On the classification of quaternary optimal Hermitian LCD codes

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Abstract

We propose a method for a classification of quaternary Hermitian LCD codes having large minimum weights. As an example, we give a classification of quaternary optimal Hermitian LCD codes of dimension 3.

1 Introduction

Linear complementary dual (LCD for short) codes are codes that intersect with their dual codes trivially. LCD codes were introduced by Massey [13] and gave an optimum linear coding solution for the two user binary adder channel. Recently, much work has been done concerning LCD codes for both theoretical and practical reasons (see e.g. [5], [6], [7], [11] and the references given therein). For example, if there is a quaternary Hermitian LCD $[n, k, d]$ code, then there is a maximal entanglement entanglement-assisted quantum error-correcting $[[n, k, d; n-k]]$ code (see e.g. [11]). From this point of view, quaternary Hermitian LCD codes play an important role in the study of maximal entanglement entanglement-assisted quantum error-correcting codes. In addition, Carlet, Mesnager, Tang, Qi and Pellikaan [6] showed that any code over $\mathbb{F}_q$ is equivalent to some Euclidean LCD code for $q \geq 4$ and any code

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over $\mathbb{F}_{q^2}$ is equivalent to some Hermitian LCD code for $q \geq 3$, where $\mathbb{F}_q$ denotes the finite field of order q and q is a prime power. This is also a motivation of our study of quaternary Hermitian LCD codes.

It is a fundamental problem to determine the largest minimum weight $d_4(n,k)$ among all quaternary Hermitian LCD $[n,k]$ codes and classify quaternary optimal Hermitian LCD $[n,k,d_4(n,k)]$ codes for a given pair (n,k). It was shown that $d_4(n,2) = \lfloor \frac{4n}{5} \rfloor$ if $n \equiv 1,2,3 \pmod 5$ and $d_4(n,2) = \lfloor \frac{4n}{5} \rfloor - 1$ otherwise for $n \geq 3$ [10] and [11]. Recently, it has been shown that $d_4(n,3) = \lfloor \frac{16n}{21} \rfloor$ if $n \equiv 5,9,13,17,18 \pmod{21}$ and $d_4(n,3) = \lfloor \frac{16n}{21} \rfloor - 1$ otherwise for $n \geq 6$ [2] and [11]. More recently, Ishizuka [8] has completed a classification of quaternary optimal Hermitian LCD codes of dimension 2.

Araya, Harada and Saito [2] gave some conditions on the nonexistence of certain quaternary Hermitian LCD codes having large minimum weights ([2, Theorem 9]). The aim of this note is to propose a method for a classification of quaternary Hermitian LCD codes having large minimum weights by following the same line as in the proof of [2, Theorem 9]. As an example, we give a classification of quaternary optimal Hermitian LCD $[n,3,d_4(n,3)]$ codes for arbitrary n. We also give an alternative classification of quaternary optimal Hermitian LCD $[n,2,d_4(n,2)]$ codes and a classification of quaternary near-optimal Hermitian LCD $[n,2,d_4(n,2)-1]$ codes for arbitrary n.

2 Preliminaries

In this section, we give some definitions, notations and basic results used in this note.

We denote the finite field of order 4 by $\mathbb{F}_4 = \{0,1,\omega,\omega^2\}$, where $\omega^2 = \omega + 1$. For any element $\alpha \in \mathbb{F}_4$, the conjugation of α is defined as $\overline{\alpha} = \alpha^2$. Throughout this note, we use the following notations. Let $\mathbf{0}_s$ and $\mathbf{1}_s$ denote the zero vector and the all-one vector of length s, respectively. Let O denote the zero matrix of appropriate size. Let I_k denote the identity matrix of order k. Let A^T denote the transpose of a matrix A. For a $k \times n$ matrix $A = (a_{ij})$, the conjugate matrix of A is defined as $\overline{A} = (\overline{a_{ij}})$. For a positive integer s and a $k \times n$ matrix A, we denote by $A^{(s)}$ the $k \times ns$ matrix $\begin{pmatrix} A & \cdots & A \end{pmatrix}$.

A *quaternary* $[n,k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_4^n$. The parameters n and k are called the *length* and *dimension* of C, respectively. A generator matrix of a quaternary $[n,k]$ code C is a $k \times n$ matrix such that the rows of the matrix generate C. The *weight* of a vector $x \in \mathbb{F}_4^n$ is the number of non-zero components of x. A vector of C is called a *codeword* of C. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C. A quaternary $[n, k, d]$ code is a quaternary $[n, k]$ code with minimum weight d. Two quaternary $[n, k]$ codes C and C' are *equivalent*, denoted $C \cong C'$, if there is an $n \times n$ monomial matrix P over $\mathbb{F}_4$ with $C' = \{xP \mid x \in C\}$. For any quaternary $[n, k, d]$ code, the Griesmer bound is given by $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{4^i} \right\rceil$. Throughout this note, we use the following notation:

$$g_4(n, k) = \max \left\{ d \in \mathbb{Z}_{\geq 0} \;\middle|\; n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{4^i} \right\rceil \right\},$$

where $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers.

The *Hermitian dual* code $C^\perp$ of a quaternary $[n, k]$ code C is defined as:

$$C^\perp = \{x \in \mathbb{F}_4^n \mid \langle x, y \rangle_H = 0 \text{ for all } y \in C\},$$

where $\langle x, y \rangle_H = \sum_{i=1}^n x_i \overline{y_i}$ for $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{F}_4^n$. A quaternary $[n, k]$ code C is called *Hermitian linear complementary dual* (Hermitian LCD for short) if $C \cap C^\perp = \{\mathbf{0}_n\}$. Note that quaternary Hermitian LCD codes are also called *zero radical* codes (see e.g. [11]). Let $d_4(n, k)$ denote the largest minimum weight among all quaternary Hermitian LCD $[n, k]$ codes. A quaternary Hermitian LCD $[n, k, d_4(n, k)]$ code is called *optimal*. In this note, we say that a quaternary Hermitian LCD $[n, k, d_4(n, k)-1]$ code is *near-optimal*. The minimum weight of the Hermitian dual code $C^\perp$ of C is called the (Hermitian) *dual distance* of C and it is denoted by $d^\perp$.

The following characterization gives a criterion for quaternary Hermitian LCD codes and is analogous to [13, Proposition 1].

Proposition 2.1 ([7, Proposition 3.5])**.** *Let C be a quaternary code and let G be a generator matrix of C. Then C is a Hermitian LCD code if and only if $G\overline{G}^T$ is nonsingular.*

Throughout this note, we use the above characterization without mentioning this.

A quaternary code C is called *Hermitian self-orthogonal* if $C \subset C^\perp$. A quaternary code C is called *even* if the weights of all codewords of C are even. A quaternary code C is Hermitian self-orthogonal if and only if C is even [12, Theorem 1]. In addition, a quaternary code C is Hermitian self-orthogonal if and only if $G\overline{G}^T = O$ for a generator matrix G of C.

3 Background materials

Let C be a quaternary Hermitian LCD $[n, k, d]$ code. Define an $[n+1, k, d]$ code $\widehat{C}$ as $\widehat{C} = \{(x, 0) \mid x \in C\}$. The following lemma was given for binary LCD codes and ternary LCD codes [1, Proposition 3]. The argument can be applied to quaternary Hermitian LCD codes trivially.

Lemma 3.1 (Ishizuka [8])**.** *Let $\mathcal{C}_{n,k,d}$ denote all equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes. Let $\mathcal{D}_{n,k,d}$ denote all equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^\perp \geq 2$. Let $\widehat{\mathcal{C}_{n-1,k,d}}$ denote all equivalence classes containing $\widehat{C_1}, \widehat{C_2}, \ldots, \widehat{C_t}$, where $C_1, C_2, \ldots, C_t$ denote representatives of $\mathcal{C}_{n-1,k,d}$ and $t = |\mathcal{C}_{n-1,k,d}|$. Then $\mathcal{C}_{n,k,d} = \mathcal{D}_{n,k,d} \cup \widehat{\mathcal{C}_{n-1,k,d}}$.*

For a classification of quaternary Hermitian LCD $[n, k, d]$ codes, by the above lemma, it is sufficient to consider a classification of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^\perp \geq 2$.

According to [11], we define the $k \times (\frac{4^k-1}{3})$ $\mathbb{F}_4$-matrices S_k by inductive constructions as follows:

$$S_1 = \begin{pmatrix} 1 \end{pmatrix},$$
$$S_k = \begin{pmatrix} S_{k-1} & \mathbf{0}^T_{\frac{4^{k-1}-1}{3}} & S_{k-1} & S_{k-1} & S_{k-1} \\ \mathbf{0}_{\frac{4^{k-1}-1}{3}} & 1 & \mathbf{1}_{\frac{4^{k-1}-1}{3}} & \omega\mathbf{1}_{\frac{4^{k-1}-1}{3}} & \omega^2\mathbf{1}_{\frac{4^{k-1}-1}{3}} \end{pmatrix} \text{ if } k \geq 2.$$

The matrix S_k is a generator matrix of the quaternary simplex $[\frac{4^k-1}{3}, k, 4^{k-1}]$ code. It is known that the quaternary simplex $[\frac{4^k-1}{3}, k, 4^{k-1}]$ code is a constant weight code. More precisely, the code contains codewords of weights 0 and 4^{k-1} only. Thus, for $k \geq 2$, the quaternary simplex $[\frac{4^k-1}{3}, k, 4^{k-1}]$ code is even. By [12, Theorem 1], the quaternary simplex $[\frac{4^k-1}{3}, k, 4^{k-1}]$ code is Hermitian self-orthogonal for $k \geq 2$.

Let $h_{k,i}$ be the i-th column of the $k \times (\frac{4^k-1}{3})$ $\mathbb{F}_4$-matrix S_k. For a vector $m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}}) \in \mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$, we define a $k \times \sum_{i=1}^{\frac{4^k-1}{3}} m_i$ $\mathbb{F}_4$-matrix $G_k(m)$, which consists of m_i columns $h_{k,i}$ for each i as follows:

$$G_k(m) = \left(h_{k,1} \cdots h_{k,1} h_{k,2} \cdots h_{k,2} \cdots h_{k,\frac{4^k-1}{3}} \cdots h_{k,\frac{4^k-1}{3}} \right). \tag{1}$$

Here $m_i = 0$ means that no column of $G_k(m)$ is $h_{k,i}$. Throughout this note, we denote by $C_k(m)$ the quaternary code with generator matrix $G_k(m)$.

Remark 3.2. By considering all vectors $m \in \mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$ with $n = \sum_{i=1}^{\frac{4^k-1}{3}} m_i$, it is possible to find representatives of all equivalence classes of quaternary $[n, k]$ codes with dual distances $d^\perp \geq 2$ as $C_k(m)$.

The following lemma was given for $k = 2, 3, 4$ in [11]. The argument can be applied to arbitrary k trivially.

Lemma 3.3. *Suppose that $k \geq 2$ and s is a positive integer. Let $m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}})$ be a vector of $\mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$ with $n = \sum_{i=1}^{\frac{4^k-1}{3}} m_i$. If $C_k(m)$ is a quaternary Hermitian LCD $[n, k, d]$ code, then the quaternary code C with generator matrix*

$$\begin{pmatrix} S_k^{(s)} & G_k(m) \end{pmatrix}$$

is a quaternary Hermitian LCD $[n + \frac{4^k-1}{3}s, k, d + 4^{k-1}s]$ code.

The following lemma was given for $k \geq 3$ [2, Lemma 7]. The argument can be applied to $k = 2$ trivially.

Lemma 3.4. *Suppose that $k \geq 2$. Let $m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}})$ be a vector of $\mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$ with $n = \sum_{i=1}^{\frac{4^k-1}{3}} m_i$. If the quaternary LCD $[n, k]$ code $C_k(m)$ has minimum weight at least d, then*

$$4d - 3n \leq m_i \leq n - \frac{4^{k-1} - 1}{3 \cdot 4^{k-2}} d, \tag{2}$$

for each $i \in \{1, 2, \ldots, \frac{4^k-1}{3}\}$.

The following lemma was given for binary LCD codes and ternary LCD codes [3, Lemmas 4.3 and 4.4]. By following the same line as in the proof of [3, Lemmas 4.3 and 4.4], we have the following lemma trivially.

Lemma 3.5. *Suppose that $\ell \geq 1$ and $k \geq 2$. Let C and C' be quaternary Hermitian LCD $[n, k]$ codes with dual distances $d(C^\perp) \geq 2$ and $d(C'^\perp) \geq 2$. Suppose that there are quaternary Hermitian LCD $[n, k]$ codes D and D' satisfying the following conditions:*

(i) *$C \cong D$ and $C' \cong D'$,*

(ii) *D and D' have generator matrices*

$$G = \left(\quad S_k^{(\ell)} \quad G_0 \quad \right) \text{ and } G' = \left(\quad S_k^{(\ell)} \quad G_0' \quad \right),$$

where G_0 and G_0' are generator matrices of some quaternary Hermitian LCD $[n - \frac{(4^k-1)\ell}{3}, k]$ codes C_0 and C_0', respectively.

Then $C \cong C'$ if and only if $C_0 \cong C_0'$.

4 Characterizations of quaternary Hermitian LCD codes

In the rest of this note, we use the following notation:

$$r_{n,k,d} = 4^{k-1}n - \frac{4^k - 1}{3}d, \tag{3}$$

for a given set of parameters n, k, d.

4.1 Theorem 4.1 and its proof

By following the same line as in the proof of Theorem 9 in [2], we have the following theorem.

Theorem 4.1. *Suppose that $4d - 3n \geq 1$ and $4r_{n,k,d} \geq k \geq 2$, where $r_{n,k,d}$ is the integer defined in (3). Then there is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^\perp \geq 2$ and equivalence classes of quaternary Hermitian LCD $[4r_{n,k,d}, k, 3r_{n,k,d}]$ codes with dual distances $d^\perp \geq 2$.*

Proof. Let C be a quaternary $[n, k, d]$ code with dual distance $d^\perp \geq 2$. Since $d^\perp \geq 2$, by Remark 3.2, there is a vector $m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}}) \in \mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$ such that $C \cong C_k(m)$ and $n = \sum_{i=1}^{\frac{4^k-1}{3}} m_i$. Since the minimum weight of $C_k(m)$ is d, we have

$$4d - 3n \leq m_i,$$

by Lemma 3.4. Thus, the generator matrix $G_k(m)$ in (1) of $C_k(m)$ consists of at least $4d - 3n$ columns $h_{k,i}$ for each $i \in \{1, 2, \ldots, \frac{4^k-1}{3}\}$. Note that

$4d - 3n \geq 1$ from the assumption. Hence, we obtain a matrix G of the following form:

$$G = \begin{pmatrix} S_k^{(4d-3n)} & G_0 \end{pmatrix}, \tag{4}$$

by permuting columns of $G_k(m)$. Here G_0 is a $k \times (n - \frac{(4^k-1)(4d-3n)}{3})$ matrix, noting that

$$n - \frac{(4^k - 1)(4d - 3n)}{3} = 4\left(4^{k-1}n - \frac{4^k - 1}{3}d\right) = 4r_{n,k,d}.$$

Since $S_k\overline{S_k}^T = O$, we have $G\overline{G}^T = G_0\overline{G_0}^T$. Since C is Hermitian LCD, we have

$$4r_{n,k,d} \geq \operatorname{rank}(G_0) \geq \operatorname{rank}(G_0\overline{G_0}^T) = \operatorname{rank}(G\overline{G}^T) = k. \tag{5}$$

Let C_0 be the quaternary code with generator matrix G_0. It follows from (5) that C_0 is a quaternary Hermitian $[4r_{n,k,d}, k]$ LCD code. From the assumption $k \geq 2$, the quaternary code C' with generator matrix $S_k^{(4d-3n)}$ is a Hermitian self-orthogonal $[n', k, d']$ code, where

$$n' = \frac{(4^k - 1)(4d - 3n)}{3} \text{ and } d' = (4d - 3n)4^{k-1}.$$

By Lemma 3.3, we have

$$d = d_0 + d' \text{ and } d_0 = 3\left(4^{k-1}n - \frac{4^k - 1}{3}d\right) = 3r_{n,k,d}.$$

Hence, if there is a quaternary Hermitian LCD $[n, k, d]$ code C with dual distance $d^\perp \geq 2$, then there is a quaternary Hermitian LCD $[n, k, d]$ code C' such that $C \cong C'$ and C' has generator matrix of form (4). In addition, G_0 is a generator matrix of some quaternary Hermitian LCD $[4r_{n,k,d}, k, 3r_{n,k,d}]$ code.

Now let C and C' be quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d(C^\perp) \geq 2$ and $d(C'^\perp) \geq 2$. By the above argument, there are quaternary Hermitian LCD $[n, k, d]$ codes D and D' satisfying the following conditions:

(i) $C \cong D$ and $C' \cong D'$,

(ii) D and D' have generator matrices

$$G = \begin{pmatrix} S_k^{(4d-3n)} & G_0 \end{pmatrix} \text{ and } G' = \begin{pmatrix} S_k^{(4d-3n)} & G'_0 \end{pmatrix},$$

where G_0 and G'_0 are generator matrices of some quaternary Hermitian LCD $[4r_{n,k,d}, k, 3r_{n,k,d}]$ codes C_0 and C'_0, respectively.

It follows from Lemma 3.5 that $C \cong C'$ if and only if $C_0 \cong C'_0$. This completes the proof. ☐

The above theorem says that for a given set of parameters n, k, d a classification of quaternary Hermitian LCD $[n, k, d]$ codes is obtained from that of quaternary Hermitian LCD $[4r_{n,k,d}, k, 3r_{n,k,d}]$ codes, where $4r_{n,k,d} \le n$.

4.2 Modification of Theorem 4.1

As the next step, by following the same line as in the proof of [3, Theorem 4.7], we modify Theorem 4.1 to the form which is used easily by adding some assumption (6) on minimum weights for our study in Section 5 (Theorem 4.3).

Assume that we write

$$n = \frac{4^k - 1}{3}s + t,$$

where $s \in \mathbb{Z}_{\ge 0}$ and $t \in \{0, 1, \ldots, \frac{4^k-1}{3} - 1\}$. In addition, assume the following:

$$\begin{aligned} &\text{the minimum weight } d \text{ is written as} \\ &d(s,t) = 4^{k-1}s + \alpha(t), \\ &\text{where } \alpha(t) \text{ is a constant depending on only } t. \end{aligned} \tag{6}$$

The condition $4d - 3n \ge 1$ in Theorem 4.1 is equivalent to that $s \ge s'_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$, where

$$s'_{(\frac{4^k-1}{3}s+t),k,d(s,t)} = \frac{4r_{(\frac{4^k-1}{3}s+t),k,d(s,t)} - t}{\frac{4^k-1}{3}} + 1. \tag{7}$$

From (3), we have

$$\begin{aligned} r_{(\frac{4^k-1}{3}s+t),k,d(s,t)} &= 4^{k-1}\left(\frac{4^k - 1}{3}s + t\right) - \frac{4^k - 1}{3}d(s,t) \\ &= 4^{k-1}t - \frac{4^k - 1}{3}\alpha(t). \end{aligned} \tag{8}$$

From (7) and (8), we have the following:

Lemma 4.2. *Both* $r_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *and* $s'_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *depend on only* k, t *and do not depend on* s*.*

From (7), we have

$$4r_{(\frac{4^k-1}{3}s+t),k,d(s,t)} = \frac{4^k-1}{3}\left(s'_{(\frac{4^k-1}{3}s+t),k,(4^{k-1}s+\alpha(t))} - 1\right) + t. \tag{9}$$

From (8) and (9), we have

$$\begin{aligned} 3r_{(\frac{4^k-1}{3}s+t),k,d(s,t)} &= 3\left(4^{k-1}t - \frac{4^k-1}{3}\alpha(t)\right) \\ &= \frac{3}{4}\left(\frac{4^k-1}{3}\left(s'_{(\frac{4^k-1}{3}s+t),k,(4^{k-1}s+\alpha(t))} - 1\right) + t\right) \\ &= 4^{k-1}\left(s'_{(\frac{4^k-1}{3}s+t),k,(4^{k-1}s+\alpha(t))} - 1\right) \\ &\quad + \frac{1}{4}\left(-\left(s'_{(\frac{4^k-1}{3}s+t),k,(4^{k-1}s+\alpha(t))} - 1\right) + 3t\right) \\ &= 4^{k-1}\left(s'_{(\frac{4^k-1}{3}s+t),k,(4^{k-1}s+\alpha(t))} - 1\right) + \alpha(t). \end{aligned} \tag{10}$$

By Lemma 4.2, (9) and (10), we have the following:

Theorem 4.3. *Write* $n = \frac{4^k-1}{3}s+t \geq k$*, where* $s \in \mathbb{Z}_{\geq 0}$ *and* $t \in \{0, 1, \dots, \frac{4^k-1}{3} - 1\}$*. Assume that* d *is written as* $d(s,t) = 4^{k-1}s+\alpha(t)$*, where* $\alpha(t)$ *is a constant depending on* t*. Let* r *denote the integer* $r_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *defined in* (3)*. Let* s' *denote the integer* $s'_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *defined in* (7)*. Suppose that* $4r \geq k \geq 2$*. Then there is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD codes with dual distances* $d^\perp \geq 2$ *and parameters*

$$[4r, k, 3r] = \left[\frac{4^k-1}{3}(s'-1) + t, k, 4^{k-1}(s'-1) + \alpha(t)\right]$$

and equivalence classes of quaternary Hermitian LCD code with dual distances $d^\perp \geq 2$ *and parameters*

$$\left[\frac{4^k-1}{3}s + t, k, 4^{k-1}s + \alpha(t)\right],$$

for every integer $s \geq s'$*.*

The above theorem says that for a given set of parameters $k, t, \alpha(t)$ a classification of quaternary Hermitian LCD $[\frac{4^k-1}{3}(s'-1)+t, k, 4^{k-1}(s'-1)+\alpha(t)]$ codes yields that of quaternary Hermitian LCD $[\frac{4^k-1}{3}s+t, k, 4^{k-1}s+\alpha(t)]$ codes for every integer $s \geq s'$. We remark that the assumption (6) on the minimum weight is automatically satisfied for our study in Section 5.

4.3 Consequence of Theorem 4.3

We end this section by giving a consequence of Theorem 4.3.

Corollary 4.4. *Write* $n = \frac{4^k-1}{3}s+t \geq k$*, where* $s \in \mathbb{Z}_{\geq 0}$ *and* $t \in \{0, 1, \dots, \frac{4^k-1}{3}-1\}$*. Assume that* d *is written as* $d(s,t) = 4^{k-1}s+\alpha(t)$*, where* $\alpha(t)$ *is a constant depending on* t*. Let* r *denote the integer* $r_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *defined in* (3)*. Let* s' *denote the integer* $s'_{(\frac{4^k-1}{3}s+t),k,d(s,t)}$ *defined in* (7)*. Suppose that* $4r \geq k \geq 2$*. If there is no quaternary Hermitian LCD code with dual distance* $d^\perp \geq 2$ *and parameters*

$$[4r, k, 3r] = \left[\frac{4^k-1}{3}(s'-1)+t, k, 4^{k-1}(s'-1)+\alpha(t)\right],$$

then there is no quaternary Hermitian LCD code with dual distance $d^\perp \geq 2$ *and parameters*

$$\left[\frac{4^k-1}{3}s+t, k, 4^{k-1}s+\alpha(t)\right],$$

for every integer s*.*

Proof. For $s \geq s'$, the assertion follows directly from Theorem 4.3.

Suppose that there is a quaternary Hermitian LCD $[\frac{4^k-1}{3}s''+t, k, 4^{k-1}s''+\alpha(t)]$ code with dual distance $d^\perp \geq 2$ for some $s'' < s'-1$. Then, by Lemma 3.3, there is a quaternary Hermitian LCD $[\frac{4^k-1}{3}(s'-1)+t, k, 4^{k-1}(s'-1)+\alpha(t)]$ code with dual distance $d^\perp \geq 2$. This is a contradiction. □

The above corollary is an improvement of [2, Theorem 9 (ii)] by adding some assumption (6) on minimum weights.

5 Quaternary optimal Hermitian LCD codes

In this section, by Theorem 4.3, we give a classification of quaternary optimal Hermitian LCD codes of dimension 2 and a classification of quaternary optimal Hermitian LCD codes of dimension 3.

5.1 Classification method

Here we suppose that $k \in \{2,3\}$. As described in Remark 3.2, it is possible to find representatives of all equivalence classes of quaternary Hermitian LCD $[n,k]$ codes with dual distances $d^{\perp} \geq 2$ as $C_k(m)$, by considering all vectors $m = (m_1, m_2, \ldots, m_{\frac{4^k-1}{3}}) \in \mathbb{Z}_{\geq 0}^{\frac{4^k-1}{3}}$ satisfying $n = \sum_{i=1}^{\frac{4^k-1}{3}} m_i$ and the condition (2). In addition, any quaternary $[n,k,d]$ code is equivalent to some code with generator matrix of form $\begin{pmatrix} I_k & A \end{pmatrix}$, where A is a $k \times (n-k)$ matrix and the weight of the first row of A is exactly $d-1$. Hence, we may assume without loss of generality that

$$m_1 \geq 1, m_2 \geq 1 \text{ and } \sum_{i \in \mathcal{S}_k} m_i = d \text{ if } k = 2, \tag{11}$$

$$m_1 \geq 1, m_2 \geq 1, m_6 \geq 1 \text{ and } \sum_{i \in \mathcal{S}_k} m_i = d \text{ if } k = 3, \tag{12}$$

where $\mathcal{S}_k$ denotes the support of the first row of S_k. In this way, we found all quaternary Hermitian LCD $[n,k,d]$ codes which must be checked further for equivalences. For calculations of determinants of $G\overline{G}^T$ for generator matrices G, the NTL function `determinant` [15] was used. To test equivalence of quaternary codes, we used the algorithm given in [9, Section 7.3.3] as follows. For a quaternary $[n,k]$ code C, define the digraph $\Gamma(C)$ with vertex set V and arc set A, where

$$\begin{aligned}
V &= C \cup (\{1,2,\ldots,n\} \times (\mathbb{F}_4 \setminus \{0\})), \\
A &= \{(c,(j,c_j)) \mid c = (c_1,c_2,\ldots,c_n) \in C, j \in \{1,2,\ldots,n\}\} \\
&\quad \cup \{((j,y),(j,\omega y)) \mid j \in \{1,2,\ldots,n\},\ y \in \mathbb{F}_4 \setminus \{0\}\}.
\end{aligned}$$

Then, two quaternary $[n,k]$ codes C and C' are equivalent if and only if $\Gamma(C)$ and $\Gamma(C')$ are isomorphic. We used NAUTY [14] for digraph isomorphism testing.

All computer calculations in this section were done by programs in the language C. In addition, few verification was done by MAGMA [4]. Let $\mathcal{C}_{n,k,d}$ denote our equivalence classes of quaternary Hermitian LCD $[n, k, d]$ codes with dual distances $d^\perp \geq 2$ obtained by the above method. Especially, we verified by MAGMA that C is a quaternary Hermitian LCD $[n, k, d]$ code with dual distance $d^\perp \geq 2$ for $C \in \mathcal{C}_{n,k,d}$, and C and C' are inequivalent for $C, C' \in \mathcal{C}_{n,k,d}$ with $C \neq C'$.

5.2 Quaternary optimal Hermitian $[n, 2]$ LCD codes

The largest minimum weights $d_4(n, 2)$ were determined in [2], where $d_4(n, 2)$ are listed in Table 1. Recently, Ishizuka [8] has completed a classification of quaternary optimal Hermitian LCD codes of dimension 2. Here we present an alternative approach to the classification by using Theorem 4.3.

For $n \geq 2$, write $n = 5s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 4\}$. Let $r = r_{5s+t,2,d_4(5s+t,2)}$ and $s' = s'_{5s+t,2,d_4(5s+t,2)}$ be the integers defined in (3) and (7), respectively. For each $5s+t$, we list $d_4(5s+t, 2)$, s' and r in Table 1. Then $d_4(5s+t, 2)$ is written as $4s+\alpha(t)$, where $\alpha(t)$ is a constant depending on only t. Since $d_4(5s+t, 2)$ satisfies the assumption (6) in Theorem 4.3, we have the following:

Proposition 5.1. *There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^\perp \geq 2$ and equivalence classes of quaternary Hermitian LCD $[5s + t, 2, d_4(5s + t, 2)]$ code with dual distances $d^\perp \geq 2$ for every integer $s \geq s'$, where r and s' are listed in Table 1.*

Table 1: $d_4(n, 2)$, s' and r

n	$d_4(n,2)$	s'	r	n	$d_4(n,2)$	s'	r
$5s$	$4s-1$	5	5	$5s+3$	$4s+2$	2	2
$5s+1$	$4s$	4	4	$5s+4$	$4s+2$	5	6
$5s+2$	$4s+1$	3	3				

By the method given in Section 5.1, our computer search completed a classification of all quaternary optimal Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^{\perp} \geq 2$ for r listed in Table 1. The numbers $N_4(4r, 2)$ of all inequivalent quaternary optimal Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^{\perp} \geq 2$ are listed in Table 2. In addition, our computer search completed a classification of all quaternary optimal Hermitian LCD $[5s + t, 2, d_4(5s + t, 2)]$ codes with dual distances $d^{\perp} \geq 2$ for $s < s'$, where s' is given in Table 1. The numbers $N_4(5s + t, 2)$ of all inequivalent quaternary optimal Hermitian LCD $[5s + t, 2, d_4(5s + t, 2)]$ codes with dual distances $d^{\perp} \geq 2$ are also listed in Table 2. From Proposition 5.1 and Table 2, we have the following:

Proposition 5.2. (i) *Suppose that $t \in \{0, 1\}$. Then there are 2 inequivalent quaternary optimal Hermitian LCD $[5s + t, 2]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 2$.*

(ii) *Suppose that $t \in \{2, 3\}$. Then there is a unique quaternary optimal Hermitian LCD $[5s + t, 2]$ codes with dual distances $d^{\perp} \geq 2$, up to equivalence, for every integer $s \geq 0$.*

(iii) *There are 5 inequivalent quaternary optimal Hermitian LCD $[5s+4, 2]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 3$.*

Of course, the above classification coincides with that given in [8].

Table 2: $N_4(n, 2)$

n	$N_4(4r, 2)$	$N_4(5s + t, 2)$ $(s < s')$		
$5s$	$N_4(20, 2) = 2$	$N_4(5, 2) = 1$	$N_4(10, 2) = 2$	$N_4(15, 2) = 2$
$5s + 1$	$N_4(16, 2) = 2$	$N_4(6, 2) = 1$	$N_4(11, 2) = 2$	
$5s + 2$	$N_4(12, 2) = 1$	$N_4(2, 2) = 1$	$N_4(7, 2) = 1$	
$5s + 3$	$N_4(8, 2) = 1$	$N_4(3, 2) = 1$		
$5s + 4$	$N_4(24, 2) = 5$	$N_4(4, 2) = 1$	$N_4(9, 2) = 3$	$N_4(14, 2) = 4$
		$N_4(19, 2) = 5$		

5.3 Quaternary Hermitian LCD $[n, 2, d_4(n, 2) - 1]$ codes

Similar to the previous subsection, here we give a classification of quaternary near-optimal Hermitian LCD $[n, 2, d_4(n, 2) - 1]$ codes. Similar to Proposition 5.1, we have the following:

Proposition 5.3. *There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^{\perp} \geq 2$ and equivalence classes of quaternary Hermitian LCD $[5s+t, 2, d_4(5s+t, 2)-1]$ code with dual distances $d^{\perp} \geq 2$ for every integer $s \geq s'$, where r and s' are listed in Table 3.*

Table 3: $d_4(n, 2) - 1$, s' and r

n	$d_4(n,2)-1$	s'	r	n	$d_4(n,2)-1$	s'	r
$5s$	$4s-2$	9	10	$5s+3$	$4s$	10	12
$5s+1$	$4s-1$	8	9	$5s+4$	$4s+1$	9	11
$5s+2$	$4s$	7	8				

Our computer search completed a classification of all quaternary near-optimal Hermitian LCD $[4r, 2, 3r]$ codes with dual distances $d^{\perp} \geq 2$ for r listed in Table 3 and all quaternary near-optimal Hermitian LCD $[5s+t, 2, d_4(5s+t, 2)-1]$ codes with dual distances $d^{\perp} \geq 2$ for $s < s'$, where s' is given in Table 3. In Table 4, we list the numbers $N'_4(4r, 2)$ of the inequivalent quaternary near-optimal Hermitian LCD $[4r, 2, 3r]$ codes and the numbers $N'_4(5s+t, 2)$ of the inequivalent quaternary near-optimal Hermitian LCD $[5s+t, 2, d_4(5s+t, 2)-1]$ codes. From Proposition 5.3 and Table 4, we have the following:

Proposition 5.4. (i) *There are* 15 *inequivalent quaternary near-optimal Hermitian LCD $[5s, 2, 4s-2]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 7$.*

(ii) *There are* 7 *inequivalent quaternary near-optimal Hermitian LCD $[5s+1, 2, 4s-1]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 5$.*

(iii) *There are* 8 *inequivalent quaternary near-optimal Hermitian LCD $[5s+2, 2, 4s]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 5$.*

(iv) *There are* 22 *inequivalent quaternary near-optimal Hermitian LCD $[5s+3, 2, 4s]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 8$.*

(v) *There are* 12 *inequivalent quaternary near-optimal Hermitian LCD $[5s+4, 2, 4s+1]$ codes with dual distances $d^{\perp} \geq 2$ for every integer $s \geq 6$.*

Table 4: $N'_4(n, 2)$

n	$N'_4(4r, 2)$	$N'_4(5s + t, 2)$ $(s < s')$		
$5s$	$N'_4(40, 2) = 15$	$N'_4(5, 2) = 1$	$N'_4(10, 2) = 5$	$N'_4(15, 2) = 9$
		$N'_4(20, 2) = 11$	$N'_4(25, 2) = 13$	$N'_4(30, 2) = 14$
		$N'_4(35, 2) = 15$		
$5s + 1$	$N'_4(36, 2) = 7$	$N'_4(6, 2) = 2$	$N'_4(11, 2) = 3$	$N'_4(16, 2) = 5$
		$N'_4(21, 2) = 6$	$N'_4(26, 2) = 7$	$N'_4(31, 2) = 7$
$5s + 2$	$N'_4(32, 2) = 8$	$N'_4(7, 2) = 2$	$N'_4(12, 2) = 4$	$N'_4(17, 2) = 6$
		$N'_4(22, 2) = 7$	$N'_4(27, 2) = 8$	
$5s + 3$	$N'_4(48, 2) = 22$	$N'_4(8, 2) = 2$	$N'_4(13, 2) = 7$	$N'_4(18, 2) = 12$
		$N'_4(23, 2) = 16$	$N'_4(28, 2) = 18$	$N'_4(33, 2) = 20$
		$N'_4(38, 2) = 21$	$N'_4(43, 2) = 22$	
$5s + 4$	$N'_4(44, 2) = 12$	$N'_4(4, 2) = 1$	$N'_4(9, 2) = 2$	$N'_4(14, 2) = 7$
		$N'_4(19, 2) = 8$	$N'_4(24, 2) = 10$	$N'_4(29, 2) = 11$
		$N'_4(34, 2) = 12$	$N'_4(39, 2) = 12$	

For each C of the quaternary codes listed in Table 4, there is a vector $(m_1, m_2, \dots, , m_5) \in \mathbb{Z}^5_{\geq 0}$ with $C_2(m) \cong C$. For

$$(n, d) = (32, 24), (36, 27), (40, 30), (44, 33), (48, 36),$$

the corresponding vectors $(m_1, m_2, \dots, m_5)$ are listed in Table 5. For the codes listed in Table 4, let V_{5s+t} be the set of the corresponding vectors $(m_1, m_2, \dots, m_5)$ for the inequivalent quaternary near-optimal Hermitian LCD $[5s+t, 2, d_4(5s+t, 2)-1]$ codes with dual distances $d^{\perp} \geq 2$. We verified that $V_{5(s-1)+t}$ is obtained as:

$$\{(m_1 - 1, m_2 - 1, \dots, m_5 - 1) \in \mathbb{Z}^5_{\geq 0} \mid (m_1, m_2, \dots, m_5) \in V_{5s+t}\}$$

for $s \leq s'$. The corresponding vectors $(m_1, m_2, \dots, , m_5)$ can be also obtained electronically from `http://www.math.is.tohoku.ac.jp/~mharada/qLCD/`.

5.4 Quaternary optimal Hermitian $[n, 3]$ LCD codes

The largest minimum weights $d_4(n, 3)$ were determined in [2], where $d_4(n, 3)$ are listed in Table 6. In this subsection, we complete a classification of quaternary optimal Hermitian LCD codes of dimension 3.

Table 5: $(m_1, m_2, m_3, m_4, m_5)$

(n,d)	$(m_1, m_2, m_3, m_4, m_5)$			
$(32, 24)$	$(3, 8, 8, 8, 5)$	$(5, 8, 5, 6, 8)$	$(6, 8, 5, 7, 6)$	$(6, 8, 7, 4, 7)$
	$(6, 8, 8, 7, 3)$	$(7, 8, 4, 5, 8)$	$(8, 8, 2, 7, 7)$	$(8, 8, 8, 7, 1)$
$(36, 27)$	$(6, 9, 7, 6, 8)$	$(6, 9, 9, 6, 6)$	$(8, 9, 3, 8, 8)$	$(8, 9, 8, 2, 9)$
	$(8, 9, 8, 4, 7)$	$(8, 9, 8, 5, 6)$	$(8, 9, 9, 6, 4)$	
$(40, 30)$	$(4, 10, 10, 7, 9)$	$(6, 10, 7, 9, 8)$	$(6, 10, 7, 10, 7)$	$(6, 10, 9, 10, 5)$
	$(7, 10, 5, 8, 10)$	$(8, 10, 7, 8, 7)$	$(8, 10, 9, 5, 8)$	$(8, 10, 9, 6, 7)$
	$(9, 10, 9, 6, 6)$	$(9, 10, 9, 8, 4)$	$(9, 10, 10, 2, 9)$	$(10, 10, 1, 9, 10)$
	$(10, 10, 9, 3, 8)$	$(10, 10, 10, 5, 5)$	$(10, 10, 10, 7, 3)$	
$(44, 33)$	$(4, 11, 9, 10, 10)$	$(7, 11, 10, 8, 8)$	$(8, 11, 8, 8, 9)$	$(8, 11, 11, 6, 8)$
	$(9, 11, 6, 10, 8)$	$(10, 11, 10, 3, 10)$	$(10, 11, 10, 6, 7)$	$(10, 11, 8, 5, 10)$
	$(10, 11, 9, 8, 6)$	$(10, 11, 11, 10, 2)$	$(11, 11, 6, 10, 6)$	$(11, 11, 8, 10, 4)$
$(48, 36)$	$(5, 12, 12, 11, 8)$	$(8, 12, 9, 8, 11)$	$(8, 12, 10, 11, 7)$	$(9, 12, 9, 12, 6)$
	$(9, 12, 10, 9, 8)$	$(9, 12, 10, 10, 7)$	$(9, 12, 11, 6, 10)$	$(10, 12, 6, 9, 11)$
	$(10, 12, 11, 5, 10)$	$(11, 12, 1, 12, 12)$	$(11, 12, 2, 11, 12)$	$(11, 12, 6, 8, 11)$
	$(11, 12, 7, 10, 8)$	$(11, 12, 10, 12, 3)$	$(11, 12, 11, 4, 10)$	$(12, 12, 3, 9, 12)$
	$(12, 12, 4, 9, 11)$	$(12, 12, 5, 7, 12)$	$(12, 12, 6, 7, 11)$	$(12, 12, 7, 9, 8)$
	$(12, 12, 9, 10, 5)$	$(12, 12, 10, 7, 7)$		

We apply Theorem 4.3 to $k = 3$. For $n \geq 3$, write $n = 21s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 20\}$. Let $r = r_{21s+t,3,d_4(21s+t,3)}$ and $s' = s'_{21s+t,3,d_4(21s+t,3)}$ be the integers defined in (3) and (7), respectively. For each $21s + t$, we list $d_4(21s+t, 3)$, s' and r in Table 6. Then $d_4(21s+t, 3)$ is written as $16s+\alpha(t)$, where $\alpha(t)$ is a constant depending on only t. Since $d_4(21s + t, 3)$ satisfies the assumption (6) in Theorem 4.3, we have the following:

Proposition 5.5. *There is a one-to-one correspondence between equivalence classes of quaternary Hermitian LCD* $[4r, 3, 3r]$ *codes with dual distances* $d^{\perp} \geq 2$ *and equivalence classes of quaternary Hermitian LCD* $[21s + t, 3, d_4(21s + t, 3)]$ *code with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq s'$.

By the method given in Section 5.1, our computer search completed a classification of all quaternary optimal Hermitian LCD $[4r, 3, 3r]$ codes with dual distances $d^{\perp} \geq 2$ for r listed in Table 6. The numbers $N_4(4r, 3)$ of all inequivalent quaternary optimal Hermitian LCD $[4r, 3, 3r]$ codes with dual distances $d^{\perp} \geq 2$ are listed in Table 7. In addition, our computer search completed a classification of all quaternary optimal Hermitian LCD $[21s + t, 3, d_4(21s + t, 3)]$ codes with dual distances $d^{\perp} \geq 2$ for $s < s'$, where s' is

Table 6: $d_4(n,3)$, s' and r

n	$d_4(n,3)$	s'	r	n	$d_4(n,3)$	s'	r
$21s$	$16s-1$	5	21	$21s+11$	$16s+7$	6	29
$21s+1$	$16s-1$	8	37	$21s+12$	$16s+8$	5	24
$21s+2$	$16s$	7	32	$21s+13$	$16s+9$	4	19
$21s+3$	$16s+1$	6	27	$21s+14$	$16s+9$	7	35
$21s+4$	$16s+2$	5	22	$21s+15$	$16s+10$	6	30
$21s+5$	$16s+3$	4	17	$21s+16$	$16s+11$	5	25
$21s+6$	$16s+3$	7	33	$21s+17$	$16s+12$	4	20
$21s+7$	$16s+4$	6	28	$21s+18$	$16s+13$	3	15
$21s+8$	$16s+5$	5	23	$21s+19$	$16s+13$	6	31
$21s+9$	$16s+6$	4	18	$21s+20$	$16s+14$	5	26
$21s+10$	$16s+6$	7	34				

given in Table 6. The numbers $N_4(21s+t,3)$ of all inequivalent quaternary optimal Hermitian LCD $[21s+t,3,d_4(21s+t,3)]$ codes with dual distances $d^{\perp} \geq 2$ are also listed in Table 7. From Proposition 5.5 and Table 7, we have the following:

Theorem 5.6. (i) *Suppose that* $t \in \{0,12\}$*. Then there are* 7 *inequivalent quaternary optimal Hermitian LCD* $[21s+t,3]$ *codes with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq 2$ *if* $t=0$ *and* $s \geq 1$ *if* $t=12$*.*

(ii) *There are* 12808 *inequivalent quaternary optimal Hermitian LCD* $[21s+1,3]$ *codes with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq 5$*.*

(iii) *Suppose that* $t \in \{2,11\}$*. Then there are* 318 *inequivalent quaternary optimal Hermitian LCD* $[21s+t,3]$ *codes with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq 3$*.*

(iv) *Suppose that* $t \in \{3,15\}$*. Then there are* 147 *inequivalent quaternary optimal Hermitian LCD* $[21s+t,3]$ *codes with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq 3$ *if* $t=3$ *and* $s \geq 2$ *if* $t=15$*.*

(v) *Suppose that* $t \in \{4,13\}$*. Then there are* 4 *inequivalent quaternary optimal Hermitian LCD* $[21s+t,3]$ *codes with dual distances* $d^{\perp} \geq 2$ *for every integer* $s \geq 1$*.*

Table 7: $N_4(n, 3)$

n	$N_4(4r,3)$	$N_4(21s+t,3)$ $(s<s')$		
$21s$	$N_4(84,3)=7$	$N_4(21,3)=5$	$N_4(42,3)=7$	$N_4(63,3)=7$
$21s+1$	$N_4(148,3)=12808$	$N_4(22,3)=1871$	$N_4(43,3)=9793$	$N_4(64,3)=12405$
		$N_4(85,3)=12781$	$N_4(106,3)=12808$	$N_4(127,3)=12808$
$21s+2$	$N_4(128,3)=318$	$N_4(23,3)=135$	$N_4(44,3)=288$	$N_4(65,3)=318$
		$N_4(86,3)=318$	$N_4(107,3)=318$	
$21s+3$	$N_4(108,3)=147$	$N_4(24,3)=73$	$N_4(45,3)=138$	$N_4(66,3)=147$
		$N_4(87,3)=147$		
$21s+4$	$N_4(88,3)=4$	$N_4(4,3)=0$	$N_4(25,3)=4$	$N_4(46,3)=4$
		$N_4(67,3)=4$		
$21s+5$	$N_4(68,3)=1$	$N_4(5,3)=0$	$N_4(26,3)=1$	$N_4(47,3)=1$
$21s+6$	$N_4(132,3)=2162$	$N_4(6,3)=2$	$N_4(27,3)=937$	$N_4(48,3)=1948$
		$N_4(69,3)=2145$	$N_4(90,3)=2162$	$N_4(111,3)=2162$
$21s+7$	$N_4(112,3)=44$	$N_4(7,3)=1$	$N_4(28,3)=30$	$N_4(49,3)=44$
		$N_4(70,3)=44$	$N_4(91,3)=44$	
$21s+8$	$N_4(92,3)=23$	$N_4(8,3)=1$	$N_4(29,3)=18$	$N_4(50,3)=23$
		$N_4(71,3)=23$		
$21s+9$	$N_4(72,3)=1$	$N_4(9,3)=1$	$N_4(30,3)=1$	$N_4(51,3)=1$
$21s+10$	$N_4(136,3)=947$	$N_4(10,3)=13$	$N_4(31,3)=589$	$N_4(52,3)=889$
		$N_4(73,4)=947$	$N_4(94,3)=947$	$N_4(115,3)=947$
$21s+11$	$N_4(116,3)=318$	$N_4(11,3)=13$	$N_4(32,3)=220$	$N_4(53,4)=309$
		$N_4(74,3)=318$	$N_4(95,3)=318$	
$21s+12$	$N_4(96,3)=7$	$N_4(12,3)=2$	$N_4(33,3)=7$	$N_4(54,3)=7$
		$N_4(75,3)=7$		
$21s+13$	$N_4(76,3)=4$	$N_4(13,3)=2$	$N_4(34,3)=4$	$N_4(55,3)=4$
$21s+14$	$N_4(140,3)=5736$	$N_4(14,3)=156$	$N_4(35,3)=3562$	$N_4(56,3)=5398$
		$N_4(77,3)=5709$	$N_4(98,3)=5736$	$N_4(119,3)=5736$
$21s+15$	$N_4(120,3)=147$	$N_4(15,3)=28$	$N_4(36,3)=118$	$N_4(57,3)=147$
		$N_4(78,3)=147$	$N_4(99,3)=147$	
$21s+16$	$N_4(100,3)=44$	$N_4(16,3)=10$	$N_4(37,3)=39$	$N_4(58,3)=44$
		$N_4(79,3)=44$		
$21s+17$	$N_4(80,3)=1$	$N_4(17,3)=1$	$N_4(38,3)=1$	$N_4(59,3)=1$
$21s+18$	$N_4(60,3)=1$	$N_4(18,3)=1$	$N_4(39,3)=1$	
$21s+19$	$N_4(124,3)=947$	$N_4(19,3)=196$	$N_4(40,3)=774$	$N_4(61,3)=930$
		$N_4(82,3)=947$	$N_4(103,3)=947$	
$21s+20$	$N_4(104,3)=23$	$N_4(20,3)=10$	$N_4(41,3)=23$	$N_4(62,3)=23$
		$N_4(83,3)=23$		

(vi) *Suppose that* $t \in \{5, 9, 17, 18\}$. *Then there is a unique quaternary optimal Hermitian LCD* $[21s+t, 3]$ *code with dual distance* $d^\perp \geq 2$, *up equivalence, for every integer* $s \geq 1$ *if* $t = 5$ *and* $s \geq 0$ *if* $t \in \{9, 17, 18\}$.

(vii) *There are* 2162 *inequivalent quaternary optimal Hermitian LCD* $[21s+6, 3]$ *codes with dual distances* $d^\perp \geq 2$ *for every integer* $s \geq 4$.

(viii) *Suppose that* $t \in \{7, 16\}$. *Then there are* 44 *inequivalent quaternary optimal Hermitian LCD* $[21s+t, 3]$ *codes with dual distances* $d^\perp \geq 2$ *for every integer* $s \geq 2$.

(ix) *Suppose that* $t \in \{8, 20\}$. *Then there are* 23 *inequivalent quaternary optimal Hermitian LCD* $[21s+t, 3]$ *codes with dual distances* $d^\perp \geq 2$ *for every integer* $s \geq 2$ *if* $t = 8$ *and* $s \geq 1$ *if* $t = 20$.

(x) *Suppose that* $t \in \{10, 19\}$. *Then there are* 947 *inequivalent quaternary optimal Hermitian LCD* $[21s+t, 3]$ *codes with dual distances* $d^\perp \geq 2$ *for every integer* $s \geq 3$.

(xi) *There are* 5736 *inequivalent quaternary optimal Hermitian LCD* $[21s+14, 3]$ *codes with dual distances* $d^\perp \geq 2$ *for every integer* $s \geq 4$.

For each C of the quaternary codes listed in Table 7, there is a vector $m \in \mathbb{Z}_{\geq 0}^{21}$ with $C_3(m) \cong C$. The vectors m can be obtained electronically from `http://www.math.is.tohoku.ac.jp/~mharada/qLCD/`. For the codes listed in Table 7, let V_{21s+t} be the set of the corresponding vectors $(m_1, m_2, \dots, m_{21})$ for the inequivalent quaternary optimal Hermitian LCD $[21s+t, 3]$ codes with dual distances $d^\perp \geq 2$. We verified that $V_{21(s-1)+t}$ is obtained as:

$$\{(m_1 - 1, m_2 - 1, \dots, m_{21} - 1) \in \mathbb{Z}_{\geq 0}^{21} \mid (m_1, m_2, \dots, m_{21}) \in V_{21s+t}\}$$

for $s \leq s'$.

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