ON THE CLOSURE OF THE BICYCLIC SEMIGROUP(1)

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Introduction. Suppose that S is a topological semigroup which contains the bicyclic semigroup B as a subsemigroup. Let T denote the closure of B in S. We investigate the structure of the semigroup T and the extent to which B determines this structure.

In §I, two properties of T are established which hold for arbitrary S; namely, that B is a discrete open subspace of T and $T\backslash B$ is an ideal of T if it is nonvoid. In §II, we introduce the notion of a topological inverse semigroup and establish several properties of such objects. Some questions are posed. In §III, it is shown that if S is a topological inverse semigroup, then $T\backslash B$ is a group with a dense cyclic subgroup. §IV contains a description of three examples of a topological semigroup which contains B as a dense proper subsemigroup. Finally, in §V, we assume that S is a locally compact topological inverse semigroup and show that either B is closed in S or T is isomorphic with the last of the examples described in §IV. A corollary about homomorphisms from B into a locally compact topological inverse semigroup is obtained which generalizes a result due to A. Weil [1, p. 96] concerning homomorphisms from the integers into a locally compact group.

All spaces are topological Hausdorff in this paper.

We state the definitions of Green's equivalence relations in a semigroup and the definition of an inverse semigroup. Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{D}$ on a semigroup S are defined by:

$a\mathscr{R}b$ if and only if $a \cup aS = b \cup bS$,

$a\mathscr{L}b$ if and only if $a \cup Sa = b \cup Sb$,

$\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$.

The notations R_a, L_a, H_a, and D_a stand for the appropriate equivalence class of a in S.

A semigroup S is an *inverse semigroup* provided each element x of S has a unique inverse; that is, an element x^{-1} of S such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. For details about inverse semigroups and Green's relations, see [2]. We assume a certain familiarity with these notions.

I. **The bicyclic semigroup.** B is the semigroup with identity 1 generated by two elements p and q subject only to the condition that $pq = 1$. The distinct elements

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of B are exhibited in the following useful array:

$$\begin{array}{ccccc} 1 & p & p^2 & p^3 & \cdots \\ q & qp & qp^2 & qp^3 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

In this array, the rows are the $\mathscr{R}$-classes of B, the columns are the $\mathscr{L}$-classes, the $\mathscr{H}$-classes are points, and the idempotents lie on the diagonal starting at 1. There is only one $\mathscr{D}$-class; that is, B is a bisimple semigroup. The following lemma follows by induction from the definition of B.

I.1. LEMMA. (i) *For each* $x \in B$, $BxB = B$.

(ii) *For each* $x, y \in B$, $\{z \mid xz = y\}$ *and* $\{z \mid zx = y\}$ *are both finite sets*; *that is, left translation by x and right translation by x are finite-to-one functions.*

I.2. COROLLARY. *The only topology on B which makes B into a topological semigroup is the discrete topology. Thus B is a discrete subspace of any topological semigroup containing it.*

Proof. In fact, suppose B is topologized so that right and left translations are continuous and suppose that $x \in B$ is a limit point of B. Let $y \in B$ and consider yB. If $\{yx\}$ is open in B, then there is an open set U about x so that $yU = \{yx\}$. However U must contain infinitely many points of B, even if B is only T_1. Since this contradicts I.1.ii. we conclude that yx is a limit point of B. Similarly xy is a limit point of B. We conclude from I.1.i. that each point of B is a limit point of B. But $\{1\} = R_1 \cap L_1 = (B \backslash qpB) \cap (B \backslash Bqp)$ is open since qpB and Bqp are retracts of B and hence closed. This contradicts our original assumption, so B must be discrete.

I.3. THEOREM. *Let B be contained densely in a topological semigroup T. Then B is open in T and $T \backslash B$ is an ideal of T provided it is nonvoid.*

Proof. Let $x \in B$. Let U be an open set in T such that $U \cap B = \{x\}$. Since B is dense in T, $U \cap B$ is dense in U^*, the closure of U in T. Thus $U^* = (U \cap B)^* = \{x\}$ and so $U = \{x\}$. Hence B is open in T. Now suppose $x \in T \backslash B$ and $y \in T$. If $xy \in B$, then there exist open sets U and V of T so that $x \in U$, $y \in V$ and $UV = \{xy\}$. Now $U \cap B$ is infinite and there is some point $w \in V \cap B$. Hence $(U \cap B)(V \cap B) \supset (U \cap B) \cdot w = \{xy\}$. This contradicts I.1.ii., so $xy \in T \backslash B$. Similarly, $yx \in T \backslash B$.

II. **Topological inverse semigroups.** Let S be a topological semigroup which is algebraically an inverse semigroup. The inversion function on S, the function taking x to x^{-1}, is always 1-1 and onto; however, it may or may not be continuous. If it is continuous, then S will be called a *topological inverse semigroup*. For example, any topological group or any topological semilattice is a topological inverse semigroup. On the other hand, the nonnegative real numbers with the usual topology and ordinary multiplication is an inverse semigroup which is not a topological inverse semigroup.

For the remainder of this section, we assume that S is a topological inverse semigroup.

II.1. Proposition. *Inversion is a homeomorphism on S.*

Proof. This follows from the facts that inversion is continuous and inversion is its own inverse.

II.2. Proposition. *Let A be an inverse subsemigroup of S. Then A and A^* are topological inverse semigroups.*

Proof. Let x_α be a net in A converging to $x \in A^*$. Then x_α^{-1} converges to x^{-1}, and so $x^{-1} \in A^*$ since $x_\alpha^{-1} \in A$ for all α. Thus A^* is an inverse subsemigroup of S. The inversion functions on A and A^* are simply the appropriate restrictions of the inversion function on S, and hence are continuous.

Let E denote the idempotents of S.

II.3. Proposition. *Let A be a dense inverse subsemigroup of S. Then $E=(E \cap A)^*$, that is, the idempotents of A are dense in the idempotents of S.*

Proof. Let $e \in E$, and let U be an open set containing e. By the continuity of multiplication there is an open set $V \subset U$ so that $e \in V$ and $V^2 \subset U$. Note that $V^{-1} \cap V = W$ is an open set containing e by the continuity of the inversion function on S, and that $W = W^{-1}$. Now since $A^* = S$, we know that there is some $x \in A \cap W$. Hence $x^{-1} \in W^{-1} = W$ and $xx^{-1} \subset W^2 \subset V^2 \subset U$. But $xx^{-1} \in E \cap A$, so $(E \cap A)^* = E$.

II.4. Proposition. *If I is a compact ideal of S, then the Rees quotient semigroup S/I (the semigroup obtained by identifying I to a point) is a topological inverse semigroup under the quotient topology on S/I. Furthermore if S is locally compact, then so is S/I.*

Proof. It is well known that S/I is a topological semigroup and that S/I is locally compact if S is locally compact.

Let $x \in S$. Then since $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$ and I is an ideal of S, it follows that $x \in I$ if and only if $x^{-1} \in I$. Hence S/I is algebraically an inverse semigroup.

To see the continuity of the inversion function on S/I, we note that the inversion function on S/I is induced by the inversion function on S; that is, the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\text{inversion}"] \arrow[d, "\text{natural map}"'] & S \arrow[d, "\text{natural map}"] \\ S/I \arrow[r, "\text{inversion}"'] & S/I \end{tikzcd}$$

Applying the induced function theorem [3, p. 126] establishes the continuity of inversion on S/I.

II.5. Proposition. *The functions* $x \to xx^{-1}$ *and* $x \to x^{-1}x$ *are continuous retractions of S onto E, the set of idempotents of S.*

Proof. Obvious.

II.6. Corollary. *The* $\mathscr{L}$ *and* $\mathscr{R}$ *relations are closed in* $S \times S$. *Further,* $\mathscr{H}$ *is closed in* $S \times S$ *and all maximal subgroups of S are closed in S.*

Proof. We show that $\mathscr{R}$ is closed. A similar argument shows that $\mathscr{L}$ is closed. The rest follows.

Let $a, b \in S$ with $R_a \neq R_b$. Note that $aa^{-1} \in R_a$ and $bb^{-1} \in R_b$, so $aa^{-1} \neq bb^{-1}$. Choose disjoint subsets of E, U and V, which are open relative to E and which contain aa^{-1} and bb^{-1} respectively. Let U' and V' be the preimages of U and V respectively under the mapping $x \to xx^{-1}$. Then, by II.5, U' and V' are disjoint open sets in S containing a and b respectively. Further if $x \in U'$ and $y \in V'$, then $xx^{-1} \neq yy^{-1}$, and so x and y are not $\mathscr{R}$ related. Thus $(U' \times V') \cap \mathscr{R} = \varnothing$. We conclude that $\mathscr{R}$ is closed in $S \times S$.

We remark that $\mathscr{D}$ need not be closed in $S \times S$, as we shall see later.

If one views topological inverse semigroups as a generalization of topological groups, then several questions arise, which can all be included in the general question: what properties of topological groups generalize to topological inverse semigroups? Stated another way: given a theorem about topological groups, find the appropriate generalization of it to topological inverse semigroups. For example, each of the results in this section with the exception of II.4. can be regarded as a generalization of a theorem about topological groups.

As an additional example, a well-known result about topological groups, due to R. Ellis [4], is that a locally compact topological semigroup which is algebraically a group is a topological group. The nonnegative real numbers under ordinary multiplication and with the usual topology provide an easy counterexample to the above statement if "inverse semigroup" is substituted for "group". We ask whether the statement is true if "bisimple inverse semigroup" is substituted for "group".

III. **The closure of B in a topological inverse semigroup.** In this section S will again be a topological inverse semigroup, B will denote the bicyclic semigroup and we assume that $B \subset S$. Let T denote the closure of B in S. Then T is a topological inverse semigroup by II.2. Further each point of B is open in T and $T \backslash B$ is a closed ideal of T provided $T \backslash B \neq \varnothing$ by I.2 and I.3 respectively.

For the following three propositions we assume that $T \backslash B \neq \varnothing$.

III.1. Proposition. *$T \backslash B$ is a group.*

Proof. Let $x \in T\backslash B$. So $e=xx^{-1} \in T\backslash B$ since $T\backslash B$ is an ideal of T. Now suppose $f^2=f \in T\backslash B$ and that $e \neq ef$. Then there exist open sets U, V, and W of T containing e, f, and ef respectively so that $U \cap W=\varnothing$ and $UV \subset W$. Let E denote the idempotents of T. By III.3 we know that $E=(E \cap B)^*$. Hence $U \cap (E \cap B)$ is infinite and $V \cap (E \cap B)$ is infinite. Let $q^n p^n \in V \cap (E \cap B)$. There is an $m>n$ such that $q^m p^m \in U \cap (E \cap B)$. Thus $q^m p^m q^n p^n = q^m p^m \in W$, a contradiction since $U \cap W=\varnothing$. Thus $ef=e$. A similar argument shows $ef=f$. Hence we obtain $(T\backslash B) \cap E=\{e\}$, and $T\backslash B$ is an inverse semigroup with only one idempotent. It follows that $T\backslash B$ is a group.

III.2. Proposition. *The idempotent e of $T\backslash B$ is the center of T. Furthermore the function ψ from B into $T\backslash B$ which takes x to ex is a continuous homomorphism.*

Proof. The first statement will follow if we can show that $ep=pe$ and $eq=qe$. For then e will commute with the elements of B and hence with the elements of $B^*=T$. Now $e \in (B \cap E)^*$, hence there is a net $q^{n_\alpha} p^{n_\alpha}$ in $B \cap E$ converging to e. Hence $qep=\lim q^{n_\alpha+1} p^{n_\alpha+1}$. The set E is closed in T, so qep is idempotent. Further $qep \in T\backslash B$. Hence $qep=e$. Multiply on the left by $q^{-1}=p$ and we obtain $ep=pe$. Multiply on the right by $p^{-1}=q$ and we obtain $qe=eq$. Thus e is in the center of T. Now for $x, y \in B$, $exy=eexy=exey$ and so ψ is a homomorphism. ψ is continuous since multiplication is continuous and the second statement is proved.

III.3. Corollary. *$T\backslash B$ is the closure of a cyclic subgroup, namely eB.*

Proof. eB is a homomorphic image of B which lies in a group. Hence eB is a cyclic subgroup of $T\backslash B$ by [2, p. 43]. Let $x \in T\backslash B$. Then $x=\lim x_\alpha$ where $x_\alpha \in B$. So $x=ex=\lim ex_\alpha$. Hence $x \in (eB)^*$ and eB is dense in $T\backslash B$.

IV. **Three examples.** In this section we shall describe three topological semigroups containing the bicyclic semigroup as a dense proper subsemigroup.

IV.1. Example. The bicyclic semigroup with a zero adjoined as a limit point.

Let $B_0=B \cup \{0\}$ where 0 is some point not in B. Extend the multiplication on B to B_0 by defining $0 \cdot x=x \cdot 0=0$ for all $x \in B_0$. B_0 is easily seen to be a semigroup.

Now topologize B_0 by taking as a basis the points of B together with the sets of the form $q^n p^n B q^n p^n \cup \{0\}$. This clearly defines a Hausdorff topology on B_0 so that $B^*=B_0$.

To see that multiplication is continuous: Let $(x, y) \in S \times S$. If $x, y \in B$, then $\{x\}$, $\{y\}$, and $\{xy\}$ are open in B_0 and continuity is no problem. If $x \in B$ and $y=0$, let $U=q^n p^n B q^n p^n \cup \{0\}$ be a basic open set about 0. Write $x=q^l p^r$ and let

$$V = q^{n+r} p^{n+r} B q^{n+r} p^{n+r} \cup \{0\}.$$

Then

$$\begin{aligned}\{x\} \cdot V &= q^l p^r (q^{n+r} p^{n+r} B q^{n+r} p^{n+r} \cup \{0\}) \\ &= q^{n+l} p^{n+r} B q^{n+r} p^{n+r} \cup \{0\} \subset U.\end{aligned}$$

A similar argument works for the case $x=0$, $y \in B$.

Finally if $x=0$, $y=0$, then we simply note that each basic open set about 0 is a subsemigroup of B_0. This completes the argument that multiplication is continuous.

IV.2. REMARKS. 1. B_0 is a topological inverse semigroup, for 0 is its own unique inverse and the inverse of a basic open set is a basic open set.

2. B_0 contains two $\mathscr{D}$-classes, namely B and $\{0\}$. Hence B_0 shows that the $\mathscr{D}$ relation need not be closed in a topological inverse semigroup.

3. B_0 is *not* locally compact. Each basic open set about 0 contains an infinite discrete closed subset, for example, some $\mathscr{R}$-class of B with a finite number of points removed.

IV.3. EXAMPLE. The bicyclic semigroup with a left zero adjoined at the end of each $\mathscr{R}$-class.

Let $L=\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ be a set so that $x_i=x_j$ if and only if $i=j$, and $B \cap L=\varnothing$. Let L have the left trivial multiplication; that is, define $xy=x$ for $x, y \in L$. This is easily seen to be a semigroup operation on L. Let $B_L=B \cup L$ and extend the operations on B and L by defining

$$x_n \cdot q^i p^j = x_n, \qquad q^i p^j \cdot x_n = x_{i-j+\max\{j,n\}}$$

for $x_n \in L$, $q^i p^j \in B$.

It is evident upon inspection that the only associativity problem occurs when $x, y \in B$ and $z \in L$. In this case, letting $x=q^n p^m$, $y=q^l p^r$ and $z=x_i$, we have on the one hand:

$$\begin{aligned}(xy)z &= (q^n p^m q^l p^r)x_i = (q^{n-m+\max\{m,l\}} p^{r-l+\max\{m,l\}})x_i \\ &= x_{n-m-r+l+\max\{r-l+\max\{m,l\},i\}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}x(yz) &= q^n p^m (q^l p^r x_i) = q^n p^m x_{l-r+\max\{r,i\}} \\ &= x_{n-m+\max\{m,l-r+\max\{r,i\}\}}.\end{aligned}$$

Hence we need only show that

$$\max\{m, l-r+\max\{r, i\}\} = l-r+\max\{r-l+\max\{m, l\}, i\}.$$

Case 1. Suppose $\max\{m, l-r+\max\{r, i\}\}=m$. Then $m \geqq l-r+\max\{r, i\} \geqq l$ and $m \geqq l-r+\max\{r, i\} \geqq l-r+i$, so

$$\begin{aligned}l-r+\max\{r-l+\max\{m, l\}, i\} &= l-r+\max\{r-l+m, i\} = l-r+(r-l+m) \\ &= m = \max\{m, l-r+\max\{r, i\}\}.\end{aligned}$$

Case 2. Suppose $\max\{m, l-r+\max\{r, i\}\}=l-r+\max\{r, i\}$. Then we need only show that $\max\{r, i\}=\max\{r-l+\max\{m, l\}, i\}$.

Subcase 1. Suppose $\max\{r, i\}=r$. Then $l=l-r+\max\{r, i\} \geqq m$ and so

$$\max\{r, i\} = \max\{r, l+\max\{m, l\}, i\}.$$

Subcase 2. Suppose $\max\{r, i\}=i$. Then if $l \geqq m$ we have

$$\max\{r, i\} = \max\{r-l+\max\{m, l\}, i\},$$

and if $m \geqq l$ we have

$$l-r+\max\{r, i\} = l-r+i \geqq m = \max\{m, l\},$$

and so $i \geqq r-l+\max\{m, l\}$, from which we conclude that

$$\max\{r-l+\max\{m, l\}, i\} = i = \max\{r, i\}.$$

This completes the proof of associativity.

Now topologize B_L by taking as a basis the points of B together with the sets of the form

$$R_{q^n}p^m \cup \{x_n\}, \quad \text{where } R_{q^n} \text{ is the } \mathscr{R}\text{-class of } q^n.$$

It is easily seen that this defines a Hausdorff topology on B_L and that $B^* = B_L$ in this topology. To check the continuity of the multiplication at (x, y) we need only verify the case where $x \in B$, $y \in L$, for all the other cases are evident.

So suppose $x = q^n p^m$ and $y = x_i$. Then $xy = x_{n-m+\max\{i,m\}}$. Let $V = R_{q^t}p^j \cup \{x_t\}$ be a basic open set about x_t where $t = n-m+\max\{i, m\}$. Now choose the basic open set $R_{q^i}p^{i+j} \cup \{x_i\}$ about x_i.

Let $q^i p^{i+j+k} \in R_{q^i}p^{i+j}$. Then

$$(q^n p^m)(q^i p^{i+j+k}) = q^t p^{j+k+\max\{m,i\}} \in R_{q^t}p^j.$$

We conclude that $\{q^n p^m\}(R_{q^i}p^{i+j} \cup \{x_i\}) \subset V$, and hence that multiplication is continuous.

IV.4. Remarks. B_L is not an inverse semigroup, because idempotents do not commute.

2. B_L is locally compact, in fact,

$$B_L = \bigcup_n (R_{q^n})^* = \bigcup_n (R_{q^n} \cup \{x_n\})$$

and each of the sets $R_{q^n} \cup \{x_n\}$ is a compact open subset of B_L.

3. A similar construction can be made by adjoining a right zero to the end of each $\mathscr{L}$-class of B.

IV.5. Example. The bicyclic semigroup with the integers adjoined.

Let B_I be the disjoint union of the bicyclic semigroup B and the integers I.

Define a relation $\mathscr{I}$ on B by $q^l p^s \mathscr{I} q^m p^n$ if and only if $s-l = n-m$. It is easily verified that $\mathscr{I}$ is a congruence on B and that the quotient semigroup $B/\mathscr{I}$ is isomorphic with the integers under the correspondence $n \leftrightarrow B_n = \{q^z p^s \mid s-l = n\}$. It is an interesting exercise to show that, in fact, $\mathscr{I}$ is the *only* congruence on B so that $B/\mathscr{I}$ is isomorphic with the integers. We shall make use of this fact in §V.

Now extend the operations on B and I to B_I by defining $n \cdot q^l p^s = n+s-l = q^l p^s \cdot n$ for $n \in I$, $q^l p^s \in B$.

To see that the extended operation is associative we need only check two possibilities, the others being evident.

Case 1. $(n\cdot q^lp^s)q^up^r=(n+s-l)q^up^r=n+s-l+v-u$. Also

$$\begin{aligned} n\cdot(q^lp^sq^up^r) &= n\cdot q^{l-s+\max\{s,u\}}p^{v-u+\max,\{s,u\}} \\ &= n+v-u+\max\{s,u\}-l+s-\max\{s,u\} \\ &= n+v-u-l+s. \end{aligned}$$

Hence $(n\cdot q^lp^s)q^up^r=n\cdot(q^lp^sq^up^r)$.

Case 2. $(n+m)\cdot q^lp^s=n+m+s-l$.

Also $n+(m\cdot q^lp^s)=n+(m+s-l)=n+m+s-l$. Hence $(n+m)\cdot q^lp^s=n+(m\cdot q^lp^s)$. This completes the proof of associativity.

Thus B_I is a semigroup. It is easily seen that B_I is an inverse semigroup and that I is the kernel of B_I.

Now topologize B_I by taking as a basis the points of B together with the sets of the form $B_{n,i}=q^iB_np^i\cup\{n\}$, where n is an integer and i is a positive integer.

It is evident that this defines a Hausdorff topology on B_I and that $B^*=B_I$ in this topology.

To establish the continuity of the operation we first prove a lemma.

LEMMA. (1) $B_{n,i}B_{m,i}\subseteq B_{n+m,i}$.

(2) If $|l-s|\leqq i\leqq\max\{2i-s,2i-l\}$, then

$$q^lp^sB_{n,2i}\cup B_{n,2i}q^lp^s\subseteq B_{n+s-l,i}.$$

Proof.

(1)
$$\begin{aligned} B_{n,i}B_{m,i} &= q^iB_np^iq^iB_mp^i = q^iB_nB_mp^i \\ &\subseteq q^iB_{n+m}p^i = B_{n+m,i}. \end{aligned}$$

(2)
$$\begin{aligned} (q^lp^s)q^{2i}B_np^{2i} &= q^{2i-s+l}B_np^{2i} = q^i(q^{i-s+l}B_np^i)p^i \\ &\subseteq q^iB_{n+s+l}p^i = B_{n+s-l,i}. \end{aligned}$$

Also

$$\begin{aligned} q^{2i}B_np^{2i}(q^lp^s) &= q^{2i}B_np^{2i-l+s} = q^i(q^iB_np^{i-l+s})p^i \\ &\subseteq q^iB_{n+s-l}p^i = B_{n+s-l,i}. \end{aligned}$$

Hence (2) is also established.

Now to estâblish continuity we need only check the cases (n,m), (n,q^lp^s), and (q^lp^s,n).

Case 1. Continuity at (n,m). Let $B_{n+m,i}\cup\{n+m\}$ be a basic open set about $n+m$. Then $B_{n,i}\cup\{n\}$ and $B_{m,i}\cup\{m\}$ are basic open sets about n and m respectively. It follows from (1) of the lemma and the definition of the operation that

$$(B_{n,i}\cup\{n\})(B_{m,i}\cup\{m\})\subseteq B_{n+m,i}\cup\{m+n\}.$$

This completes case 1.

Case 2. Continuity at (q^lp^s,n) and (n,q^lp^s). Let $B_{n+s-l,i}\cup\{n+s-l\}$ be a basic open set about $n+s-l$, where i is chosen so that $|l-s|\leqq i\leqq\max\{2i-s,2i-l\}$, (for example take $i=s+l$). Then $B_{n,2i} \cup \{n\}$ is a basic open set about n and $\{q^l p^s\}$ is open. It follows from (2) of the lemma that

$$[\{q^l p^s\} \cdot (B_{n,2i} \cup \{n\})] \cup [(B_{n,2i} \cup \{n\}) \cdot \{q^l p^s\}] \subseteq B_{n+s-l,i} \cup \{n+s-l\}$$

and this completes Case 2.

IV.6. Remarks. 1. B_I is a topological inverse semigroup, since the inverse of a basic open set is a basic open set.

2. B_I is locally compact, but not compact. Each basic open set is compact, hence B_I is locally compact. Each $\mathscr{L}$-class and each $\mathscr{R}$-class of B is closed in B_I but not compact. This is also true because B is not embeddable into a compact semigroup [5, p. 524].

3. B_I contains two $\mathscr{D}$-classes, namely B and I. Since $B^* = B_I$, we have an example of a locally compact topological inverse semigroup in which the $\mathscr{D}$ relation is not closed.

V. **The closure of B in a locally compact topological inverse semigroup.** Let S be a locally compact topological inverse semigroup containing the bicyclic semigroup B, let $T = B^*$, and assume that $T \backslash B \neq \varnothing$. Then by the results in sections I, II, and III, we know that $T \backslash B$ is a topological group with a dense cyclic subgroup. We shall show that in fact $T \backslash B$ is iseomorphic with the integers and T is iseomorphic with the last example of IV.

V.1. Proposition. *Let E denote the idempotents of T. Then E is compact.*

Proof. E is closed in T so E is locally compact. Further $E \cap (T \backslash B)$ is a single point, call it e, and $E \backslash e$ is a discrete subspace of E. Let U be an open subset of E about e with U^* compact. Note that $U^* = U$ and U is infinite, since e is the only limit point of E. Suppose $E \backslash U$ is infinite. Then for each positive integer n, there exists a positive integer $r > n$ so that $q^r p^r \in E \backslash U$ and $q^{r+1} p^{r+1} \in U$. By induction we can find an infinite set $A \subseteq U \backslash e$ such that $pAq \subseteq E \backslash U$. But U is compact and A is infinite so $e \in A^*$. Hence $e \in pA^*q \subseteq (pAq)^* \subseteq E \backslash U$, a contradiction since $e \in U$. Consequently $E \backslash U$ is finite and we conclude that E is compact.

V.2. Lemma. *The $\mathscr{L}$ ($\mathscr{R}$)-class of* 1 *in B is the $\mathscr{L}$ ($\mathscr{R}$)-class of* 1 *in T.*

Proof. Let R_1, R_1' be the $\mathscr{R}$-class of 1 in B and in T respectively. By [5, p. 524], we have that $R_1 \subseteq R_1'$. Furthermore $R_1' \subseteq B$ since $T \backslash B$ is an ideal. Now suppose $q^l p^s \in R_1'$. Then

$$T = q^l p^s T = q^l p^s (B \cup T \backslash B) = q^l p^s B \cup q^l p^s (T \backslash B)$$

and since $q^l p^s (T \backslash B) \subseteq T \backslash B$, we have that $q^l p^s B = B$. Hence $q^l p^s \in R_1$ and $R_1 = R_1'$. A similar argument shows that the $\mathscr{L}$-class of 1 in B is the $\mathscr{L}$-class of 1 in T.

V.3. Proposition. $T \backslash B \neq \{e\}$.

Proof. Assume $T\backslash B=\{e\}$. Let U be an open set containing e whose closure is compact. By II.6 and V.2 we know that L_1 and R_1 are open and closed subsets of T. Hence we may assume that $U\subseteq T\backslash(L_1 \cup R_1)$. Note that $U=U^*$. If $n>0$, then the function from E to $A_n=\{q^lp^s \mid s-l=n\} \cup \{e\}$ given by $x \to xp^n$ is 1-1, continuous, and onto. Hence A_n is homeomorphic with E for $n>0$. If $n<0$, then since $(A_{-n})^{-1}=A_n$ and inversion is a homeomorphism, A_n is also homeomorphic with E. In particular A_n is compact for each integer n.

Now since $U \cap A_n$ is a compact open subset of A_n containing e, the only limit point of T, we conclude that $A_n\backslash U$ is finite for each n. Also since $A_n \cap (L_1 \cup R_1) \neq \varnothing$, we have $A_n\backslash U\neq \varnothing$ for each n. For each $x \in A_n\backslash U$, $x=q^{i+n}p^i$ for some natural number i. Let x_n denote that $x \in A_n\backslash U$ for which i is as large as possible. Let A be the set of all these x_n's.

Consider the continuous 1-1 functions $f\colon A \to U$ and $g\colon f(A) \to A$ defined by $f(x)=qxp$ and $g(x)=pxq$. Note that $f(A)$ is an infinite subset of U and hence $e \in f(A)^*$. Therefore $e \in g(f(A)^*)\subseteq g(f(A))^*=A^*$. But $A\subset T\backslash U$, a closed set not containing e. This is a contradiction, so $T\backslash B\neq\{e\}$.

V.4. Remark. We observe that local compactness is necessary to the proof, because of Example IV.3.

V.5. Corollary. *$T\backslash B$ is iseomorphic with the integers. Further, the homomorphism ψ of* III.2 *is onto and the restriction of ψ to $R_1 \cup L_1$ is* 1-1 *and onto.*

Proof. By III.3, $T\backslash B$ is a locally compact group containing a dense cyclic subgroup. By Weil's theorem, $T\backslash B$ is iseomorphic with the integers or $T\backslash B$ is compact. But if $T\backslash B$ is compact, then the Rees quotient semigroup $T/T\backslash B$ is, by II.4, a locally compact topological inverse semigroup. Now it is easily seen that $T/T\backslash B$ contains B as a dense subsemigroup and that $(T/T\backslash B)\backslash B$ is a single point. However this contradicts the previous proposition; hence $T\backslash B$ is iseomorphic with the integers.

Now $eB=\psi(B)$ is dense in $T\backslash B$ and $T\backslash B$ is discrete. Hence $\psi(B)=T\backslash B$; that is, ψ is onto. The last statement follows since the congruence on B determined by the homomorphism ψ is precisely the congruence $\mathscr{I}$ on B described in IV.5.

Recall that $B_n=\{q^lp^s \in B \mid s-l=n\}$, for each integer n.

V.6. Lemma. *For each integer n, B_n^* is a compact open subset of T. Further $B_n^*=B_n \cup \{ep^n\}$ if $n\geqq 0$ and $B_n^*=B_n \cup \{eq^n\}$ if $n\leqq 0$.*

Proof. Note that $B_0^*=E$ by II.3 and hence is compact by V.1. Also if $n\geqq 0$, then $B_n^*=(B_0p^n)^*=B_0^*p^n$, since B_0^* is compact, and hence B_n^* is compact. Also $B_0^*=B_0 \cup \{e\}$, so $B_n^*=(B_0p^n)^*=B_0^*p^n=(B_0 \cup \{e\})p^n=B_0p^n \cup \{ep^n\}=B_n \cup \{ep^n\}$ if $n\geqq 0$ and similarly $B_n^*=B_n \cup \{eq^n\}$ if $n\leqq 0$.

To see that B_n^* is open let U be an open set containing n so that $U \cap T\backslash B=\{n\}$. U exists by V.5. By the regularity of T, we can choose an open V containing n so that $V^* \subset U$ and V^* is compact. Since n is the only limit point of V^* and $v \in V$ we have that $V = V^*$. Hence $V \cap (T \backslash B_n^*)$ is a finite open set, and therefore n lies in the interior of B_n^*. Every other point of B_n^* is open in T, so B_n^* is open.

V.7. Theorem. *T is iseomorphic with B_I, the bicyclic semigroup with the integers adjoined.*

Proof. Define $f: B_I \to T$ by

$$\begin{aligned} f(x) &= x && \text{if } x \in B \\ &= ep^n && \text{if } x = n \in I \text{ and } n \geqq 0 \\ &= eq^n && \text{if } x = n \in I \text{ and } n < 0. \end{aligned}$$

It is clear that f is a well-defined function with domain B_I and range T. f is 1-1 on B. Also f is 1-1 on I since $f(n) = \psi(p^n)$ for $n \geqq 0$ and $f(n) = \psi(q^n)$ for $n < 0$ and $\psi | L_1 \cup R_1$ is 1-1.

To see that f is a homeomorphism, consider $f | B_n \cup \{n\}$. It is not hard to see that f maps the compact open set $B_n \cup \{n\}$ of B_I in a 1-1 continuous fashion onto the compact open set $B_n^* = B_n \cup \{ep^n\}$ or $B_n^* = B_n \cup \{eq^n\}$, depending on the sign of n. Hence $f | B_n \cup \{n\}$ is a homeomorphism for each n. From this we conclude that f is a homeomorphism.

To see that f is a homomorphism, we note that $f | B$ is a homomorphism and that f is continuous on B_I. From this it readily follows that f is a homomorphism on $B^* = B_I$.

This completes the proof of V.7.

V.8. Corollary. *Let θ be a homomorphism from B into a locally compact topological inverse semigroup. Then one and only one of* (i), (ii), *and* (iii) *is true*:

(i) *$\theta(B)$ is a finite cyclic group.*

(ii) *$\theta(B)$ is the integer and either $\theta(B)$ is a closed discrete subspace of S or else the closure of $\theta(B)$ is compact.*

(iii) *θ is* 1-1 *and $\theta(B)$ is a closed discrete subspace of S or the closure of $\theta(B)$ is iseomorphic with B_I.*

Added in proof. The authors have learned that I.1, I.2, I.3, and I.5 were also proved by R. J. Koch and A. D. Wallace in *Notes on inverse semigroups*, Rev. Roumaine Math. Pures Appl. **9**(1964), 19–24.

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