

ON THE CLASSIFICATION OF SMOOTH PROJECTIVE TORIC VARIETIES

VICTOR V. BATYREV

(Received September 11, 1990, revised May 13, 1991)

Abstract. We investigate the problem of the classification of smooth projective toric varieties V of dimension d with a given Picard number ρ over an algebraically closed field. For that purpose we introduce a convenient combinatorial description of such varieties by means of primitive relations among $d + \rho$ integral generators of the associated complete regular fan of convex cones in d -dimensional real space. The main conjecture asserts that the number of the primitive relations is bounded by an absolute constant depending only on ρ . We prove this conjecture for $\rho \leq 3$ and give the classification of d -dimensional smooth complete toric varieties with $\rho = 3$.

1. Introduction. Let k be an arbitrary algebraically closed field. A d -dimensional algebraic torus T is a product of d copies of the multiplicative group k^* of k . A toric variety V is a normal algebraic variety containing T as a Zariski open dense subset with an algebraic action of T on V which extends the group law of T . Any toric variety can be described by a finite system of cones spanned by integer points in the real space \mathbf{R}^d . The reader is referred to [1] for the precise definitions.

In this paper we restrict ourselves to complete smooth toric varieties V . Moreover, we shall often assume that V is a projective toric variety.

One can notice that any description of smooth toric varieties has two sides: the combinatorial structure of the corresponding fan and unimodularity conditions on its generators. The weighted triangulations of $(d-1)$ -dimensional sphere introduced in [7] is an example of such a description. One of our objectives is to give a new description of complete smooth toric varieties.

In §2 we introduce the notion of a *primitive collection* of generators and the notion of an associated *primitive relations* among generators. We use these notions to describe toric varieties. If a toric variety V is projective we define also the *degree* of a primitive relation and the *distance* between a generator and a d -dimensional cone of the corresponding fan $\Sigma(V)$.

All these notions are used in §3 to get some properties of the combinatorial structure of a d -dimensional fan $\Sigma(V)$ associated with a toric variety V . It should be remarked that if the Picard number $\rho(V) \geq 3$ there exist combinatorial types of simplicial polytopes which do not give rise to any complete regular fan defining a smooth toric variety [2]. We prove that an arbitrary d -dimensional projective regular fan of cones has a primitive

collection $\mathcal{P} = \{x_1, \dots, x_k\}$ of its generators such that $x_1 + \dots + x_k = 0$. The last statement is a generalization of a result of Oda in [7] for $d=2$.

Our next purpose is the classification of several types of smooth complete toric varieties. This problem for $d \leq 3$ was investigated by Oda and Miyake in [7]. They obtained the list of all 3-dimensional smooth complete toric varieties with the Picard number $\rho \leq 5$ which cannot be blown down. It is easy to see that the projective space is the unique smooth complete d -dimensional toric variety with $\rho=1$. Recently Kleinschmidt [4] has classified all smooth complete d -dimensional toric varieties with $\rho=2$. It turns out that all such varieties are projectivizations of a decomposable bundle over a projective space of a smaller dimension. In this paper we give two generalizations of this result of Kleinschmidt. First in §4 we give a criterion for a smooth complete d -dimensional toric variety V to be produced from a projective space by a sequence of projectivizations of decomposable bundles. On the other hand, in §§5–6 we give the classification of all smooth complete d -dimensional toric varieties with $\rho=3$.

In §5 we prove strong combinatorial restrictions on a d -dimensional fan Σ with $d+3$ generators which generalize the result of Gretenkort, Kleinschmidt and Sturmfels [2]. After that in §6 we find all primitive relations describing Σ . Finally, in §7 we state some open questions.

ACKNOWLEDGEMENTS. I appreciate the hospitality of the Institute of Mathematics of Warsaw University, where I began to prepare this paper. I would like to thank Professors J. Jurkiewicz and J. Wisniewski for many useful conversations.

I would like to thank also the hospitality of the Mathematical Institute of Tôhoku University. Especially, I am grateful to Professor T. Oda whose numerous valuable remarks helped me to improve the paper.

2. Basic definitions. We first recall some standard definitions used in the geometry of toric varieties (see [1]).

2.1. DEFINITION. A convex subset $\sigma \subset \mathbf{R}^d$ is called a *regular d -dimensional cone* if there exists a \mathbf{Z} -basis $\{e_1, \dots, e_d\}$ of the integer lattice $\mathbf{Z}^d \subset \mathbf{R}^d$ such that

$$\sigma = \{\lambda_1 e_1 + \dots + \lambda_d e_d \mid \lambda_i \in \mathbf{R}, \lambda_i \geq 0\}.$$

In this case the elements e_1, \dots, e_d are called *generators* of σ .

2.2. DEFINITION. Let $\sigma \in \mathbf{R}^d$ be an arbitrary regular d -dimensional cone with generators $e_1, \dots, e_d \in \mathbf{Z}^d$. For any subset $E \subset \{e_1, \dots, e_d\}$ we denote by $L(E)$ the linear hull of E (if $E = \emptyset$, we let $L(E) = 0$). Then we call $\sigma' = L(E) \cap \sigma$ a *face* of σ and we write $\sigma' < \sigma$.

2.3. DEFINITION. A convex subset $\sigma' \in \mathbf{R}^d$ is called a *regular k -dimensional cone* if there exist a regular d -dimensional cone $\sigma \in \mathbf{R}^d$ and a subset E of its generators such

that $k = \dim L(E)$ and $\sigma' = L(E) \cap \sigma$ is a face of σ . In this case we call E the set of generators of σ' .

2.4. DEFINITION. A finite system $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of regular cones in \mathbf{R}^d is called a complete regular d -dimensional fan if the following conditions hold:

- (i) if $\sigma \in \Sigma$ and $\sigma' < \sigma$ then $\sigma' \in \Sigma$;
- (ii) if σ, σ' are in Σ , then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$;
- (iii) $\mathbf{R}^d = \sigma_1 \cup \dots \cup \sigma_s$.

We call any generator of a cone $\sigma \in \Sigma$ a generator of Σ .

Every complete regular d -dimensional fan Σ is associated with a smooth complete d -dimensional toric variety $V(\Sigma)$. Moreover, two smooth complete d -dimensional toric varieties $V(\Sigma)$ and $V(\Sigma')$ are isomorphic algebraic varieties if and only if the corresponding fans Σ and Σ' are isomorphic up to unimodular transformation of \mathbf{Z}^d .

2.5. DEFINITION. A complete regular d -dimensional fan Σ in \mathbf{R}^d is said to be projective if there exists a function $\varphi: \mathbf{R}^d \rightarrow \mathbf{R}$ such that

- (i) $\varphi(\mathbf{Z}^d) \subset \mathbf{Z}$;
- (ii) φ is a linear function on each cone of Σ ;
- (iii) for two arbitrary distinct d -dimensional cones σ and σ' in Σ the restrictions $\varphi|_\sigma$ and $\varphi|_{\sigma'}$ are different linear functions;
- (iv) φ is a convex function: $\varphi(x) + \varphi(y) \geq \varphi(x+y)$ for all $x, y \in \mathbf{R}^d$.

We call such a function φ a support function on Σ .

It is well-known that a smooth complete d -dimensional toric variety $V(\Sigma)$ is a projective variety if and only if the corresponding fan Σ has a support function φ (see [1], [7]).

We introduce now our new definitions.

Let Σ be a complete regular d -dimensional fan and Let $G(\Sigma)$ be the set of all generators of Σ .

2.6. DEFINITION. A nonempty subset $\mathcal{P} = \{x_1, \dots, x_k\} \subset G(\Sigma)$ is called a primitive collection if for each generator $x_i \in \mathcal{P}$ the elements of $\mathcal{P} \setminus \{x_i\}$ generate a $(k-1)$ -dimensional cone in Σ , while \mathcal{P} does not generate any k -dimensional cone in Σ .

2.7. DEFINITION. Let $\mathcal{P} = \{x_1, \dots, x_k\}$ be a primitive collection in $G(\Sigma)$. Let $S(\mathcal{P})$ denote $x_1 + \dots + x_k$. The focus $\sigma(\mathcal{P})$ of \mathcal{P} is the cone in Σ of the smallest dimension containing $S(\mathcal{P})$. (It follows from 2.4 (iii) that such $\sigma(\mathcal{P})$ exists.)

2.8. DEFINITION. Let $\mathcal{P} = \{x_1, \dots, x_k\}$ be a primitive collection in $G(\Sigma)$ and $\sigma(\mathcal{P})$ its focus. Let y_1, \dots, y_m be generators of $\sigma(\mathcal{P})$. It follows from 2.1–2.3 that there exists a unique linear combination $n_1y_1 + \dots + n_my_m$ with positive integer coefficients n_i which is equal to $x_1 + \dots + x_k$. Then the linear relation

$$x_1 + \dots + x_k - n_1y_1 - \dots - n_my_m = 0$$

is called the *primitive relation associated with \mathcal{P}* and is denoted by $\mathcal{R}(\mathcal{P})$.

Suppose that Σ is a projective regular d -dimensional fan with a support function φ .

2.9. DEFINITION. Let $\mathcal{P} = \{x_1, \dots, x_k\}$ be a primitive collection in $G(\Sigma)$ and let

$$x_1 + \dots + x_k - n_1 y_1 - \dots - n_m y_m = 0$$

be the associated primitive relation. The integer

$$\begin{aligned} D_\varphi(\mathcal{P}) &= \varphi(x_1) + \dots + \varphi(x_k) - n_1 \varphi(y_1) - \dots - n_m \varphi(y_m) \\ &= \varphi(x_1) + \dots + \varphi(x_k) - \varphi(x_1 + \dots + x_k) \end{aligned}$$

is called the *degree of \mathcal{P} relative to φ* . (It follows from 2.5 (iii), and 2.5 (iv) that $D_\varphi(\mathcal{P})$ is always a positive integer.)

2.10. DEFINITION. Let σ be an arbitrary d -dimensional cone in Σ with generators x_1, \dots, x_d and let x be an element of $G(\Sigma)$. There exists a unique linear combination $a_1 x_1 + \dots + a_d x_d$ with integer coefficients a_1, \dots, a_d which is equal to x . The integer

$$d_\varphi(x, \sigma) = \varphi(x) - a_1 \varphi(x_1) - \dots - a_d \varphi(x_d)$$

is called the *distance between x and σ* . (It follows from 2.5 (iii), and 2.5 (iv) that $d_\varphi(x, \sigma) \geq 0$, and $d_\varphi(x, \sigma) = 0$ if and only if $x \in \sigma$.)

2.11. DEFINITION. Let σ be an arbitrary d -dimensional cone in Σ with generators x_1, \dots, x_d and let x be an element of $G(\Sigma)$. We call x a *nearest generator of Σ relative to σ* if $x \notin \sigma$ and for any generator $x' \notin \sigma$, one has $d_\varphi(x, \sigma) \leq d_\varphi(x', \sigma)$. (It is possible that σ has several nearest generators.)

We recall the computation of the Picard group $\text{Pic}(V(\Sigma))$ of a smooth toric variety $V(\Sigma)$ associated with a regular fan Σ (see [1], [6], [7]).

2.12. PROPOSITION. *There exists a short exact sequence*

$$0 \longrightarrow \mathbf{Z}^d \xrightarrow{\psi} F \longrightarrow \text{Pic}(V(\Sigma)) \longrightarrow 0,$$

where F is the free abelian group whose generators are the elements of $G(\Sigma)$, and the map ψ is defined by the integer matrix Ψ whose rows consist of coordinates of the corresponding elements of $G(\Sigma)$.

2.13. COROLLARY. *If Σ is a complete regular fan, then the dual group*

$$\text{Pic}(V(\Sigma))^* = \text{Hom}(\text{Pic}(V(\Sigma)), \mathbf{Z})$$

can be identified with the group $A_1(V(\Sigma))$ of algebraic 1-cycles modulo numerical equivalence, and it consists of all possible linear relations with integer coefficients among the elements of $G(\Sigma) \subset \mathbf{Z}^d$.

2.14. REMARK. The group $\text{Pic}(V(\Sigma))$ consists of all functions $\delta: \mathbf{R}^d \rightarrow \mathbf{R}$ which satisfy 2.5 (i), (ii) modulo integral linear functions. If

$$a_1x_1 + \cdots + a_kx_k = 0$$

is an integral linear relation among generators of Σ , which is an element R of $A_1(V(\Sigma))$, then

$$\langle R, \delta \rangle = a_1\delta(x_1) + \cdots + a_k\delta(x_k)$$

is the corresponding intersection number. Obviously, this number does not change its value if we replace δ by a sum $\delta + f$, where $f: \mathbf{R}^d \rightarrow \mathbf{R}$ is an integral linear function. In particular, the degree of a primitive collection relative to a support function φ is also an intersection number.

We finish this paragraph by the following important theorem.

2.15. THEOREM. Let Σ be a projective regular d -dimensional fan of cones in \mathbf{R}^d and let $\text{Pr}(\Sigma)$ be the cone generated in $A_1(V(\Sigma)) \otimes \mathbf{R}$ by all primitive relations. Then $\text{Pr}(\Sigma)$ coincides with Mori's cone $\overline{NE}(V(\Sigma))$ of effective 1-cycles (see [9]).

The proof of this theorem is contained in [6], [8], [9].

3. Some properties. Let Σ be a complete regular d -dimensional fan of cones in \mathbf{R}^d .

3.1. PROPOSITION. Let $\mathcal{P} = \{x_1, \dots, x_k\}$ be a primitive collection in $G(\Sigma)$ with the focus $\sigma(\mathcal{P})$. Then $\mathcal{P} \cap \sigma(\mathcal{P}) = \emptyset$.

PROOF. Let $\{y_1, \dots, y_m\}$ be the generators of $\sigma(\mathcal{P})$. It is sufficient to prove that $\{x_1, \dots, x_k\} \cap \{y_1, \dots, y_m\} = \emptyset$. Assume, for instance, that $x_1 = y_1$. It follows from the definition of primitive collections that the element $x = x_2 + \cdots + x_k$ is in the interior of the $(k-1)$ -dimensional cone σ' generated by x_2, \dots, x_k . On the other hand, it follows from the equality $x_1 = y_1$ and the primitive relation

$$x_1 + \cdots + x_k - n_1y_1 - \cdots - n_my_m = 0$$

that

$$x_2 + \cdots + x_k = (n_1 - 1)y_1 + \cdots + n_my_m,$$

and the element $x = x_2 + \cdots + x_k$ is in the interior of the cone σ'' generated by y_1, \dots, y_m (if $n_1 > 1$), or by y_2, \dots, y_m , (if $n_1 = 1$). By 2.4 (ii), one has $\sigma' = \sigma''$. The last equality is possible only if $\{x_2, \dots, x_k\} = \{y_1, \dots, y_m\}$ and $n_1 = 2, n_2 = \cdots = n_m = 1$, or if $\{x_2, \dots, x_k\} = \{y_2, \dots, y_m\}$ and $n_1 = n_2 = \cdots = n_m = 1$.

If σ'' is generated by $\{y_1, \dots, y_m\}$, then y_1 must coincide with one x_2, \dots, x_k . This contradicts the assumption that x_1, \dots, x_k are different generators of Σ .

If σ'' is generated by $\{y_2, \dots, y_m\}$, then $\{x_1, \dots, x_k\} = \{y_1, \dots, y_m\}$. This contradicts the fact that y_1, \dots, y_m are generators of $\sigma(\mathcal{P})$.

Now we assume that Σ is a projective regular d -dimensional fan of cones in \mathbf{R}^d with a support function φ .

3.2. PROPOSITION. *There exists a primitive collection $\mathcal{P} = \{x_1, \dots, x_k\}$ in $G(\Sigma)$ such that the associated primitive relation is of the form*

$$x_1 + \dots + x_k = 0.$$

In the other words, the focus $\sigma(\mathcal{P}) = \{0\}$.

PROOF. Since Σ is a complete fan, there exist generators $x_1, \dots, x_m \in G(\Sigma)$ and positive integers a_1, \dots, a_m such that

$$a_1x_1 + \dots + a_mx_m = 0.$$

We can assume that the sum

$$a_1\varphi(x_1) + \dots + a_m\varphi(x_m)$$

has the smallest possible value r (by 2.5 (iii), (iv), r is a positive integer).

Now we shall prove that in fact $a_1 = \dots = a_m = 1$ and $\{x_1, \dots, x_m\}$ is a primitive collection in $G(\Sigma)$.

Obviously, x_1, \dots, x_m cannot be generators of a cone $\sigma \in \Sigma$. So, there exists a subset in $\{x_1, \dots, x_m\}$ (e.g. $\{x_1, \dots, x_q\}$) which is a primitive collection. Let

$$x_1 + \dots + x_q - b_1y_1 - \dots - b_py_p = 0$$

be the corresponding primitive relation. One has

$$\begin{aligned} r &= a_1\varphi(x_1) + \dots + a_m\varphi(x_m) \\ &= (a_1 - 1)\varphi(x_1) + \dots + (a_q - 1)\varphi(x_q) \\ &\quad + a_{q+1}\varphi(x_{q+1}) + \dots + a_m\varphi(x_m) + \varphi(x_1) + \dots + \varphi(x_q) \\ &> (a_1 - 1)\varphi(x_1) + \dots + (a_q - 1)\varphi(x_q) \\ &\quad + a_{q+1}\varphi(x_{q+1}) + \dots + a_m\varphi(x_m) + b_1\varphi(y_1) + \dots + b_p\varphi(y_p). \end{aligned}$$

On the other hand,

$$(a_1 - 1)x_1 + \dots + (a_q - 1)x_q + a_{q+1}x_{q+1} + \dots + a_mx_m + b_1y_1 + \dots + b_py_p = 0.$$

This contradicts the choice of r unless $a_1 = \dots = a_m = 1$, $q = m$ and the subset of generators $\{x_1, \dots, x_m\}$ is a primitive collection in $G(\Sigma)$.

3.3. PROPOSITION. *Let σ be a d -dimensional cone in Σ and let x_1, \dots, x_d be the generators of σ . Consider two generators $x, x' \in G(\Sigma)$ which do not belong to σ . By 2.6, there exists a primitive collection $\mathcal{P} \subset \{x, x_1, \dots, x_d\}$. Then the following hold:*

- (i) *if $\sigma(\mathcal{P})$ contains x' , then $d_\varphi(x, \sigma) > d_\varphi(x', \sigma)$;*
- (ii) *if all generators of $\sigma(\mathcal{P})$ are in σ , then $d_\varphi(x, \sigma) = D_\varphi(\mathcal{P})$;*
- (iii) *there exists at most one primitive collection $\mathcal{P} \subset \{x, x_1, \dots, x_d\}$ such that the*

focus $\sigma(\mathcal{P}) \subset \sigma$;

(iv) if x is a nearest generator in $G(\Sigma)$ relative to σ , then \mathcal{P} is a unique primitive collection in $\{x, x_1, \dots, x_d\}$, and $d_\varphi(x, \sigma) = D_\varphi(\mathcal{P})$.

PROOF. (i) We first prove that if a primitive collection \mathcal{P} (e.g., $\mathcal{P} = \{x, x_1, \dots, x_k\}$, $k < d$), gives rise to a primitive relation

$$x + x_1 + \dots + x_k - n_1 y_1 - \dots - n_m y_m = 0,$$

then

$$(1) \quad d_\varphi(x, \sigma) > n_1 d_\varphi(y_1, \sigma) + \dots + n_m d_\varphi(y_m, \sigma).$$

Let $y_i = b_{i,1}x_1 + \dots + b_{i,d}x_d$ ($b_{i,j} \in \mathbf{Z}$), and $x = a_1x_1 + \dots + a_dx_d$. Then

$$\begin{aligned} a_1 &= n_1 b_{1,1} + \dots + n_m b_{m,1} - 1, \\ &\dots \\ a_k &= n_1 b_{1,k} + \dots + n_m b_{m,k} - 1, \\ a_{k+1} &= n_1 b_{1,k+1} + \dots + n_m b_{m,k+1}, \\ &\dots \\ a_d &= n_1 b_{1,d} + \dots + n_m b_{m,d}. \end{aligned}$$

By 2.5 (iii), (iv), we get

$$\varphi(x_1) + \dots + \varphi(x_k) + \varphi(x) > \varphi(n_1 y_1 + \dots + n_m y_m).$$

It follows from 2.5 (ii) that

$$\varphi(n_1 y_1 + \dots + n_m y_m) = n_1 \varphi(y_1) + \dots + n_m \varphi(y_m).$$

Hence,

$$\begin{aligned} &\varphi(x_1) + \dots + \varphi(x_k) + d_\varphi(x, \sigma) \\ &= \varphi(x_1) + \dots + \varphi(x_k) + \varphi(x) - a_1 \varphi(x_1) - \dots - a_d \varphi(x_d) \\ &> n_1 \varphi(y_1) + \dots + n_m \varphi(y_m) - a_1 \varphi(x_1) - \dots - a_d \varphi(x_d) \\ &= n_1 (\varphi(y_1) - b_{1,1} \varphi(x_1) - \dots - b_{1,d} \varphi(x_d)) + \dots \\ &\quad + n_m (\varphi(y_m) - b_{m,1} \varphi(x_1) - \dots - b_{m,d} \varphi(x_d)) + \varphi(x_1) + \dots + \varphi(x_k) \\ &= \varphi(x_1) + \dots + \varphi(x_k) + n_1 d_\varphi(y_1, \sigma) + \dots + n_m d_\varphi(y_m, \sigma). \end{aligned}$$

This inequality implies (1). Thus, $d_\varphi(x, \sigma) > d_\varphi(x', \sigma)$, if $x' = y_i$ for some i ($1 < i < m$).

(ii) Let

$$x + x_1 + \dots + x_k - n_1 y_1 - \dots - n_m y_m = 0$$

be a primitive relation associated with the primitive collection \mathcal{P} . Then

$$D_\varphi(\mathcal{P}) = \varphi(x) + \varphi(x_1) + \dots + \varphi(x_m) - n_1 \varphi(y_1) - \dots - n_m \varphi(y_m).$$

Let y_1, \dots, y_m be generators of σ (i.e. $\{y_1, \dots, y_m\} \subset \{x_1, \dots, x_d\}$). Using 2.5 (ii), we get

$$a_1\varphi(x_1) + \dots + a_d\varphi(x_d) = n_1\varphi(y_1) + \dots + n_m\varphi(y_m) - \varphi(x_1) - \dots - \varphi(x_k),$$

where $x = a_1x_1 + \dots + a_dx_d$. Hence, $d_\varphi(x, \sigma) = D_\varphi(\mathcal{P})$.

(iii) Assume that there exist two different primitive collections \mathcal{P}_1 and \mathcal{P}_2 in

$$\{x, x_1, \dots, x_d\},$$

such that $\sigma(\mathcal{P}_1) \subset \sigma$ and $\sigma(\mathcal{P}_2) \subset \sigma$. Then, from the corresponding primitive relations, we get two different linear combinations of x_1, \dots, x_d which are equal to x . This is impossible, since x_1, \dots, x_d form a basis of \mathbf{Z}^d .

(iv) This statement is a corollary of (i), (ii) and (iii).

3.4. T -invariant Divisors. Every generator $x \in G(\Sigma)$ of a complete regular d -dimensional fan Σ in \mathbf{R}^d gives rise to a complete regular $(d-1)$ -dimensional fan Σ_x in \mathbf{R}^{d-1} corresponding to a smooth T -invariant divisor on $V(\Sigma)$. The fan Σ_x consists of images of all cones in Σ containing x via the natural projection $\mathbf{R}^d \rightarrow \mathbf{R}^{d-1} = \mathbf{R}^d/\mathbf{R}\langle x \rangle$. The following easy statement describes all primitive collections for Σ_x .

3.5. PROPOSITION. (i) *The set $G(\Sigma_x)$ of all generators for Σ_x consists of the images $\bar{x}' \in \mathbf{R}^d/\mathbf{R}\langle x \rangle$ of all generators x' such that $\{x, x'\}$ generate a 2-dimensional cone in Σ .*

(ii) *If $\{\bar{x}_1, \dots, \bar{x}_k\}$ is a primitive collection in $G(\Sigma_x)$, then*

$$\{x, x_1, \dots, x_k\}, \text{ or } \{x_1, \dots, x_k\}$$

is a primitive collection in $G(\Sigma)$.

PROOF. (i) The first statement is an immediate consequence of 3.4.

(ii) Let $\{\bar{x}_1, \dots, \bar{x}_k\}$ be a primitive collection in $G(\Sigma_x)$. By 3.4, x, x_1, \dots, x_k are not generators of a cone in Σ . Hence, there exists a primitive collection $\mathcal{P} \subset \{x, x_1, \dots, x_k\}$. Since $\{x, x_1, \dots, x_k\} \setminus \{x_i\}$ generates a cone in Σ for all i ($1 \leq i \leq k$), we get $\{x_1, \dots, x_k\} \subset \mathcal{P}$. Thus, $\mathcal{P} = \{x, x_1, \dots, x_k\}$, or $\mathcal{P} = \{x_1, \dots, x_k\}$.

4. Toric bundles. By [7], using the language of primitive collections and associated primitive relations, we get the following characterization of toric bundles.

4.1. PROPOSITION. *A regular complete d -dimensional fan Σ corresponds to a toric variety $V = V(\Sigma)$ which is a toric \mathbf{P}^k -bundle over a smooth $(d-k)$ -dimensional toric variety W if and only if there exists a primitive collection $\mathcal{P} = \{x_1, \dots, x_{k+1}\} \subset G(\Sigma)$ such that*

(i) *the corresponding primitive relation is*

$$x_1 + \dots + x_{k+1} = 0;$$

(ii) $\mathcal{P} \cap \mathcal{P}' = \emptyset$ *for any primitive collection $\mathcal{P}' \subset G(\Sigma)$ such that $\mathcal{P} \neq \mathcal{P}'$.*

4.2. DEFINITION. We say that a regular complete d -dimensional fan Σ is a *splitting*

fan if any two different primitive collections in $G(\Sigma)$ have no common elements.

4.3. THEOREM. *Let Σ be a splitting fan. Then the corresponding toric variety $V(\Sigma)$ is a projectivization of a decomposable bundle over a toric variety W which is associated with a splitting fan of a smaller dimension.*

PROOF. By 4.1, we have only to prove the existence of a primitive collection with zero focus (we cannot use 3.2 without knowing the projectivity of the fan Σ). We prove the last statement by induction of $\#G(\Sigma)$.

By 3.5 (ii), any divisor $D_{x_i} = V(\Sigma)$ corresponding to a generator $x_i \in G(\Sigma)$ on the toric variety $V(\Sigma)$ is also associated with a splitting fan. This allows us to apply the induction hypothesis.

Assume that any primitive collection in $G(\Sigma)$ has no zero focus. Choose a generator $x_0 \in G(\Sigma)$. Let $\{\bar{x}_1, \dots, \bar{x}_k\}$ be a primitive collection in $G(\Sigma_{x_0})$ having zero focus (by the induction hypothesis, it exists). By 3.5 (ii), we have to consider two cases.

CASE 1. $\mathcal{P} = \{x_0, x_1, \dots, x_k\}$ is a primitive collection in $G(\Sigma)$. It follows from our choice of the set $\{\bar{x}_1, \dots, \bar{x}_k\}$ that the sum $S(\mathcal{P}) = x_0 + x_1 + \dots + x_k$ is an integral multiple of x_0 . By 3.1, $S(\mathcal{P})$ cannot be a *positive* multiple of x_0 . Assume that $S(\mathcal{P}) = -ax_0$, where $a \in \mathbb{Z}_{>0}$. Then

$$x_1 + \dots + x_k = -(a+1)x_0.$$

Thus, $S(\mathcal{P})$ is in the interior of the cone $\sigma \in \Sigma$ generated by $\{x_1, \dots, x_k\}$. By 3.1, $\sigma \cap \sigma(\mathcal{P}) = \emptyset$, a contradiction. Hence only the next case is possible.

CASE 2. $\mathcal{P} = \{x_1, \dots, x_k\}$ is a primitive collection, and the sum $S(\mathcal{P}) = x_1 + \dots + x_k$ is an integral multiple of x_0 .

Since every primitive collection has at least two generators, the number of primitive collections for a splitting fan Σ is not greater than a half of the number of generators of Σ . So, there exist two different generators $x_i, x_j \in G(\Sigma)$ and a primitive collection $\mathcal{P} = \{x_1, \dots, x_k\}$ such that the sum $S(\mathcal{P}) = x_1 + \dots + x_k$ is an integral multiple of both x_i and x_j . This is possible only if $x_i = -x_j$. So, $\{x_i, x_j\}$ is a primitive collection with zero focus.

The statement is proved.

4.4. COROLLARY. *A smooth complete toric variety V is produced from a projective space by a sequence of projectivizations of decomposable bundles if and only if the corresponding fan $\Sigma(V)$ is a splitting fan.*

4.5. REMARK. One can notice that any complete smooth toric variety with Picard number 2 is associated with a splitting fan [4].

5. Toric varieties with $\rho = 3$: the number of primitive collections. Kleinschmidt and Sturmfels [5] have proved that an arbitrary smooth complete toric variety V of dimension d with Picard number $\rho = 3$ is projective. Consequently, for any complete

regular d -dimensional fan with $d+3$ generators there exists a strictly convex support function $\varphi : \mathbf{R}^d \rightarrow \mathbf{R}$ as in 2.5. Thus, the notions of the degree and the distance introduced in §2 are well-defined.

5.1. Let $X = \{x_1, \dots, x_{d+3}\}$ be an arbitrary set consisting of $d+3$ elements. We divide X into m nonempty subsets X_0, X_1, \dots, X_{m-1} without common elements, where $m = 2p + 3$ and p is a nonnegative integer. We can assume that

$$\begin{aligned} X_0 &= \{x_1, \dots, x_{s_0}\} \\ X_1 &= \{x_{s_0+1}, \dots, x_{s_1}\} \\ &\dots \\ X_{m-1} &= \{x_{s_{m-1}+1}, \dots, x_{s_m}\}, \end{aligned}$$

where $s_0 < s_1 < \dots < s_{m-1} = d+3$ and $\#X_i = s_i - s_{i-1}$ for $i > 0$. It is more convenient in the sequel to assume that the index i for X_i is an element of the residue ring $\mathbf{Z}/m\mathbf{Z}$. We denote by \mathcal{X}_i the union

$$X_i \cup X_{i+1} \cup \dots \cup X_{i+p}.$$

5.2. PROPOSITION. Let Σ be an arbitrary complete regular d -dimensional fan with $d+3$ generators. Then there exists a nonnegative integer p such that the set

$$G(\Sigma) = X = \{x_1, \dots, x_{d+3}\}$$

of all generators of Σ can be represented as a union of subsets X_0, X_1, \dots, X_{m-1} without common elements (see 5.1) and the corresponding subsets \mathcal{X}_i ($i \in \mathbf{Z}/m\mathbf{Z}$) are exactly all primitive collections of the generators of Σ .

PROOF. This statement is a simple translation of the well-known description of combinatorial types of d -polytopes with $d+3$ vertices from the Gale-transform language (see [3], [8]) to the one of primitive collections.

5.3. COROLLARY. Let $x_a \in X_\alpha, x_b \in X_\beta, x_c \in X_\gamma$ be three of $d+3$ generators of a fan Σ as in 5.2. Then the elements of $X \setminus \{x_a, x_b, x_c\}$ generate a d -dimensional cone of Σ if and only if the zero point 0 of the complex plane \mathbf{C} is in the interior of the triangle with the vertices $e^{2\pi i \alpha/m}, e^{2\pi i \beta/m}$ and $e^{2\pi i \gamma/m}$.

5.4. PROPOSITION. In the situation as in 5.2, one has $m \leq 7$.

PROOF. Assume that $m > 7$. Since m is odd, we have $m > 9$. Choose three generators $x_a, x_b, x_c \in X$ such that $x_a \in X_0, x_b \in X_t, x_c \in X_{2t}$, where $m = 3t + t', t' < 1$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a d -dimensional cone σ of Σ . By 5.2, for each $x_i \in \{x_a, x_b, x_c\}$ there exist at least two primitive collections which contain only x_i and generators of σ . This contradicts 3.3 (iv), since at least one generator from the set $\{x_a, x_b, x_c\}$ is a nearest generator relative to σ .

5.5. PROPOSITION. In the situation as in 5.2, one has $m \neq 7$.

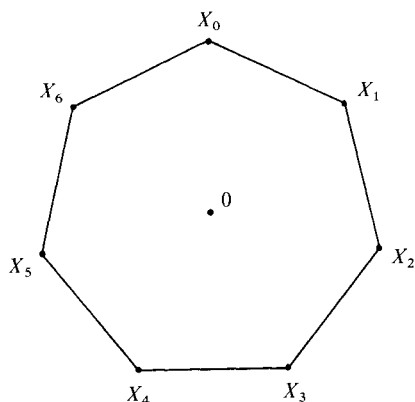


FIGURE 1.

PROOF. Assume that $m=7$. We have seen primitive relations

$$\mathcal{R}(\mathcal{X}_r): \sum_{x_i \in \mathcal{X}_r} x_i - \sum_{x'_j \in \sigma(\mathcal{X}_r)} a_{r,j} x'_j = 0,$$

where $a_{r,j}$ are positive integers and $r \in \mathbb{Z}/7\mathbb{Z}$. It is convenient to use a picture of heptagon with the vertices $ie^{2\pi ir/7} \in \mathbb{C}$ (see Figure 1).

5.6. LEMMA. For any $\alpha \in \mathbb{Z}/7\mathbb{Z}$, one has

$$\sigma(X_\alpha) \cap G(\Sigma) \subset X_{\alpha+4} \cup X_{\alpha+5}.$$

PROOF OF LEMMA 5.6. Choose $x_a \in X_{\alpha+1}$, $x_b \in X_{\alpha+3}$, $x_c \in X_{\alpha+6}$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a d -dimensional cone σ in Σ . By 3.3 (iv), in $\{x_a, x_b, x_c\}$ only x_a can be a nearest generator relative to σ , since $x_b \in \mathcal{X}_{\alpha+2} \cap \mathcal{X}_{\alpha+3}$ and $x_c \in \mathcal{X}_{\alpha+4} \cap \mathcal{X}_{\alpha+5}$. By 3.3 (i), $\sigma(\mathcal{X}_\alpha)$ does not contain x_b and x_c . But we can choose an arbitrary element in \mathcal{X}_α as x_b . So, $\sigma(\mathcal{X}_\alpha) \cap X_{\alpha+3} = \emptyset$. Similarly, $\sigma(\mathcal{X}_\alpha) \cap X_{\alpha+6} = \emptyset$. By 3.1, $\sigma(\mathcal{X}_\alpha) \cap (X_\alpha \cup X_{\alpha+1} \cup X_{\alpha+2}) = \emptyset$. Thus, the lemma is proved.

We return to 5.5.

We can take $\alpha \in \mathbb{Z}/7\mathbb{Z}$ such that

$$D_\varphi(\mathcal{X}_\alpha) = \max\{D_\varphi(\mathcal{X}_\beta) \mid \beta \in \mathbb{Z}/7\mathbb{Z}\}.$$

Choose again $x_a \in X_{\alpha+1}$, $x_b \in X_{\alpha+3}$, $x_c \in X_{\alpha+6}$. Using 5.6 and 3.3 (ii), we get $D_\varphi(\mathcal{X}_\alpha) = d_\varphi(x_a, \sigma)$, where σ is generated by $X \setminus \{x_a, x_b, x_c\}$. We have already seen in the proof of 5.6 that in $\{x_a, x_b, x_c\}$ only x_a can be a nearest generator relative to $\sigma \in \Sigma$. So, $d_\varphi(x_a, \sigma) < d_\varphi(x_b, \sigma)$ and $d_\varphi(x_a, \sigma) < d_\varphi(x_c, \sigma)$. Assume, for instance, that $d_\varphi(x_b, \sigma) \leq d_\varphi(x_c, \sigma)$. Applying 5.6 after the cyclic permutation $\alpha \mapsto \alpha+2$, one has $x_a \notin \sigma(\mathcal{X}_{\alpha+2})$. Since $x_b \in \mathcal{X}_{\alpha+2}$, it follows from 3.3 (i) that $x_c \notin \sigma(\mathcal{X}_{\alpha+2})$. Hence, by 3.3 (ii),

we have $D_\varphi(\mathcal{X}_{\alpha+2}) = d_\varphi(x_b, \sigma)$. Consequently, $D_\varphi(\mathcal{X}_{\alpha+2}) > D_\varphi(\mathcal{X}_\alpha)$. This contradicts the choice of $\alpha \in \mathbf{Z}/7\mathbf{Z}$. Thus, the case $m=7$ is impossible.

Propositions 5.4 and 5.5 imply the following theorem.

5.7. THEOREM. *If Σ is a complete regular d -dimensional fan with $d+3$ generators, then the number of primitive collections of its generators is equal to 3 or 5.*

If Σ has exactly three primitive collections in $G(\Sigma)$, then we come to a particular case of 4.3. In this case the associated smooth toric variety $V(\Sigma)$ is isomorphic to a projectivization of a decomposable bundle over a smooth toric variety W of a smaller dimension with Picard number 2. Hence, we have to investigate only the case of five primitive collections in $G(\Sigma)$. This is the object of the next section.

6. Toric varieties with $\rho=3$: the classification of primitive relations. Let Σ be a complete regular d -dimensional fan of cones in \mathbf{R}^d with $d+3$ generators and with a support function φ .

We use the notation of the previous section and assume that $G(\Sigma)$ contains exactly five primitive collections $\mathcal{X}_\alpha = X_\alpha \cup X_{\alpha+1}$, where $\alpha \in \mathbf{Z}/5\mathbf{Z}$. In our investigation it is convenient to use a picture of the pentagon with vertices $ie^{2\pi i\alpha/5} \in \dot{C}$ (see Figure 2).

6.1. PROPOSITION. *Suppose that $\sigma(X_\alpha) \cap G(\Sigma) \subset X_{\alpha+3}$ for all $\alpha \in \mathbf{Z}/5\mathbf{Z}$. Then for any $\alpha \in \mathbf{Z}/5\mathbf{Z}$ at least one of the following statements hold:*

- (i) $\sigma(\mathcal{X}_{\alpha+2}) \cap G(\Sigma) = X_\alpha$;
- (ii) $\sigma(\mathcal{X}_{\alpha+3}) \cap G(\Sigma) = X_{\alpha+1}$.

PROOF. It follows from our conditions that $\sigma(\mathcal{X}_{\alpha+2}) \cap G(\Sigma) \subset X_\alpha$ and $\sigma(\mathcal{X}_{\alpha+3}) \cap G(\Sigma) \subset X_{\alpha+1}$. Assume that there exist $x_a \in X_\alpha$ and $x_b \in X_{\alpha+1}$ such that $x_a \notin \sigma(\mathcal{X}_{\alpha+2})$ and $x_b \notin \sigma(\mathcal{X}_{\alpha+3})$. Choose an arbitrary element $x_c \in X_{\alpha+3}$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a

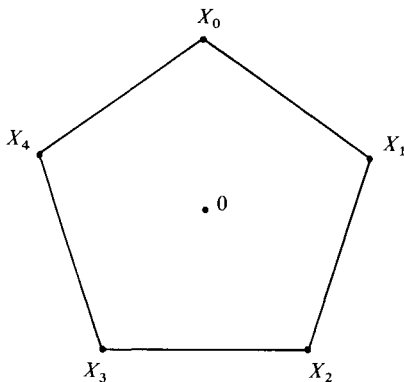


FIGURE 2.

d -dimensional cone $\sigma \in \Sigma$. Thus, we have two primitive collections $\mathcal{X}_{\alpha+2}, \mathcal{X}_{\alpha+3} \subset X \setminus \{x_a, x_b\}$ such that $\sigma(\mathcal{X}_{\alpha+2}), \sigma(\mathcal{X}_{\alpha+3}) \subset \sigma$. This contradicts 3.3 (iii).

The sum $S(\mathcal{X}_\alpha)$ of all generators in \mathcal{X}_α is denoted by S_α . Let P_α be the sum of all generators in X_α .

6.2. PROPOSITION. *Suppose that $\sigma(\mathcal{X}_\alpha) \cap G(\Sigma) \subset X_{\alpha+3}$ for all $\alpha \in \mathbb{Z}/5\mathbb{Z}$. Then up to a cyclic permutation of indices, one has $S_0=0, S_1=P_4, S_2=0, S_3=P_1, S_4=P_2$.*

PROOF. Using 6.1 for all $\alpha \in \mathbb{Z}/5\mathbb{Z}$, one can easily conclude that there exists $\beta \in \mathbb{Z}/5\mathbb{Z}$ such that

$$\sigma(\mathcal{X}_{\beta+2}) \cap G(\Sigma) = X_\beta \quad \text{and} \quad \sigma(\mathcal{X}_\beta) \cap G(\Sigma) = X_{\beta+3}.$$

Thus, we have

$$P_{\beta+2} + P_{\beta+3} = S_{\beta+2} = P_\beta + P'_\beta, \quad P_\beta + P_{\beta+1} = S_\beta = P_{\beta+3} + P'_{\beta+3},$$

where $P'_\beta \in \sigma(\mathcal{X}_{\beta+2})$ and $P'_{\beta+3} \in \sigma(\mathcal{X}_\beta)$. It follows from these two equalities that

$$P_{\beta+1} + P_{\beta+2} = P'_\beta + P'_{\beta+3}.$$

By 5.3, $X_\beta \cup X_{\beta+3}$ is contained in a d -dimensional cone $\sigma \in \Sigma$. So, the focus $\sigma(\mathcal{X}_{\beta+1})$ is generated by a subset in $X_\beta \cup X_{\beta+3}$. On the other hand, it follows from our conditions that $(\sigma(\mathcal{X}_{\beta+1}) \cap G(\Sigma)) \subset X_{\beta+4}$. Consequently, P'_β and $P'_{\beta+3}$ must be zero and $S_{\beta+2} = P_\beta, S_{\beta+2} = P_\beta, S_{\beta+1} = 0$. Using again 6.1, we get

$$\sigma(\mathcal{X}_{\beta+4}) \cap G(\Sigma) = X_{\beta+2}, \quad \text{or} \quad \sigma(\mathcal{X}_{\beta+3}) \cap G(\Sigma) = X_{\beta+1}.$$

In the first case, we can repeat the above arguments relative to

$$\sigma(\mathcal{X}_{\beta+4}) \cap G(\Sigma) = X_{\beta+2} \quad \text{and} \quad \sigma(X_{\beta+2}) \cap G(\Sigma) = X_\beta.$$

As a result, we obtain $S_{\beta+3} = 0, S_{\beta+4} = P_{\beta+2}$. In the second case, applying the same arguments to

$$\sigma(\mathcal{X}_{\beta+3}) \cap G(\Sigma) = X_{\beta+1} \quad \text{and} \quad \sigma(X_\beta) \cap G(\Sigma) = X_{\beta+3},$$

we get $S_{\beta+3} = P_{\beta+1}, S_{\beta+4} = 0$. Thus, the statement is proved.

6.3. PROPOSITION. *Suppose that a cone $\sigma(\mathcal{X}_\alpha)$ contains a generator $x_a \in X_{\alpha+2}$. Then the following statements hold:*

- (i) $X_\alpha \cap (\sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_{\alpha+3})) = \emptyset$;
- (ii) $S_{\alpha+2} = 0$;
- (iii) $\sigma(\mathcal{X}_{\alpha+1}) \cap G(\Sigma) = X_{\alpha+4}$;
- (iv) $\sigma(\mathcal{X}_{\alpha+3}) \cap G(\Sigma) = X_{\alpha+1}$;
- (v) $S_{\alpha+1} = P_{\alpha+4}, S_{\alpha+3} = P_{\alpha+1}$.

PROOF. (i) Choose arbitrary $x_b \in X_\alpha, x_c \in X_{\alpha+4}$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a

d -dimensional cone σ in Σ . By 3.3 (i), $d_\varphi(x_b, \sigma) > d_\varphi(x_a, \sigma)$. By 3.3 (iv), x_a is not a nearest generator relative to σ . Consequently, $d_\varphi(x_a, \sigma) > d_\varphi(x_c, \sigma)$ and $x_b \notin \sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_\alpha)$ (see 3.3 (i)). Thus, $X_\alpha \cap (\sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_{\alpha+3})) = \emptyset$, since x_b is an arbitrary element of X_α .

(ii) Assume that there exists $x_b \in X_{\alpha+1}$ such that $x_b \in \sigma(\mathcal{X}_{\alpha+2})$. Take an element $x_c \in X_{\alpha+4}$. Then $X \setminus \{x_a, x_b, x_c\}$ is the set of generators of a d -dimensional cone $\sigma \in \Sigma$. By 3.3 (i), it follows from $x_a \in \sigma(\mathcal{X}_\alpha)$ that $d_\varphi(x_b, \sigma) > d_\varphi(x_a, \sigma)$. Similarly, $x_b \in \sigma(\mathcal{X}_{\alpha+2})$ implies $d_\varphi(x_a, \sigma) > d_\varphi(x_b, \sigma)$. This is a contradiction. So, $\sigma(\mathcal{X}_{\alpha+2}) \cap X_{\alpha+1} = \emptyset$. Using 3.1, one has $\sigma(\mathcal{X}_{\alpha+2}) \cap (X_{\alpha+2} \cup X_{\alpha+3}) = \emptyset$. By 6.3 (i), one has $\sigma(\mathcal{X}_{\alpha+2}) \cap X_\alpha = \emptyset$. It suffices to prove that $\sigma(\mathcal{X}_{\alpha+2}) \cap X_{\alpha+4} = \emptyset$.

Assume that there exists a generator $x_d \in X_{\alpha+4}$ such that $x_d \in \sigma(\mathcal{X}_{\alpha+4})$. Using 6.3 (i) after the cyclic permutation $\alpha \mapsto \alpha + 2$, we get

$$X_{\alpha+2} \cap (\sigma(\mathcal{X}_{\alpha+3}) \cup \sigma(\mathcal{X}_{\alpha+4}) \cup \sigma(\mathcal{X}_\alpha)) = \emptyset.$$

This contradicts $x_a \in \sigma(\mathcal{X}_\alpha)$.

(iii) By 6.3 (i) and 3.1, $\sigma(\mathcal{X}_{\alpha+1}) \cap (X_{\alpha+1} \cup X_{\alpha+2} \cup X_\alpha) = \emptyset$. Assume that there exists $x_b \in X_{\alpha+3} \cap \sigma(\mathcal{X}_{\alpha+1})$. Using 6.3 (ii) after the cyclic permutation $\alpha \mapsto \alpha + 1$, one has $\sigma(\mathcal{X}_{\alpha+3}) = 0$. This contradicts 3.3 (iii), since we have $\sigma(\mathcal{X}_{\alpha+2}) = \sigma(\mathcal{X}_{\alpha+3}) = 0$. Thus, $\sigma(\mathcal{X}_{\alpha+1}) \cap G(\Sigma) \subset X_{\alpha+4}$.

Suppose that there exists $x_b \in X_{\alpha+4}$ such that $x_b \notin \sigma(\mathcal{X}_{\alpha+4})$. Take an element $x_c \in X_\alpha$. Then $X \setminus \{x_a, x_b, x_c\}$ is the set of generators of a d -dimensional cone $\sigma \in \Sigma$. We get two primitive collections $\mathcal{X}_{\alpha+1}$ and $\mathcal{X}_{\alpha+2}$ in $G(\Sigma) \setminus \{x_b, x_c\}$ such that $\sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \subset \sigma$. This contradicts 3.3 (iii).

(iv) By 6.3 (i) and 3.1, $\sigma(\mathcal{X}_{\alpha+3}) \cap (X_{\alpha+3} \cup X_{\alpha+4} \cup X_\alpha) = \emptyset$. Assume that there exists $x_b \in X_{\alpha+2} \cap \sigma(\mathcal{X}_{\alpha+3})$. Using 6.3 (ii) after the symmetry $\alpha + \beta \mapsto \alpha - \beta$ of pentagon and the cyclic permutation $\alpha \mapsto \alpha + 1$, one has $\sigma(\mathcal{X}_{\alpha+1}) = 0$. This contradicts 3.3 (iii), since we have $\sigma(\mathcal{X}_\alpha) = \sigma(\mathcal{X}_{\alpha+1}) = 0$. Thus, $\sigma(\mathcal{X}_{\alpha+3}) \cap G(\Sigma) \subset X_{\alpha+1}$.

Suppose that there exists $x_b \in X_{\alpha+1}$ such that $x_b \notin \sigma(\mathcal{X}_{\alpha+3})$. Take elements $x_c \in X_\alpha$ and $x_d \in X_{\alpha+3}$. Then $X \setminus \{x_b, x_c, x_d\}$ is the set of generators of a d -dimensional cone $\sigma \in \Sigma$. We get two primitive collections $\mathcal{X}_{\alpha+2}$ and $\mathcal{X}_{\alpha+3}$ in $G(\Sigma) \setminus \{x_b, x_c\}$ such that $\sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_{\alpha+3}) \subset \sigma$. This contradicts 3.3 (iii).

(v) By 6.3 (iii) and 6.3 (iv), one has

$$P_{\alpha+1} + P_{\alpha+2} = S_{\alpha+1} = P_{\alpha+4} + P'_{\alpha+4}, \quad P_{\alpha+3} + P_{\alpha+4} = S_{\alpha+3} = P_{\alpha+1} + P'_{\alpha+1},$$

where $P'_{\alpha+4} \in \sigma(\mathcal{X}_{\alpha+1})$ and $P'_{\alpha+1} \in \sigma(\mathcal{X}_{\alpha+3})$. It follows from these two equalities that

$$P_{\alpha+2} + P_{\alpha+3} = P'_{\alpha+1} + P'_{\alpha+4}.$$

Thus, $\sigma(\mathcal{X}_{\alpha+2}) \cap G(\Sigma) \subset (X_{\alpha+1} \cup X_{\alpha+4})$. On the other hand, we have $S_{\alpha+2} = 0$ (see 6.3 (ii)). So, $P'_{\alpha+1} = P'_{\alpha+4} = 0$ and $S_{\alpha+1} = P_{\alpha+4}$, $S_{\alpha+3} = P_{\alpha+1}$. The statement is proved.

6.4. COROLLARY. *Suppose that a cone $\sigma(\mathcal{X}_\alpha)$ contains a generator $x_a \in X_{\alpha+2}$. Then one has*

$$(\sigma(\mathcal{X}_\alpha) \cup \sigma(\mathcal{X}_{\alpha+4})) \cap G(\Sigma) \subset X_{\alpha+2} \cup X_{\alpha+3}.$$

PROOF. Assume, for instance, that there exists $x_b \in X_{\alpha+1} \cap \sigma(\mathcal{X}_{\alpha+4})$. By 6.3. (ii), after the cyclic permutation $\alpha \mapsto \alpha + 4$, one has $\sigma(\mathcal{X}_{\alpha+1}) = 0$. This contradicts 6.3 (v).

Now we assume that there exists $x_b \in X_{\alpha+4} \cap \sigma(\mathcal{X}_\alpha)$. By 6.3 (ii), after the symmetry $\alpha + \beta \mapsto \alpha - \beta$ and the cyclic permutation $\alpha \mapsto \alpha + 1$, one has $\sigma(\mathcal{X}_{\alpha+3}) = 0$. This again contradicts 6.3 (v).

Using 3.1, we finish our proof.

6.5. PROPOSITION. *Suppose that a cone $\sigma(\mathcal{X}_\alpha)$ contains a generator $x_d \in X_\alpha$. Then at least one and only one of the following statements hold:*

- (i) $X_{\alpha+3} \subset \sigma(\mathcal{X}_\alpha) \cap G(\Sigma)$;
- (ii) $X_{\alpha+2} \subset \sigma(\mathcal{X}_{\alpha+4}) \cap G(\Sigma)$.

PROOF. We first assume that there exist $x_b \in X_{\alpha+2}$ and $x_c \in X_{\alpha+3}$ such that $x_b \notin \sigma(\mathcal{X}_{\alpha+4})$ and $x_c \notin \sigma(\mathcal{X}_\alpha)$. Choose an arbitrary element $x_d \in X_\alpha$. By 5.3, $X \setminus \{x_b, x_c, x_d\}$ generates a d -dimensional cone $\sigma \in \Sigma$. Thus, we have two primitive collections $\mathcal{X}_{\alpha+4}, \mathcal{X}_\alpha \subset X \setminus \{x_b, x_c\}$ such that $\sigma(\mathcal{X}_{\alpha+4}), \sigma(\mathcal{X}_\alpha) \subset \sigma$. This contradicts 3.3 (iii). Hence, the ‘‘at least one’’ part is proved.

Assume then, for instance, that (i) holds. Since $X_{\alpha+2} \cup X_{\alpha+3}$ is a primitive collection, at least one element $x_b \in X_{\alpha+2}$ is not a generator of $\sigma(\mathcal{X}_\alpha)$. So, we have

$$P_\alpha + P_{\alpha+1} = S_\alpha = P_{\alpha+3} + P,$$

where $P \in \sigma(\mathcal{X}_\alpha)$ is a linear combination of $(X_{\alpha+2} \cup X_{\alpha+3}) \setminus \{x_b\}$ with nonnegative integral coefficients. On the other hand, it follows from 6.3 (v) that

$$P_{\alpha+3} + P_{\alpha+4} = P_{\alpha+1}.$$

These two equalities imply

$$P_{\alpha+4} + P_\alpha = P.$$

Hence, $\sigma(\mathcal{X}_{\alpha+4}) \subset \sigma(\mathcal{X}_\alpha)$. This shows that $x_b \notin \sigma(\mathcal{X}_{\alpha+4})$ and $X_{\alpha+2} \not\subset \sigma(\mathcal{X}_{\alpha+4}) \cap G(\Sigma)$.

We can now finish our classification of primitive relations.

6.6. THEOREM. *Let us assume that $\mathcal{X}_\alpha = X_\alpha \cup X_{\alpha+1}$, where $\alpha \in \mathbf{Z}/5\mathbf{Z}$,*

$$\begin{aligned} X_0 &= \{v_1, \dots, v_{p_0}\}, & X_1 &= \{y_1, \dots, y_{p_1}\}, & X_2 &= \{z_1, \dots, z_{p_2}\}, \\ X_3 &= \{t_1, \dots, t_{p_3}\}, & X_4 &= \{u_1, \dots, u_{p_4}\}, \end{aligned}$$

and $p_0 + p_1 + p_2 + p_3 + p_4 = d + 3$. Then any complete regular d -dimensional fan Σ with the set of generators $G(\Sigma) = \bigcup X_\alpha$ and five primitive collections \mathcal{X}_α can be described up to a symmetry of the pentagon by the following primitive relations with nonnegative integral coefficients $c_2, \dots, c_{p_2}, b_1, \dots, b_{p_3}$:

$$v_1 + \dots + v_{p_0} + y_1 + \dots + y_{p_1} - c_2 z_2 - \dots - c_{p_2} z_{p_2} - (b_1 + 1)t_1 - \dots - (b_{p_3} + 1)t_{p_3} = 0,$$

$$\begin{aligned}
 & y_1 + \dots + y_{p_1} + z_1 + \dots + z_{p_2} - u_1 - \dots - u_{p_4} = 0, \\
 & z_1 + \dots + z_{p_2} + t_1 + \dots + t_{p_3} = 0, \\
 & t_1 + \dots + t_{p_2} + u_1 + \dots + u_{p_3} - y_1 - \dots - y_{p_1} = 0, \\
 & u_1 + \dots + u_{p_4} + v_1 + \dots + v_{p_0} - c_2 z_2 - \dots - c_{p_2} z_{p_2} - b_1 t_1 - \dots - b_{p_3} t_{p_3} = 0.
 \end{aligned}$$

PROOF. One of the following two conditions hold:

- (i) $\sigma(\mathcal{X}_\alpha) \cap G(\Sigma) \subset X_{\alpha+3}$ for all $\alpha \in \mathbf{Z}/5\mathbf{Z}$,
- (ii) up to a symmetry of the pentagon there exists a cone $\sigma(\mathcal{X}_\alpha)$ containing a generator $x_a \in \mathcal{X}_{\alpha+2}$.

In the first case, we can use 6.1 and get the above primitive relations for $\alpha=0$, where $c_1 = \dots = c_{p_2} = b_2 = \dots = b_{p_3} = 0$.

In the second case, we can use 6.3–6.5 and get the above primitive relations, where $z_1 = x_b, \alpha=0$,

$$P = c_2 z_2 + \dots + c_{p_2} z_{p_2} + b_1 t_1 + \dots + b_{p_3} t_{p_3}$$

(We use the notation in the “only one” part in the proof of 6.5).

We can take the set

$$\{v_1, \dots, v_{p_0}, y_2, \dots, y_{p_1}, z_2, \dots, z_{p_2}, t_1, \dots, t_{p_3}, u_2, \dots, u_{p_4}\}$$

as a basis of \mathbf{Z}^d . Thus, t_1, y_1, v_1 are defined by

$$\begin{aligned}
 z_1 &= -z_2 - \dots - z_{p_2} - t_1 - \dots - t_{p_3}, \\
 y_1 &= -y_2 - \dots - y_{p_1} + z_1 + \dots + z_{p_2} - u_1 - \dots - u_{p_4}, \\
 u_1 &= -u_2 - \dots - u_{p_4} - v_1 - \dots - v_{p_0} + c_2 z_2 + \dots + c_{p_2} z_{p_2} + b_1 t_1 + \dots + b_{p_3} t_{p_3}.
 \end{aligned}$$

7. Open questions. The most interesting problem related to smooth complete projective toric varieties seems to me the following:

7.1. MAIN CONJECTURE. *For any d -dimensional smooth complete toric variety with Picard number ρ defined by a complete regular fan Σ , there exists a constant $N(\rho)$ depending only on ρ such that the number of primitive collections in $G(\Sigma)$ is always not more than $N(\rho)$.*

It is easy to see that $N(1)=1, N(2)=2$. Using our result in §5, we get $N(3)=5$. For 2-dimensional toric variety with $\rho+2$ generators the number of primitive collections equals $(\rho-1)(\rho+2)/2$. In connection with the conjecture, it is interesting to ask the following:

7.2. QUESTION. *Does there exist for $\rho > 1$ a complete regular d -dimensional fan Σ with $\rho+d$ generators such that the set $G(\Sigma)$ contains more than*

$$(\rho-1)(\rho+2)/2$$

primitive collections?

REFERENCES

- [1] V. I. DANILOV, The geometry of toric varieties, *Uspechi Math. Nauk* (2), 33 (1978), 85–134.
- [2] J. GRETENKORT, P. KLEINSCHMIDT AND B. STURMFELS, On the existence of certain smooth toric varieties, *Discrete Comput. Geom.* 5(1990), 255–262.
- [3] B. GRÜNBAUM, *Convex Polytopes*, John Wiley and Sons, London-New York-Sydney, 1967.
- [4] P. KLEINSCHMIDT, A classification of toric varieties with few generators., *Aequationes Math.* 35 (1988), 254–266.
- [5] P. KLEINSCHMIDT AND B. STURMFELS, Smooth toric varieties with small Picard number are projective, *MIP*, preprint, Univ. Passau, June, 1989.
- [6] T. ODA, *Convex Bodies and Algebraic Geometry—An Introduction to the Theory of Toric Varieties*, *Ergebnisse der Math.* (3) 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo 1988.
- [7] T. ODA, *Lectures on Torus Embeddings and Applications (Based on Joint Work with Katsuya Miyake)*, *Tata Inst. Fund. Research* 58, Springer-Verlag, Berlin, Heidelberg, New York, 1978.
- [8] T. ODA AND H. S. PARK, Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions, *Tôhoku Math. J.* 43 (1991), 375–399.
- [9] M. REID, Decomposition of toric morphisms, in *Arithmetic and Geometry*, papers dedicated to I. R. Shafarevich on the occasion of his 60th birthday (M. Artin and J. Tate, eds.), vol. II, *Geometry*, *Progress in Math.* 36, Birkhäuser, Boston, Basel, Stuttgart, (1983), 395–418.

CHAIR OF ALGEBRA
DEPARTMENT OF MECHANICS AND MATHEMATICS
MOSCOW STATE UNIVERSITY
LENINSKIE GORY
119899, MOSCOW
USSR
and
FB 6 MATHEMATIK
UNIVERSITÄT-GESAMTHOCHSCHULE ESSEN
UNIVERSITÄTSSTRASSE 2
D-4300 ESSEN
FEDERAL REPUBLIC OF GERMANY

