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ON THE CLIFFORD THEOREM FOR SURFACES

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Abstract. We give two generalizations of the Clifford theorem to algebraic surfaces. As an application, we obtain some bounds for the number of moduli of surfaces of general type.

Introduction. The classical Brill-Noether theory is to study special divisors or linear systems on an algebraic curve, and the Clifford theorem is the first step of the theory (cf. [1]). The main purpose of this paper is to generalize the Clifford theorem to algebraic surfaces.

Let X be a smooth projective complex surface and L a divisor on it. One of the fundamental problems in the surface case is to study the adjoint linear system $|K_X + L|$. Roughly speaking, the behavior of this linear system depends on the positivity of L. When L is positive, we have a celebrated method of Reider [15] (see also [4] and [16]). When L is zero, the canonical system has also been studied systematically by Beauville [2]. When L is negative, the linear system corresponds to the special divisors on a curve. Exactly, we say a divisor D on X a special divisor if it is effective and $h^0(K_X - D) > 0$. However, for surfaces, we have no general method to study such special divisors. In order to find a powerful method to study special linear systems in the surface case, we need to establish first a Clifford-type theorem.

One easy generalization of the Clifford theorem is as follows. Let L be a special divisor on X. From $h^0(L) + h^0(K_X - L) \le h^0(K_X) + 1$ and the Riemann-Roch theorem, we get

$$h^1(L) \le q + \frac{1}{2}L(K_X - L)\,, \tag{1}$$

where q is the irregularity of X. If $L = 0$ or K_X, the equality holds. As in the curve case, the nontrivial problem is to characterize the equality. Our first result describes such conditions on the surface and on the divisor L. We can assume that $L \not\sim 0$ and $K_X - L \not\sim 0$.

THEOREM 0.1. *If the equality in* (1) *holds then either L contains a divisor of the movable part of $|K_X|$, or L is contained in the fixed part of $|K_X|$, or one of the following cases occurs.*

1. *$|K_X|$ is composed of a rational pencil, and the movable part of $|L|$ is a sum of some fibers of the pencil.*
2. *$|K_X|$ is composed of a irrational pencil of elliptic curves. The corresponding elliptic fibration is $f : X \to C$ with $g(C) \ge 2$. There are two line bundles A and B on C such that*

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f^*A and f^*B are respectively the movable part of $|L|$ and $|K_X - L|$. The Clifford index of C is less than 2. Exactly, we have the following possible cases:

(a) $0 \le \chi(\mathcal{O}_X) \le 2$, *$C$ is hyperelliptic and one of A and B is a multiple of g_2^1;*

(b) $\chi(\mathcal{O}_X) = 0$, $q = g(C)$, *C is a smooth plane quintic and both of A and B are hyperplane sections;*

(c) $\chi(\mathcal{O}_S) = 0$, $q = g(C)$, *C is trigonal and one of A and B is g_3^1.*

This theorem can be considered as a generalization of the Clifford theorem. We have another type of generalization as follows.

THEOREM 0.2. *Let X be a smooth minimal complex projective surface of general type. Let L be a special divisor on X such that $L \not\sim K_X$, then $h^0(L) \le K_X L/2 + 1$. If the equality holds, then one of the following cases occurs.*

1. *$h^0(L) = 1$ and L is a sum of (-2)-curves.*
2. *The movable part of $|L|$ has no base points and $\varphi_L : X \to \boldsymbol{P}^1$ is a projective surjective morphism, whose general fiber is an irreducible smooth curve of genus 2.*
3. *The movable part of $|L|$ has no base points and φ_L is generically 2 to 1 onto a surface of minimal degree in $\boldsymbol{P}^{h^0(L)-1}$.*

The two theorems will be proved in Sections 1 and 2, respectively.

The organization of the paper is as follows. In Section 1, we prove Theorem 0.1. In Section 2, we will give some Clifford type inequalities on a surface (Propositions 2.1 and 2.3) and prove Theorem 0.2. In Section 3, we use these two inequalities to define two indices $\alpha(X)$ and $\beta(X)$ on X like the Clifford index in the case of a curve. We study some basic properties of $\alpha(X)$ and $\beta(X)$ and give some bounds for them (Propositions 3.6 and 3.8). In Section 4, we give a detailed description of X, when α and β are zero (Theorems 4.1 and 4.2). In Section 5, we use our inequalities to give some bounds for the number of moduli of surfaces (Theorem 5.2).

Throughout the paper, we let X be a smooth complex projective surface and K_X be its canonical divisor. p_g and q denote, respectively, $h^0(K_X)$ and $h^1(\mathcal{O}_X)$. For a divisor L on X, we let φ_L be the rational map defined by the linear system $|L|$. $|L|$ is said to be composed of a pencil if $\dim \varphi_L(X) = 1$. Numerical equivalence between divisors is denoted by $\equiv$ and linear equivalence by $\sim$. g_d^r denotes a linear system of degree d and dimension r on a smooth projective curve. If E is a vector space we will denote by $\boldsymbol{P}E$ the space of one-dimension subspaces of E.

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1. Proof of Theorem 0.1. In this section, we will prove Theorem 0.1. In the first place, we need the following key lemma.

LEMMA 1.1. *Suppose Z is a projective variety. Let L and D be cartier divisors on Z. Assume Y is an irreducible and reduced closed subscheme of Z and denote $\mathcal{I}_Y$ the ideal sheaf* *of Y in Z. If $h^0(L) - h^0(\mathcal{I}_Y(L)) > 0$ and $h^0(D) - h^0(\mathcal{I}_Y(D)) > 0$, then we have*

$$h^0(L) - h^0(\mathcal{I}_Y(L)) + h^0(D) - h^0(\mathcal{I}_Y(D)) \leq h^0(L + D) - h^0(\mathcal{I}_Y(L + D)) + 1 .$$

PROOF. For any Cartier divisor A on Z, we have the standard exact sequence

$$0 \to \mathcal{I}_Y(A) \to \mathcal{O}_Z(A) \xrightarrow{r_Y} \mathcal{O}_Y(A) \to 0 ,$$

where r_Y is the restriction map. We consider the linear system $r_Y|A|$ on Y:

$$r_Y|A| = \boldsymbol{P} r_Y(H^0(A)) \subset \boldsymbol{P}H^0(\mathcal{O}_Y(A)) .$$

We then define a map

$$\begin{aligned} \mu : r_Y|L| \times r_Y|D| &\to r_Y|L + D| , \\ (L_1, D_1) &\mapsto L_1 + D_1 . \end{aligned}$$

It is easy to check that μ is well defined. But every element of $r_Y|L + D|$ has finite components, thus μ is finite. Hence

$$\dim(\mathrm{Im}(\mu)) = \dim(r_Y|L| \times r_Y|D|) = h^0(L) - h^0(\mathcal{I}_Y(L)) - 1 + h^0(D) - h^0(\mathcal{I}_Y(D)) - 1 .$$

We know that

$$h^0(L + D) - h^0(\mathcal{I}_Y(L + D)) - 1 = \dim r_Y|L + D| \geq \dim(\mathrm{Im}(\mu)) .$$

We get our desired inequality. □

REMARK 1.2. If we take Y to be an irreducible and reduced divisor, then the inequality is $h^0(L) - h^0(L - Y) + h^0(D) - h^0(D - Y) \leq h^0(L + D) - h^0(L + D - Y) + 1$. Furthermore, if Y is ample enough, such that $h^0(L-Y) = h^0(D-Y) = h^0(L+D-Y) = 0$, then we get $h^0(L) + h^0(D) \leq h^0(L + D) + 1$ which is well known.

PROOF OF THEOREM 0.1. If $h^0(K_X - L) = 1$ or $h^0(L) = 1$, we have $h^0(L) = p_g$ or $h^0(K_X - L) = p_g$, respectively. Our conclusions are obvious. Hence we assume $h^0(L) \geq 2$ and $h^0(K_X - L) \geq 2$. In particular, X is either an elliptic surface or a surface of general type.

Let $|L| = |M| + V$ be the decomposition into its movable and fixed parts. We claim that $h^0(K_X - L) > h^0(K_X - L - M)$. This is because if $h^0(K_X - L) = h^0(K_X - L - M)$, then

$$h^0(K_X - L) + h^0(M) = h^0(K_X - L - M) + h^0(M) \leq h^0(K_X - L) + 1 .$$

This implies $h^0(M) \leq 1$. It is absurd. Hence we proved the claim.

If $\dim \varphi_L(X) = 2$, then $h^0(L) \geq 3$ and the general member of $|M|$ is reduced and irreducible. Since $h^0(K_X - L) > h^0(K_X - L - M)$ and $h^0(L) - h^0(L - M) = h^0(L) - 1 \geq 2$, the conditions of Lemma 1.1 are satisfied. Hence by Lemma 1.1, we have

$$h^0(L) - h^0(L - M) + h^0(K_X - L) - h^0(K_X - L - M) \leq p_g + 1 - h^0(K_X - M) .$$

Since $h^0(L) + h^0(K_X - L) = p_g + 1$, we get $h^0(K_X - M) \leq h^0(K_X - L - M) + 1$. This implies $h^0(K_X - L - M) \geq h^0(K_X - M) - 1 \geq 1$. Thus we conclude that

$$h^0(K_X - M) - 1 + h^0(M) \leq h^0(K_X - L - M) + h^0(M) \leq h^0(K_X - L) + 1 ,$$

i.e., $h^0(K_X - M) + h^0(M) \le h^0(K_X - L) + 2$. Since $h^0(K_X - M) \ge h^0(K_X - L)$, we obtain $h^0(M) \le 2$. It contradicts that $h^0(M) = h^0(L) \ge 3$. Therefore $|L|$ is composed of a pencil. Similarly, $|K_X - L|$ is also composed of a pencil.

Since $h^0(L) + h^0(K_X - L) = p_g + 1$, i.e., $\dim|K_X| = \dim|L| + \dim|K_X - L|$, we can write every divisor in $|K_X|$ as a divisor in $|L|$ plus a divisor in $|K_X - L|$. Hence $|K_X|$ is composed of a pencil. Let $\pi : \widetilde{X} \to X$ be a composite of blowing-ups such that the movable part of $|\pi^*K_X|$ is base point free. We can assume that π is the shortest among those with such a property. Let $\widetilde{X} \xrightarrow{f} C \xrightarrow{\varepsilon} \boldsymbol{P}^{p_g-1}$ be the Stein factorization of $\varphi_{\pi^*K_X}$. Then there are two base point free divisors A and B on C such that f^*A, f^*B and $f^*(A+B)$ are respectively the movable part of π^*L, $\pi^*(K_X - L)$ and π^*K_X. Thus $h^0(L) = h^0(\pi^*L) = h^0(f^*A) = h^0(A)$, $h^0(K_X - L) = h^0(B)$ and $h^0(A+B) = p_g$. If $g(C) = 1$, we have $h^0(A) + h^0(B) = \deg(A+B) = h^0(A+B)$. This implies $h^0(L) + h^0(K_X - L) = p_g$ which contradicts our assumptions. Hence $g(C) \ne 1$. When X is of general type, we know that $g(C) = 0$, $q \le 2$ by Xiao's estimate in [18] and, therefore, $|K_X|$ is composed of a rational pencil.

When X is not of general type, it must be an elliptic surface. It follows that the movable part of $|K_X|$ is base point free, $\widetilde{X} = X$ and the general fiber of f is an elliptic curve. If $h^1(A) = 0$, then $h^0(A) = \deg A - g(C) + 1$. Thus we obtain

$$h^0(B) = h^0(A+B) + 1 - h^0(A) = \deg B + 1 + h^1(A+B) \ge \deg B + 1\,.$$

Hence $g(C) = 0$. Similarly, if $h^1(B) = 0$, we also have $g(C) = 0$. Next we assume that both of A and B are special divisors and $g(C) \ge 2$. By the Clifford theorem, we have

$$p_g + 1 = h^0(A) + h^0(B) \le \frac{\deg(A+B)}{2} + 2 \le \frac{\deg f_*\omega_X}{2} + 2 = \frac{p_g + g(C) - 1}{2} + 2\,.$$

Hence we obtain $p_g \le g(C) + 1 \le q + 1$, i.e., $\chi(\mathcal{O}_X) \le 2$. If $h^0(A) = \deg A/2 + 1$ or $h^0(B) = \deg B/2 + 1$, we get the case (a) immediately. If $h^0(A) \le (\deg A + 1)/2$ and $h^0(B) \le (\deg B + 1)/2$, then we have

$$p_g + 1 = h^0(A) + h^0(B) \le \frac{\deg(A+B)}{2} + 1 \le \frac{p_g + g(C) - 1}{2} + 1\,.$$

This implies $p_g \le g(C) - 1 \le q - 1$, i.e., $\chi(\mathcal{O}_X) \le 0$. Therefore we know that $\chi(\mathcal{O}_X) = 0$, $g(C) = q = p_g + 1$, $h^0(A) = (\deg A + 1)/2$ and $h^0(B) = (\deg B + 1)/2$. By the classical knowledge of algebraic curves, we get the cases (b) and (c). □

2. Proof of Theorem 0.2. In this section, firstly we will give some Clifford type inequalities. Let L be a divisor on a smooth minimal complex projective surface X of general type. Let $|L| = |M| + V$ be the decomposition into its movable and fixed parts, and W the image of φ_L.

PROPOSITION 2.1. *If $LK_X \ge 0$, we have*

$$h^0(L) \le \max\left\{\frac{K_X L}{2} + 1, \frac{(K_X L)^2}{2K_X^2} + 2\right\}.$$

PROOF. We can first assume that $h^0(L) \geq 3$.

Case A. $\dim W = 1$. We can write $L \sim \sum_{i=1}^{a} F_i + V \equiv aF + V$, where $a \geq h^0(L) - 1$, the $F_i's$ are the fibers of φ_L and $F^2 \geq 0$. Because of the nefness of K_X we see that

$$LK_X = aFK_X + VK_X \geq (h^0(L) - 1)FK_X\,.$$

This implies $LK_X \geq 2FK_X$. When $FK_X \geq 2$, we get $h^0(L) \leq LK_X/2 + 1$.

When $FK_X = 1$, we have $F^2K_X^2 \leq (FK_X)^2 = 1$ and $LK_X \geq 2$. But since $FK_X \equiv F^2 (\mathrm{mod}\ 2)$, this implies $F^2 = 1$ and $K_X^2 = 1$. Hence

$$h^0(L) \leq LK_X + 1 \leq \frac{(LK_X)^2}{2} + 1 = \frac{(LK_X)^2}{2K_X^2} + 1\,.$$

When $FK_X = 0$, we get $F^2 \leq 0$ by Hodge's index theorem. Thus we have $F^2 = 0$ and $F \equiv 0$. Hence $F = 0$. It is absurd.

Case B. $\dim W = 2$. In this case, we have

$$M^2 \geq (\deg \varphi_L)(\deg W) \geq (\deg \varphi_L)(h^0(L) - 2)\,.$$

When $\deg \varphi_L \geq 2$, we obtain $M^2 \geq 2h^0(L) - 4$. When $\deg \varphi_L = 1$, because X is a surface of general type, W is not a ruled surface. Hence $\deg W \geq 2n - 2 = 2h^0(L) - 4$. This implies $M^2 \geq 2h^0(L) - 4$. We obtain

$$LK_X = MK_X + VK_X \geq MK_X \geq \sqrt{M^2K_X^2} \geq \sqrt{(2h^0(L) - 4)K_X^2}\,.$$

Therefore we conclude that $h^0(L) \leq (K_XL)^2/2K_X^2 + 2$. □

The following Castelnuovo type inequality is standard (cf. [12, Lemma 2.1]).

LEMMA 2.2. *Let S be a smooth projective surface, D a divisor on S such that $|D|$ defines a birational map of S onto the image. If $|D|$ has no fixed part and $(K_S - D)D \geq 0$, then $D^2 \geq 3h^0(D) - 7$.*

PROPOSITION 2.3. *If $K_XL \geq K_X^2$, then $h^0(L) \leq (K_XL)^2/2K_X^2 + 2$. If $0 \leq K_XL \leq K_X^2$, then $h^0(L) \leq K_XL/2 + 2$. If one of the conditions holds, then φ_L is generically 2 to 1 onto a surface of minimal degree in $\boldsymbol{P}^{h^0(L)-1}$.*

PROOF. Case 1. $K_XL \geq K_X^2$. This implies $(K_XL)^2/2K_X^2 + 2 \geq K_XL/2 + 2$. By Proposition 2.1, we have $h^0(L) \leq (K_XL)^2/2K_X^2 + 2$. When the equality holds, from the proof of Proposition 2.1, we obtain $\dim W = 2$, $M^2 = 2h^0(L) - 4$, $(MK_X)^2 = M^2K_X^2$ and $VK_X = 0$. Hence $|M|$ is base point free, V is a sum of some (-2)-curves and $M \equiv rK_X$ for some rational number r.

Assume $\deg \varphi_L = 1$ and $h^0(M) \geq 4$. Then by Lemma 2.2, we have $2h^0(M) - 4 = M^2 \geq 3h^0(M) - 7$, i.e., $h^0(M) \leq 3$. This is a contradiction.

Assume $\deg \varphi_L = 1$ and $h^0(M) \leq 3$. Then since $\dim W = 2$, we have $h^0(M) = 3$ and $W = \boldsymbol{P}^2$. Hence X is a rational surface. It contradicts our assumption on X.

Therefore $\deg \varphi_L = 2$ and $\deg W = h^0(L) - 2$. Thus W is a surface of minimal degree.

Case 2. $K_XL \leq K_X^2$. In this case we have $(K_XL)^2/2K_X{}^2 + 2 \leq K_XL/2 + 2$. By Proposition 2.1, we obtain $h^0(L) \leq K_XL/2 + 2$. When the equality holds, we also have $\dim W = 2$. Therefore $(K_XL)^2/2K_X{}^2 + 2 = K_XL/2 + 2$, i.e., $K_XL = K_X^2$. Thus we can finish our proof similarly as Case 1. □

Now we will prove Theorem 0.2.

PROOF OF THEOREM 0.2. Since $h^0(K_X - L) = h^2(L) > 0$, we have $(K_X - L)K_X \geq 0$. By Proposition 2.3, we get $h^0(L) \leq K_XL/2 + 2$.

If $h^0(L) = K_XL/2 + 2$, we have $K_X^2 = M^2 = 2h^0(L) - 4$ and $(MK_X)^2 = M^2K_X^2$. Therefore $M \equiv K_X$. But since $h^0(K_X - M) \geq h^0(K_X - L) > 0$, we know that $M \sim K_X$. Hence $h^0(-V) = h^0(M - L) = h^0(K_X - L) > 0$. This implies $V = 0$ and $L = M \sim K_X$. It contradicts the assumption $L \nsim K_X$. Therefore we obtain $h^0(L) \leq (K_XL - 1)/2 + 2$.

If $h^0(L) = (K_XL - 1)/2 + 2$, we have $K_XL = 2h^0(L) - 3$. When $\dim W = 1$, we obtain

$$K_XL = 2h^0(L) - 3 \geq (h^0(L) - 1)FK_X\,.$$

This implies $FK_X = 1$. Since $F^2K_X{}^2 \leq (FK_X)^2 = 1$ and $FK_X \equiv F^2 \pmod 2$, we have $F^2 = K_X^2 = FK_X = 1$. Thus $F^2K_X^2 = (FK_X)^2 = 1$. This implies $F \equiv K_X$. Since $h^0(K_X - F) \geq h^0(K_X - L) > 0$, we know that $F \sim K_X$. Hence $L \sim K_X$. It also contradicts the assumption $L \nsim K_X$. When $\dim W = 2$, we have $M^2 \geq 2h^0(L) - 4 = K_XL - 1 \geq K_XM - 1$. Since $M^2 \equiv MK_X \pmod 2$, we get $M^2 \geq K_XM$. Because $\dim W = 2$, we can find a reduced and irreducible curve in $|M|$. Hence M is a nef divisor. Since $h^0(K_X - M) \geq h^0(K_X - L) > 0$, we have $(K_X - M)M \geq 0$ and $(K_X - M)K_X \geq 0$. It follows that $M^2 \leq K_XM \leq K_X^2$. Hence $M^2 = K_XM \leq K_X^2$. By Hodge's index theorem, we get $M^2K_X^2 \leq (K_XM)^2 = (M^2)^2$, i.e., $K_X^2 \leq M^2$. Therefore $K_X^2 = M^2 = K_XM$ and $M^2K_X^2 = (K_XM)^2$. Thus $M \equiv K_X$. Because $h^0(K_X - M) > 0$ and $M \sim K_X$, we know that $M \sim K_X \sim L$. It contradicts the assumption $L \nsim K_X$ again. Hence we conclude that

$$h^0(L) \leq \frac{K_XL - 2}{2} + 2 = \frac{K_XL}{2} + 1\,.$$

Now we assume the equality holds, i.e., $K_XL = 2h^0(L) - 2$. If $h^0(L) = 1$, then $K_XL = 0$. Hence L is a sum of (-2)-curves.

When $\dim W = 1$, we have

$$2h^0(L) - 2 = LK_X = aFK_X + VK_X \geq (h^0(L) - 1)FK_X\,.$$

This implies $K_XF \leq 2$. If $K_XF = 1$, by Hodge's index theorem, we have $F^2K_X^2 \leq (FK_X)^2 = 1$. This implies $F^2 = K_X^2 = 1$. But $K_X^2 \geq K_XL = 2h^0(L) - 2 \geq 2$. It is impossible. Hence we have $K_XF = 2$. It follows that $F^2K_X^2 \leq (FK_X)^2 = 4$. Since $K_X^2 \geq 2$ and $FK_X \equiv F^2 \pmod 2$, we obtain $F^2 = 0$ or $F^2 = 2$.

If $F^2 = 2$, then $K_X^2 = K_XL = 2$. Thus $F^2K_X^2 = (K_XF)^2 = 4$. By Hodge's index theorem, we know that $F \equiv K_X$. This implies $V \sim 0$ and $K_X \sim L \sim F$. It contradicts the assumption $L \nsim K_X$.

If $F^2 = 0$, then the movable part of $|L|$ is base point free. Since $K_X F = 2$, we conclude that $a = h^0(L) - 1$, $W \cong \boldsymbol{P}^1$ and $g(F) = (F^2 + FK_X)/2 + 1 = 2$. Therefore, the general fiber of $\varphi_L : X \to W \cong \boldsymbol{P}^1$ is an irreducible smooth curve of genus 2.

When $\dim W = 2$, we have $h^0(L) \geq 3$ and $K_X L = 2h^0(L) - 2 \geq 4$. Since $M^2 \geq 2h^0(L) - 4 = K_X L - 2 \geq K_X M - 2$, $K_X M \geq M^2$ and $M^2 - K_X M$ is even, we know that $M^2 = K_X M$ or $M^2 = K_X M - 2$.

If $M^2 = K_X M$, the inequality $(K_X M)^2 \geq K_X^2 M^2$ implies that $M^2 \geq K_X^2$. Since $K_X^2 \geq K_X M = M^2$, we have $K_X^2 = M^2 = K_X M$. By Hodge's index theorem, we obtain $M \equiv K_X$. Since $h^0(K_X - M) > 0$, we obtain $L \sim M \sim K_X$. It contradicts the assumption $L \nsim K_X$.

If $M^2 = K_X M - 2 = 2h^0(M) - 4$, we have that $|M|$ is base point free and φ_L is generically 2 to 1 onto a surface of minimal degree in $\boldsymbol{P}^{h^0(L)-1}$. $\square$

3. Clifford type indices on a surface. For a smooth connected projective curve, we have an invariant, the Clifford index, introduced by Martens [14]. It plays an important role in the study of curves. Because of Theorems 0.2 and 0.1, we can define two indices of Clifford type on a smooth minimal surface X of general type.

DEFINITION 3.1. For a divisor L on X, we define two indices $\alpha(L)$ and $\beta(L)$ by

$$\begin{aligned}\alpha(L) &= K_X L - 2h^0(L) + 2\,,\\ \beta(L) &= q + \frac{1}{2}L(K_X - L) - h^1(L)\,.\end{aligned}$$

Note that by the Serre duality theorem, we have $\beta(L) = \beta(K_X - L)$ and by the Reimann-Roch theorem, we have $h^0(L) + h^0(K_X - L) = 1 + p_g - \beta(L)$ and

$$\begin{aligned}\alpha(L) + \alpha(K_X - L) &= K_X^2 - 2(h^0(L) + h^0(K_X - L)) + 4\\ &= K_X^2 - 2(1 + p_g - \beta(L)) + 4\\ &= K_X^2 - 2p_g + 2\beta(L) + 2\,.\end{aligned}$$

Next we define indices $\alpha(X)$ and $\beta(X)$ for the surface X.

DEFINITION 3.2. Let $\mathcal{S} = \{L \in \mathrm{Pic}(X); h^0(L) \geq 2, h^0(K_X - L) \geq 2\}$, we define $\alpha(X)$ and $\beta(X)$ by

$$\alpha(X) = \begin{cases}\min\limits_{L\in\mathcal{S}} \alpha(L) & \mathcal{S} \neq \emptyset\\ \infty & \mathcal{S} = \emptyset\end{cases}$$

$$\beta(X) = \begin{cases}\min\limits_{L\in\mathcal{S}} \beta(L) & \mathcal{S} \neq \emptyset\\ \infty & \mathcal{S} = \emptyset\,.\end{cases}$$

Similarly as in the curve case, we say that L computes the index $\alpha(X)$ or $\beta(X)$, if $\alpha(X) = \alpha(L)$ or $\beta(X) = \beta(L)$, respectively.

REMARK 3.3. When L computes $\alpha(X)$ or $\beta(X)$, we can always assume $|L|$ has no fixed part. This assumption is convenient for our work. The reason is as follows. Let $|L| = |M| + V$ be the decomposition into its movable and fixed parts. If L computes $\alpha(X)$, we have $h^0(L) = h^0(M)$ and $VK_X \geq 0$. Therefore $K_X L - 2h^0(X, L) + 2 \geq K_X M - 2h^0(X, M) + 2$, i.e., $\alpha(L) \geq \alpha(M)$. If L computes $\beta(X)$, we have $h^0(L) + h^0(K_X - L) = 1 + p_g - \beta(X)$. But since $h^0(L) + h^0(K_X - L) \leq h^0(M) + h^0(K_X - M) \leq 1 + p_g - \beta(X)$, we have $h^0(M) + h^0(K_X - M) = 1 + p_g - \beta(X)$. Hence M computes $\beta(X)$ too.

EXAMPLE 3.4. Let S_d be a generic hypersurface of degree d in $\boldsymbol{P}^3$. H denote the hyperplane section of S_d. When $d \geq 5$, S_d is a minimal surface of general type and $K_{S_d} = (d-4)H$. In this case, by the Noether-Lefschetz theorem, we have $\mathrm{Pic}(S_d) \cong \boldsymbol{Z}H$. Hence $\alpha(S_5) = \beta(S_5) = \infty$.

Now we assume $d \geq 6$. Let n be an integer such that $1 \leq n \leq d-5$. Then we have

$$h^0(nH) = \frac{1}{6}(n+1)(n+2)(n+3)\,.$$

Thus we obtain

$$\begin{aligned}\alpha(nH) &= nHK_{S_d} - 2h^0(nH) + 2\\ &= nd(d-4) - \frac{1}{3}(n+1)(n+2)(n+3) + 2.\end{aligned}$$

Hence $\alpha(S_d) = \min_{1\leq n\leq d-5}\alpha(nH) = \alpha(H) = d(d-4) - 6$. We also have

$$\begin{aligned}\beta(nH) &= p_g(S_d) + 1 - h^0(nH) - h^0((d-4-n)H)\\ &= -\frac{1}{2}d(n^2 - (d-4)n).\end{aligned}$$

Therefore $\beta(S_d) = \min_{1\leq n\leq d-5}\beta(nH) = \beta(H) = d(d-5)/2$.

For surfaces with $\alpha = \infty$, we have the following theorem.

THEOREM 3.5. *If S is a surface with $\alpha(S) = \infty$, then $\alpha(S') = \infty$ for every small deformation S' of S.*

PROOF. Let $f : \mathcal{X} \to \Delta$ be a small deformation of $\mathcal{X}_0 = S$, $0 \in \Delta$, such that the Picard scheme $\mathrm{Pic}_{\mathcal{X}/\Delta}$ and the Poincaré line bundle $\mathcal{L}$ on $\mathcal{X} \times \mathrm{Pic}_{\mathcal{X}/\Delta}$ exist (cf. [13]). Put

$$W_{m,n} = \{y \in \mathrm{Pic}_{\mathcal{X}/\Delta}\,;\, h^0(\mathcal{L}_y) \geq m, h^2(\mathcal{L}_y) \geq n\}\,.$$

By the semicontinuity theorem [9, Theorem 12.8], we know that $W_{m,n}$ is a closed subscheme of $\mathrm{Pic}_{\mathcal{X}/\Delta}$. Consider the natural morphism $\pi : \mathrm{Pic}_{\mathcal{X}/\Delta} \to \Delta$. Then $\{p \in \Delta\,;\, W_{m,n} \cap \pi^{-1}(p) = \emptyset\}$ is an open subset of Δ. Since $\alpha(S) = \infty$, we have $\{L \in \mathrm{Pic}(S)\,;\, h^0(L) \geq 2, h^2(L) \geq 2\} = \emptyset$. Hence $W_{2,2} \cap \pi^{-1}(0) = \emptyset$ and $\{p \in \Delta\,;\, W_{2,2} \cap \pi^{-1}(p) = \emptyset\} \neq \emptyset$. Thus for every $p \in \{p \in \Delta\,;\, W_{2,2} \cap \pi^{-1}(p) = \emptyset\}$, we have $\alpha(f^{-1}(p)) = \infty$. This completes the proof of the theorem. $\square$

The above theorem tells us the surfaces with $\alpha = \infty$ form an open subset of the moduli of surfaces. We now give some bounds for $\alpha(X)$ and $\beta(X)$ as follows:

PROPOSITION 3.6. *If* $\alpha(X) \neq \infty$, *then* $0 \leq \alpha(X) \leq K_X^2 - 2\chi(\mathcal{O}_X) + 6$.

PROOF. $\alpha(X) \geq 0$ is an easy consequence of Theorem 0.2. Suppose that L computes $\alpha(X)$. Then we have $\alpha(X) = K_X L - 2h^0(X, L) + 2$. By Remark 3.3, we can assume $|L|$ has no fixed part. Let W be the image of φ_L.

Case A. $\dim W = 1$. In this case, we have $L \sim \sum_{i=1}^{a} F_i \equiv aF$, where the $F_i's$ are the fibers of φ_L. Since

$$a \geq h^0(L) - 1 = \frac{K_X L - \alpha(X)}{2} = \frac{aK_X F - \alpha(X)}{2},$$

we have

$$2a + \alpha(X) \geq aK_X F. \tag{2}$$

By the Riemann-Roch theorem, we obtain

$$h^0(L) + h^0(K_X - L) \geq \frac{a^2}{2}F^2 - \frac{a}{2}K_X F + \chi(\mathcal{O}_X). \tag{3}$$

Since $h^0(L) = (K_X L - \alpha(X))/2 + 1$ and $h^0(K_X - L) \leq (K_X(K_X - L) - \alpha(X))/2 + 1$, we get from (3) the inequality

$$K_X^2 - 2\chi(\mathcal{O}_X) + 4 + aK_X F - a^2F^2 \geq 2\alpha(X). \tag{4}$$

If $F^2 \geq 1$, we have $2a \leq a^2 + 1 \leq a^2F^2 + 1$. This and (2) imply $aK_X F - a^2F^2 \leq \alpha(X) + 1$. Hence by (4), we get $\alpha(X) \leq K_X^2 - 2\chi(\mathcal{O}_X) + 5$.

If $F^2 = 0$, then $|L|$ has no base points. We can assume that F is a smooth and irreducible curve. When $h^0(L) = 2$, we have $W = \boldsymbol{P}^1$, and $\alpha(X) + 2 = aK_X F$. By (4), we get our conclusion immediately. When $h^0(L) \geq 3$, from the standard exact sequence

$$0 \to \mathcal{O}_X(L - F) \to \mathcal{O}_X(L) \to \mathcal{O}_F \to 0,$$

it follows that $h^0(L - F) \geq h^0(L) - 1 \geq 2$. Therefore

$$\frac{1}{2}((a-1)K_X F - \alpha(X)) + 1 \geq h^0(L - F) \geq h^0(L) - 1 = \frac{1}{2}(aK_X F - \alpha(X)).$$

This implies $K_X F \leq 2$. Since $F^2 = 0$, we have $K_X F = 2$. Hence $h^0(L - F) = h^0(L) - 1 = a - \alpha(X)/2$, for a general fiber F. Inductively, we can get $h^0(L - iF) = h^0(L) - i$, for $1 \leq i \leq h^0(L) - 1$. Let $k = h^0(L) - 2$, then $h^0(L - kF) = 2$. On one hand, by the Riemann-Roch theorem, we obtain

$$\begin{aligned} h^0(L - kF) + h^0(K_X - L + kF) &\geq \frac{1}{2}(L - kF)(L - kF - K_X) + \chi(\mathcal{O}_X) \\ &= k - a + \chi(\mathcal{O}_X). \end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned} h^0(L - kF) + h^0(K_X - L + kF) &\leq 2 + \frac{1}{2}K_X(K_X - L + kF) - \frac{1}{2}\alpha(X) + 1 \\ &= \frac{1}{2}K_X^2 + k - a - \frac{1}{2}\alpha(X) + 3. \end{aligned}$$

Combining these two inequalities, we can get $\alpha(X) \leq K_X^2 - 2\chi(\mathcal{O}_X) + 6$.

Case B. $\dim W = 2$. This case implies that $L^2 \geq 2h^0(L) - 4 = K_X L - \alpha(X) - 2$. Hence by the Riemann-Roch theorem, we obtain

$$\begin{aligned} h^0(L) + h^0(K_X - L) &\geq \frac{1}{2}L^2 - \frac{1}{2}K_X L + \chi(\mathcal{O}_X) \\ &\geq -\frac{1}{2}\alpha(X) - 1 + \chi(\mathcal{O}_X)\,. \end{aligned} \tag{6}$$

Since

$$\begin{aligned} h^0(L) + h^0(K_X - L) &\leq \frac{1}{2}(K_X L - \alpha(X)) + 1 + \frac{1}{2}(K_X(K_X - L) - \alpha(X)) + 1 \\ &= \frac{1}{2}K_X^2 - \alpha(X) + 2\,, \end{aligned}$$

we obtain $K_X^2/2 - \alpha(X) + 2 \geq -\alpha(X)/2 - 1 + \chi(\mathcal{O}_X)$, i.e., $\alpha(X) \leq K_X^2 - 2\chi(\mathcal{O}_X) + 6$. □

We can see easily the following corollary.

COROLLARY 3.7. *If $\alpha(X) = K_X^2 - 2\chi(\mathcal{O}_X) + 6$, L computes $\alpha(X)$ and $|L|$ has no fixed part, then $|L|$ is base point free and one of the following cases occurs.*

1. *$h^0(L) = 2$, $h^1(L) = 0$ and $K_X - L$ also computes $\alpha(X)$.*
2. *$h^0(L) \geq 3$ and $|L|$ is composed of a pencil of genus 2.*
3. *$h^1(L) = 0$, $K_X - L$ also computes $\alpha(X)$ and φ_L is generically 2 to 1 onto a surface of minimal degree in $\boldsymbol{P}^{h^0(L)-1}$.*

PROPOSITION 3.8. *If $\alpha(X) \neq \infty$, then $0 \leq \beta(X) \leq \alpha(X)/2 + q + 1$.*

PROOF. $\beta(X) \geq 0$ is an easy consequence of Theorem 0.1. The following proof is similar to that of Proposition 3.6. Keep the notation as in the proof of Proposition 3.6. We assume L computes $\alpha(X)$ and $|L|$ has no fixed part. Then we have $\alpha(X) = K_X L - 2h^0(L) + 2$ and

$$h^0(L) + h^0(K_X - L) = 1 + p_g - \beta(L) \leq 1 + p_g - \beta(X)\,. \tag{7}$$

Case A. $\dim W = 1$. By (3) and (7), we obtain

$$\beta(X) \leq q + \frac{a}{2}K_X F - \frac{a^2}{2}F^2\,. \tag{8}$$

If $F^2 \geq 1$, we have $2a \leq a^2 + 1 \leq a^2F^2 + 1$. This and (2) imply $aK_X F - a^2F^2 \leq \alpha(X) + 1$. From (8), it follows that $\beta(X) \leq \alpha(X)/2 + q + 1/2$.

If $F^2 = 0$, then $|L|$ has no base points. When $h^0(L) = 2$, then $\alpha(X) + 2 = aK_X F$. It follows from (8) that $\beta(X) \leq \alpha(X)/2 + q + 1$. When $h^0(L) \geq 3$, similarly as in the proof of Proposition 3.6, we have $K_X F = 2$. Hence by (5), we get

$$h^0(L - kF) + h^0(K_X - L + kF) \geq k - a + \chi(\mathcal{O}_X)\,,$$

where $k = h^0(L) - 2 = (K_X L - \alpha(X))/2 - 1$. On the other hand, $h^0(L - kF) + h^0(K_X - L + kF) \leq 1 + p_g - \beta(X)$. Combining them, we obtain

$$\begin{aligned}\beta(X) \leq q + a - k &= q + a - \frac{1}{2}(K_X L - \alpha(X)) + 1 \\ &= q + \frac{1}{2}\alpha(X) + 1 + a - \frac{aK_X F}{2} \\ &= q + \frac{1}{2}\alpha(X) + 1\,.\end{aligned}$$

Case B. $\dim W = 2$. By (6) and (7), we get $\beta(X) \leq \alpha(X)/2 + q + 1$ immediately. □

Similarly as in the case of Corollary 3.7, we have the following corollary.

COROLLARY 3.9. *If $\beta(X) = \alpha(X)/2 + q + 1$, L computes $\alpha(X)$ and $|L|$ has no fixed part, then $|L|$ is base point free and one of the following cases occurs.*

1. *$h^0(L) = 2$, $h^1(L) = 0$ and L computes $\beta(X)$.*
2. *$h^0(L) \geq 3$ and $|L|$ is composed of a pencil of genus 2.*
3. *$h^1(L) = 0$, L computes $\beta(X)$ and φ_L is generically 2 to 1 onto a surface of minimal degree in $\boldsymbol{P}^{h^0(L)-1}$.*

4. Surfaces with $\alpha = 0$ or $\beta = 0$. It is natural to ask what will happen when these indices α and β are small. The answers for $\alpha = 0$ and $\beta = 0$, respectively, are given in the following theorems. We always assume L computes $\alpha(X)$ and $|L|$ has no fixed part.

THEOREM 4.1. *If $\alpha(X) = 0$, then $|L|$ has no base point and one of the following occurs.*

1. *There exists a projective surjective morphism $f : X \to \boldsymbol{P}^1$, whose general fiber is an irreducible smooth curve of genus 2.*
2. *X is the minimal resolution of a double covering of $\boldsymbol{P}^2$, whose branch locus is a reduced curve of degree 10 with only one infinitely near triple point as its essential singularity. In this case, $K_X^2 = 7$, $p_g = 5$, $q = 0$ and $K_X \sim 2L - Z$, where Z is an effective divisor with $LZ = 0$ and $K_X L = 2L^2 = 4$.*
3. *X is the smooth minimal model of a double covering of Σ_2, whose branch locus is a reduced curve of $|8\Delta_0 + 14\Gamma|$ with at worst negligible singularities. In this case, $K_X^2 = 9$, $p_g = 6$, $q = 0$ and $K_X \sim 3D$, where $2D = L$.*
4. *X is the minimal resolution of a double covering of $\boldsymbol{P}^2$, whose branch locus is a reduced curve of degree 10 with at worst negligible singularities. In this case, $K_X^2 = 8$, $p_g = 6$, $q = 0$ and $K_X \sim 2L$.*

PROOF. Let W be the image of φ_L. Since L computes $\alpha(X)$, we get $K_X L - 2h^0(L) + 2 = \alpha(X) = 0$.

When $\dim W = 1$, by Theorem 0.2, we have $|L|$ is base point free and the general fiber of $\varphi_L : X \to W \cong \boldsymbol{P}^1$ is an irreducible smooth curve of genus 2. Thus X is the surface of type 1 in the theorem. When $\dim W = 2$, by Theorem 0.2, we know that $|L|$ is base point free, $\varphi_L : X \to W$ is generically 2 to 1 and

$$L^2 = K_X L - 2 = 2h^0(L) - 4 \geq 2\,. \tag{9}$$

By Hodge's index theorem, we obtain

$$K_X^2 L^2 \leq (K_X L)^2 = (L^2+2)^2 = (L^2)^2 + 4L^2 + 4\,.$$

This implies

$$K_X^2 \leq L^2 + \frac{4}{L^2} + 4\,. \tag{10}$$

Since $2 \leq h^0(K_X - L) \leq K_X(K_X - L)/2 + 1$, we have

$$K_X^2 \geq K_X L + 2 = L^2 + 4\,. \tag{11}$$

Combining (10) and (11), we obtain

$$L^2 + 4 \leq K_X^2 \leq L^2 + \frac{4}{L^2} + 4 \leq L^2 + 6\,.$$

Thus we get three possible cases A: $K_X^2 = L^2+4$, B: $K_X^2 = L^2+5$ and C: $K_X^2 = L^2+6$.

Case A. $K_X^2 = L^2 + 4$. This implies that $K_X^2 = K_X L + 2$. Then

$$2 \leq h^0(K_X - L) \leq \frac{1}{2}K_X(K_X - L) + 1 = \frac{1}{2}({K_X}^2 - K_X L) + 1 = 2\,.$$

Thus $h^0(K_X - L) = 2$. Let $|K_X - L| = |M'| + V'$ be the decomposition into its movable and fixed parts. Let $\phi : X \dashrightarrow \boldsymbol{P}^1$ be the rational map defined by $|K_X - L|$. Then there exists an irreducible reduced curve F', such that $F'^2 \geq 0$, $M' \equiv bF'$ and $b \geq h^0(K_X - L) - 1 = 1$. Since

$$2 = (K_X - L)K_X = M'K_X + V'K_X \geq bF'K_X \geq F'K_X\,,$$

we can get $F'K_X = 2$. Hence $b = 1$, $F'^2 = 0$ or 2. If $F'^2 = 2$, then $(K_X - L)F' = M'F' + V'F' \geq F'^2 = 2$. Thus $LF' \leq K_X F' - 2 = 0$. By Hodge's index theorem, we get $F'^2 \leq 0$. It is impossible. It follows that $F'^2 = 0$, and $|M'|$ is base point free. Hence $g(F') = (F'^2 + K_X F')/2 + 1 = 2$ and $M' \sim F'$. We know that the general fiber of $\phi : X \to \boldsymbol{P}^1$ is an irreducible smooth curve of genus 2. Therefore X is the surface of type 1 in the theorem.

Case B. $K_X^2 = L^2 + 5$. Since $2 \leq h^0(K_X - L) \leq K_X(K_X - L)/2 + 1 = 5/2$, we get $h^0(K_X - L) = 2$. By (10), we have $L^2 + 5 \leq L^2 + 4/L^2 + 4$. This implies $2 \leq L^2 \leq 4$. Since $L^2 = 2h^0(L) - 4$ is an even number, there are two cases B-I: $L^2 = 2$ and B-II: $L^2 = 4$.

Case B-I. We have $K_X^2 = 7$, $K_X L = 4$ and $h^0(L) = 3$. By Theorem 0.1, we have $p_g(X) = h^0(K_X) \geq h^0(L) + h^0(K_X - L) = 5$. Using Noether's inequality, we know that $7 = K_X^2 \geq 2p_g(X) - 4$, i.e., $p_g(X) \leq 5$. Thus we get $p_g(X) = 5$ and $K_X^2 = 7 < 10 = 2p_g(X)$. Since $K_X^2 \geq 2p_g(X)$, when X is irregular (See [7]), we conclude that $q(X) = 0$.

Since $h^0(L) = 3$, we know that $\varphi_L : X \to W = \boldsymbol{P}^2$ is generically 2 to 1. Let $X \to X' \xrightarrow{f} \boldsymbol{P}^2$ be the Stein factorization of φ_L, $\widetilde{X}$ the canonical resolution of the double covering and m_i the multiplicity of the corresponding singularity. R and B denote, respectively, the ramification divisor and the branch locus of φ_L. If H denotes a line on $\boldsymbol{P}^2$, then we have $K_X = \varphi_L^*(-3H) + R = -3L + R$. By the theory of double covering (See [10, §2], [11, III, §2] or [19, §1.3]), there exists an effective divisor Z on X, such that $2R = \varphi_L^* B - 2Z$ and $LZ = 0$. Thus

$$BH = \frac{1}{2}\varphi_L^* B \varphi_L^* H = (R+Z)L = RL = (K_X + 3L)L = K_X L + 3L^2 = 10\,.$$

Hence $B \sim 10H$ and $K_X \sim 2L - Z$. Now we can compute the invariants of $\widetilde{X}$. We have

$$\begin{aligned}\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\widetilde{X}}) &= \frac{1}{4}B\left(K_{\boldsymbol{P}^2} + \frac{1}{2}B\right) + 2\chi(\mathcal{O}_{\boldsymbol{P}^2}) - \sum_i \frac{1}{2}\left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right)\\ &= 7 - \frac{1}{2}\sum_i \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right),\end{aligned}$$

$$K_{\widetilde{X}}^2 = 2\left(K_{\boldsymbol{P}^2} + \frac{1}{2}B\right)^2 - \sum_i 2\left(\left[\frac{m_i}{2}\right] - 1\right)^2 = 8 - 2\sum_i \left(\left[\frac{m_i}{2}\right] - 1\right)^2.$$

From the equality $q(\widetilde{X}) = q(X) = 0$, it follows that

$$p_g(X) = 6 - \frac{1}{2}\sum_i \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right).$$

Since $p_g(X) = 5$, we have $[m_i/2] = 2$ for only one index and $K_{\widetilde{X}}^2 = 6$. It follows that $\widetilde{X}$ has a (-1)-curve. Therefore X is the surface of type 2 in the theorem.

Case B-II. We have $K_X^2 = L^2 + 5 = 9$ and $K_X L = L^2 + 2 = 6$ by (9). Thus $L^2 K_X^2 = 36 = (K_X L)^2$. Using Hodge's index theorem, we have $L \equiv (2/3)K_X$. It follows from (9) that $h^0(L) = 4$. By Theorem 0.1, we have $p_g(X) = h^0(K_X) \geq h^0(L) + h^0(K_X - L) = 6$. By Noether's inequality, we obtain $9 = K_X^2 \geq 2p_g(X) - 4$, i.e., $p_g(X) \leq 6$. Hence $p_g(X) = 6$ and $K_X^2 = 9 < 12 = 2p_g(X)$. It follows that $q(X) = 0$.

Since $\deg W = L^2/2 = 2$, either $W \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ or W is a quadric cone.

Assume $W \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Two rulings of W allow us to write $L \sim D_1 + D_2$ with divisors D_i satisfying $D_i^2 = 0$ $(i = 1, 2)$. Since $6 = K_X L = K_X D_1 + K_X D_2$, we may assume that $K_X D_1$ is an even integer not greater than 3. Hence $K_X D_1 = 2$. But, this is absurd, because $LD_1 = (2/3)K_X D_1 = 4/3$.

Now we assume that W is a quadric cone. In this case, by the same argument as in the proof of [10, Lemma 2, Case II b], we have $L \sim 2D + G$, where $|D|$ is a pencil and G is an effective divisor with $LG = 0$. From the equality $4 = L^2 = L(2D + G)$, it follows that $LD = 2$. Since $L \equiv (2/3)K_X$, we have $K_X D = 3$. From $LD = 2$, we get $(2D + G)D = 2D^2 + DG = 2$, hence $D^2 = 0$ or 1. But $3 = K_X D \equiv D^2 \pmod 2$, hence $D^2 = 1$ and $DG = 0$. The equality $0 = LG = 2DG + G^2$ implies that $G^2 = 0$. Then by Hodge's index theorem, we have $G = 0$. Thus $L \sim 2D$ and $K_X \equiv 3D$.

Since $h^0(D) \leq (1/2)K_X D + 1 = 5/2$, we have $h^0(D) = 2$. By $D^2 = 1$, we know $|D|$ has one base point P. Let $\sigma : \widehat{X} \to X$ be the blowing-up with center P and put $E = \sigma^{-1}(P)$.

Then the movable part $\widehat{D}$ of $|\sigma^*D|$ defines a holomorphic map $g : \widehat{X} \to \boldsymbol{P}^1$. Since $|L|$ has no base point, there exists $\eta \in H^0(\widehat{X}, \mathcal{O}_{\widehat{X}}(\sigma^*L))$ which does not vanish on E. Take $\xi \in H^0(\widehat{X}, \mathcal{O}_{\widehat{X}}(2E))$ such that $(\xi) = 2E$. Then ξ/η is a meromorphic section of $\mathcal{O}_{\widehat{X}}(-2\widehat{D})$. Then g and ξ/η define a rational map $h : \widehat{X} \to \Sigma_2$. Since η does not vanish on E, h is defined everywhere so that $h^*\Delta_0 = 2E$. We consider the linear system $|\Delta_0 + 2\Gamma|$ on Σ_2. This give rise to a morphism $q : \Sigma_2 \to \boldsymbol{P}^3$ whose image coincides with W up to an automorphism of $\boldsymbol{P}^3$. Then by the construction, we have the following commutative diagram.

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{h} & \Sigma_2 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\varphi_L} & W \end{array}$$

Let H be a plane on $\boldsymbol{P}^3$. R and B denote, respectively, the ramification divisor and the branch locus of h. Then there exists an effective divisor Z on $\widehat{X}$, such that $2R = h^*B - 2Z$ and Z is contracted by h. Since $q^*H = \Delta_0 + 2\Gamma = -(1/2)K_{\Sigma_2}$, we have $\sigma^*(2D) = \sigma^*L = \sigma^*\varphi_L^*H = h^*q^*H = h^*(\Delta_0 + 2\Gamma)$. The equality $h^*\Delta_0 = 2E$ implies that $h^*\Gamma = \sigma^*D - E$. Thus we obtain

$$K_{\widehat{X}} = h^*(-2\Delta_0 - 4\Gamma) + R = -4E - 4(\sigma^*D - E) + R = -2\sigma^*L + R\,.$$

As in Case B-I, we have

$$B\Delta_0 = Rh^*\Delta_0 = (K_{\widehat{X}} + 2\sigma^*L)(2E) = 2(\sigma^*K_X + E + 2\sigma^*L)E = -2\,,$$

$$B\Gamma = Rh^*\Gamma = (\sigma^*K_X + E + 2\sigma^*L)(\sigma^*D - E) = K_XD + 2LD - E^2 = 8\,.$$

Hence $B \sim 8\Delta_0 + 14\Gamma$. This equality implies

$$K_{\widehat{X}} = h^*(4\Delta_0 + 7\Gamma) - Z - 2\sigma^*L = 3\sigma^*D + E - Z\,,$$

i.e., $K_X \sim 3D - \sigma_*Z$. Since $K_X \equiv 3D$, we get $K_X \sim 3D$ and $Z = 0$. Let $\overline{X}$ be the canonical resolution of the double covering h and m_i the multiplicity of the corresponding singularity. By the standard theory of double covering, we obtain

$$\begin{aligned}\chi(\mathcal{O}_{\overline{X}}) = \chi(\mathcal{O}_{\widehat{X}}) &= \frac{1}{4}B\Big(-2\Delta_0 - 4\Gamma + \frac{1}{2}B\Big) + 2 - \sum_i \frac{1}{2}\Big[\frac{m_i}{2}\Big]\Big(\Big[\frac{m_i}{2}\Big] - 1\Big) \\ &= 7 - \frac{1}{2}\sum_i \Big[\frac{m_i}{2}\Big]\Big(\Big[\frac{m_i}{2}\Big] - 1\Big),\end{aligned}$$

$$K_{\overline{X}}^2 = 2\Big(K_{\Sigma_2} + \frac{1}{2}B\Big)^2 - \sum_i 2\Big(\Big[\frac{m_i}{2}\Big] - 1\Big)^2 = 8 - 2\sum_i \Big(\Big[\frac{m_i}{2}\Big] - 1\Big)^2.$$

The equality $q(\widehat{X}) = q(\overline{X}) = q(X) = 0$ implies that

$$p_g(X) = 6 - \frac{1}{2}\sum_i \Big[\frac{m_i}{2}\Big]\Big(\Big[\frac{m_i}{2}\Big] - 1\Big).$$

Since $p_g(X) = 6$, we have $[m_i/2]([m_i/2] - 1) = 0$ for all indices. Thus $B \in |8\Delta_0 + 14\Gamma|$ is a reduced curve with at worst negligible singularities. Therefore, in this case, X is the surface of type 3 in the theorem.

Case C. $K_X^2 = L^2 + 6$. From (10), it follows that $L^2 + 6 = K_X^2 \leq L^2 + 4/L^2 + 4$. This implies $L^2 \leq 2$. Thus $L^2 = 2$ and $K_X^2 = 8$. By (9), we get $h^0(L) = 3$ and $K_X L = 4$. Hence $K_X^2 L^2 = 16 = (K_X L)^2$. By Hodge's index theorem, we conclude that $K_X \equiv 2L$.

Since $h^0(L) = 3$, we know that $\varphi_L : X \to W = \boldsymbol{P}^2$ is generically 2 to 1. Let m_i be the multiplicity of the corresponding singularity. R and B denote, respectively, the ramification divisor and the branch locus of φ_L. If H denotes a line on $\boldsymbol{P}^2$, then we have $K_X = \varphi_L^*(-3H) + R = -3L + R$. By the theory of double covering, there exists an effective divisor Z on X such that $2R = \varphi_L^* B - 2Z$ and $LZ = 0$. We get $BH = RL = (K_X + 3L)L = 10$, i.e., $B \sim 10H$. Thus $K_X \sim -3L + 5L - Z \sim 2L - Z$. Since $K_X \equiv 2L$, we obtain $K_X \sim 2L$ and $Z = 0$. By [10, Lemma 5], we know that $B \in |10H|$ is a reduced curve with at worst negligible singularities. Therefore X is the surface of type 4 in the theorem and $p_g(X) = 6 - (1/2)\sum[m_i/2]([m_i/2] - 1) = 6$. □

Now, we assume L computes $\beta(X)$ and $|L|$ has no fixed part.

THEOREM 4.2. *If $\beta(X) = 0$, then $|K_X|$ is composed of a rational pencil and $|L|$ is a sum of some fibers of the pencil.*

PROOF. It is just a special case of Theorem 0.1. □

When α or β increase, the surface become more and more complicated and we can not hope to give a detailed description of the surface.

5. The number of moduli of a surface.

DEFINITION 5.1. For a surface of general type S, we define $M(S)$, which is the number of moduli of S, to be the dimension of its Kuranishi space B, i.e., the maximum of the dimensions of the irreducible components of B (cf. [5]).

Hence we have

$$10\chi(\mathcal{O}_S) - 2K_S^2 = h^1(T_S) - h^2(T_S) \leq M(S) = \dim B \leq h^1(T_S)\,.$$

By [3], we have $h^0(T_S) = h^0(\Omega_S^1(-K_S)) = 0$. By Serre duality, $h^2(T_S) = h^0(\Omega_S^1(K_S))$, and we have

$$M(S) \leq h^1(T_S) = 10\chi(\mathcal{O}_S) - 2K_S^2 + h^0(\Omega_S^1(K_S))\,. \tag{12}$$

Hence one can give an upper bound for $M(S)$ by giving an upper bound for $h^0(\Omega_S^1(K_S))$. The following theorem improves the inequality given in [6, Theorem B].

THEOREM 5.2. *Let X be a smooth minimal complex projective surface of general type. We have the inequality $M(X) \leq 10\chi(\mathcal{O}_X) + (5/2)K_X^2 + 4$. Furthermore, if $q(X) > 0$, then $M(X) \leq 10\chi(\mathcal{O}_X) + (1/2)K_X^2 + 4$.*

PROOF. We can assume $h^0(\Omega_X^1(K_X)) > 0$. We know that $\Omega_X^1(K_X)$ is K_X-semistable (cf. [8, Corollary 1.2], [3] or [17]). Thus we can find an invertible sheaf $\mathcal{O}_X(L)$ such that it is an maximal invertible subbundle of $\Omega_X^1(K_X)$ of maximal slope. Then $\Omega_X^1(K_X)/\mathcal{O}_X(L)$ is torsion free and $(\Omega_X^1(K_X)/\mathcal{O}_X(L))^{\vee\vee} \cong \mathcal{O}_X(3K_X - L)$. Hence we obtain

$$h^0(\Omega_X^1(K_X)) \leq h^0(L) + h^0((\Omega_X^1(K_X)/\mathcal{O}_X(L))) \leq h^0(L) + h^0(3K_X - L)\,. \tag{13}$$

Case 1. $K_X L < K_X^2$. The inequality implies $K_X(3K_X - L) > 2K_X^2$. By the assumption $h^0(\Omega_X^1(K_X)) > 0$, we have $K_X L \geq 0$. Thus by Proposition 2.3, we get $h^0(L) \leq K_X L/2 + 2$ and $h^0(3K_X - L) \leq (K_X(3K_X - L))^2/2K_X^2 + 2$. It follow from (13) that

$$\begin{aligned}
h^0(\Omega_X^1(K_X)) &\leq \frac{K_X L}{2} + 2 + \frac{(K_X(3K_X - L))^2}{2K_X^2} + 2\\
&= \frac{(K_X L - (5/2)K_X^2)^2 - (25/4)(K_X^2)^2}{2K_X^2} + \frac{9}{2}K_X^2 + 4\\
&\leq \frac{(0 - (5/2)K_X^2)^2 - (25/4)(K_X^2)^2}{2K_X^2} + \frac{9}{2}K_X^2 + 4\\
&= \frac{9}{2}K_X^2 + 4\,.
\end{aligned}$$

Hence $M(X) \leq 10\chi(\mathcal{O}_X) - 2K_X^2 + h^0(\Omega_X^1(K_X)) \leq 10\chi(\mathcal{O}_X) + (5/2)K_X^2 + 4$.

Case 2. $K_X L \geq K_X^2$. By Proposition 2.3, we get $h^0(L) \leq (K_X L)^2/2K_X^2 + 2$. Since $\Omega_X^1(K_X)$ is K_X-semistable, we have $K_X L \leq (3/2)K_X^2$. Hence $K_X(3K_X - L) \geq (3/2)K_X^2$. By Proposition 2.3, we obtain $h^0(3K_X - L) \leq (K_X(3K_X - L))^2/2K_X^2 + 2$. From (13), it follows that

$$\begin{aligned}
h^0(\Omega_X^1(K_X)) &\leq \frac{(K_X L)^2}{2K_X^2} + 2 + \frac{(K_X(3K_X - L))^2}{2K_X^2} + 2\\
&= \frac{(K_X L - (3/2)K_X^2)^2 - (9/4)(K_X^2)^2}{K_X^2} + \frac{9}{2}K_X^2 + 4\\
&\leq \frac{(K_X^2 - (3/2)K_X^2)^2 - (9/4)(K_X^2)^2}{K_X^2} + \frac{9}{2}K_X^2 + 4\\
&= \frac{5}{2}K_X^2 + 4\,.
\end{aligned}$$

Hence $M(X) \leq 10\chi(\mathcal{O}_X) - 2K_X^2 + h^0(\Omega_X^1(K_X)) \leq 10\chi(\mathcal{O}_X) + (1/2)K_X^2 + 4$.

When $q(X) = h^0(\Omega_X^1) > 0$, we know that $\mathcal{O}_X(K_X) \subset \Omega_X^1(K_X)$. Thus $K_X L \geq K_X^2$. Therefore, by Case 2, we have $M(X) \leq 10\chi(\mathcal{O}_X) + (1/2)K_X^2 + 4$. □

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