* T_{EFCD} *equal 0.*
5) $s = \pm 1,\ 0 < t < 1$,

$$\left.\begin{aligned} T_{1122} = ia, \quad T_{1221} = b, \quad T_{2112} = -b, \quad T_{2211} = -ia\,, \\ \textit{all } H_{EFCD} \textit{ and other } T_{EFCD} \textit{ equal } 0 \textit{ and} \end{aligned}\right\} \qquad (1.15)$$

a) $a = \cos\phi, b = \sin\phi$ (*one parameter family for* $0 \leqq \phi < \pi$) *or*
b) $a = b = 0$.

6) $s = 1$, *all* H_{EFCD} *and* T_{EFCD} *equal* 0.
6) $s = -1$:
the first case:

$$\left.\begin{aligned} H_{1111} = -(a+bi), \quad H_{1122} = a+bi, \quad H_{2112} = -2bi \\ \textit{other } H_{EFCD} \textit{ and all } T_{EFCD} \textit{ equal } 0 \textit{ and} \end{aligned}\right\} \qquad (1.16)$$

a) $a=\cos\phi,\ b=\sin\phi$ (*one parameter family for* $0\leqq\phi<\pi$) *or*
b) $a=b=0$;

the second case:

$$\left.\begin{array}{cc} H_{1212}=a+bi, & T_{1111}=-(a^2+b^2)/2\,, \\ T_{1221}=-(a^2+b^2)/2, & T_{2112}=(a^2+b^2)/2\,, \\ \multicolumn{2}{r}{\textit{other } H_{EFCD} \textit{ and } T_{EFCD} \textit{ equal } 0 \textit{ and}} \end{array}\right\} \tag{1.17}$$

$a=1,\ b=0.$

7) $s=\pm1,\ 0<t<1$, *all* H_{EFCD} *and* T_{EFCD} *equal* 0.

Remark. 1.7. The classical Poincaré group is obtained in the case 1), $s=1$, $t=1,\ H=0,\ T=0$. The quantum Poincaré group of [4] corresponds (in spinorial setting) to 1), $s=1,\ t=1$,

$$H_{1111}=-H_{1122}=\frac{1}{2}H_{1221}=\frac{1}{2}H_{2112}=-H_{2211}=H_{2222}=ih/2, \qquad h\in\mathbf{R}\,,$$

other H_{EFCD} and all T_{EFCD} equal 0. The quantum Poincaré group of [1] corresponds to 1), $s=1,\ t>0,\ H=0,\ T=0$ (t is denoted by q there). The so called soft deformations correspond to 1), $s=\pm1,\ t=1,\ H=0,\ T_{ab}=-T_{ba}\in i\mathbf{R}$.

Remark. 1.8. In the remaining cases 1), $s=1,\ t=1$ and 5), $s=\pm1,\ t=1$, one can consider T_{mn} defined as in Theorem 1.5 and

$$Z_{ij,k}=\eta_i(\Lambda_{jk})=V^{-1}_{i,AB}\,V^{-1}_{j,CD}(H_{ABCE}\delta_{DF}-\overline{H_{BADF}}\delta_{CE})V_{EF,k}$$

(then $H_{ABCE}=\frac{1}{2}V_{AB,i}V_{CD,j}Z_{ij,k}V^{-1}_{k,ED}$). In the case 1), $s=1, t=1$ a pair (Z,T) corresponds to a quantum Poincaré group if and only if

$$T_{mn}=-T_{nm}\in i\mathbf{R}, \qquad Z_{ij,s}g_{sk}=-Z_{ik,s}g_{sj}\in i\mathbf{R}\,, \tag{1.18}$$

$$\left.\begin{array}{l} \{[(\tau-\mathbf{1}^{\otimes2})\otimes\mathbf{1}][(\mathbf{1}\otimes Z)Z-(Z\otimes\mathbf{1})Z]\}_{ijm,n} \\ \quad =-\frac{1}{4}t_0(\delta_{in}g_{jm}-\delta_{jn}g_{im}), \qquad t_0\in\mathbf{R}\,, \\ \qquad\qquad A_3(Z\otimes\mathbf{1})T=0 \end{array}\right\} \tag{1.19}$$

where $g_{00}=1, g_{11}=g_{22}=g_{33}=-1$, other $g_{ij}=0$,

$$\begin{aligned} A_3 &= \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}-\tau\otimes\mathbf{1}-\mathbf{1}\otimes\tau+(\tau\otimes\mathbf{1})(\mathbf{1}\otimes\tau) \\ &\quad +(\mathbf{1}\otimes\tau)(\tau\otimes\mathbf{1})-(\tau\otimes\mathbf{1})(\mathbf{1}\otimes\tau)(\tau\otimes\mathbf{1}) \end{aligned}$$

is the classical (not normalized) antisymmetrizer. In the case 5), $s=\pm1,\ t=1$ in addition to these conditions we assume

$$T_{i_1i_2}=0 \quad \text{for } \#\{k: i_k\in\{1,2\}\}=1\,,$$

$$Z_{i_1i_2,i_3}=0 \quad \text{for } (-1)^{\#\{k:i_k\in\{1,2\}\}}=s\,,$$

and get in that way all quantum Poincaré groups (up to an isomorphism but not necessarily nonisomorphic).

Let us set $\eta=g,\ a=-iT_{mn}e_m\wedge e_n,\ b=-iZ_{ij,s}g_{sk}e_i\wedge\Omega_{j,k}$ and $c=0$ (see [16]). Then (1.19) (using (1.18)) is equivalent to (3)–(4) of [16] where t_0 is identified with t of (3)–(4) of [16]. Thus the table in [16] gives many examples of quantum Poincaré groups (cf. also the remarks at the end of [16]). The proofs of these statements involve the above formulae and the results obtained in the proof of Theorem 1.6 (with $\lambda = -\frac{1}{2}t_0$ in the case 1), $s = 1$, $t = 1$ and $\lambda = -\frac{1}{2}it_0$ in the case 5), $s = \pm 1$, $t = 1$).

We denote by d_n the number of monomials of n^{th} degree in 4 variables,

$$d_n = \#\{(a,b,c,d) \in \mathbf{N}^{\otimes 4} : a+b+c+d = n\} .$$

Theorem 1.9. *Let $\mathscr{B}$ correspond to a quantum Poincaré group G and $\mathscr{A}, w, p$ be as in Theorem* 1.4. *We set*

$$\mathscr{B}^N = \mathscr{A} \cdot span\{p_{i_1} \cdot \cdots \cdot p_{i_n} : i_1, \ldots, i_n \in \mathscr{I}, \qquad n = 0, 1, \ldots, N\} .$$

Then $\mathscr{B}^N$ is a free left $\mathscr{A}$-module and $\dim_{\mathscr{A}} \mathscr{B}^N = \sum_{n=0}^{N} d_n$.

We denote by $l : P \times M \to M$ the action of Poincaré group on Minkowski space, $\mathscr{C} = \mathrm{Poly}(M)$ denotes the unital algebra generated by coordinates $x_i (i \in \mathscr{I})$ of the Minkowski space $M = \mathbf{R}^4$. The only relations in $\mathscr{C}$ are $x_i x_j = x_j x_i$. The coaction $\Psi : \mathscr{C} \to \mathscr{A} \otimes \mathscr{C}$ and $*$ in $\mathscr{C}$ are given by $(\Psi f)(x,y) = f(l(x,y)), f^*(y) = \overline{f(y)}$, $x \in P, y \in M$.

Let $x = (g,a) \in P, y \in M, f \in \mathscr{C}$. One has

$$\begin{aligned}(\Psi x_i)((g,a),y) &= x_i(\lambda_g y + a) = (\lambda_g y)_i + a_i = (\lambda_g)_{ij} y_j + a_i \\ &= \Lambda_{ij}(g,a) x_j(y) + p_i(g,a) = (\Lambda_{ij} \otimes x_j + p_i \otimes I)((g,a),y), \text{ hence}\end{aligned}$$

$$\Psi x_i = \Lambda_{ij} \otimes x_j + p_i \otimes I . \tag{1.20}$$

One gets

6) $\mathscr{C}$ is a unital $*$-algebra generated by $x_i, i \in \mathscr{I}$, and $\Psi : \mathscr{C} \to \mathscr{B} \otimes \mathscr{C}$ is a unital $*$-homomorphism such that $(\varepsilon \otimes \mathrm{id})\Psi = \mathrm{id}, (\mathrm{id} \otimes \Psi)\Psi = (\Delta \otimes \mathrm{id})\Psi$, $x_i^* = x_i$ and (1.20) holds.

Let $\Psi W \subset \mathscr{A} \otimes W$ for a linear subspace $W \subset \mathscr{C}, f \in W, y, a \in \mathbf{R}^4$. Then $f(y+a) = f(l((e,a),y)) = (\Psi f)((e,a),y) = (\Psi f)((e,0),y) = f(l((e,0),y)) = f(y)$ ($k(e,a) = k(e,0)$ for $k \in \mathscr{A}$), $f = f(0)I \in \mathbf{C}I$ (in fact we have used the translation homogeneity of M). Therefore

7) if $\Psi W \subset \mathscr{A} \otimes W$ for a linear subspace $W \subset \mathscr{C}$ then $W \subset \mathbf{C}I$.

Let us consider $(\mathscr{C}', \Psi')$ which also satisfies 6)–7) for some $x_i' \in \mathscr{C}'$. Then

$$\Psi(x_i' x_l' - x_l' x_i') = \Lambda_{ij} \Lambda_{lm} \otimes (x_j' x_m' - x_m' x_j') .$$

Setting $W = \mathrm{span}\{x_i' x_l' - x_l' x_i' : i, l \in \mathscr{I}\}$ and using 7), one gets $x_i' x_l' - x_l' x_i' = a_{il} I$, $a_{il} \in \mathbf{C}$. Thus $a = (a_{il})_{i,l \in \mathscr{I}}$ is an invariant vector of $\Lambda \oplus \Lambda$, i.e. $a = c \cdot g$, where $c \in \mathbf{C}, g_{00} = 1, g_{11} = g_{22} = g_{33} = -1, g_{ij} = 0$ for $i \neq j$. But $a_{il} = -a_{li}$, hence $c = 0, x_i' x_l' = x_l' x_i'$ and we fix the proper choice of $(\mathscr{C}, \Psi)$ by means of

8) if $(\mathscr{C}', \Psi')$ also satisfies 6)–7) for some $x_i' \in \mathscr{C}'$, then there exists a unital $*$-homomorphism $\rho : \mathscr{C} \to \mathscr{C}'$ such that $\rho(x_i) = x_i'$ and $(\mathrm{id} \otimes \rho)\Psi = \Psi' \rho$ (universality of $(\mathscr{C}, \Psi)$).

Definition 1.10. *We say that $(\mathscr{C}, \Psi)$ describes a quantum Minkowski space associated with a quantum Poincaré group G,* $\mathrm{Poly}(G) = (\mathscr{B}, \Delta)$, *if* 6)–8) *are satisfied.*

Remark. 1.11. This definition doesn't depend on the choice of Λ (see Proposition S5.7).

Theorem 1.12. *Let G be a quantum Poincaré group with w, p as in Theorem 1.4. Then there exists a unique (up to a $*$-isomorphism) pair $(\mathscr{C}, \Psi)$ describing associated Minkowski space:*

$\mathscr{C}$ is the universal unital $*$-algebra generated by $x_i, i = 0,1,2,3,$ satisfying $x_i^* = x_i$ and

$$(R - \mathbf{1}^{\otimes 2})_{ij,kl}(x_k x_l - \eta_k(\Lambda_{lm})x_m + T_{kl}) = 0\,, \tag{1.21}$$

and Ψ is given by (1.20). *Moreover,*

$$\dim \mathscr{C}^N = \sum_{n=0}^{N} d_n\,, \tag{1.22}$$

where $\mathscr{C}^N = span\{x_{i_1} \cdot \cdots \cdot x_{i_n} : i_1, \ldots, i_n \in \mathscr{I},\ n = 0,1,\ldots,N\}$.

We set $m = (V^{-1} \otimes V^{-1})(\mathbf{1} \otimes X \otimes \mathbf{1})(E \otimes \tau E), Z_{ij,k} = \eta_i(\Lambda_{jk})$,

$$R_P = \begin{pmatrix} R & Z & -R \cdot Z & (R - \mathbf{1}^{\otimes 2})T \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad m_P = \begin{pmatrix} 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.23}$$

Theorem 1.13. *Let G be a quantum Poincaré group with w, p as in Theorem 1.4. Then*

1) $\mathrm{Mor}(\mathscr{P} \oplus \mathscr{P}, \mathscr{P} \oplus \mathscr{P}) = \mathbf{C}\mathrm{id} \oplus \mathbf{C}R_P \oplus \mathbf{C}m_p$.

2) *Let us consider the cases listed in Theorem 1.6. Then $W \in \mathrm{Mor}(\mathscr{P} \oplus \mathscr{P}, \mathscr{P} \oplus \mathscr{P})$ and*

$$(W \otimes \mathbf{1})(\mathbf{1} \otimes W)(W \otimes \mathbf{1}) = (\mathbf{1} \otimes W)(W \otimes \mathbf{1})(\mathbf{1} \otimes W)\,, \tag{1.24}$$

if and only if

a) $W = x \cdot \mathrm{id}$ $(x \in \mathbf{C}_*)$ *or*

b) $W = y \cdot R_P + z \cdot m_P$ $(y, z \in \mathbf{C}$, *for* 4), $s = 1$, $b \neq 0$ *one must have* $y = 0)$.

Those W are invertible if and only if we have the case a) or b) with $y \neq 0$.

2. Proof of the Classification

In this section we prove the theorems of Sect. 1.

Let H be a quantum Lorentz group, i.e. $\mathrm{Poly}(H) = (\mathscr{A}, \Delta)$ satisfies the conditions i.–iv. of Sect. 1. According to [14], we can choose w in such a way that $\mathscr{A}$ is the univeral $*$-algebra generated by $w_{AB}, A, B = 1,2$, satisfying (1.4)–(1.6), where $X = \tau Q', Q' = \alpha Q$ and

1) $E = e_1 \otimes e_2 - q e_2 \otimes e_1, E' = -q^{-1} e^1 \otimes e^2 + e^2 \otimes e^1$, Q is given by (13)–(19) of [14], $q \in \mathbf{C}\backslash\{0, i, -i\}$, or

2) $E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1$, $E' = -e^1 \otimes e^2 + e^2 \otimes e^1 + e^2 \otimes e^2$, Q is given by (20)–(21) of [14], we set $q = 1$ in that case,

$e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}, e^1 = (1,0), e^2 = (0,1)$ (due to the remarks before formula (1) in [14], $E'E \neq 0$, which means $q \neq \pm i$). In all these cases X is invertible, Δ is given by $\Delta w_{ij} = w_{ik} \otimes w_{kj}$ and $(\mathscr{A}, \Delta)$ corresponds to a quantum Lorentz group.

The numbers $\alpha \neq 0$ are not essential now and are chosen in such a way that

$$(X \otimes \mathbf{1})(\mathbf{1} \otimes X)(E \otimes \mathbf{1}) = \mathbf{1} \otimes E \tag{2.1}$$

(see (5) of [14]). Then (we use (6) of [14] and direct computations)

$$\tau \bar{X} \tau = \beta^{-1} X \tag{2.2}$$

for some $\beta \in \{1, -1, i, -i\}$. We set $\tilde{E} = \tau \bar{E} \in \mathrm{Mor}(I, \bar{w} \oplus \bar{w}), \tilde{E}' = \overline{E'}\tau \in \mathrm{Mor}(\bar{w} \oplus \bar{w}, I)$, where $e_i \otimes e_j, e^j \otimes e^i, i, j = 1, 2$, are treated as reals. Using (2.1), (2.2) and

$$(E' \otimes \mathbf{1})(\mathbf{1} \otimes E) = \mathbf{1}, \qquad (\mathbf{1} \otimes E')(E \otimes \mathbf{1}) = \mathbf{1}$$

(see (3) of [14]; matrices of E' and E are inverse one to another), one obtains (using e.g. diagram notation)

$$(\mathbf{1} \otimes X^{-1})(X^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes E) = E \otimes \mathbf{1}\,, \tag{2.3}$$

$$\beta^{-2}(\mathbf{1} \otimes X)(X \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{E}) = \tilde{E} \otimes \mathbf{1}\,, \tag{2.4}$$

$$\beta^{2}(X^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes X^{-1})(\tilde{E} \otimes \mathbf{1}) = \mathbf{1} \otimes \tilde{E}\,, \tag{2.5}$$

$$(\mathbf{1} \otimes E')(X \otimes \mathbf{1})(\mathbf{1} \otimes X) = E' \otimes \mathbf{1}\,, \tag{2.6}$$

$$(E' \otimes \mathbf{1})(\mathbf{1} \otimes X^{-1})(X^{-1} \otimes \mathbf{1}) = \mathbf{1} \otimes E'\,, \tag{2.7}$$

$$\beta^{-2}(\tilde{E}' \otimes \mathbf{1})(\mathbf{1} \otimes X)(X \otimes \mathbf{1}) = \mathbf{1} \otimes \tilde{E}'\,, \tag{2.8}$$

$$\beta^{2}(\mathbf{1} \otimes \tilde{E}')(X^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes X^{-1}) = \tilde{E}' \otimes \mathbf{1}\,. \tag{2.9}$$

Proposition 2.1 (cf. Theorem 6.3 of [8], Remark 2 on page 229 of [14]). *Let $q \in \mathbf{C} \backslash \{0$, roots of unity$\}$ (we treat $q = \pm 1$ as* **not** *a root of unity). Then*

1) *there exist representations $w^s (s \in \mathbf{N}/2)$ of H such that $w^0 = I$, $w^{1/2} = w$, $\dim w^s = 2s + 1$ and*

$$w^s \oplus w^{s'} \simeq w^{|s-s'|} \oplus w^{|s-s'|+1} \oplus \cdots \oplus w^{s+s'} \quad (s, s' \in \mathbf{N}/2)\,.$$

2) *$w^s \oplus \overline{w^{s'}} (s, s' \in \mathbf{N}/2)$ are all unequivalent irreducible representations of H.*

3) *$w^s \oplus \overline{w^{s'}} \simeq \overline{w^{s'}} \oplus w^s (s, s' \in \mathbf{N}/2)$.*

4) *Each representation of H is completely reducible.*

Proof. Let $\mathscr{A}_{\mathrm{hol}}$ be the subalgebra of $\mathscr{A}$ generated by matrix elements of w. Then $\mathrm{Poly}(H_{\mathrm{hol}}) = (\mathscr{A}_{\mathrm{hol}}, \Delta|_{\mathscr{A}_{\mathrm{hol}}})$ is a Hopf subalgebra of $\mathrm{Poly}(H) = (\mathscr{A}, \Delta)$. According to Proposition 4.1.1 of [14], $\mathscr{A}_{\mathrm{hol}}$ is the universal algebra generated by matrix elements of w satisfying the relations (1.4) and (1.5). Due to Theorem 4.2 of [13] and the facts given in cases I, III of the Introduction to [13] (cf. (1.9), (1.30) and Theorem 1.15 of [3]), 1) holds and matrix elements of $w^s (s \in \mathbf{N}/2)$ form a linear basis of $\mathscr{A}_{\mathrm{hol}}$. Using Proposition 4.1.2–3 of [14], matrix elements of $w^s \oplus \overline{w^{s'}} (s, s' \in \mathbf{N}/2)$ form a linear basis of $\mathscr{A}$. Now Proposition 4.1 of [13] (see also Proposition A.2 of [7]) gives 2) and 4). The condition iii. of Sect. 1 implies $(\mathrm{Tr}\ w)(\mathrm{Tr}\ \bar{w}) = (\mathrm{Tr}\ \bar{w})(\mathrm{Tr}\ w)$. That and 1) give that $\mathrm{Tr}\ v (v \in \mathrm{Irr}\ H)$ commute among themselves. By virtue of Proposition B.4 of [3] (cf. also Proposition 5.11 of [12]), one obtains 3). □

Proof of Theorem 1.4. We have the Hopf $*$-algebra $\mathscr{B}$, its Hopf $*$-subalgebra $\mathscr{A}$ and two-dimensional representation w of $\mathscr{A}$ which satisfy the conditions i.–iv., (1.2) and 1.–6. of Sect. 1. We shall use the results of Sect. 1 of [10] with A replaced by $\mathscr{L} = w \oplus \bar{w}$, $\mathscr{L}_{AB,CD} = w_{AC} w_{BD}{}^*$. Hence we deal with $p_{AB} = V_{AB,i} p_i$ instead of p_i. By virtue of (S1.3), it suffices to check (S1.5) for the generators: $a = w_{AB}$ or $a = w_{AB}{}^*$. Inserting such a into (S1.5), we get

$$G_{\mathscr{L}} \in \mathrm{Mor}(w \oplus w \oplus \bar{w}, w \oplus \bar{w} \oplus w), \quad \tilde{G}_{\mathscr{L}} \in \mathrm{Mor}(\bar{w} \oplus w \oplus \bar{w}, w \oplus \bar{w} \oplus \bar{w}),$$

where

$$(G_{\mathscr{L}})_{ABC,DEF} = f_{AB,EF}(w_{CD}), \quad (\tilde{G}_{\mathscr{L}})_{ABC,DEF} = f_{AB,EF}(w_{CD}{}^*)\,. \tag{2.10}$$

Thus $G_{\mathscr{L}} = (\mathbf{1} \otimes X)A$, $\tilde{G}_{\mathscr{L}} = B(X^{-1} \otimes \mathbf{1})$, where A is an intertwiner of $w \oplus w \oplus \bar{w} \simeq w^1 \oplus \bar{w} \oplus \bar{w}$, B is an intertwiner of $w \oplus \bar{w} \oplus \bar{w} \simeq w \oplus \overline{w^1} \oplus w$. But $w^1 \oplus \bar{w}, \bar{w}, w \oplus \overline{w^1}, w$ are irreducible (we use Propositions 4.1 and 4.2 of [14]), hence

$$\mathrm{Mor}(w \oplus w \oplus \bar{w}, w \oplus w \oplus \bar{w}) = \mathbf{C}EE' \otimes \mathbf{1} \oplus \mathbf{C}\mathbf{1}^{\otimes 3}\,,$$
$$\mathrm{Mor}(w \oplus \bar{w} \oplus \bar{w}, w \oplus \bar{w} \oplus \bar{w}) = \mathbf{C}\mathbf{1} \otimes \tilde{E}\tilde{E'} \oplus \mathbf{C}\mathbf{1}^{\otimes 3}\,.$$

Therefore $A = L \otimes \mathbf{1}$, $B = \mathbf{1} \otimes \tilde{L}$, where

$$L = a\mathbf{1}^{\otimes 2} + bEE', \qquad \tilde{L} = \tilde{a}\mathbf{1}^{\otimes 2} + \tilde{b}\tilde{E}\tilde{E}', \qquad a, \tilde{a}, b, \tilde{b} \in \mathbf{C}\,. \tag{2.11}$$

According to (S1.3), $f : \mathscr{A} \to M_4(\mathbf{C})$ should be a unital homomorphism. It means that f should preserve the relations (1.4), (1.4)*, (1.5), (1.5)*, (1.6) ((1.4)* denotes the relation conjugated to (1.4) etc.), i.e.

$$(G_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes G_{\mathscr{L}})(E \otimes \mathbf{1}^{\otimes 2}) = \mathbf{1}^{\otimes 2} \otimes E\,, \tag{2.12}$$

$$(\tilde{G}_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{G}_{\mathscr{L}})(\tilde{E} \otimes \mathbf{1}^{\otimes 2}) = \mathbf{1}^{\otimes 2} \otimes \tilde{E}\,, \tag{2.13}$$

$$(\mathbf{1}^{\otimes 2} \otimes E')(G_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes G_{\mathscr{L}}) = E' \otimes \mathbf{1}^{\otimes 2}\,, \tag{2.14}$$

$$(\mathbf{1}^{\otimes 2} \otimes \tilde{E'})(\tilde{G}_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{G}_{\mathscr{L}}) = \tilde{E'} \otimes \mathbf{1}^{\otimes 2}\,, \tag{2.15}$$

$$(\tilde{G}_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes G_{\mathscr{L}})(X \otimes \mathbf{1}^{\otimes 2}) = (\mathbf{1}^{\otimes 2} \otimes X)(G_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{G}_{\mathscr{L}})\,. \tag{2.16}$$

Using (2.3), (2.5), (2.6) and (2.8), Eqs. (2.12)–(2.15) are equivalent to

$$(L \otimes \mathbf{1})(\mathbf{1} \otimes L)(E \otimes \mathbf{1}) = \mathbf{1} \otimes E\,, \tag{2.17}$$

$$\beta^{-2}(\tilde{L} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{L})(\tilde{E} \otimes \mathbf{1}) = \mathbf{1} \otimes \tilde{E}\,, \tag{2.18}$$

$$(\mathbf{1} \otimes E')(L \otimes \mathbf{1})(\mathbf{1} \otimes L) = E' \otimes \mathbf{1}\,, \tag{2.19}$$

$$\beta^{-2}(\mathbf{1} \otimes \tilde{E'})(\tilde{L} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{L}) = \tilde{E'} \otimes \mathbf{1}\,. \tag{2.20}$$

Using (2.11), computing $a, b, \tilde{a}, \tilde{b}$, and inserting them into (2.11), one gets that the solutions of (2.17)–(2.20) are

$$L = L_i, \qquad \tilde{L} = \beta\tau\overline{L_j^{-1}}\tau, \qquad i, j = 1, 2, 3, 4\,, \tag{2.21}$$

where

$$L_i = q_i(\mathbf{1}^{\otimes 2} + q_i^{-2}EE')\,, \tag{2.22}$$

$q_{1,2} = \pm q^{\frac{1}{2}}, q_{3,4} = \pm q^{-\frac{1}{2}}$. Using these relations, (2.1), (2.4), (2.6) and (2.8), we get that (2.16) is satisfied. Therefore, the solutions of (S1.3), (S1.5) are given by (2.10), where

$$G_{\mathscr{L}} = (\mathbf{1} \otimes X)(L \otimes \mathbf{1}), \qquad \tilde{G}_{\mathscr{L}} = (\mathbf{1} \otimes \tilde{L})(X^{-1} \otimes \mathbf{1})\,, \tag{2.23}$$

$L, \tilde{L}$ are given by (2.21)–(2.22) (in general 16 solutions). Moreover,

$$\begin{aligned}(R_{\mathscr{L}})_{ABCD,EFGH} &= f_{AB,GH}(\mathscr{L}_{CD,EF}) = f_{AB,MN}(w_{CE})f_{MN,GH}(w_{DF}{}^{*})\\ &= (G_{\mathscr{L}})_{ABC,EMN}(\tilde{G}_{\mathscr{L}})_{MND,FGH}\,,\end{aligned}$$

$$R_{\mathscr{L}} = (G_{\mathscr{L}} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{G}_{\mathscr{L}}) = (\mathbf{1} \otimes X \otimes \mathbf{1})(L \otimes \tilde{L})(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})\,. \tag{2.24}$$

We know that $\dot{p}_{AB} \otimes \dot{p}_{CD}$ form a basis of $(\dot{\Gamma}_2)_{\text{inv}}$ which transforms under Δ_{2L} according to $\mathscr{L} \circledT \mathscr{L}$. It is easy to check that the decomposition into irreducible unequivalent components

$$\mathscr{L} \circledT \mathscr{L} \simeq w \circledT w \circledT \bar{w} \circledT \bar{w} \simeq w^1 \circledT \overline{w^1} \oplus w^1 \oplus \overline{w^1} \oplus I$$

corresponds to

$$(\dot{\Gamma}_2)_{\text{inv}} = W_{1\bar{1}} \oplus W_1 \oplus W_{\bar{1}} \oplus W_0\,, \tag{2.25}$$

where

$$\begin{aligned}W_{1\bar{1}} &= \text{span}\{(\phi \otimes \psi)(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})(\dot{p} \otimes \dot{p})\colon \phi, \psi \in (\mathbf{C}^2 \otimes \mathbf{C}^2)',\\ &\qquad \phi E = 0, \psi \tilde{E} = 0\}\,,\\ W_1 &= \{(\phi \otimes \tilde{E}')(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})(\dot{p} \otimes \dot{p})\colon \phi \in (\mathbf{C}^2 \otimes \mathbf{C}^2)', \qquad \phi E = 0\}\,,\\ W_{\bar{1}} &= \{(E' \otimes \psi)(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})(\dot{p} \otimes \dot{p})\colon \psi \in (\mathbf{C}^2 \otimes \mathbf{C}^2)', \qquad \psi \tilde{E} = 0\}\,,\\ W_0 &= \mathbf{C}(E' \otimes \tilde{E}')(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})(\dot{p} \otimes \dot{p})\end{aligned}$$

(indices as in the matrix multiplication rule have been omitted). But $R^T_{\mathscr{L}}$ is the matrix of ρ in the basis $\dot{p}_{AB} \otimes \dot{p}_{CD}$ (see the remark after (S1.13)). Using (2.24), we get that (2.25) corresponds to

$$\rho = \beta q_i \overline{q_j}^{-1} \oplus -\beta q_i \overline{q_j}^3 \oplus -\beta q_i^{-3} \overline{q_j}^{-1} \oplus \beta q_i^{-3} \overline{q_j}^3\,.$$

Comparing the condition 6. with Proposition S1.6, we get $\dim K = 6$. Therefore $K_{\text{inv}} = W_1 \oplus W_{\bar{1}}$. But Proposition S1.4 implies $K \subset \ker(\rho + \text{id})$, hence $\beta q_i \overline{q_j}^3 = \beta q_i^{-3} \overline{q_j}^{-1} = 1$. Remembering that $\beta \in \{1, -1, i, -i\}$, $q \neq \pm i$, we get $q = \pm 1$. Thus we can (and will) omit L_3, L_4. We obtain $\beta = q$, $i = j$ or $\beta = -q$, $i \neq j (q \in \{1, -1\}$, $i, j \in \{1, 2\})$. In all these cases $\rho = 1 \oplus -1 \oplus -1 \oplus 1$, hence $K = \ker(\rho + \text{id})$. Moreover, $\rho^2 = \text{id}, R^2 = \mathbf{1}^{\otimes 4}$. By virtue of Proposition 2.1 the conditions a)–c) of Sect. 2 of [10] are satisfied and we can use the results of Sects. 1–4 of [10]. In particular, Corollary S4.2 implies the first statement of the theorem.

Let us pass from $\mathscr{L}$ to $\Lambda = V^{-1}\mathscr{L}V$ (see (1.2)). Since $\bar{V} = \tau V, \bar{\Lambda} = \Lambda$. We replace p_{AB}, $A, B = 1, 2$, corresponding to $\mathscr{L}$ by $p_i = V^{-1}_{i,AB} p_{AB}, f_{AB,CD}$ by $f_{ij} = V^{-1}_{i,AB} f_{AB,CD} V_{CD,j}$ (cf. (S1.2)), $R_{\mathscr{L}}, G_{\mathscr{L}}$ and $\tilde{G}_{\mathscr{L}}$ by $R = (V^{-1} \otimes V^{-1})R_{\mathscr{L}}(V \otimes V)$,

$G=(V^{-1}\otimes \mathbf{1})G_{\mathscr{L}}(\mathbf{1}\otimes V), \tilde{G}=(V^{-1}\otimes \mathbf{1})\tilde{G}_{\mathscr{L}}(\mathbf{1}\otimes V)$. Then (2.10) gives

$$f_{ij}(w_{CD})=G_{iC,Dj}, \qquad f_{ij}(w_{CD}{}^{*})=\tilde{G}_{iC,Dj}\,. \tag{2.26}$$

Now we pass to a new p_i such that $p_i^*=p_i$ without change of $\tilde{\mathscr{B}}^N, \dot{\Gamma}_2, \xi, \rho, K, f_{ij}, R, G$ and $\tilde{G}$ (see Proposition S4.5.1. and (S1.2)). We redefine p_{AB} accordingly. By virtue of Proposition S4.5.2, (S4.10) holds. Setting $a=w_{EF}{}^{*}$ and passing back to $\mathscr{L}$ one has $f_{BA,DC}(w_{EF}^{-1})=\overline{f_{AB,CD}(w_{EF}{}^{*})}$. It means

$$(G_{\mathscr{L}}^{-1})_{EBA,DCF}=\overline{(\tilde{G}_{\mathscr{L}})_{ABE,FCD}}\,, \tag{2.27}$$

i.e. $(L_i^{-1}\otimes \mathbf{1})(\mathbf{1}\otimes X^{-1})=(L_j^{-1}\otimes \mathbf{1})(\mathbf{1}\otimes X^{-1})$ (we used (2.23), (2.21), (2.2)). Thus $L_i=L_j$, $i=j$. Consequently, $\beta=q=\pm 1$ and $i=j=1,2$. Conversely, this condition gives (S4.10) for $a=w_{EF}{}^{*}$ and (using $S\circ * = *\circ S^{-1}$) $a=w_{EF}^{-1}$, hence for all $a\in\mathscr{A}$. The list of X such that $\beta=q=\pm 1$ is provided in the formulation of Theorem 1.4 (they contain the factor α which is computed in such a way that (2.1) is satisfied, we also restricted the range of parameters according to remarks on p. 220 of [14]). For E, E', X as in Theorem 1.4 and f_{ij} computed above (S1.3), (S1.5) and (S4.10) (for A) are satisfied.

According to Proposition 2.1, the only 2-dimensional irreducible representations of H are $UwU^{-1}, U\bar{w}U^{-1}, U\in GL(2,\mathbf{C})$. Thus if $\phi\colon \mathscr{A}_1\to\mathscr{A}_2$ is an isomorphism of Hopf $*$-algebras $\mathscr{A}_1, \mathscr{A}_2$ included in our list, then

$$(1)\ \phi(w)=UwU^{-1} \quad \text{or} \quad (2)\ \phi(w)=U\bar{w}U^{-1}\,.$$

Let us consider the case (1). We denote E, E', X for A_i, $i=1,2$, by E_i, E_i', X_i. Applying ϕ to (1.4)–(1.6) for $\mathscr{A}_1$, one gets

$$(U^{-1}\otimes U^{-1})E_1=k^{-1}E_2\,, \tag{2.28}$$

$$E_1'(U\otimes U)=k'E_2'\,, \tag{2.29}$$

$$(\bar{U}^{-1}\otimes U^{-1})X_1(U\otimes \bar{U})=lX_2 \tag{2.30}$$

for some $k, k', l\in\mathbf{C}_*$. Considering (2.28)–(2.29), one gets $E_1=E_2=E$ and

$$U\in GL(2,\mathbf{C}), k=k'=\det U \ \text{ for } E=e_1\otimes e_2-e_2\otimes e_1\,,$$

$$U\in\left\{\begin{pmatrix} m & x\\ 0 & m\end{pmatrix} : m\in\mathbf{C}_*, \quad x\in\mathbf{C}\right\}, \quad k=k'=m^2$$

for $E=e_1\otimes e_2-e_2\otimes e_1+e_1\otimes e_1$,

$$U\in\left\{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}, \begin{pmatrix} 0 & x\\ y & 0\end{pmatrix} : x, y\in\mathbf{C}_*\right\}, \qquad k=k'=xy$$

for $E=e_1\otimes e_2+e_2\otimes e_1$.

Inserting such U in (2.30), one gets (for $E=e_1\otimes e_2-e_2\otimes e_1$ see Sect. 5.1 of [14]) $X_1=X_2=X, l=1$, so

$$(\bar{U}^{-1}\otimes U^{-1})X(U\otimes\bar{U})=X \tag{2.31}$$

and in particular cases:

1) $t = 1: U \in GL(2, \mathbf{C})$

$$0 < t < 1: U \in \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{C}_* \right\},$$

2) $U = m \begin{pmatrix} e^{i\phi} & x \\ 0 & e^{-i\phi} \end{pmatrix}, m \in \mathbf{C}_*, x \in \mathbf{C}, \phi \in \mathbf{R},$

3) $U = m \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, m \in \mathbf{C}_*, x \in \mathbf{C},$

4) $U = m \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix}, m \in \mathbf{C}_*, x \in \mathbf{R},$

5) $U \in \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{C}_* \right\},$

6) $U \in \left\{ m \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} : \ m \in \mathbf{C}_*, \phi \in \mathbf{R} \right\},$

7) $U \in \left\{ m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : m \in \mathbf{C}_* \right\}.$

Next, let us consider the case (2). Then

$$(\overline{U^{-1}} \otimes \overline{U^{-1}})\tilde{E}_1 = \bar{k}^{-1} E_2 \,, \tag{2.32}$$

$$\tilde{E_1'}(\bar{U} \otimes \bar{U}) = \bar{k'} E_2' \,, \tag{2.33}$$

$$(\bar{U}^{-1} \otimes U^{-1}) X_1 (U \otimes \bar{U}) = l^{-1} X_2^{-1} \tag{2.34}$$

for some $k, k', l \in \mathbf{C}_*$. Considering (2.32)–(2.33), one gets $E_1 = E_2 = E$,

$$U \in GL(2, \mathbf{C}), \ k = k' = -\det U \quad \text{for } E = e_1 \otimes e_2 - e_2 \otimes e_1 \,,$$

$$U \in \left\{ \begin{pmatrix} m & x \\ 0 & -m \end{pmatrix} : m \in \mathbf{C}_*, \quad x \in \mathbf{C} \right\}, \ k = k' = m^2 \,,$$

for $E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1$,

$$U \in \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{C}_* \right\}, \qquad k = k' = xy \,,$$

for $E = e_1 \otimes e_2 + e_2 \otimes e_1$.

Inserting such U in (2.34), it is possible only for $X_1 = X_2 = X$ in the following cases:

1) $t = 1: U \in GL(2, \mathbf{C}), l = 1,$

3) $r = 0: U = m \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, m \in \mathbf{C}_*, x \in \mathbf{C}, l = 1,$

4) $U = m \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix}, m \in \mathbf{C}_*, x \in \mathbf{R}, l = 1,$

5) $t = 1 : U \in \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{C}_* \right\}, l = -1,$

(l is computed for normalization of X as in Theroem 1.4, which includes α).

In particular, all considered $(\mathscr{A}, \Delta)$ are nonisomorphic. □

Remark. 2.2. Using (2.17)–(2.21) for $i = j = 1, 2$, $\beta = q = \pm 1$, one also gets

$$(\mathbf{1} \otimes L)(L \otimes \mathbf{1})(\mathbf{1} \otimes E) = E \otimes \mathbf{1}, \tag{2.35}$$

$$(\mathbf{1} \otimes \tilde{L})(\tilde{L} \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{E}) = \tilde{E} \otimes \mathbf{1}, \tag{2.36}$$

$$(E' \otimes \mathbf{1})(\mathbf{1} \otimes L)(L \otimes \mathbf{1}) = \mathbf{1} \otimes E', \tag{2.37}$$

$$(\tilde{E}' \otimes \mathbf{1})(\mathbf{1} \otimes \tilde{L})(\tilde{L} \otimes \mathbf{1}) = \mathbf{1} \otimes \tilde{E}'. \tag{2.38}$$

Let us repeat that $K = \ker(\rho + \mathrm{id})$, $\rho^2 = \mathrm{id}$, the conditions a)–c) of Sect. 2 of [10] are satisfied and we can use the results of Sect. 1–4 of [10]. We notice that $L, \tilde{L}, G, \tilde{G}$ and R given before Theorem 1.5 and in the proof of Theorem 1.4 coincide ($i = j = 1$ corresponds to $s = 1$, while $i = j = 2$ to $s = -1$). They correspond to Λ as in (1.2), $\bar{\Lambda} = \Lambda$.

Proof of Theorem 1.5. Using Theorem 1.4 and Corollary S3.8.a, $\mathscr{B}$ is the universal $*$-algebra with I generated by w_{AB} and p_i satisfying (1.4)–(1.9). Next, (S2.6) coincides with a), (S4.10)–(S4.11) imply b), (2.26) gives the first formula of c). The next two formulae in c) can be treated as definitions of H_{EFCD} and T_{EFCD}. Since $w, \bar{w}$ and $\mathscr{P}$ are representations, formulae concerning the Hopf structure follow. Uniqueness of f, η, T and the $*$-Hopf structure is obvious.

Let $\mathscr{B}, \hat{\mathscr{B}}$ describe two quantum Poincaré groups and $\mathscr{A}, \Delta, \Lambda, p, f, \eta, T$, $\hat{\mathscr{A}}, \hat{\Delta}, \hat{\Lambda}, \hat{p}, \hat{f}, \hat{\eta}, \hat{T}$, be the corresponding objects as in Theorems 1.4 and 1.5. Assume that $\phi : \mathscr{B} \to \hat{\mathscr{B}}$ is an isomorphism of Hopf $*$-algebras. According to Proposition S4.4, one has $\phi(\mathscr{A}) = \hat{\mathscr{A}}$ and we put $\phi_{\mathscr{A}} = \phi|_{\mathscr{A}} : \mathscr{A} \to \hat{\mathscr{A}}$. Due to the proof of Theorem 1.5, one has

$$(1)\ \phi(w) = U\hat{w}U^{-1} \quad \text{or} \quad (2)\ \phi(w) = U\bar{\hat{w}}U^{-1}.$$

Using (1.2), one gets $\phi(\Lambda) = M\hat{\Lambda}M^{-1}$, where

$$\left.\begin{aligned} M &= V^{-1}(U \otimes \bar{U})V && \text{in the case (1)}, \\ M &= q^{1/2}V^{-1}(U \otimes \bar{U})XV && \text{in the case (2)}. \end{aligned}\right\} \tag{2.39}$$

Using (1.3) and (2.2), one gets $\bar{M} = M(\overline{q^{1/2}} = \beta q^{1/2}$ since $\beta = q = \pm 1)$.

By virtue of (S4.2), one has

$$\hat{f}_{ij}(\hat{w}_{CD}) = U_{CA}^{-1}(M^{-1})_{il} f_{lm}(w_{AB}) M_{mj} U_{BD} \quad \text{in the case (1)},$$

$$\hat{f}_{ij}(\hat{w}_{CD}) = \bar{U}_{CA}^{-1}(M^{-1})_{il} f_{lm}(w_{AB}{}^*) M_{mj} \bar{U}_{BD} \quad \text{in the case (2)}.$$

Using (2.28)–(2.30) or (2.32)–(2.34), we get $\hat{L} = L$ in all cases. Thus there are no isomorphisms between quantum Poincaré groups with different s. □

Using the computer MATHEMATICA program, we made several computations performed in

Proof of Theorem 1.6. Let $\mathscr{B}, \mathscr{A}, \Delta, \Lambda, p, f, \eta, T$ describe a quantum Poincaré group. According to Propositions S4.4 and S4.5.3, it is always possible to replace η by $\hat{\eta} = \eta + fh - \varepsilon h$, where $h_i \in \mathbf{R}$. We put $\mathscr{A} = \hat{\mathscr{A}}, w = \hat{w}, c = 1, M = \mathbf{1}_4, \phi_{\mathscr{A}} = \text{id}$, and f doesn't change. Thus we substitute H_{EFCD} by

$$\hat{H}_{EFCD} = V_{EF,i}\hat{\eta}_i(\hat{w}_{CD}) = H_{EFCD} + f_{EF,AB}(w_{CD})h_{AB} - \delta_{CD}h_{EF} ,$$

where $f_{EF,AB}(w_{CD})$ are given by (2.10) and (2.23), $h_{EF} = V_{EF,i}h_i$ (i.e. $h_{11}, h_{22} \in \mathbf{R}$, $\overline{h_{12}} = h_{21} \in \mathbf{C}$). In each equivalence class obtained by such substitutions we restrict ourselves to exactly one H singled out by the following constraints:

no constraints for 1), $s = 1,\ t = 1$,

$H_{1111} \in i\mathbf{R},\ H_{2222} \in i\mathbf{R}, H_{1222} = 0$ for 1), $s = 1,\ t \neq 1$,

$H_{1112} = 0,\ H_{2112} \in i\mathbf{R}$ for 2), $s = 1$,

$H_{1111} = 0,\ H_{1112} \in i\mathbf{R},\ H_{1211} \in i\mathbf{R}$ for 3), $s = 1$,

$H_{2111} \in i\mathbf{R},\ H_{1122} \in \mathbf{R},\ H_{1112} \in i\mathbf{R}$ for 4), $s = 1$,

$H_{1111} \in i\mathbf{R},\ H_{2111} = 0, H_{2222} \in i\mathbf{R}$ for 5), $s = 1,\ t \neq 1$,

$H_{1111} \in i\mathbf{R},\ H_{2222} \in i\mathbf{R}$ for 5), $s = 1,\ t = 1$,

$H_{1122} \in i\mathbf{R},\ H_{1112} = 0,\ H_{2211} \in i\mathbf{R}$ for 6), $s = 1$,

$H_{1122} \in i\mathbf{R},\ H_{2222} \in i\mathbf{R},\ H_{1222} = 0$ for 7), $s = 1$,

$H_{1111} \in i\mathbf{R},\ H_{1222} = 0,\ H_{2222} \in i\mathbf{R}$ for 1), $s = -1$,

$H_{1122} \in i\mathbf{R},\ H_{1211} = 0,\ H_{2211} \in i\mathbf{R}$ for 2), $s = -1$,

$H_{2111} = 0,\ H_{1111} \in i\mathbf{R},\ H_{2211} \in i\mathbf{R}$ for 3), $s = -1$,

$H_{2211} \in i\mathbf{R},\ H_{1222} = 0,\ H_{1111} \in i\mathbf{R}$ for 4), $s = -1$,

$H_{1222} = 0,\ H_{1111} \in i\mathbf{R},\ H_{2222} \in i\mathbf{R}$ for 5), $s = -1, t \neq 1$,

$H_{1222} = 0$ for 5), $s = -1,\ t = 1$,

$H_{1211} = 0,\ H_{2112} \in i\mathbf{R}$ for 6), $s = -1$,

$H_{1122} \in i\mathbf{R},\ H_{1222} = 0,\ H_{2222} \in i\mathbf{R}$ for 7), $s = -1$.

We also may and will assume (S3.50).

By virtue of the theory presented in Sect. 1–4 of [10] (see e.g. Theorem S3.1 and Proposition S4.5) H_{EFCD} and T_{EFCD} give a quantum Poincaré group if and only if (S1.5), (S2.6), (S2.14), (S3.1), (S3.2), (S4.10), (S4.11) and (S4.12) are satisfied (cf. the proof of Theorem 1.4). We shall investigate subsequent conditions and dealing with the next ones we assume that previously investigated conditions are satisfied. We already know that f is a unital homomorphism satisfying (2.26), (S1.5) and (S4.10). Thus (S2.6) means that applying η_i to the relations (1.4), (1.4)*, (1.5), (1.5)* and (1.6) (* means that we conjugate the relation) and using (S2.5), one gets relations on $H^{w}_{iA,B} = \eta_i(w_{AB})$ and $H^{\bar{w}}_{iA,B} = \eta_i(w_{AB}{}^*)$, which should be satisfied. They read as follows:

$$\{(G\otimes\mathbf{1})(\mathbf{1}\otimes H^{w})+(H^{w}\otimes\mathbf{1})\}E=0\,, \tag{2.40}$$

$$\{(\tilde G\otimes\mathbf{1})(\mathbf{1}\otimes H^{\bar w})+(H^{\bar w}\otimes\mathbf{1})\}\tilde E=0\,, \tag{2.41}$$

$$(\mathbf{1}\otimes E')\{(H^{w}\otimes\mathbf{1})+(G\otimes\mathbf{1})(\mathbf{1}\otimes H^{w})\}=0\,, \tag{2.42}$$

$$(\mathbf{1}\otimes \tilde E')\{(H^{\bar w}\otimes\mathbf{1})+(\tilde G\otimes\mathbf{1})(\mathbf{1}\otimes H^{\bar w})\}=0\,, \tag{2.43}$$

$$\left.\begin{aligned}&(\mathbf{1}^{\otimes 2}\otimes X)\{(H^{w}\otimes\mathbf{1})+(G\otimes\mathbf{1})(\mathbf{1}\otimes H^{\bar w})\}\\ &\quad=\{(H^{\bar w}\otimes\mathbf{1})+(\tilde G\otimes\mathbf{1})(\mathbf{1}\otimes H^{w})\}X\,.\end{aligned}\right\} \tag{2.44}$$

Setting $a=w_{EF}{}^*$ in (S4.11), one gets

$$(H^{\bar w})_{iE,F}=\overline{\eta_i(w_{EF}^{-1})}=-\overline{f_{ij}(w_{EL}^{-1})}\cdot\overline{\eta_j(w_{LF})}=-\overline{G^{-1}}_{Ei,jL}\cdot\overline{H^{w}_{jL,F}} \tag{2.45}$$

(we used (S2.5), (S1.4) and (2.26)). Conversely, (2.45) gives (S4.11) for $a=w_{EF}{}^*$ and (using $S\circ *\circ S\circ *=\mathrm{id}$) $a=w_{EF}^{-1}$, hence for all $a\in\mathscr{A}$ (it suffices to check the conditions (S4.10)–(S4.11) on generators of $\mathscr{A}$ as an algebra: they are equivalent to Theorem 1.5.b for a^*).

Using the 16 relations (2.1), (2.3)–(2.9), (2.17)–(2.20) and (2.35)–(2.38), one gets that (2.40) is equivalent to (2.42), (2.41) is equivalent to (2.43). Moreover, (2.40) is equivalent to (2.41) (one conjugates (2.41) and uses (2.27), (2.45)). Thus (2.41)–(2.43) are superfluous. The remaining equations: (2.44) (with inserted (2.45)) and (2.40) give a set of $\mathbf{R}$-linear equations on $H_{EFCD}=V_{EF,i}H^{w}_{iC,D}$.

Next, (S3.50) gives a set of linear equations on $T_{EFCD}=V_{EF,i}V_{CD,j}T_{ij}$. By virtue of (S3.50) and (S4.14), one obtains ($\tilde T$ was defined after (S4.12)) $R\tilde T=-\tilde T$, $RD=-D$, where $D=\tilde T-T$. Therefore D corresponds to a subrepresentation of $\Lambda\oplus\Lambda$ equivalent to $w^1\oplus\overline{w^1}$. But (S4.12) means that D is an invariant vector of $\Lambda\oplus\Lambda$, hence $D=0$, $\tilde T=T$ (conversely, this implies (S4.12)). This gives a set of $\mathbf{R}$-linear conditions on T_{EFCD}.

According to Proposition S3.13 and Corollary S4.9, we may replace (S2.14) by (S3.55) for $b=w_{AB}$. But this is equivalent to $M\in\mathrm{Mor}(w,\Lambda\oplus\Lambda\oplus w)$, where $M_{ijC,B}=\tau^{ij}(w_{CB})$. Using $[(R+\mathbf{1}_4^{\otimes 2})\otimes\mathbf{1}]M=0$ (see (S3.54)), one gets $M=[(V^{-1}\otimes V^{-1})(\mathbf{1}\otimes X\otimes\mathbf{1})\otimes\mathbf{1}]N$, where $N\in\mathrm{Mor}(w,w\oplus w\oplus\bar w\oplus\bar w\oplus w)$, $(L_1\otimes\tilde L_1\otimes\mathbf{1})N=-N(L\otimes L$ doesn't depend on s, one can put $s=1$). Thus $N_{ABCDF,G}=P_{ABF,G}\tilde E_{CD}$ with $P\in\mathrm{Mor}(w,w\oplus w\oplus w)$, $(L_1\otimes\mathbf{1})P=q^{1/2}P$. It means $P=\lambda\mathbf{1}\otimes E+\mu E\otimes\mathbf{1}$, $(EE'\otimes\ \mathbf{1})P=0$. Hence $\mu=\frac{1}{2}q\lambda$, $P=\lambda(\mathbf{1}\otimes E+\frac{1}{2}qE\otimes\mathbf{1})$. On the other hand,

$$\begin{aligned}\tau^{ij}(w_{CB})=(R-\mathbf{1}^{\otimes 2})_{ij,kl}(&\eta_l(w_{CA})\eta_k(w_{AB})-\eta_k(\Lambda_{ls})\eta_s(w_{CB})+T_{kl}\delta_{CB}\\ &-f_{ln}(w_{CA})f_{km}(w_{AB})T_{mn})\,.\end{aligned}$$

Using (1.2), (S2.5), (2.45) and (2.26), one gets a set of equations containing terms bilinear in Re H_{ABCD}, Im H_{ABCD}, terms linear in Re T_{ABCD}, Im T_{ABCD} and terms linear in Re λ, Im λ.

We shall prove that (S3.1) is equivalent to $\lambda\in q^{1/2}\mathbf{R}$. One has (see (1.2)) $\tilde F=(V^{-1}\otimes V^{-1}\otimes V^{-1})JV$, where $\tilde F=((R-\mathbf{1}_4^{\otimes 2})\otimes\mathbf{1})F$ and

$$J_{QRTVAB,CD}=V_{QR,i}V_{TV,j}\tau^{ij}(w_{AC}w_{BD}{}^*)\,.$$

Using Proposition S3.13,

$$\tau^{ij}(w_{AC}{w_{BD}}^*) = \tau^{ij}(w_{AC})\delta_{BD} + G_{jA,Es}G_{iE,Cm}\tau^{ms}({w_{BD}}^*)\,.$$

But by virtue of Proposition S4.8,

$$\tau^{ms}({w_{BD}}^*) = \overline{\tau^{sm}(w_{BD}^{-1})}\,.$$

Using once again Proposition S3.13 for $a = w_{LS}$, $b = w_{SD}^{-1}$, multiplying both sides by $G_{Bs,iP}^{-1}G_{Pm,jL}^{-1}$ and conjugating both sides, one gets

$$\overline{\tau^{sm}(w_{BD}^{-1})} = -\tilde{G}_{sB,Pi}\tilde{G}_{mP,Lj}\overline{\tau^{ij}(w_{LD})}$$

(see (2.27)). Inserting all these data, after some calculations (using the 16 relations), one obtains

$$J = \bar{\lambda}qA + \lambda B + \frac{1}{2}(\bar{\lambda} + \lambda q)C\,,$$

$$\begin{aligned}\text{where } A &= (\mathbf{1}\otimes X\otimes X\otimes \mathbf{1})(E\otimes X\otimes \tilde{E})\,,\\ B &= \mathbf{1}\otimes(\mathbf{1}\otimes X^{-1}\otimes\mathbf{1})(\tilde{E}\otimes E)\otimes\mathbf{1},\\ C &= (\mathbf{1}\otimes X\otimes\mathbf{1})(E\otimes\tilde{E})\otimes\mathbf{1}\otimes\mathbf{1}\,.\end{aligned}$$

We shall also use $D = \mathbf{1}\otimes\mathbf{1}\otimes(\mathbf{1}\otimes X\otimes\mathbf{1})(E\otimes\tilde{E})$. Using (2.24) and the 16 relations, one has

$$\begin{aligned}&(R_{\mathscr{L}}\otimes\mathbf{1})A = -A - qC, \qquad (R_{\mathscr{L}}\otimes\mathbf{1})B = -B - qC\,,\\ &(R_{\mathscr{L}}\otimes\mathbf{1})C = C, \qquad (R_{\mathscr{L}}\otimes\mathbf{1})D = D + qA + qB + C\,,\\ &(\mathbf{1}\otimes R_{\mathscr{L}})A = -A - qD, \qquad (\mathbf{1}\otimes R_{\mathscr{L}})B = -B - qD\,,\\ &(\mathbf{1}\otimes R_{\mathscr{L}})C = C + qA + qB + D, \qquad (\mathbf{1}\otimes R_{\mathscr{L}})D = D\,.\end{aligned}$$

In particular, $(R_{\mathscr{L}}\otimes\mathbf{1})J = -J$ (it also follows from (S3.54)). Thus we can compute

$$\begin{aligned}-2(V\otimes V\otimes V)A_3FV^{-1} &= (V\otimes V\otimes V)A_3\tilde{F}V^{-1}\\ &= [\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1} - \mathbf{1}\otimes R_{\mathscr{L}} - R_{\mathscr{L}}\otimes\mathbf{1} + (\mathbf{1}\otimes R_{\mathscr{L}})(R_{\mathscr{L}}\otimes\mathbf{1})\\ &\quad + (R_{\mathscr{L}}\otimes\mathbf{1})(\mathbf{1}\otimes R_{\mathscr{L}}) - (R_{\mathscr{L}}\otimes\mathbf{1})(\mathbf{1}\otimes R_{\mathscr{L}})(R_{\mathscr{L}}\otimes\mathbf{1})]J\\ &= 2[J - (\mathbf{1}\otimes R_{\mathscr{L}})J + (R_{\mathscr{L}}\otimes\mathbf{1})(\mathbf{1}\otimes R_{\mathscr{L}})J]\\ &= 3(\bar{\lambda}q - \lambda)(A - B)\,.\end{aligned}$$

But $A \neq B$ ($\mathrm{im}[(\mathbf{1}\otimes X^{-1}\otimes X^{-1}\otimes\mathbf{1})A] = \mathrm{im}\,E\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathrm{im}\,\tilde{E}$ while $\mathrm{im}\,[(\mathbf{1}\otimes X^{-1}\otimes X^{-1}\otimes\mathbf{1})B] = \mathbf{C}^2\otimes W_0\otimes\mathbf{C}^2$, where $\dim\,\mathrm{im}\,E = \dim\mathrm{im}\,\tilde{E} = \dim W_0 = 1$), hence $A_3F = 0$ if and only if $\bar{\lambda} = q\lambda$, i.e. $\lambda\in q^{1/2}\mathbf{R}$.

We notice that

$$\Lambda\oplus\Lambda\oplus\Lambda \simeq w\oplus w\oplus w\oplus\bar{w}\oplus\bar{w}\oplus\bar{w} \simeq (w\oplus w\oplus w^{3/2})\oplus(\bar{w}\oplus\bar{w}\oplus\overline{w^{3/2}})\,,$$

hence

$$\mathrm{Mor}(I,\Lambda\oplus\Lambda\oplus\Lambda) = \{0\}\,. \tag{2.46}$$

Therefore (S3.2) is equivalent to $A_3(Z \otimes \mathbf{1}_4 - \mathbf{1}_4 \otimes Z)T = 0$. This gives a set of equations, which are $\mathbf{R}$-linear in $\operatorname{Re} H_{ABCD} \cdot \operatorname{Re} T_{EFGH}$, $\operatorname{Im} H_{ABCD} \cdot \operatorname{Re} T_{EFGH}$, $\operatorname{Re} H_{ABCD} \cdot \operatorname{Im} T_{EFGH}$ and $\operatorname{Im} H_{ABCD} \cdot \operatorname{Im} T_{EFGH}$. Our strategy is as follows: we set constraints for H_{ABCD} as before, solve $\mathbf{R}$-linear equations, insert these data into $\mathbf{R}$-bilinear equations and finally use the condition for λ and the last set of equations. In the cases 1), $t = 1$, $s = 1$ and 5), $t = 1$, $s = \pm 1$, we haven't solved the $\mathbf{R}$-bilinear equations (but see Remark 1.8). In other cases one gets the following solutions (with the parameters being real numbers):

(1.10) with $T_{ab} = -T_{ba} \in i\mathbf{R}$ for 1), $s = -1, t = 1$,

(1.11) for 1), $s = \pm 1$, $0 < t < 1$,

(1.12) or (1.13) for 2), $s = 1$,

(1.12) with $a = b = 0$ for 2), $s = -1$,

(1.14) for 4), $s = 1$,

(1.15) for 5), $s = \pm 1$, $0 < t < 1$,

(1.16) or (1.17) for 6), $s = -1$,

in the remaining cases all H_{EFCD} and T_{EFCD} must equal 0. Moreover, $\lambda = 8b^2$ in the case 4), $s = 1$ and $\lambda = 0$ in other solved cases.

Let us remark that for fixed w_{AB} and $p_i : \phi_i, \eta_i$ and H_{EFCD} are uniquely determined (cf. (S1.6)). Moreover, T_{EFCD} satisfying (S3.50) are also uniquely determined: if T' would also satisfy (S3.46) and (S3.50), then for $L = T - T'$ we would have

$$0 = (R - \mathbf{1}_4^{\otimes 2})(L - (A \oplus A)L) = (R - \mathbf{1}_4^{\otimes 2})L - (A \oplus A)(R - \mathbf{1}_4^{\otimes 2})L$$
$$= -2(L - (A \oplus A)L),$$

$L \in \operatorname{Mor}(I, A \oplus A)$, but $RL = -L$ gives that L corresponds to the subrepresentation $w^1 \oplus \overline{w^1}$ of $A \oplus A$, $L = 0$, $T = T'$.

It remains to check which pairs (H, T) as above give isomorphic objects. By virtue of Propositions S4.4 and S4.5 and above remarks it would mean that $(\hat{H}, \hat{T})$ is obtained from (H, T) via formulae (S4.3)–(S4.4) with $c, h_i \in \mathbf{R}$, $c \neq 0$, M as in (2.39). After some calculations one can choose one pair (H, T) in each equivalence class (for each considered case). The results are presented in the formulation of the theorem. □

Proof of Theorem 1.9. By virtue of Corollary S3.6 it suffices to prove $\dim S_n = d_n$. Taking $A = \mathscr{L} = w \oplus \bar{w}$, one has the projection $S_n = \frac{1}{n!}\sum_{\pi \in \Pi_n} R_\pi$, where $R_\pi = (R_{\mathscr{L}})_{i_1} \cdot \dots \cdot (R_{\mathscr{L}})_{i_k}$ for a minimal decomposition $\pi = t_{i_1} \cdot \dots \cdot t_{i_k}$, $R_{\mathscr{L}} = (\mathbf{1} \otimes X \otimes \mathbf{1})(L \otimes \tilde{L})(\mathbf{1} \otimes X^{-1} \otimes \mathbf{1})$. Putting

$$K_X = (\mathbf{1}^{\otimes n-1} \otimes X \otimes \mathbf{1}^{\otimes n-1})(\mathbf{1}^{\otimes n-2} \otimes X \otimes X \otimes \mathbf{1}^{\otimes n-2}) \cdot \dots \cdot$$
$$(\mathbf{1} \otimes X \otimes \dots \otimes X \otimes \mathbf{1})(X \otimes \dots \otimes X),$$

and K_τ defined similarly with X replaced by τ, one can define $S'_n = K_X S_n K_X^{-1}$ and $S''_n = K_\tau^{-1} S'_n K_\tau$. Therefore $\dim S_n = \operatorname{tr} S_n = \operatorname{tr} S'_n = \operatorname{tr} S''_n$. One gets the formula for S''_n as for S_n but with $R_{\mathscr{L}}$ replaced by $R''_{\mathscr{L}} = (\mathbf{1} \otimes \tau \otimes \mathbf{1})(L \otimes \tilde{L})(\mathbf{1} \otimes \tau^{-1} \otimes \mathbf{1})$ (we use the 16 relations).

Moreover, $L \otimes \tilde{L} = L_0 \otimes \tau L_0 \tau$, where $L_0 = \mathbf{1}^{\otimes 2} + q^{-1}EE'$, $E = e_1 \otimes e_2 - qe_2 \otimes e_1 + t_0 e_1 \otimes e_1$, $E' = -qe^1 \otimes e^2 + e^2 \otimes e^1 + t_0 e^2 \otimes e^2$, $q = \pm 1$, $t_0 = 0, 1$ (for $q = -1$ one has $t_0 = 0$). Replacing e_1 by ce_1, $c \neq 0$, one has to replace e^1 by $c^{-1}e^1$, L_0 by L_0 with t_0 replaced by $c \cdot t_0$. Thus (for $q = 1$) $\mathrm{tr}\, S''_n$ doesn't depend on $t_0 \in \mathbf{C}$ (for $t_0 \neq 0$ and also for $t_0 = 0$ in limit). So we may put $t_0 = 0$. Then $L_0 e_1 \otimes e_1 = e_1 \otimes e_1$, $L_0 e_2 \otimes e_2 = e_2 \otimes e_2$, $L_0 e_1 \otimes e_2 = qe_2 \otimes e_1$, $L_0 e_2 \otimes e_1 = qe_1 \otimes e_2$. Setting $A_{\alpha\beta} = e_\alpha \otimes e_\beta$, one has

$$R''_{\mathscr{L}} A_{\alpha\beta} \otimes A_{\gamma\delta} = q^{\alpha+\beta+\gamma+\delta} A_{\gamma\delta} \otimes A_{\alpha\beta} .$$

It is easy to show that $S''_n (R''_{\mathscr{L}})_k = (R''_{\mathscr{L}})_k S''_n = S''_n, S''_n$ is a projection,

$$S''_n (A_{11}^{\otimes a} \otimes A_{12}^{\otimes b} \otimes A_{21}^{\otimes c} \otimes A_{22}^{\otimes d}), \qquad a + b + c + d = n ,$$

form a basis of $\mathrm{im}\, S''_n$. We get

$$\begin{aligned} \dim S_n &= \mathrm{tr}\, S''_n = \dim \mathrm{im}\, S''_n \\ &= \#\{(a,b,c,d) \in \mathbf{N}^{\otimes 4} : a + b + c + d = n\} = d_n . \qquad \square \end{aligned}$$

Proof of Theorem 1.12. We know that $\Lambda \oplus \Lambda \simeq I \oplus w^1 \oplus \overline{w^1} \oplus w^1 \oplus \overline{w^1}$, where $\ker(R + \mathbf{1}_4^{\otimes 2})$ corresponds to $w^1 \oplus \overline{w^1}$. Therefore (S5.2) holds. Moreover, (2.46) coincides with (S5.4). Using Theorem S5.6, we get the first statement. The second statement follows from Proposition S5.3, Proposition S5.5 and $\dim S_n = d_n$ (see the proof of Theorem 1.9). $\square$

Proof of Theorem 1.13. We know that (S3.59) holds (see (S3.2) and (2.46)) and $R \neq \pm \mathbf{1}_4^{\otimes 2}$ (see the proof of Theorem 1.4). Moreover, $(\Lambda \oplus \Lambda)m' = m'$ means that m' is proportional to m. According to the proof of Theorem 1.6, $\tilde{F} = 0$ if and only if $\lambda = 0$ (otherwise, using $\bar{\lambda} = q\lambda, A + B + qC = 0$, acting $\mathbf{1} \otimes R_{\mathscr{L}}, C = D$, $V_0 \otimes \mathbf{C}^4 = \mathrm{im}\, C = \mathrm{im}\, D = \mathbf{C}^4 \otimes V_0$, where $V_0 = \mathrm{im}[(\mathbf{1} \otimes X \otimes \mathbf{1})(E \otimes \tilde{E})]$, $\dim V_0 = 1$, contradiction), which means $b = 0$ in the case 4), $s = 1$ and no condition in other cases listed in Theorem 1.6. Then we use Proposition S3.14. $\square$

Remark. 2.3. According to Corollary S3.8.b, $\mathscr{B}$ is the universal unital algebra generated by $\mathscr{A}$ and $p_i (i \in \mathscr{I})$ satisfying $I_{\mathscr{B}} = I_{\mathscr{A}}$, (S3.48) and (S3.47) for w and $\tilde{w}$ (cf. Remark S3.10).

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Note added in proof. We take that opportunity to make corrections in our paper ref. 8:

1) At the beginning of Theorem 4.6 add in a separate paragraph:

Assume that the Haar measure on G_c is faithful.

2) on page 417, line 6 up, replace $\mu^{-1/2}$ by $\mu^{+1/2}$

3) on page 390, line 3 up, replace $w^1 \circ w^2$ by $w^1 \oplus w^2$

References

1. Chaichian, M., Demichev, A.P.: Quantum Poincaré group, Phys. Lett. **B304**, 220–224 (1993) Cf. also Schirrmacher, A.: Varieties on quantized spacetime. In: Symmetry methods in physics. A.N. Sissakin et al. (eds.), vol. 2, Moscow: Dubna 1994, pp. 463–470
2. Dobrev, V.K.: Canonical q-deformations of noncompact Lie (super-) algebras. J. Phys. A: Math. Gen. **26**, 1317–1334 (1993)
3. Kondratowicz, P., Podleś, P.: Irreducible representations of quantum $SL_q(2)$ groups at roots of unity. hep-th 9405079
4. Lukierski, J., Nowicki, A., Ruegg, H.: New quantum Poincaré algebra and κ-deformed field theory. Phys. Lett. **B293**, 344–352 (1992); Zakrzewski, S.: Quantum Poincaré group related to the κ-Poincaré algebra. J. Phys. A: Math. Gen. **27**, 2075–2082 (1994); Cf. also Lukierski, J., Nowicki, A., Ruegg, H., Tolstoy, V.N.: q-deformation of Poincaré algebra. Phys. Lett. **B264**, 331–338 (1991)
5. Majid, S.: Braided momentum in the q-Poincaré group. J. Math. Phys. **34**, 2045–2058 (1993)
6. Ogievetsky, O., Schmidke, W.B., Wess, J., Zumino, B.: q-Deformed Poincaré algebra. Commun. Math. Phys. **150**, 495–518 (1992)
7. Podleś, P.: Complex quantum groups and their real representations. Publ. RIMS, Kyoto University **28**, 709–745 (1992)
8. Podleś, P., Woronowicz, S.L.: Quantum deformation of Lorentz group. Commun. Math. Phys. **130**, 381–431 (1990)
9. Podleś, P., Woronowicz, S.L.: Inhomogeneous quantum groups. Submitted to Proceedings of First Caribbean School of Mathematics and Theoretical Physics in Guadeloupe, 1993
10. Podleś, P., Woronowicz, S.L.: On the structure of inhomogeneous quantum groups. hep-th 9412058, UC Berkeley preprint PAM-631
11. Schlieker, M., Weich, W., Weixler, R.: Inhomogeneous quantum groups, Z. Phys. C.–Particles and Fields **53**, 79–82 (1992); Inhomogeneous quantum groups and their universal enveloping algebras. Lett. Math. Phys. **27**, 217–222 (1993)
12. Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613–665 (1987)
13. Woronowicz, S.L.: New quantum deformation of $SL(2, \mathbf{C})$. Hopf algebra level. Rep. Math. Phys. **30**(2), 259–269 (1991)
14. Woronowicz, S.L., Zakrzewski, S.: Quantum deformations of the Lorentz group. The Hopf $*$-algebra level. Comp. Math. **90**, 211–243 (1994)
15. Zakrzewski, S.: Geometric quantization of Poisson groups–diagonal and soft deformations. Contemp. Math. **179**, 271–285 (1994)
16. Zakrzewski, S.: Poisson Poincaré groups. Submitted to Proceedings of Winter School of Theoretical Physics, Karpacz 1994, hep-th 9412099 (Some errors are corrected in the last reference); Cf. also Poisson homogeneous spaces. Submitted to Proceedings of Winter School of Theoretical Physics, Karpacz 1994, hep-th 9412101; Poisson structures on the Poincaré group, in preparation

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