On the closed-loop Volterra method for analyzing time series

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Abstract

The main focus of this paper is to approximate time series data based on the closed-loop Volterra series representation. Volterra series expansions are a valuable tool for representing, analyzing, and synthesizing nonlinear dynamical systems. However, a major limitation of this approach is that as the order of the expansion increases, the number of terms that need to be estimated grows exponentially, posing a considerable challenge. This paper considers a practical solution for estimating the closed-loop Volterra series in stationary nonlinear time series using the concepts of Reproducing Kernel Hilbert Spaces (RKHS) and polynomial kernels. We illustrate the applicability of the suggested Volterra representation by means of simulations and real data analysis. Furthermore, we apply the Kolmogorov-Smirnov Predictive Accuracy (KSPA) test, to determine whether there exists a statistically significant difference between the distribution of estimated errors for concurring time series models, and secondly to determine whether the estimated time series with the lower error based on some loss function also has exhibits a stochastically smaller error than estimated time series from a competing method. The obtained results indicate that the closed-loop Volterra method can outperform the ARFIMA, ETS, and Ridge regression methods in terms of both smaller error and increased interpretability. Keywords: Time series analysis, Volterra series, Closed-loop method. MSC Classification: 37M10, 62M10, 62M20.

1 Introduction

Nonlinear systems with memory are frequently encountered in time series analysis. One of the primary objectives of time series analysis is to infer the functional relationship between the input and output of these systems based on observations. The first approach to a nonparametric characterization of nonlinear systems which is similar to the Taylor series dates back to Volterra (1887). Volterra extended the standard convolution definition of linear systems using a series of polynomial integral operators with increasing degree of nonlinearity. The Volterra series expansion, proposed by Priestley (1988), forms a model for the system's output as a polynomial in the delayed inputs. Boyd and Chua (1985) showed that this model can provide a good representation for a wide class of nonlinear systems. During the last years, many studies have been done in diverse fields such as nonlinear differential equations, neuroscience, fluid dynamics, or electrical

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engineering, which can be represented by the Volterra operators, see for example (Giannakis and Serpedin, 2001; Mathews and Sicuranza, 2000; Rugh, 1981).

Consider a dynamical system with an input variable x_t and an output variable y_t , observed at discrete time points t = 1, ..., T. The discrete Volterra series shares a conceptual similarity with the Taylor polynomial expansion of y_t in terms of $(x_t, x_{t-1}, ..., x_{t-m+1})^T$, where m represents the memory of the system. This study focuses on a specific process where the output y_t can be modeled by incorporating the lagged values of the original process. To accomplish this, we employ a closed-loop version of the Volterra series, which captures the relationship between the output y_t and the delayed inputs, denoted as x_t (equivalent to y_{t-1}), for further details refer to Barahona and Poon (1996). In this framework, we analyze time series using a discrete Volterra series of order p and memory m as a model to approximate time series at time t = 1, ..., T. Let $Y_T = (y_1, ..., y_T)$ denote a time series of length T, where every y_t is a function of m-lagged vectors $(x_t, x_{t-1}, ..., x_{t-m+1})^T$ and as mentioned, $x_t \equiv y_{t-1}$. Using this definition, y_t can be approximated by applying a discrete Volterra series of order p as follows:

$$\hat{y}_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \dots + a_m y_{t-m} + a_{m+1} y_{t-1}^2 + a_{m+2} y_{t-1} y_{t-2} + \dots + a_{M-1} y_{t-m}^p, \quad (1)$$

where M = M(m, p) is equal to the total dimension of the Volterra model. Therefore, each model is characterized by its memory, m, and degree of nonlinearity, p. However, as shown in Equation (7) below, the value of M increases exponentially with increasing m and p, which is a primary challenge associated with the Volterra method. In this paper, the emphasis is to alleviate this problem for closed-loop Volterra system by reformulating the series as operators in Reproducing Kernel Hilbert Spaces (RKHS). This technique involves mapping the input variables to a new space, so that the original nonlinear system becomes a linear system. From a practical standpoint, the rich structure of the Volterra series expansion enables us to carry out inner product operations efficiently, regardless of the dimensionality of the associated RKHS. In fact, the dimension of these spaces can even be infinite.

The rest of the present paper is organized as follows. Section 2 considers the theory of Volterra method for nonlinear systems and the discrete Volterra series for representing nonlinear time series. Section 3 is devoted to estimating Volterra series using linear regression in RKHS. Section 4 presents empirical evidence from applications to simulated and real data, where the performance of the Volterra method is compared to different approximation methods, and the paper concludes in Section 5.

2 Volterra theory of stationary nonlinear systems

A closed system can be defined as a map between output y_t and an *m*-lagged input vector $(y_{t-1}, ..., y_{t-m})^T$, in the form

$$y_t = f(y_{t-1}, \dots, y_{t-m}),$$
 (2)

where f is a system operator that maps the delayed-input vector $\mathbf{x}_t = (y_{t-1}, ..., y_{t-m})^T$ to the corresponding output value y_t . The system is typically assumed to be continuous and time invariant, which means if the *m*-lagged input vector is time shifted then so is the output. Regarding to traditional systems theory, f is restricted to be a sufficiently well-behaved compact linear operator

 H_1 , such that y_t can be represented by a convolution of \mathbf{x}_t as

$$y_t = H_1 \mathbf{x}_t = \int h^{(1)}(m) y_{t-m} \, dm,$$
 (3)

where $h^{(1)}(m)$ is a linear kernel or impulse response. An extension of this convolution expression is the Volterra series expansion

$$y_t = V \mathbf{x}_t = H_0 \mathbf{x}_t + H_1 \mathbf{x}_t + H_2 \mathbf{x}_t + \dots + H_n \mathbf{x}_t + \dots,$$
(4)

where the zero-order kernel is simply the time average, i.e., $H_0 \mathbf{x}_t = h^{(0)} = \bar{y}_t$ of the output function. The *n*th-order Volterra operator can be defined in the form

$$H_n \mathbf{x}_t = \int h^{(n)}(m_1, ..., m_n) y_{t-m_1} ... y_{t-m_n} dm_1 ... dm_n,$$
(5)

where the integral kernels $h^{(n)}(m_1, ..., m_n)$ are the Volterra kernels and $1 \le m_i \le m$. Depending on the system, the integral can be defined over a finite or infinite time domain. The memory of the system defines the support of the Volterra kernels, i.e., it specifies the time interval in which past inputs can affect the current system output. As mentioned before, the Volterra series can be thought of as a Taylor series with memory, except that the Taylor series only represents systems that map input to output instantaneously, while the Volterra series represents systems in which the output also depends on past inputs.

2.1 Discrete Volterra system

In applied signal processing, a discrete form of the Volterra system can be used for a finite sample of data. Consider the nonlinear, discrete-time, and time-invariant input-output (I/O) relationship as Equation (2). As such a nonlinear mapping can have infinite memory, finite memory truncation is used in practice to yield $\hat{y}_t = f(\mathbf{x}_t)$, where the input data is given as a delayed vector $\mathbf{x}_t = (y_{t-1}, y_{t-2}..., y_{t-m})^T \in \mathbb{R}^m$ of finite dimension (memory) m. The vectorial data can be generated by a sliding window over a discretized time series. Then the *n*th-order closed-loop version of the discretized Volterra operator in which the output y_t feeds back as delayed input, is defined as the function

$$H_n(\mathbf{x}) = \sum_{i_1=1}^m \dots \sum_{i_n=1}^m h_{i_1,\dots,i_n}^{(n)} y_{i_1}\dots y_{i_n},\tag{6}$$

with a finite number of m^n coefficients $h_{i_1,...,i_n}^{(n)}$ of Volterra kernel, see Alper (1965). This equation is composed of a linear combination of all ordered *n*th-order monomials of the delayed vectors $\mathbf{x}_t = (y_{t-1}, y_{t-2}..., y_{t-m})^T$ up to degree p, i.e., $0 \le n \le p$. Such a model has been shown to provide a good representation for a wide class of nonlinear systems, see for example (Mathews and Sicuranza, 2000; Palm and Poggio, 1977). It is usually assumed that the Volterra kernels are symmetric with respect to permutations of indices; which means the products in Equation (6) remain constant when two different indices are permuted. To obtain a unique representation of Equation (6), it is necessary to keep only one of these permutations. After discarding the redundant coefficients, the dimension of $h^{(n)}$ and the $y_i(t)$'s is reduced to $\binom{m+n-1}{n}$. By utilizing these redundancies across all orders, it is possible to reduce the overall dimensionality to a level given by:

$$M = \sum_{n=0}^{p} \binom{m+n-1}{n} = \binom{m+p}{p} = \binom{m+p}{m},\tag{7}$$

which still grows rapidly as p and m increase. For example with m = 10 and p = 10, taking into account the symmetry in the coefficients, it is still required to estimate $\frac{(10+10)!}{10!10!} = 184,756$ parameters. So, in order to minimize the number of parameters that need to be estimated, it is useful to redefine the Volterra series as operators within an RKHS.

3 Estimating closed-loop Volterra series using linear regression in RKHS

To simplify the estimation of the closed-loop Volterra series, we can construct a Hilbert space of functions that corresponds to the series. This Hilbert space will be shown to be a RKHS, providing a convenient framework for estimation that is computationally feasible. The goal here is to estimate $h_{i_1,...,i_n}^{(n)}$ for n = 0, 1, ..., p, $i_j = 1, 2, ..., m$ and the given delayed-input and output samples $\{\mathbf{x}_t, y_t\}_{t=1}^N$, where \mathbf{x}_t is as defined in Section 2. As the number of terms in higher-order kernels grows exponentially, we transform the closed-loop Volterra series into a suitable form for regression in the RKHS framework.

3.1 Regression in RKHS

Let us embed a one-dimensional time series $Y_T = \{y_1, y_2, ..., y_T\}$ with a length of T into the multidimensional series $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}$ with vectors $\mathbf{x}_t = (y_{t-1}, ..., y_{t-m}) \in \mathbb{R}^m$, where N = T - m and $t = \{m + 1, ..., T\} \in \mathbb{R}^N$. For a given set of observations $(\mathbf{x}_1, y_{m+1}), ..., (\mathbf{x}_N, y_T)$:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_m & y_{m+1} \\ y_2 & y_3 & \cdots & y_{m+1} & y_{m+2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ y_N & y_{N+1} & \cdots & y_{T-1} & y_T \end{pmatrix},$$
(8)

the estimation of y_t , last column of (8), as a function of row delayed-vectors \mathbf{x}_t , using linear regression is of the form

$$y_t = f(\mathbf{x}_t) = \sum_{j=1}^M \gamma_j \varphi_j(\mathbf{x}_t), \tag{9}$$

where $\gamma_j \in \mathbb{R}, \varphi_j : \mathbb{R}^m \to \mathbb{R}$ and M is as defined in Equation (7). In the case of *p*th-order Volterra series, the φ_j 's consist of all monomials of \mathbf{x} up to order p. Utilizing a quadratic loss function l, the φ_j can be found by minimizing the mean squared error over the dataset as follows,

$$l((\mathbf{x}_1, y_1, f(\mathbf{x}_1)), ..., (\mathbf{x}_N, y_N, f(\mathbf{x}_N)) = \frac{1}{N} \sum_{j=1}^N (f(\mathbf{x}_j) - y_j)^2,$$
(10)

where for ease of notation, we use indexes $\{1, ..., N\}$ instead of $\{m + 1, ..., T\}$. Now consider the case in which instead of applying the monomials as basis functions, each φ_j is specified in terms of a kernel function k in the form $\varphi_j(\mathbf{x}) = k(\mathbf{x}, \mathbf{x}_j)$ for j = 1, ..., M. Especially, it is considered that kernels are positive definite, i.e., for all choices of the $\mathbf{x}_1, ..., \mathbf{x}_N$ from the input domain, the Gram matrix $K = [K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1}^N$ is positive definite. Such kernels can be represented as a dot product in an associated linear space \mathbb{F} which means there is a map (into feature space) Φ such that $k(\mathbf{x}, \mathbf{x}') = \Phi(\mathbf{x})^T \Phi(\mathbf{x}')$. This feature space consists of all possible polynomials in the \mathbf{x} up to, and including, order p. For example consider the case of p = 2 and m = 2 for $\mathbf{x} = \{x_1, x_2\}$, then the feature expansion is $\Phi(\mathbf{x}) = (1, x_1, x_2, x_1 x_2, x_1^2, x_2^2)^T$. For a fixed \mathbf{x} and the kernel $k(\mathbf{x}, .)$, \mathbb{F} can be expressed with a space of functions with the property of an RKHS as

$$f(\mathbf{x}) = \sum_{j=1}^{M} \gamma_j k(\mathbf{x}, \mathbf{x}_j), \tag{11}$$

which allows for the application of the so-called *Representer theorem*, see Theodoridis (2015, ch.11). This theorem allows the empirical optimization of the loss function to be performed based on a finite set of samples in a very efficient way, even if the estimated function belongs to very high (even infinite) dimensional RKHS \mathbb{F} . The theorem states: Let Ω be an arbitrary strictly monotonically increasing function on \mathbb{R}_+ , c be an arbitrary loss function and $\|.\|_{\mathbb{F}}$ represents the norm of RKHS. Then each minimizer $f \in \mathbb{F}$ of the regularized minimization

$$argmin\sum_{i=1}^{N} c(y_i, f(\mathbf{x}_i)) + \Omega(\|f\|_{\mathbb{F}}),$$
(12)

over γ_i and \mathbf{x}_i , admits a representation of the form

$$f(\mathbf{x}) = \sum_{i=1}^{N} \gamma_i k(\mathbf{x}, \mathbf{x}_i), \qquad \gamma_i \in \mathbb{R},$$
(13)

as a solution. By utilizing the quadratic loss function defined in Equation (10) and considering a regularizer Ω of zero, the solution for the vector $\gamma = (\gamma_1, ..., \gamma_N)$ can be computed by setting the derivative of Equation (10) with respect to γ equal to zero. Then the result takes the form $\gamma = K_p^{-1} \mathbf{y}$, where $\mathbf{y} = (y_1, ..., y_N)^T$, hence

$$f(\mathbf{x}) = \gamma^T \mathbf{k}(\mathbf{x}) = \mathbf{y}^T K_p^{-1} \mathbf{k}(\mathbf{x}), \tag{14}$$

where $\mathbf{k}(\mathbf{x}) = (k(\mathbf{x}, \mathbf{x}_1), ..., k(\mathbf{x}, \mathbf{x}_N))^T \in \mathbb{R}^N$ and denotes the **x**-th column of K_p .

3.2 Volterra series as a linear operator in RKHS

As previously mentioned, estimating coefficients in Equation (6) becomes challenging due to the exponentially increasing number of coefficients for higher-order Volterra kernels. This problem can be addressed by converting the Volterra series into a suitable form for regression in RKHS.

Utilizing the discretized version of Volterra operators from Equation (6), the *n*th-order Volterra operator is a sum of all *n*th-order monomials of the input vector \mathbf{x} . We define the map Φ_n as

$$\Phi_0(\mathbf{x}) = 1 \qquad and \qquad \Phi_n(\mathbf{x}) = (x_1^n, x_1^{n-1} x_2, ..., x_1 x_2^{n-1}, x_2^n, ..., x_m^n), \tag{15}$$

which contains all m^n ordered monomials of degree n evaluated at \mathbf{x} , such that Φ_n maps the input $\mathbf{x} \in \mathbb{R}^m$ into a vector $\Phi_n(\mathbf{x}) \in \mathbb{F}^n = \mathbb{R}^{m^n}$ (remember here that \mathbf{x} is a delayed-input vector). In Equation (6), the *n*th-order Volterra operator can be expressed using Φ_n as a scalar product in \mathbb{F}_n :

$$H_n(\mathbf{x}) = \eta_n^T \Phi_n(\mathbf{x}),\tag{16}$$

where $\eta_n = (h_{1,1,...,1}^{(n)}, h_{1,2,...,1}^{(n)}, h_{1,3,...,1}^{(n)}, ...)^T \in \mathbb{F}^n$. Schölkopf and Smola (2018) showed that

$$\Phi_n(\mathbf{x}_1)^T \Phi_n(\mathbf{x}_2) = (\mathbf{x}_1^T \mathbf{x}_2)^T = k_n(\mathbf{x}_1, \mathbf{x}_2).$$
(17)

To express the estimation problem as a scalar product in \mathbb{F}^p , the idea used to express the *n*th-order Volterra operator in terms of a scalar product is extended to the entire *p*th-order Volterra series. This results in the following representation:

$$y(\mathbf{x}) = \sum_{n=0}^{p} H_n(\mathbf{x}) = (\eta^{(p)})^T \Phi^{(p)}(\mathbf{x}),$$
(18)

where $\Phi^{(p)}(\mathbf{x})$ can be obtained by stacking the maps Φ_n into a single map $\Phi^{(p)}(\mathbf{x}) = (\Phi_0(\mathbf{x}), \Phi_1(\mathbf{x}), \dots, \Phi_p(\mathbf{x}))^T$ and $\eta^{(p)} \in \mathbb{F}^{(p)}$. The associated scalar product can easily be computed as

$$\Phi^{(p)}(\mathbf{x}_1)^T \Phi^{(p)}(\mathbf{x}_2) = \sum_{n=0}^p (\mathbf{x}_1^T \mathbf{x}_2)^n = k^{(p)}(\mathbf{x}_1, \mathbf{x}_2).$$
(19)

There are different types of kernels where a specific case of this kernel is the inhomogeneous polynomial kernel used in Dodd and Harrison's Volterra estimation approach (Dodd and Harrison, 2002),

$$k^{(p)}(\mathbf{x}_1, \mathbf{x}_2) = (1 + \mathbf{x}_1^T \mathbf{x}_2)^p = \sum_{n=0}^p \binom{p}{n} (\mathbf{x}_1^T \mathbf{x}_2)^n,$$
(20)

which corresponds to a mapping into the space of all possible polynomials of order up to p. For infinite Volterra series, the kernel can be obtained as

$$k^{(\infty)}(\mathbf{x}_1, \mathbf{x}_2) = e^{\mathbf{x}_1^T \mathbf{x}_2} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{x}_1^T \mathbf{x}_2)^n.$$
 (21)

Therefore, it can be concluded that the Volterra series in both finite and infinite discrete states can be expressed as linear operators in an RKHS.

The space of functions $\Phi_n(\mathbf{x})$, n = 0, ..., p has an RKHS structure, which implies that the estimation of Equation (18) can be expressed in terms of kernels, as follows according to the representer theorem:

$$\hat{y}(\mathbf{x}) = \sum_{n=0}^{p} H_n(\mathbf{x}) = \mathbf{y}^T (K_p + \lambda I_N)^{-1} \mathbf{k}^{(p)}(\mathbf{x}),$$
(22)

where the Gram matrix K_p and the coefficient vector $\mathbf{k}^{(p)}(\mathbf{x})$ are computed using the kernel from Equation (19), $\mathbf{1} = (1, 1, ...)^T \in \mathbb{R}^N$, and λ controls the trade-off between data fit and penalty term. It is clear that using the RKHS representation, can avoid the need to compute the possibly large number of coefficients explicitly.

The individual nth-order Volterra operators can be recovered in principle from Equation (22) by collecting all terms containing monomials of the desired order and summing them up as follows

$$H_n(\mathbf{x}) = \mathbf{y}^T K_p^{-1} \mathbf{k}_n(\mathbf{x}), \tag{23}$$

where $\mathbf{k}_n(\mathbf{x}) = ((\mathbf{x}_1^T \mathbf{x})^n, (\mathbf{x}_2^T \mathbf{x})^n, ..., (\mathbf{x}_N^T \mathbf{x})^n)^T$. Furthermore, the coefficient vector $\eta_n = (h_{1,1,...,1}^{(n)}, h_{1,2,...,1}^{(n)}, h_{1,3,...,1}^{(n)}, ...)^T$ of the Volterra operator can be obtained as

$$\eta_n = \Phi_n^T \ K_p^{-1} \ \mathbf{y},\tag{24}$$

where $\Phi_n = (\Phi_n(\mathbf{x_1}), \Phi_n(\mathbf{x_2}), ..., \Phi_n(\mathbf{x_N}))^T$ is a matrix containing all monomials corresponding to the *n*th-order Volterra operator. Using the latter and stacking (18) for all input vectors $\mathbf{X} = {\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}}$ and output vector $\mathbf{y} = (y_1, y_2, ..., y_N)^T$, one arrives at the equation

$$\mathbf{y} = \mathbf{E}^T \mathbf{\Phi} + \mathbf{e},\tag{25}$$

where $\mathbf{\Phi} = (\Phi^{(p)}(\mathbf{x}_1) \dots \Phi^{(p)}(\mathbf{x}_N))^T$ is a matrix of dimension $N \times M$ where M is defined in Equation (7) and $\mathbf{E} = (\eta_0 \ \eta_1 \dots \eta_p)^T$. An estimate for \mathbf{E} can be obtained by minimizing the mean squared estimation error. In other words, we select the coefficients $\hat{\mathbf{E}}$ that minimize the following expression among all possible choices of \mathbf{E} :

$$\mathbb{E}\{\mathbf{e}\mathbf{e}^T\} = \mathbb{E}\{(\mathbf{y} - \mathbf{E}^T\mathbf{\Phi})(\mathbf{y} - \mathbf{E}^T\mathbf{\Phi})^T\},\tag{26}$$

where \mathbb{E} denotes the statistical expectation. To obtain the optimum coefficients \mathbf{E} we use the orthogonality principle as follows.

Theorem 3.1 (Orthogonality). In the pth-degree closed-loop Volterra system with square integrable components of the monomial vectors $\mathbf{\Phi}$, the estimation error $(\mathbf{y} - \hat{\mathbf{E}}^T \mathbf{\Phi})$ is orthogonal to all monomial vectors $\mathbf{\Phi}$. This implies that the expected inner product between the monomial vector $\mathbf{\Phi}$ and the estimation error is zero:

$$\mathbb{E}\left[\left\langle \mathbf{\Phi}, \mathbf{y} - \hat{\mathbf{E}}^T \mathbf{\Phi} \right\rangle\right] = \mathbf{0}.$$
(27)

The square integrability condition ensures that the inner product and the integral involved in the expectation are well-defined mathematical quantities.

Proof. The squared error defined in Equation (26), has the unique minimum which can be found by differentiating with respect to **E** and setting the results equal to zero at point $\mathbf{E} = \hat{\mathbf{E}}$ as follows:

$$\frac{\partial}{\partial \mathbf{E}} \left[\left(\mathbf{y} - \mathbf{E}^T \mathbf{\Phi} \right)^2 \right] = \frac{\partial}{\partial \mathbf{E}} \left[(\mathbf{y} - \mathbf{E}^T \mathbf{\Phi})^T (\mathbf{y} - \mathbf{E}^T \mathbf{\Phi}) \right]$$
$$= -2\mathbf{\Phi} (\mathbf{y} - \mathbf{E}^T \mathbf{\Phi})^T = -2\mathbf{\Phi}^T (\mathbf{y} - \mathbf{E}^T \mathbf{\Phi}).$$

Setting this derivative equal to zero, we obtain:

$$\hat{\mathbf{E}} = \mathbf{\Phi}^T (\mathbf{\Phi} \mathbf{\Phi}^T)^{-1} \mathbf{y}, \tag{28}$$

and vector \mathbf{y} in Equation (25) can be estimated as

$$\hat{\mathbf{y}} = \hat{\mathbf{E}}^T \boldsymbol{\Phi}.$$
 (29)

Substituting the optimal value of \mathbf{E} , $(\hat{\mathbf{E}})$, back into the expression for the expected inner product between $\boldsymbol{\Phi}$ and the estimation error, we get:

$$\begin{split} & \mathbb{E}\left[\langle \boldsymbol{\Phi}, \mathbf{y} - \hat{\mathbf{E}}^{T} \boldsymbol{\Phi} \rangle\right] = \mathbb{E}\left[\langle \boldsymbol{\Phi}, \mathbf{y} - \boldsymbol{\Phi}^{T} (\boldsymbol{\Phi} \boldsymbol{\Phi}^{T})^{-1} \mathbf{y} \boldsymbol{\Phi} \rangle\right] \\ & = \mathbb{E}\left[\langle \boldsymbol{\Phi}, (\mathbf{I} - \boldsymbol{\Phi}^{T} (\boldsymbol{\Phi} \boldsymbol{\Phi}^{T})^{-1} \boldsymbol{\Phi}) \mathbf{y} \rangle\right] \\ & = \mathbf{0}, \end{split}$$

where the last equality follows from the fact that $(\mathbf{I} - \mathbf{\Phi}(\mathbf{\Phi}^T \mathbf{\Phi})^{-1} \mathbf{\Phi}^T)$ is a projection matrix that projects onto the orthogonal complement of the column space of $\mathbf{\Phi}$. Therefore, the estimation error is orthogonal to all monomial vectors $\mathbf{\Phi}$ with square integrable components, completing the proof.

Theorem 3.2. The closed-loop volterra estimator $\hat{\mathbf{E}}$ defined in Equation (28), as a function of memory length (m), is unbiased if \mathbf{e} is zero mean and \mathbf{e} and $\mathbf{\Phi}$ are stochastically independent.

Proof. From Equations (28) and (25), we find that

$$\hat{\mathbf{E}}(m) = \mathbf{\Phi}^T (\mathbf{\Phi} \mathbf{\Phi}^T)^{-1} \mathbf{y} = \mathbf{\Phi}^T (\mathbf{\Phi} \mathbf{\Phi}^T)^{-1} (\mathbf{E}^T \mathbf{\Phi} + \mathbf{e}) = \mathbf{E} + \mathbf{\Phi}^T (\mathbf{\Phi} \mathbf{\Phi}^T)^{-1} \mathbf{e},$$
(30)

for all values of m. Taking the expectation from both sides of (30) and using the fact that **e** and Φ are stochastically independent, it follows that for all value of m

$$\mathbb{E}(\mathbf{\hat{E}}(m)) = \mathbf{E}.$$
(31)

Remark 1. It is clear that if $\hat{\mathbf{E}}(m)$ converges to \mathbf{E} in mean-squared, i.e., $\lim_{m\to\infty} \mathbb{E}[(\hat{\mathbf{E}}(m) - \mathbf{E})^2] = 0$, then it converges to \mathbf{E} in probability, and, therefore, it can be concluded that $\hat{\mathbf{E}}(m)$ is a consistent estimator of \mathbf{E} . To establish L_2 convergence of $\hat{\mathbf{E}}(m)$, the following conditions are required:

- 1. Both $\hat{\mathbf{E}}(m)$ and \mathbf{E} should be bounded in the L_2 norm.
- 2. The squared values of $\mathbf{E}(m)$ and \mathbf{E} should have finite expected values.

Satisfying these conditions ensures L_2 convergence of $\hat{\mathbf{E}}(m)$ to \mathbf{E} , supporting the conclusion that $\hat{\mathbf{E}}(m)$ is a consistent estimator of \mathbf{E} .

Corollary 1. Given the conditions in Theorems 3.1 and 3.2 and Equation (25), if $\mathbf{\hat{E}}(m)$ converges to \mathbf{E} in probability, it can be easily deduced that $\lim_{m\to\infty} \{\hat{\mathbf{y}}(m) - \mathbf{y}\} \to 0$ in probability, where $\hat{\mathbf{y}}(m)$ is defined as $\hat{\mathbf{y}}(m) = \hat{\mathbf{E}}^T(m) \Phi$.

The underlying assumption in the convergence analysis and consistency results presented in Remark 1 and Corollary 1 is that the sample size N implicitly grows along with the memory m. This assumption ensures that as the memory increases, a sufficient number of observations are available in the sample to support the estimation of a model with a large memory.

Furthermore, similar to (12), the optimal solution for Equation (28) can be expressed as

$$\hat{\mathbf{E}} = \mathbf{\Phi}^T (\mathbf{\Phi} \mathbf{\Phi}^T + \lambda I_N)^{-1} \mathbf{y}, \tag{32}$$

where λ is a regularizing penalty term.

In the case of vector first-order autoregressive model, when $\mathbf{y}_t = \mathbf{B}^T \mathbf{y}_{t-1} + \varepsilon_t$, Anderson and Taylor (1979) showed that the least square estimate of \mathbf{B} is strongly consistent due to the strong consistency of the moment matrix. As an implication from Lemma 2 in Anderson and Taylor (1979) and Remark 1, the obtained estimation of \mathbf{E} is also consistent in the stronger sense, which directly follows from the strong consistency of the matrix $\boldsymbol{\Phi}$ with respect to m. The following theorem shows that under a set of defined conditions, $\hat{\mathbf{E}}_N \to \mathbf{E}$, where N is the number of windows taken from time series of length T as defined in Section. 3.1.

Theorem 3.3 (Strong Consistency). Consider a given set of observations defined in (8) as a vector autoregressive model VAR(m) with N variables. Let \mathbf{y}_0 is a N-vector of constants serving the intercept of the model and \mathbf{e}_t is a N-vector of error terms. Let $\mathbf{y}_t = \mathbf{E}^T \mathbf{\Phi}_t + \mathbf{e}_t$, \mathcal{F}_t be the σ -algebra generated by $(\mathbf{e}_t, \mathbf{y}_t)$, t = 1, 2, ..., and \mathcal{F}_0 be the σ -algebra generated by \mathbf{y}_0 . Assume that $\mathbb{E}(\mathbf{e}_t \mid \mathcal{F}_{t-1}) = \mathbf{0}$, $\mathbb{E}(\mathbf{e}_t \mathbf{e}_t^T \mid \mathcal{F}_{t-1}) = \mathbf{\Sigma}$, $\mathbb{E}(\mathbf{y}_0) = \mathbf{0}$, and $\mathbb{E}(\mathbf{y}_0 \mathbf{y}_0^T) = \mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is the solution of $\mathbf{\Gamma} - \mathbf{E}^T \mathbf{\Gamma} \mathbf{E} = \mathbf{\Sigma}$ and $\mathbf{\Sigma}$ is positive definite. Further assume that $\mathbb{E}(\mathbf{y}_t^4) < \infty$ and $\mathbb{E}(\mathbf{e}_t^4) < \infty$, then:

- (i) $N^{-1} \lim_{N \to \infty} \mathbf{y}_t \mathbf{y}_t^T = \mathbf{\Gamma}$ with probability one.
- (ii) $\hat{\mathbf{E}}_N$ is a strong consistent estimator of \mathbf{E} with probability one.

Proof. Using the Cauchy-Schwarz inequality, the fourth-order condition on $\{\mathbf{e}_t\}$ implies that $\sum_{t=1}^{\infty} \mathbb{E} t^{-2} (e_{it}^2 e_{jt}^2) < \infty$, where e_{it} is the *i*th component of \mathbf{e}_t . Hence by the law of large numbers for martingales, see Feller (1991, ch.7), $\sum_{t=1}^{N} t^{-1} e_{it} e_{jt}$ converges with probability one and $N^{-1} \sum_{t=1}^{N} e_{it} e_{jt} \to \sigma_{ij}$ with probability one, where σ_{ij} is an element of Σ , and $N^{-1} \sum_{t=1}^{N} \mathbf{e}_t \mathbf{e}_t^T \to \Sigma$

with probability one, as well. Further,

$$N^{-1}\sum_{t=1}^{N} \mathbf{e}_{t} \mathbf{e}_{t}^{T} = N^{-1}\sum_{t=1}^{N} (\mathbf{y}_{t} - \mathbf{E}^{T} \mathbf{\Phi}_{t}) (\mathbf{y}_{t} - \mathbf{E}^{T} \mathbf{\Phi}_{t})^{T}$$
(33)
$$= N^{-1} \Big[\sum_{t=1}^{N} \mathbf{y}_{t} \mathbf{y}_{t}^{T} - \mathbf{E}^{T} \sum_{t=1}^{N} \mathbf{\Phi}_{t} \mathbf{y}_{t}^{T} - \sum_{t=1}^{N} \mathbf{y}_{t} \mathbf{\Phi}_{t} \mathbf{E} + \mathbf{E}^{T} \sum_{t=1}^{N} \mathbf{\Phi}_{t} \mathbf{\Phi}_{t}^{T} \mathbf{E} \Big]$$
$$= N^{-1} \sum_{t=1}^{N} \mathbf{y}_{t} \mathbf{y}_{t}^{T} - \mathbf{E}^{T} (N^{-1} \sum_{t=1}^{N} \mathbf{y}_{t} \mathbf{y}_{t}^{T}) \mathbf{E} + \mathbf{E}^{T} (N^{-1} \sum_{t=1}^{N} \mathbf{\Phi}_{t} \mathbf{e}_{t}^{T})$$
$$+ (N^{-1} \sum_{t=1}^{N} \mathbf{e}_{t} \mathbf{\Phi}_{t}^{T}) \mathbf{E} - \mathbf{E}^{T} N^{-1} (\mathbf{y}_{0} \mathbf{y}_{0}^{T} - \mathbf{y}_{N} \mathbf{y}_{N}^{T}) \mathbf{E}.$$

Under the condition $\sum_{t=1}^{N} y_{t-i}e_t$ is a martingale and using Kronecker's lemma in Shiryaev (1996, p.390), this expression converges to zero with probability one. Furthermore, the condition $\mathbb{E}(\mathbf{y}_t^4) < \infty$ implies $\mathbb{E}(y_{ti}^2y_{tj}^2)$ and hence $\sum_{t=1}^{N} y_{ti}y_{tj}$ converges with probability one and $y_{ti}y_{tj}/t$ converges to zero with probability one, where y_{it} is the *i*th component of \mathbf{y}_t . These results indicate that the last three terms in Equation (33) converges to zero with probability one. Thus

$$(\lim_{N \to \infty} N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{y}_t^T) - \mathbf{E}^T (\lim_{N \to \infty} N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{y}_t^T) \mathbf{E} = \mathbf{\Sigma},$$
(34)

with probability one. Hence $\lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{y}_t^T = \mathbf{\Gamma}$, which is the proof of part (*i*). To show that $\hat{\mathbf{E}}_N$ is a strongly consistent estimator of \mathbf{E} , it can be observed that since $\boldsymbol{\Sigma}$ is positive definite, $\boldsymbol{\Gamma}$ is also positive definite, which implies $\lim_{N\to\infty} N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{y}_t^T$ is positive definite. Consequently, $\hat{\mathbf{E}}_N - \mathbf{E} = (N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{y}_t^T)^{-1} N^{-1} \sum_{t=1}^{N} \mathbf{y}_t \mathbf{e}_t^T$ converges to zero with probability one. Hence, strong consistency of the estimator is proved.

It is noteworthy that the Theorem 3.3 emphasizes the behavior of the estimator in relation to the sample size N, and establishes conditions for strong consistency. However, it does not explicitly consider the influence of the memory parameter m on the estimator.

3.2.1 Model Selection

Equation (4) describes the full Volterra kernel model, which, while comprehensive, is not always the most interpretable representation of an input-output system. When a model has an excessive number of coefficients, it can result in fitting noise rather than the underlying signal in the dataset, which leads to overfitting. This can result in poor generalization to new data, inaccurate predictions, and difficulty in interpreting the system's performance. So, in practice a truncated version of the Volterra representation of the time series is considered by selecting an optimal subset of model parameters $(p, m \text{ and } \lambda)$. This procedure, known as model selection, helps to balance the risk of underfitting or overfitting, leading to reliable and accurate models. For example different choices for the weight λ , which controls the trade-off between performance smoothness and fitting error, can lead to overfitting or underfitting. Figure 1 illustrates how the kernel width σ affects the fit of ridge regression with a Gaussian kernel, $k(x, y) = exp(\frac{-||x-y||^2}{\sigma})$, for an arbitrary time series. Choosing σ too large results in a very smooth function that barely follows the shape of the underlying data, in other words, we are underfitting. By choosing σ too small, there is a strong preference for accommodating small fluctuations in the data due to noise, at the expense of smoothness, in this case, we are overfitting. Finally, a good choice of σ lead to a regression curve which fits the underlying trend without being overly affected by noise. In this regard, the following statistical model selection method is used for configuration and estimation. First, split



Figure 1: Effect of different choice of σ on the fit of Gaussian kernel ridge regression.

data into a training set of size n_{tr} and a test set of size $n_{te} = N - n_{tr}$. The training set is used to estimate the model and the test set is kept to validate the results of the training set. These two subsets are called in-sample and out-of-sample, respectively. Then, break the training set into kequally sized chunks, each of size $n_k = n_{tr}/k$. The first fold is treated as a validation set, and for each triple $q = (\lambda, m, p)$, the Volterra model is fitted on the remaining k - 1 folds and used to estimate performance measures. The first subset is returned to the training set, and the procedure repeats with the second subset held out, and so on. Finally the optimum $q^* = (\lambda^*, m^*, p^*)$ triple can be estimated by minimizing the lowest average validation error. This approach is called k-fold cross-validation.

Theorem 3.4 below shows that the k-fold cross-validation method can help to reduce the risk of overfitting or underfitting, leading to reliable and accurate models.

Theorem 3.4. In the model defined by Equation (25), we assume that $\mathbf{e} \sim N(\mathbf{0}, \sigma^2)$, and the matrix $\mathbf{\Phi}$ has a covariance matrix $\mathbf{\Sigma}$, with the minimum and maximum eigenvalues denoted by $\underline{\lambda} \in (0, \sigma^2]$ and $\overline{\lambda} < \infty$, respectively. We are given a finite set of candidate models $\mathcal{M} = \{1, 2, ..., M_c\}$, where for each model $q \in \mathcal{M}$, $f = \mathbf{E}^T \mathbf{\Phi}$ is estimated by $\hat{f}q = \hat{\mathbf{E}}_q^T \mathbf{\Phi}$ using the training dataset. Let us suppose that the k-fold cross-validation criterion is constructed as $CV_k(q) = 1/n_{te} \sum \ell(\hat{f}_q, \mathbf{y})$. We define the model that achieves the best average prediction performance across all folds as $q^* = \underset{q \in \mathcal{M}}{\operatorname{argmin}} CV_k(q)$. Then, as n_{tr} increases, for all $q \in \mathcal{M}$, we have $\mathbb{P}(\ell(\hat{f}_{q^*}, \mathbf{y}) \leq \ell(\hat{f}_q, \mathbf{y})) \to 1$. Here,

 ℓ is the mean squared error (MSE) loss function, which measures the average squared difference between the predicted and true outputs over all input-output pairs in the dataset.

Proof. For two candidate models q and q', the difference of squared prediction error of a test data points is

$$\xi_{q,q'} = (\mathbf{y} - \hat{\mathbf{E}}_q^T \Phi)^2 - (\mathbf{y} - \hat{\mathbf{E}}_{q'}^T \Phi)^2 =$$

$$2\mathbf{e}\Phi(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q'}^T) + (\Phi(\hat{\mathbf{E}}_q^T - \mathbf{E}^T))^2 - (\Phi(\hat{\mathbf{E}}_{q'}^T - \mathbf{E}^T))^2,$$
(35)

then by applying the Cauchy-Schwarz inequality and taking expectation from both side of (35) we have

$$\mathbb{E}(\xi_{q,q'}) \leq$$

$$2\mathbb{E}(\mathbf{e})\mathbb{E}(\mathbf{\Phi}(\hat{\mathbf{E}}_{q}^{T} - \hat{\mathbf{E}}_{q'}^{T})) + \mathbb{E}(\mathbf{\Phi}(\hat{\mathbf{E}}_{q}^{T} + \hat{\mathbf{E}}_{q'}^{T} - 2\mathbf{E}^{T}))\mathbb{E}(\mathbf{\Phi}(\hat{\mathbf{E}}_{q}^{T} - \hat{\mathbf{E}}_{q'}^{T})) \leq$$

$$\bar{\lambda}\mathbb{E}(\hat{\mathbf{E}}_{q}^{T} + \hat{\mathbf{E}}_{q'}^{T} - 2\mathbf{E}^{T})\mathbb{E}(\hat{\mathbf{E}}_{q}^{T} - \hat{\mathbf{E}}_{q'}^{T}) \leq c\bar{\lambda}\mathbb{E}(\hat{\mathbf{E}}_{q}^{T} - \hat{\mathbf{E}}_{q'}^{T}) = c_{0},$$
(36)

for some fixed constants c and c_0 . On the other hand

$$var(\xi_{q,q'}) \ge var(2\mathbf{e}\boldsymbol{\Phi}(\hat{\mathbf{E}}_{q}^{T} - \hat{\mathbf{E}}_{q'}^{T})) = 4\sigma^{2}\mathbb{E}(\Sigma^{1/2}(\mathbf{E}_{q}^{T} - \mathbf{E}_{q'}^{T}))^{2}$$

$$\ge 4\underline{\lambda}^{2}\mathbb{E}(\mathbf{E}_{q}^{T} - \mathbf{E}_{q'}^{T})^{2}.$$
(37)

Let $q' = q^*$, combining Equations (36) and (37) we have

$$\mathbb{P}\left(\frac{\xi_{q,q^*} - \mathbb{E}(\xi_{q,q^*})}{\sqrt{var(\xi_{q,q^*})}} \le c_1\right) \to 1,\tag{38}$$

for some constant c_1 and, uniformly over all $q \in \mathcal{M}$.

By consistency of the *closed-loop* Volterra estimator $\hat{\mathbf{E}}$ from Theorem 3.3, it can be concluded that $\sup_{q \in \mathcal{M}} \left\| \hat{\mathbf{E}}_q - \mathbf{E}_q \right\|_2 \leq \delta$. For large enough n_{tr} , $q \in \mathcal{M}$ and using the *Cauchy–Schwarz* inequality, we have

$$\hat{\mu}_{q,q^*} = \mathbb{E}(\xi_{q,q^*}) = \mathbb{E}((\mathbf{\Phi}(\hat{\mathbf{E}}_q^T - \mathbf{E}^T))^2 - (\mathbf{\Phi}(\hat{\mathbf{E}}_{q^*}^T - \mathbf{E}^T))^2) \ge$$

$$(\hat{\mathbf{E}}_q^T - \mathbf{E}^T)^T \Sigma(\hat{\mathbf{E}}_q^T - \mathbf{E}^T) - (\hat{\mathbf{E}}_{q^*}^T - \mathbf{E}^T)^T \Sigma(\hat{\mathbf{E}}_{q^*}^T - \mathbf{E}^T) \ge$$

$$\underline{\lambda}min[(\hat{\mathbf{E}}_q^T - \mathbf{E}^T)^T (\hat{\mathbf{E}}_q^T - \mathbf{E}^T)] - \underline{\lambda}\delta = \delta^*.$$
(39)

Now we need to provide an upper bound for σ_{q,q^*} :

$$var(\xi_{q,q^*}) =$$

$$var[2\mathbf{e}\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T)] + var[\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T + \hat{\mathbf{E}}_{q^*}^T - 2\mathbf{E}^T)\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T)] =$$

$$4\sigma^2(\hat{\mathbf{E}}_q^T - \mathbf{E}_{q^*}^T)^T \Sigma(\hat{\mathbf{E}}_q^T - \mathbf{E}_{q^*}^T) + \mathbb{E}[(\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T + \hat{\mathbf{E}}_{q^*}^T - 2\mathbf{E}^T))(\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T)]^2 -$$

$$\mathbb{E}^2[(\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T + \hat{\mathbf{E}}_{q^*}^T - 2\mathbf{E}^T))(\boldsymbol{\Phi}(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T)] = 4\sigma^2(\hat{\mathbf{E}}_q^T - \mathbf{E}_{q^*}^T)^T \Sigma(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T) +$$

$$\mathbb{E}\left[(\hat{\mathbf{E}}_q^T + \hat{\mathbf{E}}_{q^*}^T - 2\mathbf{E}^T)^T(\boldsymbol{\Phi}\boldsymbol{\Phi}^T - \Sigma)(\hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T)\right]^2 \leq C\bar{\lambda}^2 = c_2,$$

$$(40)$$

for some constant c_2 , and C. The last inequality is obtained as follow: Let $r = \hat{\mathbf{E}}_q^T + \hat{\mathbf{E}}_{q^*}^T - 2\mathbf{E}^T$ and $v = \hat{\mathbf{E}}_q^T - \hat{\mathbf{E}}_{q^*}^T$, then

$$\begin{split} & \mathbb{E}\left[r^T(\mathbf{\Phi}\mathbf{\Phi}^T - \Sigma)v\right]^2 = \mathbb{E}\left[(r^T\mathbf{\Phi})(v^T\mathbf{\Phi}) - r^T\Sigma v\right]^2 \leq 2\mathbb{E}\left[(r^T\mathbf{\Phi})^2(v^T\mathbf{\Phi})^2\right] + 2\mathbb{E}\left[(r^T\Sigma v)^2\right] \\ & \leq \mathbb{E}\left[(r^T\mathbf{\Phi})^4 + (v^T\mathbf{\Phi})^4\right]\bar{\lambda}^2 + 2\left[(r^Tv)\right]^2\bar{\lambda}^2 \leq \left[(r^T)^4 + (v^T)^4\right]\bar{\lambda}^2 + 2\left[(r^Tv)\right]^2\bar{\lambda}^2 = \\ & \left[((r^T)^4 + (v^T)^4) + 2(r^Tv)\right]\bar{\lambda}^2 = C\bar{\lambda}^2. \end{split}$$

Combining Equations (39) and (40), we have

$$\mathbb{P}\left(\sqrt{n_{tr}}\frac{\mathbb{E}(\xi_{q,q^*})}{\sqrt{var(\xi_{q,q^*})}} \ge \frac{\delta^*}{\sqrt{c_2}} = c_3\right) \to 1,\tag{41}$$

which means the probability of choosing model q^* versus model q approaches one as n_{tr} increases and, the proof is done.

Theorem 3.4 introduces model selection through cross-validation, allowing us to strike a balance between capturing relevant nonlinear effects and ensuring consistent and accurate predictions by carefully choosing the appropriate values for the parameters $q = (\lambda, m, p)$.

4 Application

In this section, in terms of Root Mean Square Error (RMSE) as defined in Equation (42), we illustrate the performance of the proposed closed-loop Volterra method for approximating time series in comparison with the following methods: Ridge regression, Autoregressive Fractionally Integrated Moving Average (ARFIMA) model (see Hyndman and Khandakar, 2008), which is suitable for long memory processes that display a long-term dependencies, and Exponential Smoothing (ETS), (see Hyndman and Athanasopoulos, 2021) that can capture a variety of trend and seasonal structures (additive or multiplicative) and combinations of those. The (RMSE) criterion is used for computing the difference between the approximated values and the observations as follows:

RMSE =
$$\sqrt{\frac{1}{N} \sum_{t=1}^{N} (y_t - \hat{y}_t)^2},$$
 (42)

where \hat{y}_t is the estimated value at time t.

Moreover, for comparing the predictive accuracy of two sets of approximations, a Kolmogorov-Smirnov Predictive Accuracy test is considered. In terms of approximated errors, the two-sample, two-sided KSPA test hypothesis can be approximately represented as follows. Let ε_{d_1} and ε_{d_2} are the absolute or squared approximated errors from two approximating models d_1 and d_2 with unknown continuous empirical cumulative distribution functions (ECDFs), then the two-sided KSPA test will test the hypothesis:

$$\begin{cases} H_0: F_{\varepsilon_{d_1}}(z) = F_{\varepsilon_{d_2}}(z) \\ H_1: F_{\varepsilon_{d_1}}(z) \neq F_{\varepsilon_{d_2}}(z) \end{cases}, \tag{43}$$

which determine if there is a significant statistical difference between the distribution of predictive errors or not. The next purpose of KSPA test is to determine whether the model which reports the lowest error based on some loss function also reports a stochastically lower error against the corresponding model. This is a one-sided KSPA test will test the hypothesis:

$$\begin{cases} H_0: F_{\varepsilon_{d_1}}(z) \le F_{\varepsilon_{d_2}}(z) \\ H_1: F_{\varepsilon_{d_1}}(z) > F_{\varepsilon_{d_2}}(z) \end{cases}$$

$$\tag{44}$$

Rejecting the null hypothesis in this case indicates that the ECDF of approximated errors from model d_1 is shifted towards the left and is above the ECDF of approximated errors from model d_2 . In particular the acceptance of the alternate hypothesis confirms that model d_1 reports a lower stochastic error than model d_2 , for more details see Hassani and Silva (2015).

In following the performance of the proposed closed-loop Volterra method is evaluated in terms of the RMSE criterion and KSPA test, by applying that to various real and simulated time series data.

4.1 Simulated series

Example 1. We consider three autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA) processes as follows:

- $P_1 \colon y_t = 0.5y_{t-1} + \varepsilon_t$
- $P_2: y_t = \varepsilon_t 0.9\varepsilon_{t-1}$
- $P_3: y_t = y_{t-1} 0.9y_{t-1} + \varepsilon_t 0.8\varepsilon_{t-1}.$

For each model P_1, P_2 , and P_3 , we consider time series of length T = 100 and 100 simulation runs. For all processes, we used ε_t as white noise, means $\varepsilon_t \sim N(0,1)$. Optimal values for the order p, the memory m and regularization parameter λ are computed based on cross-validation. The maximum memory in the Volterra representation is set equal to 10, and the maximum order is set equal to $p_{max} = 5$. We report our simulation results in terms of Root Mean Square Error (RMSE) in Table 1. Table 1 reports the average RMSE values (averaged over the 100 Monte Carlo

Model	р	m	Volterra	Ridge regression	ARFIMA	ETS
P_1	5	10	0.18	0.06	1.37	1.3
	3	8	0.09	0.06	0.88	0.94
P_2	5	10	1.08e-4	6.39e-6	0.99	1.3
	3	8	4e-3	4e-4	1.02	1.28
P_3	5	10	1.87e-7	7.12e-9	0.99	5.34
	3	8	1.45e-5	4.22e-6	0.96	3.05

Table 1: RMSE results for models P_1, P_2, P_3 .

repetitions of the simulation) attained under each of the four aforementioned simulation methods.

It becomes apparent from Table 1 that the closed-loop Volterra method has a higher accuracy in approximating the models P_1, P_2 and P_3 than ARFIMA and ETS methods, and very close results to the Ridge regression method, for all considered choices of the parameters m and p.

Furthermore, in terms of the defined KSPA tests (Equations (43) and (44)) and for d_1 as closedloop Volterra method, the obtained results are given in Table 2. In order to account for multiplicity in this study, within each of the three scenarios, we control the FWER, i.e., the probability of making at least one false discovery among the 6 considered hypotheses. The FWER is controlled at the level of $\alpha = 0.05$ using the Bonferroni correction, as described by Bonferroni (1935). Based

Model	Test hypothesis	Volterra		
	rest ny potnosis	Ridge regression	ARFIMA	ETS
P_1	Two-sided	_	*	*
	One-sided	_	*	*
P_2	Two-sided	*	*	*
	One-sided	—	*	*
P_3	Two-sided	*	*	*
	One-sided	_	*	*

Table 2: Comparison of adjusted p-values (\tilde{p}) for Example 1 (m = 8, p = 3): Significance: * $(\tilde{p} < 0.05)$, - (non-significant). P-values were adjusted within each scenario via the Bonferroni correction.

on the results in Table 2 at a 95% confidence level, it can be concluded that the two-sided KSPA tests show statistically significant differences between the distribution of estimated time series errors from the Volterra method and other methods in in all models P_1 , P_2 , and P_3 , except for the Ridge regression method in model P_1 . Next, we applied the one-sided KSPA test to find out whether estimated time series errors from Volterra method (which have the lower RMSE in most cases) report a lower stochastic error than other estimated time series errors. The one-sided KSPA tests indicate that the Volterra method provides lower stochastic error than other methods, and provides supplementary evidence to the conclusion from the two-sided KSPA test for the existence of a statistically significant difference between the two estimated time series. While for model P_1 , both Volterra and Ridge regression methods have obtained very close results and for the model P_3 , it seems that the ridge regression model provides less stochastic error than the Volterra method for estimating time series.

4.2 Real data

Example 2. As the first real data we use the Death series of length 72, which shows the monthly accidental deaths in the USA between 1973 and 1978. This dataset can be found in many time series books (see for example Brockwell and Davis, 2016) and in every R software installation.

Example 3. As a second real data example, we consider a time series of length 100, which measures the annual flow of the Nile River at Aswan between 1871 and 1970. This data can be found in many time series books (see for example Durbin and Koopman, 2012) and in every R software installation.

Table 3 displays the RMSE results for comparing all methods for examples 2 and 3, for m = 10 and p = 5. From Table 3, it can be seen that based on the RMSE values, the Volterra method

Data	Volterra	Ridge regression	ARFIMA	ETS
Example 2	8.46e-06	6.82e-05	1.65	1.28
Example 3	2.62e-07	9.42e-07	1.65	7.96

Table 3: Root Mean Square Error (RMSE) Results for Death and Nile series.

and the ridge regression method have close results, while the error produced by the closed-loop Volterra method is much less compared to the ARFIMA and ETS methods.

Furthermore, the obtained results from two-sided and one-sided KSPA tests for Examples 2 and 3, show that *p*-values for all comparisons are less than 5%. It means at 5% significance level, based on the two-sided KSPA test, there are statistically significant differences between the closed-loop Volterra method against Ridge regression, ETS, and ARFIMA methods, and based on the results of the one-sided KSPA test, it can be concluded that the closed-loop Volterra method outperforms the other methods in terms of accuracy in estimation, as it consistently produces estimates with smaller errors. Following a subset of results obtained from comparing the closed-loop Volterra method via other methods are represented in Figures 2 and 3.



Figure 2: Histogram of errors and ECDFs of errors for USA Death series data in example 2.



Figure 3: Histogram of errors and ECDFs of errors for Nile series data in example 3.

Figures 2 and 3 display the histograms of errors and the ECDFs for examples 2 and 3 obtained via Volterra and Ridge regression methods. The distribution of the absolute errors from Volterra and Ridge regression can be seen in Figures 2 and 3 (left and middle). However, without a formal statistical test it is not possible to determine whether there exists a statistically significant difference between the distribution of these errors. Now to identify if one method does indeed provide a lower stochastic error than the other method, we look at the ECDF's plot which is shown in Figures 2 and 3 (right). In this case it is clear that based on the ECDF, the closed-loop Volterra method provides a lower stochastic error than the Ridge regression method.

5 Conclusion

In this paper, we have presented the application of closed-loop Volterra from the field of kernel methods for approximating the time series. Especially, we have utilized the discrete closed-loop Volterra theory by applying polynomial kernels in a regularized regression framework. Through a simulation study, comparisons between closed-loop Volterra and Ridge regression, ARFIMA, and ETS methods for estimating time series were carried out using both simulated and real data using the RMSE criterion. The results indicated that the proposed closed-loop Volterra method outperforms other defined methods generally for different levels of m and p.

Furthermore, the results of the KSPA test were considered to evaluate the efficiency of the closedloop Volterra method. The results clearly indicate that the proposed closed-loop Volterra method outperforms competing methods such as Ridge regression, ARFIMA, and ETS in terms of lower stochastic errors.

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