On the cluster category of a marked surface without punctures

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We study the cluster category $\mathscr{C}_{(S,M)}$ of a marked surface (S, M) without punctures. We explicitly describe the objects in $\mathscr{C}_{(S,M)}$ as direct sums of homotopy classes of curves in (S, M) and one-parameter families related to noncontractible closed curves in (S, M). Moreover, we describe the Auslander–Reiten structure of the category $\mathscr{C}_{(S,M)}$ in geometric terms and show that the objects without self-extensions in $\mathscr{C}_{(S,M)}$ correspond to curves in (S, M) without self-intersections. As a consequence, we establish that every rigid indecomposable object is reachable from an initial triangulation.

1. Introduction

We study the cluster category of a marked surface without punctures: Consider a compact connected oriented 2-dimensional bordered Riemann surface S and a finite set of marked points M lying on the boundary ∂S of S with at least one marked point on each boundary component. The condition $M \subset \partial S$ means that we do not allow the marked surface (S, M) to have punctures. (While some of the background results we need are valid in the more general context of punctured surfaces, the main result we need from [Assem et al. 2010] works only without punctures.) In [Fomin et al. 2008] a cluster algebra $\mathscr{A}(S, M)$ is associated to the marked surface (S, M). The initial seed of $\mathscr{A}(S, M)$ corresponds to a triangulation Γ of (S, M), and the mutation of a cluster variable corresponds to the flip of an arc in the triangulation (see [Fomin and Zelevinsky 2002] for the definition of a cluster algebra). The cluster algebra, defined by iterated mutations, is thus independent of the chosen triangulation Γ of (S, M).

The cluster category $\mathscr{C}_{(S,M)}$ providing a categorification of the algebra $\mathscr{A}(S, M)$ was defined in [Amiot 2009]. In fact, in [Labardini-Fragoso 2009a; Assem et al. 2010] a quiver with potential (Q_Γ, W_Γ) has been defined for each triangulation Γ

of (S, M). Since the Jacobian algebra $J(Q_\Gamma, W_\Gamma)$ is finite-dimensional, one can use [Amiot 2009] to define the cluster category $\mathscr{C}_\Gamma$ associated to Γ. By [Keller and Yang 2009; Fomin et al. 2008; Labardini-Fragoso 2009a], this category $\mathscr{C}_\Gamma$ does not depend on the triangulation Γ of (S, M) and is just denoted by $\mathscr{C}_{(S,M)}$ (more precisely, $\mathscr{C}_\Gamma$ is triangle equivalent to $\mathscr{C}_{\Gamma'}$ if Γ' is another triangulation of (S, M)).

Cluster categories associated to quivers are object of intense investigation (see for example [Geiss et al. 2005; Caldero et al. 2006; Buan et al. 2006; 2008; Caldero and Chapoton 2006; Geiß et al. 2006; Assem et al. 2008]) and are understood explicitly in terms of representations of the quiver. The category $\mathscr{C}_{(S,M)}$, defined by a quiver with potential, is however quite difficult to compute and only a few cases of surfaces allow direct calculations. The aim of this paper is to provide an explicit description of the objects and the irreducible morphisms in the category $\mathscr{C}_{(S,M)}$ in terms of the surface (S, M), independent of the choice of a triangulation. In fact, the category $\mathscr{C}_{(S,M)}$ is closely related to module categories of string algebras, and a well-know classification theorem [Butler and Ringel 1987] describes all modules over a string algebra by listing the indecomposable modules in a list of string modules and a collection of band modules. Similarly, the indecomposable objects of $\mathscr{C}_{(S,M)}$ can be described as belonging to two classes which we call string objects and band objects. The cluster categories $\mathscr{C}_{(S,M)}$ are k-categories, where k denotes a fixed algebraically closed field. Moreover they are Krull–Schmidt; thus it is sufficient to classify indecomposable objects up to isomorphism.

Theorem 1.1. *A parametrization of the isoclasses of indecomposable objects in* $\mathscr{C}_{(S,M)}$ *is given by "string objects" and "band objects", where*

(1) *the string objects are indexed by the homotopy classes of noncontractible curves in (S, M) which are not homotopic to a boundary segment of (S, M), subject to the equivalence relation $\gamma \sim \gamma^{-1}$;*

(2) *the band objects are indexed by $k^* \times \Pi_1^*(S, M)/\!\sim$, where $k^* = k \setminus \{0\}$ and $\Pi_1^*(S, M)/\!\sim$ is given by the nonzero elements of the fundamental group of (S, M) subject to the equivalence relation generated by $a \sim a^{-1}$ and cyclic permutation.*

Remark 1.2. The curves we consider in Theorem 1.1 are allowed to have self-intersections.

The boundary of the surface S consists of a collection of disjoint circles, each one inheriting an orientation from the orientation of S. Given any curve γ in (S, M), we denote by $_s\gamma$ the curve obtained from γ by moving its starting point clockwise to the next marked point on the boundary. Likewise, moving the ending point of γ clockwise along the boundary to the next marked point, one obtains a curve γ_e from γ. These moves establish the irreducible morphisms between string

objects, as explained in the following theorem. Note that in the special case where S is a disc, this description of irreducible morphisms has been given in [Caldero et al. 2006], establishing one of the first categorifications of a cluster algebra (of type A).

In the next few statements, we use the abbreviation AR to refer to the Auslander–Reiten structure.

Theorem 1.3. *Let γ be a noncontractible curve in (S, M) that is not homotopic to a boundary segment of (S, M). Then there is an AR-triangle in $\mathscr{C}_{(S,M)}$ as follows, where $_s\gamma$ or γ_e are zero objects in $\mathscr{C}_{(S,M)}$ if they are boundary segments:*

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Moreover, all AR-triangles between string objects in $\mathscr{C}_{(S,M)}$ are of this form.

We obtain also the AR-translation in the cluster category $\mathscr{C}_{(S,M)}$:

Proposition 1.4. *The AR-translation in $\mathscr{C}_{(S,M)}$ is given by the simultaneous counterclockwise rotation of the starting and ending points of a curve to the next marked points at the boundary.*

Moreover, from Theorem 1.3 one can deduce the shape of the AR-components in the category $\mathscr{C}_{(S,M)}$. In fact, each boundary component of S with t marked points gives rise to a tube of rank t in $\mathscr{C}_{(S,M)}$. All other components formed by string objects are composed of meshes with exactly two middle terms.

We also study the effect of a change of the triangulation. It is well-known that any two triangulations of (S, M) can be transformed into each other by a sequence of flips $f_i(\Gamma)$ which locally change one arc in Γ. On the other hand, each triangulation yields a cluster-tilting object T in $\mathscr{C}_{(S,M)}$, and there one can apply mutations $\mu_i(T)$ locally changing one summand. We show that these two operations are compatible:

Theorem 1.5. *Each triangulation Γ of (S, M) yields a cluster-tilting object T_Γ in $\mathscr{C}_{(S,M)}$, and*

$$\mu_i(T_\Gamma) = T_{f_i(\Gamma)}.$$

We then study the relation between extensions in the category $\mathscr{C}_{(S,M)}$ and intersections of the corresponding curves. Given any two curves in (S, M) with intersections, we explicitly construct one or two new curves, sometimes resolving the intersection, and sometimes increasing the winding number. These new curves serve as middle term of certain nonsplit short exact sequences which allow to prove the following theorem and corollary:

Theorem 1.6. *Curves in (S, M) without self-intersections correspond to the indecomposable objects without self-extensions in $\mathscr{C}_{(S,M)}$.*

Corollary 1.7. *There is a bijection between triangulations of (S, M) and cluster-tilting objects of $\mathscr{C}_{(S,M)}$. In particular, each indecomposable object without self-extension is reachable from the cluster-tilting object T_Γ (the initial cluster-tilting object).*

This paper is organized as follows: we first describe in Section 2 the indecomposable objects in $\mathscr{C}_{(S,M)}$, using the description of modules over a string algebra. In Section 3 we describe the irreducible morphisms in $\mathscr{C}_{(S,M)}$ and study the shape of its AR-components. Section 4 is devoted to compare the effect of a flip of an arc in the triangulation and a mutation of the corresponding cluster-tilting object. We also compare it to results obtained for decorated representations of quivers with potential. Finally, in Section 5 we prove that curves without self-intersections correspond to objects without self-extensions in $\mathscr{C}_{(S,M)}$ and we establish the bijection between triangulations and cluster-tilting objects.

2. Indecomposable objects in $\mathscr{C}_{(S,M)}$

In this section, we choose a triangulation Γ of (S, M), and by studying the corresponding Jacobian algebra, we give a geometric characterization of the indecomposable objects in $\mathscr{C}_{(S,M)}$. By a curve in (S, M), we mean a continuous function $\gamma : [0, 1] \to S$ with $\gamma(0), \gamma(1) \in M$. A closed curve is one with $\gamma(0) = \gamma(1)$, and a simple curve is one where γ is injective, except possibly at the endpoints. For a curve γ, we denote by γ^{-1} the inverse curve $\gamma^{-1} : [0, 1] \to S$, $t \mapsto \gamma(1-t)$. We always consider curves up to homotopy, and for any collection of curves we implicitly assume that their mutual intersections are minimal possible in their respective homotopy classes. We recall from [Fomin et al. 2008] the definition of a triangulation:

Definition 2.1. An arc δ in (S, M) is a simple noncontractible curve in (S, M). The boundary of S is a disjoint union of circles, which are subdivided by the points in M into boundary segments. We call an arc δ a *boundary arc* if it is homotopic to such a boundary segment. Otherwise, δ is said to be an *internal arc*. A *triangulation* of (S, M) is a maximal collection Γ of arcs that do not intersect except at their endpoints. We call a triangle $\triangle$ in Γ an *internal triangle* if all edges of $\triangle$ are internal arcs.

Proposition 2.2 [Fomin et al. 2008]. *In each triangulation of (S, M), the number of internal arcs is*

$$n = 6g + 3b + c - 6$$

where g is the genus of S, b is the number of boundary components, and $c = |M|$ is the number of marked points.

The quiver with potential (Q_Γ, W_Γ). We recall from [Assem et al. 2010; Labardini-Fragoso 2009a] that each triangulation Γ of (S, M) yields a quiver Q_Γ with potential W_Γ:

(1) $Q_\Gamma = (Q_0, Q_1)$, where the set of vertices Q_0 is given by the internal arcs of Γ, and the set of arrows Q_1 is defined as follows: whenever there is a triangle $\triangle$ in Γ containing two internal arcs a and b, there is an arrow $\rho : a \to b$ in Q_1 if a is a predecessor of b with respect to clockwise orientation at the joint vertex of a and b in $\triangle$.

(2) Every internal triangle $\triangle$ in Γ gives rise to an oriented cycle $\alpha_\triangle \beta_\triangle \gamma_\triangle$ in Q, unique up to cyclic permutation of the factors $\alpha_\triangle$, $\beta_\triangle$, $\gamma_\triangle$. We define

$$W_\Gamma = \sum_\triangle \alpha_\triangle \beta_\triangle \gamma_\triangle$$

where the sum runs over all internal triangles $\triangle$ of Γ.

Unless we compare two different triangulations, we omit the subscript and denote the quiver with potential defined by Γ just with (Q, W). We refer to [Derksen et al. 2008] for more details on quivers with potentials, such as the definition of the Jacobian algebra $J(Q, W)$ associated to (Q, W). From [Assem et al. 2010] we know that $J(Q, W)$ is a finite-dimensional string algebra provided (Q, W) is defined by a triangulation of a marked surface as above.

Moreover, we denote by $\Lambda_{(Q,W)}$ the corresponding Ginzburg dg-algebra (see [Ginzburg 2006] for more details), and denote by $D(\Lambda_{(Q,W)})$ its derived category [Keller 2006]. The *cluster category* $\mathscr{C}_\Gamma = \mathscr{C}_{(Q,W)}$ associated to (Q, W) is defined in [Amiot 2009] as the quotient of triangulated categories Per $\Lambda_{(Q,W)}/D^b(\Lambda_{(Q,W)})$ where Per $\Lambda_{(Q,W)}$ is the thick subcategory of $D(\Lambda_{(Q,W)})$ generated by $\Lambda_{(Q,W)}$ and $D^b(\Lambda_{(Q,W)})$ is the full subcategory of $D(\Lambda_{(Q,W)})$ of the dg-modules whose homology is of finite total dimension.

Theorem 2.3 [Amiot 2009; Koenig and Zhu 2008]. *Suppose (Q, W) is a quiver with potential whose Jacobian algebra $J(Q, W)$ is finite-dimensional.*

(1) *$\mathscr{C}_\Gamma$ is 2-Calabi–Yau, Hom-finite and the image T_Γ of the free module $\Lambda_{(Q,W)}$ in the quotient Per $\Lambda_{(Q,W)}/D^b(\Lambda_{(Q,W)})$ is a cluster-tilting object.*

(2) *$\mathscr{C}_\Gamma/T_\Gamma$ is equivalent to mod $J(Q, W)$, the category of finite dimensional modules over $J(Q, W)$. Moreover, the projection functor $\mathscr{C}_\Gamma \to$ mod $J(Q, W)$ is given by $\mathrm{Ext}^1_{\mathscr{C}_\Gamma}(T_\Gamma, -)$.*

As explained in the introduction, the category $\mathscr{C}_\Gamma$ is (up to triangle equivalence) independent of the choice of a triangulation of (S, M), and is therefore denoted by $\mathscr{C}_{(S,M)}$. Moreover, it is a Krull–Schmidt category, so it is sufficient to describe all indecomposable objects up to isomorphism in order to describe its objects. From

Theorem 2.3(2), the indecomposable objects in $\mathcal{C}_{(S,M)}$ are either indecomposable modules over the Jacobian algebra $J(Q, W)$ or one of the $|Q_0| = 6g + 3b + |M| - 6$ summands of T_Γ. In order to give a description of all indecomposable objects in $\mathcal{C}_{(S.M)}$, we first study the Jacobian algebra $J(Q, W)$.

***The string algebra $J(Q, W)$.*** We now recall some basic definitions related to string algebras and prove Theorem 1.1. Recall from [Butler and Ringel 1987] that a finite-dimensional algebra A is a *string algebra* if there is a quiver Q and an admissible ideal I such that $A = kQ/I$ and the following conditions hold:

(S1) At each vertex of Q start at most two arrows and stop at most two arrows.

(S2) For each arrow α there is at most one arrow β and at most one arrow δ such that $\alpha\beta \notin I$ and $\delta\alpha \notin I$.

Given an arrow $\beta \in Q_1$, let $s(\beta)$ be its starting point and $e(\beta)$ its ending point. We denote β^{-1} the formal inverse of β with $s(\beta^{-1}) = e(\beta)$ and $e(\beta^{-1}) = s(\beta)$. A word $w = \alpha_n \alpha_{n-1} \cdots \alpha_1$ of arrows and their formal inverses is called a *string* if $\alpha_{i+1} \neq \alpha_i^{-1}$, $e(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i \leq n-1$, and no subword nor its inverse belongs to I. Thus a string w can be viewed as a walk in the quiver Q avoiding the zero relations defining the ideal I:

$$w: \quad x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots x_{n-1} \xrightarrow{\alpha_{n-1}} x_n \xrightarrow{\alpha_n} x_{n+1}$$

where x_i are vertices of Q and α_i are arrows in either direction. We denote by $s(w) = s(\alpha_1)$ and $e(w) = e(\alpha_n)$ the starting point and the ending point of w, respectively. For technical reasons, we also consider the empty string which we also call zero string. A string w is called *cyclic* if the first vertex x_1 and the last vertex x_{n+1} coincide. A *band* $b = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$ is defined to be a cyclic string b such that each power b^m is a string, but b itself is not a proper power of any string. Thus b can be viewed as a cyclic walk:
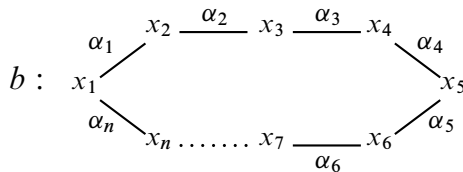

We recall from [Butler and Ringel 1987] that each string w in A defines a *string module* $M(w)$ in mod A. The underlying vector space of $M(w)$ is obtained by replacing each x_i in w by a copy of the field k. The action of an arrow α of Q on $M(w)$ is induced by the relevant identity morphisms if α lies on w, and is zero otherwise. For the zero string 0, we let $M(0)$ be the zero module. Each band b defines a family of band modules $M(b, n, \phi)$ with $n \in \mathbb{N}$ and $\phi \in \text{Aut}(k^n)$ by replacing each x_i in b a copy of the vector space k^n, and the action of an arrow α

on $M(b, n, \phi)$ is induced by identity morphisms if $\alpha = \alpha_j$ for $j = 1, 2 \ldots n - 1$ and by ϕ if $\alpha = \alpha_n$ (see [Butler and Ringel 1987]).

Let Γ be a triangulation of the marked surface (S, M), and denote by (Q, W) the corresponding quiver with potential. In [Assem et al. 2010] the strings and bands of $J(Q, W)$ are related to the noncontractible curves and simple noncontractible closed curves respectively in (S, M): For two curves γ', γ in (S, M) we denote by $I(\gamma', \gamma)$ the minimal intersection number of two representatives of the homotopic classes of γ' and γ. For each curve γ in (S, M) with

$$d = \sum_{\gamma' \in \Gamma} I(\gamma', \gamma)$$

we fix an orientation of γ, and let $x_1, x_2, \ldots, x_d$ be the internal arcs of Γ that intersect γ in the fixed orientation of γ, as in the figure:
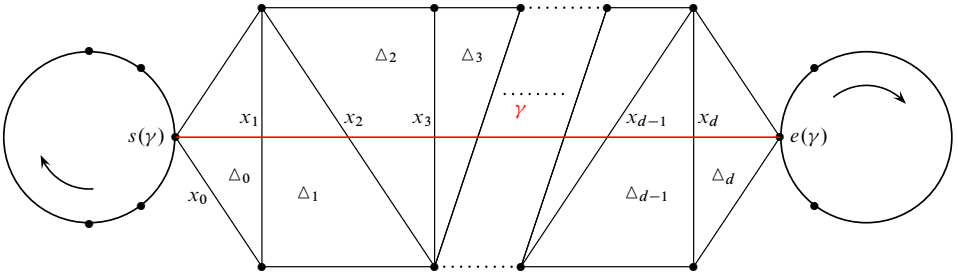
Here we denote by $s(\gamma) = \gamma(0)$ the starting point of γ and by $e(\gamma) = \gamma(1)$ its endpoint. Both points lie on the boundary, indicated by the circles in the preceding figure. Along its way, the curve γ is passing through (not necessarily distinct) triangles $\triangle_0, \triangle_1, \ldots, \triangle_d$. Thus we obtain a string $w(\gamma)$ in $J(Q, W)$:

$$w(\gamma): \quad x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots x_{d-2} \xrightarrow{\alpha_{d-2}} x_{d-1} \xrightarrow{\alpha_{d-1}} x_d.$$

We recall the following result concerning the map $\gamma \mapsto w(\gamma)$:

Theorem 2.4 [Assem et al. 2010]. *Given a triangulation Γ of a marked surface (S, M), the map $\gamma \mapsto w(\gamma)$ is a bijection between the homotopy classes of noncontractible curves in (S, M) not homotopic to an arc in Γ and the strings of $J(Q, W)$.*

Similarly, each noncontractible closed curve b in S defines a cyclic walk in $J(Q, W)$. If b is not a proper power of any element in $\Pi_1^*(S, M)$, then it defines a band $w(b)$.

Remark 2.5. It will be convenient later to assume that each element $b \in \Pi_1^*(S, M)$ is given by a noncontractible closed curve b with endpoints $b(0) = b(1)$ lying on a marked point. Remark that as an element in $\Pi_1^*(S, M)$, b does not depend on the choice of the marked point.

Example 2.6. We consider an annulus with two marked points on each boundary and a triangulation Γ with internal arcs 1, 2, 3, 4, 5 as follows:
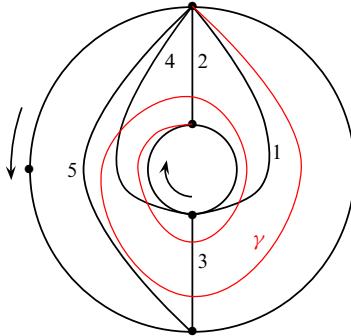
The associated quiver Q_Γ is cluster-tilted of type $\tilde{A}_4$ with potential $W_\Gamma = \alpha\beta\theta$:
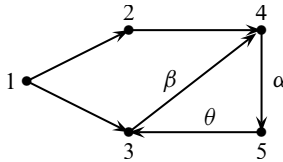
From the diagram at the top of the page we read off

$$w(\gamma): \quad 3 \longrightarrow 4 \longleftarrow 2 \longleftarrow 1 \longrightarrow 3 \longrightarrow 4$$

where γ is the curve in red, and

where b is a simple closed curve representing a generator of the fundamental group of the annulus.

Proof of Theorem 1.1. Let Γ be a triangulation of (S, M). As explained on page 533, the indecomposable objects in $\mathscr{C}_{(S,M)}$ are either given by indecomposable modules over the Jacobian algebra $J(Q, W)$, or they correspond to the indecomposable summands of T_Γ, thus to the internal arcs in Γ. The (finite-dimensional) indecomposable modules over a string algebra A are classified in [Butler and Ringel 1987]: Each indecomposable A-module is (isomorphic to) a string or a band module. The string module $M(w)$ is isomorphic to the string module $M(w^{-1})$ defined by the inverse string w^{-1}, and the band module $M(b, n, \phi)$ is isomorphic to $M(b', n, \phi)$ whenever b' is obtained from b by inversion or cyclic permutation. Apart from that, there are no isomorphisms between string or band modules.

Thus each noncontractible curve γ that is not homotopic to a boundary segment of (S, M) corresponds to an indecomposable object in $\mathscr{C}_{(S,M)}$. If γ is not an internal

arc in Γ, it corresponds to the string module $M(w(\gamma))$. From Theorem 2.4 and what we have discussed above we conclude that two such curves γ, δ are isomorphic as objects in $\mathscr{C}_{(S,M)}$ precisely when γ is homotopic to δ or to its inverse δ^{-1}. We refer to these objects as the string objects or curves in (S, M), as described in part (1) of Theorem 1.1.

The remaining indecomposable objects in $\mathscr{C}_{(S,M)}$ correspond to the band modules $M(b, n, \phi)$ over $J(Q, W)$; we refer to them as band objects. They are parametrized by a positive integer n, an automorphism ϕ of k^n which is given by an element of k^* since k is algebraically closed, and a band b of $J(Q, W)$. The fundamental group $\Pi_1(S, M)$ is a free group with a finite number of generators which are given by simple noncontractible closed curves in S (see [Küçük 2005]). In order to avoid counting curves with opposite orientation twice we consider the elements in $\Pi_1(S, M)$ up to the equivalence relation $a \sim a^{-1}$.

Moreover, to comply with the definition of a band module, we write each element a in $\Pi_1(S, M)$ as $a = b^n$ (multiplicatively written) for some $b \in \Pi_1(S, M)$ which itself is not a proper power of an element in $\Pi_1(S, M)$. Furthermore, we consider the elements of $\Pi_1(S, M)$ up to cyclic permutation of their factors. Then it is clear that the band modules $M(b, n, \phi)$ over $J(Q, W)$ correspond bijectively to $k^* \times \Pi_1^*(S, M)/\sim$, where $\Pi_1^*(S, M)/\sim$ is given by the nonzero elements of the fundamental group of (S, M) subject to the equivalence relation generated by $a \sim a^{-1}$ and cyclic permutation. This is the description of the band objects in $\mathscr{C}_{(S,M)}$ given in part (2) of Theorem 1.1. $\qquad\square$

3. Irreducible morphisms in $\mathscr{C}_{(S,M)}$

Based on the geometric characterization of the indecomposable objects in $\mathscr{C}_{(S,M)}$ in the previous section, we study in this section the irreducible morphisms in $\mathscr{C}_{(S,M)}$.

The AR-quiver of a string algebra. We first recall some basic definitions from [Butler and Ringel 1987]. Let $A = kQ/I$ be a finite-dimensional string algebra with $Q = (Q_0, Q_1)$ and $\mathscr{S}$ the set of all strings in A. A string w *starts (or ends) on a peak* if there is no arrow $\alpha \in Q_1$ with $w\alpha \in \mathscr{S}$ (or $\alpha^{-1}w \in \mathscr{S}$); likewise, a string w *starts (or ends) in a deap* if there is no arrow $\beta \in Q_1$ with $w\beta^{-1} \in \mathscr{S}$ (or $\beta w \in \mathscr{S}$).

A string $w = \alpha_1\alpha_2 \cdots \alpha_n$ with all $\alpha_i \in Q_1$ is called *direct string*, and a string of the form w^{-1} where w is a direct string is called *inverse string*. Strings of length zero are both direct and inverse. For each arrow $\alpha \in Q_1$, let $N_\alpha = U_\alpha \alpha V_\alpha$ be the unique string such that U_α and V_α are inverse strings and N_α starts in a deep and ends on a peak. (See figure.)
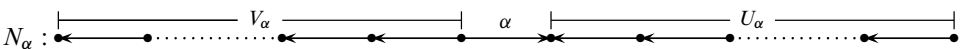
If the string w does not start on a peak, we define $w_h = w\alpha V_\alpha$ and say that w_h is obtained from w by adding a hook on the starting point $s(w)$. Dually, if w does not end on a peak, we define $_h w = V_\alpha^{-1}\alpha^{-1}w$ and say that $_h w$ is obtained from w by *adding a hook* on the ending point $e(w)$. (See figure.)
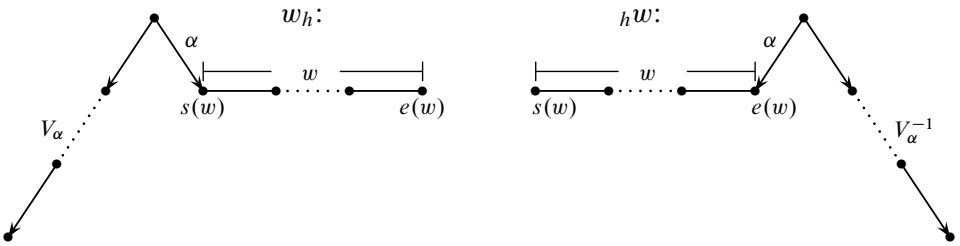
Suppose now that the string w starts on a peak. If w is not a direct string, we can write $w = w_c\beta^{-1}\gamma_1\gamma_2\cdots\gamma_r = w_c\beta^{-1}U_\beta^{-1}$ for some $\beta \in Q_1$ and $r \geq 0$. We say in this case that w_c is obtained from w by *deleting a cohook* on $s(w)$. If w is a direct string, we define $w_c = 0$. Dually, assume that w ends on a peak. Then, if w is not an inverse string, we can write $w = \gamma_r^{-1}\cdots\gamma_2^{-1}\gamma_1^{-1}\beta_c w = U_\beta\beta_c w$ for some $\beta \in Q_1$ and $r \geq 0$. We say that $_c w$ is obtained from w by deleting a cohook on $e(w)$, and if w is an inverse string, we define $_c w = 0$. (See figure.)
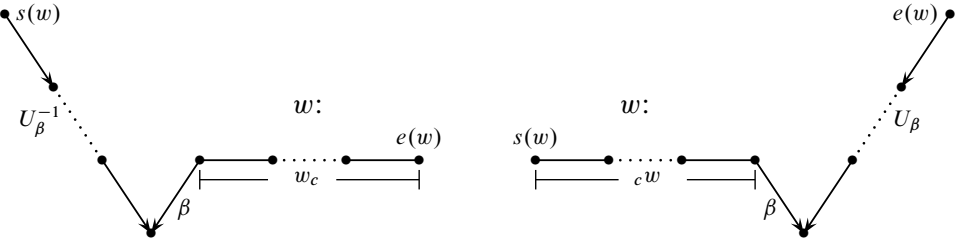
Theorem 3.1 [Butler and Ringel 1987]. *For a string algebra A, let w be a string such that $M(w)$ is not an injective A-module. Then the AR-sequence starting in $M(w)$ is given,*

(1) *if w neither starts nor ends on a peak, by*

$$0 \longrightarrow M(w) \longrightarrow M(w_h) \oplus M(_h w) \longrightarrow M(_h w_h) \longrightarrow 0;$$

(2) *if w does not start but ends on a peak, by*

$$0 \longrightarrow M(w) \longrightarrow M(w_h) \oplus M(_c w) \longrightarrow M(_c w_h) \longrightarrow 0;$$

(3) *if w starts but does not end on a peak, by*

$$0 \longrightarrow M(w) \longrightarrow M(w_c) \oplus M(_h w) \longrightarrow M(_h w_c) \longrightarrow 0;$$

(4) *if w both starts and ends on a peak, by*

$$0 \longrightarrow M(w) \longrightarrow M(w_c) \oplus M(_c w) \longrightarrow M(_c w_c) \longrightarrow 0.$$

Remarks 3.2. (1) Each arrow $\alpha \in Q_1$ defines a string $N_\alpha = U_\alpha \alpha V_\alpha$ that starts in a deep and ends on a peak. Then $w = U_\alpha$ does not start but ends on a peak, hence we have an AR-sequence by Theorem 3.1:

$$0 \longrightarrow M(w) \longrightarrow M(w_h) \oplus M(_cw) \longrightarrow M(_cw_h) \longrightarrow 0.$$

Here $M(_cw) = 0$ since w is an inverse string, and $w_h = N_\alpha$ and $_cw_h = _c(N_\alpha) = V_\alpha$ by definition. Hence, for each $\alpha \in Q_1$, there is an AR-sequence with an indecomposable middle term in mod A:

$$0 \longrightarrow M(U_\alpha) \longrightarrow M(N_\alpha) \longrightarrow M(V_\alpha) \longrightarrow 0.$$

In fact, the AR-sequences between string modules which admit only one direct summand in the middle term are indexed by the arrows in Q (see [Butler and Ringel 1987]).

(2) If the string module $M(w)$ is injective, then w both starts and ends on a peak. Thus we can write

$$w = \gamma_r^{-1} \gamma_{r-1}^{-1} \cdots \gamma_1^{-1} \beta_1 \beta_2 \cdots \beta_{s-1} \beta_s \quad \text{or}$$
$$w = U_{\beta_1} \beta_1 \beta_2 \cdots \beta_{s-1} \beta_s = \gamma_r^{-1} \gamma_{r-1}^{-1} \cdots \gamma_1^{-1} U_{\gamma_1}^{-1},$$

where $r + s \geq 1$; hence $_cw = \gamma_r^{-1} \gamma_{r-1}^{-1} \cdots \gamma_2^{-1}$, $w_c = \beta_2 \cdots \beta_{s-1} \beta_s$, and $M(w)/\text{Soc}(M(w)) = M(_cw) \oplus M(w_c)$ which yields two irreducible morphisms $M(w) \rightarrow M(_cw)$ and $M(w) \rightarrow M(w_c)$. Moreover, $_cw_c = _c(w_c) = (_cw)_c$ is the empty string, hence $M(_cw_c)$ is the zero module. Thus one might say that the case where M(w) is injective is contained in the case (4) of Theorem 3.1:

(3) The number of indecomposable summands in the middle term of an AR-sequence between string modules is at most two.

Irreducible morphisms in $J(Q, W)$. We now fix a triangulation $\Gamma = \{\tau_1, \tau_2, \ldots \tau_n, \tau_{n+1}, \ldots, \tau_{n+m}\}$ of the marked surface (S, M), where $\tau_1, \ldots, \tau_n$ are internal arcs and $\tau_{n+1}, \ldots, \tau_{n+m}$ are boundary arcs. The aim of this subsection is to describe the AR-quiver of the string algebra $J(Q, W)$ in terms of objects in $\mathscr{C}_{(S,M)}$, which we identified with noncontractible curves and noncontractible closed curves in (S, M). A curve γ in (S, M) defines a string $w(\gamma)$ in $J(Q, W)$ (which is empty if γ is contractible or homotopic to an arc in Γ), and consequently a string module $M(w(\gamma))$ in mod $J(Q, W)$ (which is zero if γ is contractible or homotopic to an arc in Γ). We use the notation $M(\gamma)$ for $M(w(\gamma))$ and from the discussion in Section 2 we know that the string module $M(\gamma)$ is the image of γ under the projection functor $\text{Ext}^1_{\mathscr{C}_\Gamma}(T_\Gamma, -): \mathscr{C}_{(S,M)} \rightarrow \text{mod } J(Q, W)$.

To define elementary moves on curves in (S, M) we use the fact that the orientation of S induces an orientation on each boundary component of S: For any curve

γ in (S, M) we denote by ${}_s\gamma$ the *pivot elementary move* of γ on its starting point, meaning that the curve ${}_s\gamma$ is obtained from γ by moving the starting point $s(\gamma)$ clockwise to the next marked point b on the same boundary (note that $b = s(\gamma)$ if there is only one marked point lying on the same boundary). Similarly, we denote by γ_e the pivot elementary move of γ on its ending point. (See figure.)
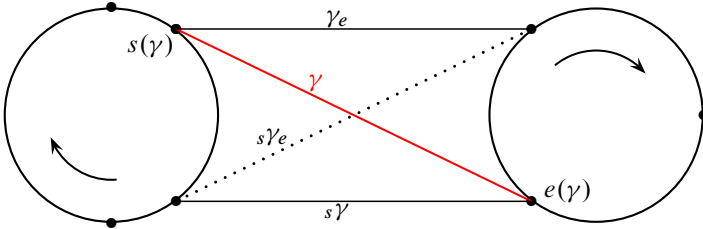
Iterated pivot elementary moves are denoted ${}_s\gamma_e = {}_s(\gamma_e) = ({}_s\gamma)_e$, ${}_{s^2}\gamma = {}_s({}_s\gamma)$ and $\gamma_{e^2} = (\gamma_e)_e$, respectively.

Remark 3.3. If $c = d^n$ is a noncontractible closed curve, where $n \geq 1$ and d is a simple closed curve, we view c as a closed curve in (S, M) with endpoints on the boundary as in Remark 2.5. By the definition of pivot elementary moves we obtain that ${}_sc_e$ is again a closed curve whose endpoints are moved to the next point on the boundary; thus ${}_sc_e = d^n = c$ as elements in $\Pi_1^*(S, M)$. See the picture for an example.
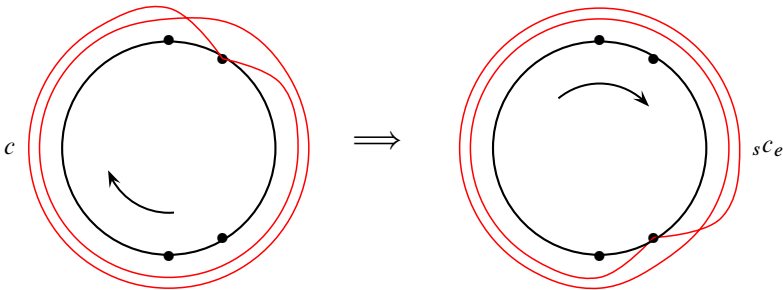
Lemma 3.4. *Let Γ be a triangulation of a marked surface (S, M), and let γ be a curve in (S, M) such that the string $w(\gamma)$ of $J(Q, W)$ is nonempty.*

(1) *If $w(\gamma)$ does not start on a peak, then $w({}_s\gamma)$ is obtained by adding a hook on $s(w(\gamma))$. Thus $w({}_s\gamma) = w(\gamma)_h$ and there is an irreducible morphism in $\mathrm{mod}\, J(Q, W) : M(\gamma) \longrightarrow M({}_s\gamma)$.*

(2) *If $w(\gamma)$ does not end on a peak, then $w(\gamma_e)$ is obtained by adding a hook on $e(w(\gamma))$. Thus $w(\gamma_e) = {}_hw(\gamma)$ and there is an irreducible morphism in $\mathrm{mod}\, J(Q, W) : M(\gamma) \longrightarrow M(\gamma_e)$.*

Proof. We only prove (1), since (2) is obtained dually.
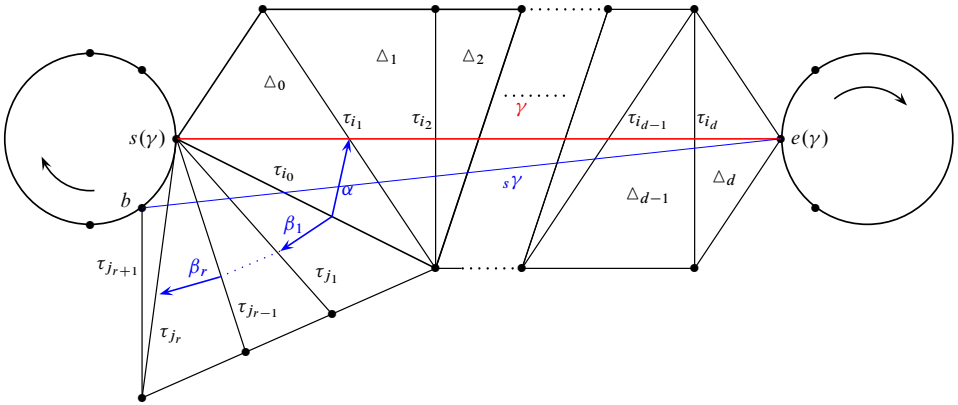
As shown in the figure above, we denote by $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ the internal arcs of Γ that intersect γ. Let b be the marked point lying clockwise next to $s(\gamma)$ on the same boundary component. Since γ does not start on a peak, there exits an arrow $\alpha : \tau_{i_0} \to \tau_{i_1}$ in Q such that $w(\gamma)\alpha$ is a string in $J(Q, W)$, for some internal arc $\tau_{i_0} \in \Gamma$. Let $\tau_{j_1}, \tau_{j_2}, \ldots, \tau_{j_r}$ be all internal arcs in Γ which intersect τ_{i_0} in the vertex $s(\gamma)$ and which are successors of τ_{i_0} with respect to clockwise orientation at the common vertex $s(\gamma)$. We denote by $\beta_1, \beta_2, \ldots, \beta_r$ the arrows of Q induced by the internal arcs $\tau_{i_0}, \tau_{j_1}, \ldots, \tau_{j_r}$. Then

$$w(_s\gamma) = w(\gamma)\alpha\beta_1^{-1}\beta_2^{-1} \cdots \beta_r^{-1} = w(\gamma)\alpha V_\alpha,$$

where

$$V_\alpha = \beta_1^{-1}\beta_2^{-1} \cdots \beta_r^{-1}.$$

This means that $w(_s\gamma)$ is obtained from $w(\gamma)$ by adding a hook on $s(w(\gamma))$. By Theorem 3.1 there is an irreducible morphism $M(\gamma) \to M(_s\gamma)$ in mod $J(Q, W)$. $\square$

Lemma 3.5. *Let Γ be a triangulation of a marked surface (S, M), and let γ be a curve in (S, M) such that the string $w(\gamma)$ of $J(Q, W)$ is nonempty.*

(1) *If $w(\gamma)$ starts on a peak, then $w(_s\gamma)$ is obtained by deleting a cohook on $s(w(\gamma))$. Thus $w(_s\gamma) = w(\gamma)_c$ and if $w(\gamma)_c$ is nonempty there is an irreducible morphism in mod $J(Q, W) : M(\gamma) \longrightarrow M(_s\gamma)$.*

(2) *If $w(\gamma)$ ends on a peak, then $w(\gamma_e)$ is obtained by deleting a cohook on $e(w(\gamma))$. Thus $w(\gamma_e) = {}_cw(\gamma)$ and if ${}_cw(\gamma)$ is nonempty there is an irreducible morphism in mod $J(Q, W) : M(\gamma) \longrightarrow M(\gamma_e)$.*

Proof. As before we only prove part (1) of the lemma.

We denote by b the marked point lying clockwise next to $s(\gamma)$ on the same boundary. We further denote by $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ the internal arcs that intersect γ
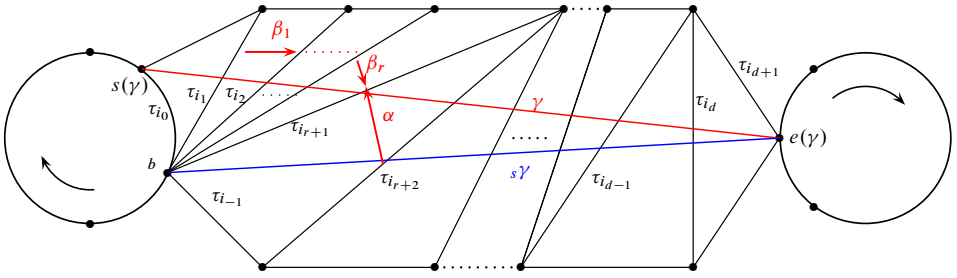
in the order indicated in the figure above. Since $w(\gamma)$ starts on a peak, there is an $r \geq 1$ such that the arcs $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_{r+1}}$ intersect in the vertex b and induce arrows $\beta_1, \beta_2, \ldots, \beta_r$ in Q_1 as shown in the figure. We choose r to be maximal and distinguish two cases:

(i) If $r + 1 = d$ then $w(\gamma)$ is a direct string and $w({}_s\gamma) = w(\gamma)_c$ is the empty string.

(ii) If $r + 1 < d$ then by maximality of r there is an arrow α in Q_1 from $\tau_{i_{r+2}}$ to $\tau_{i_{r+1}}$. Since γ starts on a deep, τ_{i_0} is a boundary arc which implies $U_\alpha^{-1} = \beta_r\beta_{r-1} \cdots \beta_1$. Moreover, we know $w(\gamma) = w_0\alpha^{-1}\beta_r\beta_{r-1}\cdots\beta_1 = w_0\alpha^{-1}U_\alpha^{-1}$. Thus $w({}_s\gamma) = w_0$ which means that $w({}_s\gamma)$ is obtained by deleting a cohook on $s(w(\gamma))$. If $w({}_s\gamma)$ is nonempty, we obtain by Theorem 3.1 that there is an irreducible morphism in mod $J(Q, W) : M(\gamma) \longrightarrow M({}_s\gamma)$.   □

Theorem 3.6. *Let Γ be a triangulation of a marked surface (S, M), and let γ be a curve in (S, M). Then each irreducible morphism in mod $J(Q, W)$ starting in $M(\gamma)$ is obtained by pivot elementary moves on endpoints of γ. Moreover, all AR-sequences between string modules in mod $J(Q, W)$ are of the form*

$$0 \longrightarrow M(\gamma) \longrightarrow M({}_s\gamma) \oplus M(\gamma_e) \longrightarrow M({}_s\gamma_e) \longrightarrow 0$$

for some curve γ in (S, M).

Proof. The irreducible morphisms and AR-sequences between string modules are described in Theorem 3.1. Among the four cases listed there, we only consider the second case here, the others being similar: Suppose $w(\gamma)$ does not start but ends on a peak. Then we get from Lemma 3.4 and Lemma 3.5 two irreducible morphisms, $M(\gamma) \to M({}_s\gamma)$ and $M(\gamma) \to M(\gamma_e)$. Moreover, the construction of ${}_s\gamma$ does not change any information of γ on $e(\gamma)$, that is $w({}_s\gamma)$ also ends on a peak if it is nonempty. By Lemma 3.5 again we get an irreducible morphism $M({}_s\gamma) \to M({}_s\gamma_e)$ in case $w({}_s\gamma_e)$ is nonempty, and similarly, there is an irreducible morphism $M(\gamma_e) \to M({}_s\gamma_e)$ induced by γ_e. This completes the proof since the number of summands in the middle term of an AR-sequence between string modules is at most two.   □

To complete the description of irreducible morphisms in mod $J(Q, W)$, we recall from [Butler and Ringel 1987] that each band in $J(Q, W)$ yields a k^*-family of homogeneous tubes in the AR-quiver of $J(Q, W)$, given by an embedding mod $k[t, t^{-1}] \to$ mod $J(Q, W)$.

Corollary 3.7. *The AR-translation in* mod $J(Q, W)$ *is given by simultaneous counterclockwise rotation of starting and ending point of a curve to the next marked points at the boundary, that is*

$$\tau^{-1}(M(\gamma)) = M(_s\gamma_e)$$

if $M(\gamma)$ is not an injective $J(Q, W)$-module.

Proof. For string modules this is shown in Theorem 3.6 above. For any band b in $J(Q, W)$ we have $b = _sb_e$ as elements in $\Pi_1(S, M)$. Moreover, the corresponding band modules $M(b, n, \phi)$ lie on homogeneous tubes and thus satisfy $\tau^{-1}(M(b, n, \phi)) = M(b, n, \phi)$. □

Example 3.8. We reconsider the curve γ from Example 2.6. From the diagram defining Γ it is easy to obtain the following descriptions:

$$M_\gamma = \begin{smallmatrix} & 1 & \\ 3 & 2 & 3 \\ & 4 & \quad 4 \end{smallmatrix} \qquad\qquad M(_s\gamma) = \begin{smallmatrix} & 1 & & 1 \\ 2 & 3 & 2 & 3 \\ & 4 & 4 & \quad 4 \end{smallmatrix}$$

$$M(\gamma_e) = \begin{smallmatrix} & 1 & \\ 3 & 2 & 3 & 2 \\ & 4 & \quad 4 \end{smallmatrix} \qquad\qquad M(_s\gamma_e) = \begin{smallmatrix} & 1 & & 1 \\ 2 & 3 & 2 & 3 & 2 \\ & 4 & 4 & \quad 4 \end{smallmatrix}$$

Hence, we have an AR-sequence:

$$0 \longrightarrow M(\gamma) \longrightarrow M(_s\gamma) \oplus M(\gamma_e) \longrightarrow M(_s\gamma_e) \longrightarrow 0.$$

Remark 3.9. If $M(\gamma)$ is an injective $J(Q, W)$-module, it follows from the description in Remark 3.2(2) that $_s\gamma_e$ is an internal arc in Γ which implies $M(_s\gamma_e) = 0$. Dually, if $\gamma \in \Gamma$ is an internal arc, $M(_s\gamma_e)$ is a projective module in mod $J(Q, W)$.

Irreducible morphisms in $\mathcal{C}_{(S,M)}$. Recall that $\mathcal{C} := \mathcal{C}_\Gamma$ is a triangulated Hom-finite k-category which is 2-Calabi–Yau. We know that $\mathcal{C}$ has a cluster-tilting object $T_\Gamma = \tau_1 \oplus \tau_2 \oplus \cdots \oplus \tau_n$, where $\tau_1, \dots \tau_n$ are the internal arcs in Γ. Denote by [1] the suspension functor of the triangulated category $\mathcal{C}$. Moreover, since $\mathcal{C}$ is 2-Calabi–Yau, $\mathcal{C}$ has AR-triangles and the AR-translation is given by $\tau = [1]$ (see [Reiten and Van den Bergh 2002]).

Curves in (S, M) which are noncontractible and not homotopic to a boundary arc give a parametrization of the isoclasses of string objects in $\mathcal{C}$, we refer to them as non-boundary curves. We identify contractible curves and boundary arcs with the zero object in $\mathcal{C}$. We further say that a curve γ is not in Γ and write $\gamma \notin \Gamma$ if

it is noncontractible and not homotopic to an arc in the triangulation Γ. Since the non-boundary arcs in Γ yield the cluster-tilting object T_Γ, the string module $M(\gamma)$ is nonzero if $\gamma \notin \Gamma$. The following lemma will be used frequently:

Lemma 3.10 [Keller and Reiten 2007; Koenig and Zhu 2008]. *Let γ be a curve in (S, M) such that $\gamma \notin \Gamma$.*

(1) *$M(\gamma)$ is projective in $\mod J(Q, W)$ if and only if $\gamma[1] \in \Gamma$.*

(2) *$M(\gamma)$ is injective in $\mod J(Q, W)$ if and only if $\gamma[-1] \in \Gamma$.*

From [Koenig and Zhu 2008], we know that any sink (or source) map in $\mathscr{C}$ is again a sink (or source) map in $\mod J(Q, W)$, therefore each AR-triangle in $\mathscr{C}$

$$\gamma' \longrightarrow \bigoplus \gamma_i \longrightarrow \gamma'' \longrightarrow \gamma'[1]$$

with $\gamma' \notin \Gamma$ and $\gamma'[-1] \notin \Gamma$ yields an AR-sequence in $\mod J(Q, W)$:

$$0 \longrightarrow M(\gamma') \longrightarrow \bigoplus M(\gamma_i) \longrightarrow M(\gamma'') \longrightarrow 0.$$

Moreover, all AR-sequences in $\mod J(Q, W)$ are obtained in this way and the AR-translation in $\mod J(Q, W)$ is induced by that in $\mathscr{C}$. Combining this with Corollary 3.7 and Remark 3.9, we can easily get this:

Proposition 3.11. *The AR-translation in $\mathscr{C}_{(S,M)}$ is given by simultaneous counterclockwise rotation of starting and ending point of a curve to the next marked points at the boundary. That is $\gamma[-1] = {}_s\gamma_e$ if γ is a string object and $(\lambda, b^n)[-1] = (\lambda, {}_sb_e^n)$ if (λ, b^n) is a band object in $\mathscr{C}_{(S,M)}$.*

Proof. Assume γ is a string object. If γ is a non-boundary curve in (S, M) such that $M(\gamma)$ is not injective, then $M(\gamma[-1]) = M(\tau^{-1}(\gamma)) = \tau^{-1}(M(\gamma)) = M({}_s\gamma_e)$, which implies $\gamma[-1] = {}_s\gamma_e$. If $M(\gamma)$ is injective, then Remark 3.9 shows that ${}_s\gamma_e \in \Gamma$ is an internal arc, and by Lemma 3.10 and the discussion in Section 2 we obtain $\gamma[-1] = {}_s\gamma_e$. Similarly, if $\gamma \in \Gamma$ is an internal arc, then $\gamma[-1] = {}_s\gamma_e$.

Assume (λ, b^n) is a band object, where $\lambda \in k^*$, $b^n \in \Pi_1^*(S, M)$ with $n \geq 1$ and b a noncontractible closed curve. Then $M((\lambda, b^n)[-1]) = M(\tau^{-1}(\lambda, b^n)) = \tau^{-1}(M((\lambda, b^n))) = \tau^{-1}(M(w(b), n, \phi_\lambda)) = M(w(b), n, \phi_\lambda) = M(w({}_sb_e), n, \phi_\lambda) = M((\lambda, {}_sb_e^n))$, where $\phi_\lambda \in \mathrm{Aut}(k^n)$ is induced by λ. Hence $(\lambda, b^n)[-1] = (\lambda, {}_sb_e^n)$ as a band object in $\mathscr{C}_{(S,M)}$. $\square$

Lemma 3.12. *Let $\gamma \notin \Gamma$ be a curve in (S, M) such that $M(\gamma_e)$ is a nonzero projective-injective module in $\mod J(Q, W)$. There is a source map in $\mod J(Q, W)$:*

$$M({}_s\gamma_e) \longrightarrow M({}_{s^2}\gamma_e).$$

Proof. Assume $M(\gamma_e)$ is a projective-injective module. Then $w(\gamma_e)$ must be an inverse (or a direct) string which both starts and ends on a peak and also both starts and ends in a deep. Without loss of generality, assume $w(\gamma_e) = \beta_1^{-1} \cdots \beta_r^{-1}\beta_{r+1}^{-1}$, where $r \geq 0$.
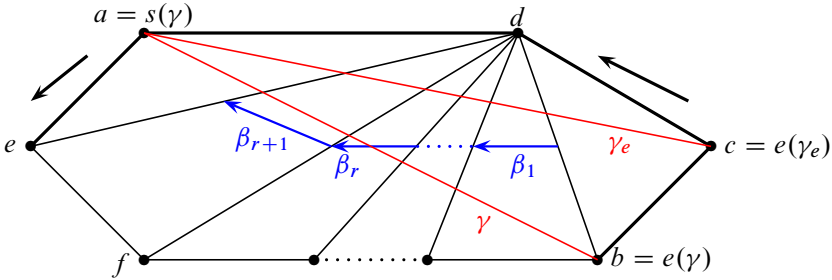
Let $a = s(\gamma) = s(\gamma_e)$, $b = e(\gamma)$, $c = e(\gamma_e)$, and let d, e, f be marked points as in the figure above. Since $w(\gamma_e)$ both starts and ends in a deep, bc and da are boundary arcs. Similarly, ae and dc are boundary arcs since γ_e both starts and ends on a peak, therefore b, c, d, a, e lie consecutively on the same boundary.

(1) If $M(_s\gamma_e)$ is an injective module, that is $w(_s\gamma_e) = \beta_1^{-1} \cdots \beta_r^{-1}$ both starts and ends on a peak, then ef is a boundary arc. Then the definition of pivot elementary move implies that

$$M(_{s^2}\gamma_e) = M(\beta_1^{-1} \cdots \beta_{r-1}^{-1}) = M(_s\gamma_e)/\operatorname{Soc}(M(_s\gamma_e))$$

and $_s\gamma_{e^2} = de \in \Gamma$ is an internal arc. Hence by Remark 3.2(2) there is a source map: $M(_s\gamma_e) \to M(_{s^2}\gamma_e)$.

(2) If $M(_s\gamma_e)$ is not an injective module, that is $w(_s\gamma_e) = \beta_1^{-1} \cdots \beta_r^{-1}$ does not start on a peak, then there exists $\alpha \in Q_1$ such that $w(_s\gamma_e)\alpha$ is a string. By Theorem 3.6, there is an AR-sequence:

$$0 \longrightarrow M(_s\gamma_e) \longrightarrow M(_{s^2}\gamma_e) \oplus M(_s\gamma_{e^2}) \longrightarrow M(_{s^2}\gamma_{e^2}) \longrightarrow 0,$$

where $w(_s\gamma_e) = \beta_1^{-1} \cdots \beta_r^{-1} = U_\alpha$, $w(_{s^2}\gamma_e) = w(_s\gamma_e)_h = N_\alpha$ by Lemma 3.4, and $_s\gamma_{e^2} = de \in \Gamma$ is an internal arc in Γ. Thus there is a source map: $M(_s\gamma_e) \to M(_{s^2}\gamma_e)$ in mod $J(Q, W)$. $\square$

Proposition 3.13. *Let γ be a non-boundary curve and*

$$0 \longrightarrow M(\gamma) \longrightarrow M(_s\gamma) \oplus M(\gamma_e) \longrightarrow M(_s\gamma_e) \longrightarrow 0 \qquad (3\text{-}1)$$

an AR-sequence in mod $J(Q, W)$ *with $M(_s\gamma) \neq 0 \neq M(\gamma_e)$. Then there is an AR-triangle with two middle terms in $\mathscr{C}$ as follows:*

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Proof. Since AR-sequences in mod $J(Q, W)$ are induced by AR-triangles in $\mathscr{C}$, we assume that

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \oplus \sigma \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1] \qquad (3\text{-}2)$$

is an AR-triangle in $\mathscr{C}$ with $M(\sigma) = 0$. It suffices to prove that $\sigma = 0$.

Otherwise, σ contains a direct summand ρ which is an internal arc in Γ. Then ${}_s\gamma[1] \in \Gamma$ or $\gamma_e[1] \in \Gamma$ since $M(\rho[1]) \neq 0$ in mod $J(Q, W)$ by Lemma 3.10 and since the number of summands in the middle term of an AR-sequence in $J(Q, W)$ is at most two. Similarly, we conclude ${}_s\gamma[-1] \in \Gamma$ or $\gamma_e[-1] \in \Gamma$. Hence there are four cases; we consider only two of them, since the others are similar.

(1) Assume ${}_s\gamma[1] \in \Gamma$ and $\gamma_e[-1] \in \Gamma$. Then Lemma 3.10 implies that $M({}_s\gamma)$ is projective and $M(\gamma_e)$ is injective. Hence $M(\gamma)$ must be a direct summand of rad $M({}_s\gamma)$ and $M({}_s\gamma_e)$ must be a direct summand of $M(\gamma_e)/\operatorname{Soc} M(\gamma_e)$. This yields a contradiction by comparing the dimensions in (3-1).

(2) Assume $\gamma_e[1] \in \Gamma$ and $\gamma_e[-1] \in \Gamma$. Then Lemma 3.10 implies that $M(\gamma_e)$ is a projective-injective module, and so by Lemma 3.12 there is a source map

$$M({}_s\gamma_e) \longrightarrow M({}_{s^2}\gamma_e) \qquad (3\text{-}3)$$

in mod $J(Q, W)$. But after shifting (3-2) by $[-1]$, there is an AR-sequence in mod $J(Q, W)$:

$$0 \longrightarrow M({}_s\gamma_e) \longrightarrow M({}_{s^2}\gamma_e) \oplus M({}_s\rho_e) \longrightarrow M({}_{s^2}\gamma_{e^2}) \longrightarrow 0.$$

with $M({}_s\rho_e) = M(\rho[-1]) \neq 0$ a projective module. This contradicts the fact that the source map (3-3) ends with just one indecomposable module. $\qquad \square$

Since neither injective modules nor projective modules occur in homogeneous tubes, the following corollary related to band objects can be obtained similarly.

Corollary 3.14. *Let* (λ, b^n) *be a band object in* $\mathscr{C}$, *where* $\lambda \in k^*$, $b^n \in \Pi_1^*(S, M)$ *with* $n \geq 1$ *and* b *itself is not a proper power of an element in* $\Pi_1^*(S, M)$. *Then there is an AR-triangle in* $\mathscr{C}$:

$$(\lambda, b^n) \longrightarrow (\lambda, b^{n+1}) \oplus (\lambda, b^{n-1}) \longrightarrow (\lambda, b^n) \longrightarrow (\lambda, b^n)[1],$$

where (λ, b^0) *is the zero object in* $\mathscr{C}$.

Therefore, the band objects are closed under irreducible morphisms and the corresponding AR-components in $\mathscr{C}$ are homogeneous tubes. It remains to consider the string objects in $\mathscr{C}$.

Lemma 3.15. *If* γ *is an internal arc in* Γ, *the AR-triangle in* $\mathscr{C}$ *starting in* γ *is of the form*

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Proof. Suppose γ is an internal arc in Γ. Then $M({}_s\gamma_e)$ is projective in mod $J(Q,W)$ by Lemma 3.10. We consider $\gamma[-1] = {}_s\gamma_e$.

(1) If $M({}_s\gamma_e)$ is not injective, then Theorem 3.6 induces an AR-sequence in mod $J(Q,W)$:

$$0 \longrightarrow M({}_s\gamma_e) \longrightarrow M({}_{s^2}\gamma_e) \oplus M({}_s\gamma_{e^2}) \longrightarrow M({}_{s^2}\gamma_{e^2}) \longrightarrow 0. \qquad (3\text{-}4)$$

The definition of a cluster-tilting object guarantees that ${}_{s^2}\gamma_e$ and ${}_s\gamma_{e^2}$ cannot be internal arcs in Γ. If neither ${}_{s^2}\gamma_e$ nor ${}_s\gamma_{e^2}$ is boundary arc (thus $M({}_{s^2}\gamma_e) \neq 0 \neq M({}_s\gamma_{e^2})$), then Proposition 3.13 induces an AR-triangle

$$ {}_s\gamma_e \longrightarrow {}_{s^2}\gamma_e \oplus {}_s\gamma_{e^2} \longrightarrow {}_{s^2}\gamma_{e^2} \longrightarrow {}_s\gamma_e[1]$$

which yields the AR-triangle

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Assume now that one of ${}_{s^2}\gamma_e$ and ${}_s\gamma_{e^2}$ is a boundary arc. Without loss of generality, let ${}_{s^2}\gamma_e$ be a boundary arc, then by definition of a cluster-tilting object, $\mathrm{Hom}_{\mathscr{C}}({}_s\gamma_e, \sigma) = 0$ for any $\sigma \in \Gamma$, hence (3-4) induces an AR-triangle of the form

$$ {}_s\gamma_e \longrightarrow {}_{s^2}\gamma_e \oplus {}_s\gamma_{e^2} \longrightarrow {}_{s^2}\gamma_{e^2} \longrightarrow {}_s\gamma_e[1]$$

with ${}_{s^2}\gamma_e = 0$. The definition of a pivot elementary move implies that γ_e is also a boundary arc, hence there is an AR-triangle of the form

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1]$$

where $\gamma_e = 0$ is a boundary arc in $\mathscr{C}$.

(2) If $M({}_s\gamma_e) = M(\gamma[-1])$ is injective, then $M({}_s\gamma_e)$ is a projective-injective module by Lemma 3.10. By definition of a cluster-tilting object and Remark 3.2(3) we can assume that the AR-triangle starting in γ is of the form

$$\gamma \longrightarrow \delta_1 \oplus \delta_2 \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1]$$

where neither δ_1 nor δ_2 are internal arcs in Γ. Since $M({}_s\gamma_e)$ is a projective-injective module in mod $J(Q,W)$, one of δ_1 and δ_2 must be a boundary arc. Without loss of generality, assume δ_2 is a boundary arc; then Lemma 3.12 implies

$$M(\delta_1) = \mathrm{rad}\, M({}_s\gamma_e) = M({}_s\gamma);$$

that is, $\delta_1 = {}_s\gamma$ and $\delta_2 = \gamma_e$ is a boundary arc. Therefore the AR-triangle is of the form

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1]$$

where $\gamma_e = 0$ is a boundary arc.   □

As shown in Remark 3.2(3), the middle terms of AR-sequences in mod $J(Q, W)$ contain at most two indecomposable summands. The following proposition establishes the same result for AR-triangles in $\mathscr{C}$.

Proposition 3.16. *The number of indecomposable summands in the middle term of an AR-triangle in $\mathscr{C}_{(S,M)}$ is at most two.*

Proof. We only need to consider string objects in $\mathscr{C}$. Let therefore γ be a non-boundary curve, and let

$$\gamma \longrightarrow \bigoplus_{i=1}^{r} \delta_i \oplus \bigoplus_{j=1}^{s} \tau_j \longrightarrow \gamma[-1] \longrightarrow \gamma[1] \qquad (3\text{-}5)$$

be an AR-triangle in $\mathscr{C}$ starting in γ with $\delta_i \notin \Gamma$ and where τ_j are internal arcs in Γ. Remark 3.2(3) implies $r \le 2$ and $s \le 2$. If $\gamma \in \Gamma$ or $\gamma[-1] \in \Gamma$, then by Lemma 3.15, there is nothing to prove. We suppose now $\gamma \notin \Gamma$ and $\gamma[-1] \notin \Gamma$. If $r = 2$, Proposition 3.13 implies $s = 0$. If $r = 1$ and $s = 2$, then (3-5) induces an AR-sequence with one middle term in mod $J(Q, W)$:

$$0 \longrightarrow M(\gamma) \longrightarrow M(\delta_1) \longrightarrow M(\gamma[-1]) \longrightarrow 0,$$

hence there exists $\alpha \in Q_1$ such that $w(\delta_1) = N_\alpha$ by Remark 3.2(1). On the other hand, if we shift (3-5) by [1] and [−1], then Remark 3.2(3) implies that $\delta_1[1]$ and $\delta_1[-1]$ must be the internal arcs in Γ, since

$$M(\tau_1[1]) \neq 0 \neq M(\tau_2[1]) \quad \text{and} \quad M(\tau_1[-1]) \neq 0 \neq M(\tau_2[-1])$$

in mod $J(Q, W)$ by Lemma 3.10. Thus $M(\delta_1) = M(N_\alpha)$ is a projective-injective module in mod $J(Q, W)$. The definition of N_α implies that the quiver Q is of type A_2. But this is impossible, since the cluster category of type A_2 is well-known, and there is no AR-triangle with more than two summands in the middle term. $\square$

Now we consider the AR-sequences with just one middle term in mod $J(Q, W)$.

Lemma 3.17. *Let γ be a non-boundary curve in (S, M) such that $w(\gamma) = U_\alpha$, for some $\alpha \in Q$. Then $w(_s\gamma) = N_\alpha$ and $w(\gamma_e) = 0$ is a zero string, and the AR-triangle in $\mathscr{C}$ starting in γ is of the form*

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Proof. Since $w(\gamma) = U_\alpha$ does not start but ends on a peak, we know by Lemma 3.4 that $w(_s\gamma) = w(\gamma)_h = N_\alpha$ and by Lemma 3.5 that

$$w(\gamma_e) = {}_cw(\gamma) = {}_cU_\alpha = 0$$

is a zero string. Hence γ_e might be an internal arc in Γ or a boundary arc in (S, M),

as in the figure:
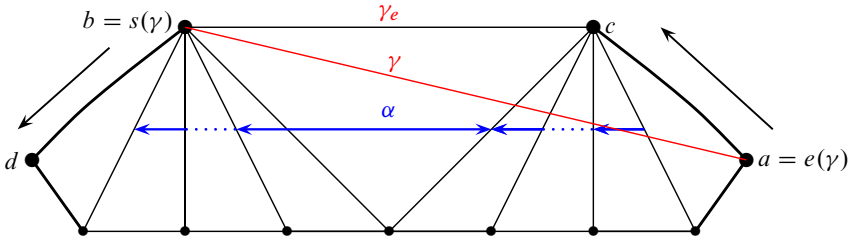
Let $b = s(\gamma)$ and $a = e(\gamma)$ be the endpoints of γ and let d be the marked point lying clockwise next to b on the same boundary. Since U_α ends on a peak, ac is a boundary arc. The definition of pivot elementary move implies $c = e(\gamma_e)$, hence there is an AR-sequence with one middle term in mod $J(Q, W)$ (see Remark 3.2(1)):

$$0 \longrightarrow M(\gamma) \longrightarrow M(_s\gamma) \longrightarrow M(_s\gamma_e) \longrightarrow 0. \qquad (3\text{-}6)$$

(1) If $\gamma_e \in \Gamma$, Lemma 3.10 implies that $M(\gamma_e[1])$ is an injective module. Remark 3.2(2) induces an irreducible morphism $M(\gamma_e[1]) \to M(\gamma)$ in mod $J(Q, W)$; hence there are two irreducible morphisms in $\mathscr{C}$:

$$\gamma_e[1] \to \gamma \quad \text{and} \quad \gamma \to \gamma_e[1][-1] = \gamma_e.$$

Combining this with (3-6) and Proposition 3.16, we get the AR-triangle in $\mathscr{C}$:

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

(2) If γ_e is a boundary arc, then $w(\gamma)$ starts in a deep and ends on a peak which implies $M(\gamma) = M(U_\alpha)$ is projective in mod $J(Q, W)$; thus $\gamma[1] \in \Gamma$ by Lemma 3.10. By (3-6) and the definition of a cluster-tilting object, the AR-triangle starting in γ is of the form

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1]$$

where $\gamma_e = 0$ in $\mathscr{C}$. $\qquad\qquad\square$

By the following theorem we finally obtain the results formulated in Theorem 1.3 in the introduction:

Theorem 3.18. *Let (S, M) be a marked surface without punctures, and let γ be a non-boundary curve in (S, M). Then each irreducible morphism in $\mathscr{C}_{(S,M)}$ starting in γ is obtained by pivot elementary moves on endpoints of γ. The AR-triangle starting in γ is of the form*

$$\gamma \longrightarrow {}_s\gamma \oplus \gamma_e \longrightarrow {}_s\gamma_e \longrightarrow \gamma[1].$$

Proof. Lemma 3.15 implies the case when γ or $\gamma[-1] = {}_s\gamma_e$ is an internal arc in Γ. Proposition 3.13 and Lemma 3.17 yield the remaining case.   □

AR-components in $\mathscr{C}_{(S,M)}$. The aim of this subsection is to describe the AR-components in $\mathscr{C}_{(S,M)}$. From Corollary 3.14 we know that all band objects of $\mathscr{C}_{(S,M)}$ lie in homogeneous tubes, so we focus on string objects from now on. It follows from Theorem 3.18 that each string object γ in $\mathscr{C}_{(S,M)}$ is starting point of a mesh with two middle terms in the AR-quiver of $\mathscr{C}_{(S,M)}$ except when one of ${}_s\gamma$ or γ_e is a boundary arc. The situation where one of ${}_s\gamma$ or γ_e is a boundary arc is explicitly described in the following corollary.

Corollary 3.19. *Let C be a boundary component of S with t marked points. If we choose a numbering and an orientation of the boundary arcs $\delta^1, \ldots, \delta^t$ such that all δ^i are clockwise oriented and $e(\delta^i) = s(\delta^{i+1})$ for all i, then the objects $\delta^i_{e^j}$ with $i = 1, \ldots, t$ and $j \geq 1$ form a tube of rank t in $\mathscr{C}_{(S,M)}$. Moreover, there is a bijection between the boundary components of S and the tubes in $\mathscr{C}_{(S,M)}$ which are not formed by band objects.*

Proof. The proof follows easily from the description of the AR-triangles given in Theorem 3.18 once the orientation of the boundary arcs is chosen. We illustrate the situation where $t = 3$:
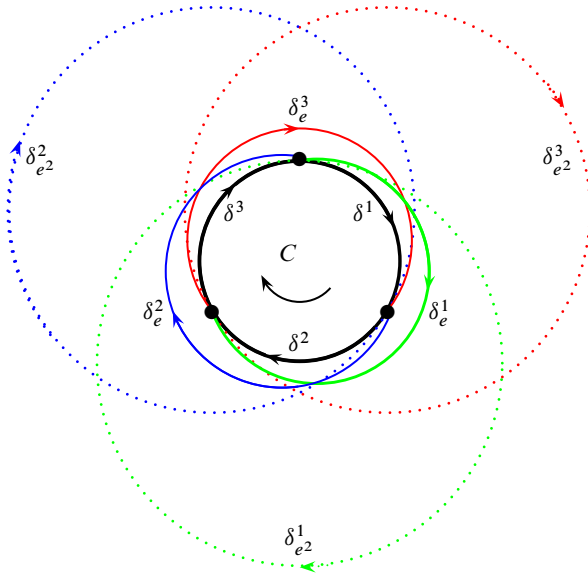
Note that $\delta^i_{e^{t-1}}$ is a noncontractible closed curve for all surfaces except a disc, where all closed curves are contractible. Thus, for a disc the objects $\delta^i_{e^{t-1}}$ are zero, and the remaining objects $\delta^i_{e^j}$ with $j > t - 1$ have to be identified accordingly, see the description in [Caldero et al. 2006].   □

The preceding corollary yields a description of all AR-components (formed by string objects) which contain some meshes with only one middle term. The remaining AR-components in $\mathscr{C}_{(S,M)}$ are formed by meshes with exactly two middle terms. If Λ is one such component, one can choose one object γ in Λ and then the component is formed by all ${}_{s^i}\gamma_{e^j}$ with $i, j \geq 1$. In case there are no identifications of the objects, one obtains thus components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$. If S is neither a disc nor an annulus, then the Jacobian algebra associated to a triangulation will in general be of nonpolynomial growth (see [Assem et al. 2010]), and there are plenty of components of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ in the AR-quiver of $\mathscr{C}_{(S,M)}$. If S is an annulus, the cluster category $\mathscr{C}_{(S,M)}$ is of type $\widetilde{\mathbb{A}}$ and there is no component of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$.

4. Flips and mutations

In the previous sections we fixed one triangulation Γ of (S, M) and studied the irreducible morphisms and AR-components of the cluster category $\mathscr{C}_\Gamma$ defined by the quiver with potential (Q_Γ, W_Γ). The aim of this section is to study the effect of a change in the triangulation on the cluster category $\mathscr{C}_\Gamma$.

If τ_i is an internal arc in Γ, then there exists exactly one internal arc $\tau_i' \neq \tau_i$ in (S, M) such that $f_{\tau_i}(\Gamma) := (\Gamma \backslash \{\tau_i\}) \cup \{\tau_i'\}$ is also a triangulation of (S, M). In fact, the internal arc τ_i is a diagonal in the quadrilateral formed by the two triangles of Γ containing τ_i, and τ_i' is the other diagonal in that quadrilateral, see [Fomin et al. 2008]. We denote τ_i' by $f_\Gamma(\tau_i)$ and say that $f_{\tau_i}(\Gamma)$ is obtained from Γ by applying a flip along τ_i. In fact, by applying iterated flips one can obtain all triangulations of (S, M):

Theorem 4.1 [Fomin et al. 2008]. *For any two triangulations of (S, M) there is a sequence of flips which transforms one triangulation into the other.*

As shown in Theorem 1.1, we can view the non-boundary curves in (S, M) as objects in $\mathscr{C}_\Gamma$. If we denote all internal arcs of Γ by $\tau_1, \ldots, \tau_n$, their direct sum $T_\Gamma = \tau_1 \oplus \tau_2 \oplus \cdots \oplus \tau_n$ is a cluster-tilting object in $\mathscr{C}_\Gamma$. The following theorem is adapted from its source to our setup.

Theorem 4.2 [Iyama and Yoshino 2008]. *If $\tau_i \in \Gamma$ is an internal arc, there is a curve τ_i^* in (S, M) (unique up to homotopy) which is not homotopic to τ_i such that the object $\mu_{\tau_i}(T_\Gamma)$ obtained from T_Γ by replacing τ_i with τ_i^* is also a cluster-tilting object in $\mathscr{C}_\Gamma$.*

The object $\mu_{\tau_i}(T_\Gamma)$ is called the *mutation* of T_Γ in τ_i, and (τ_i, τ_i^*) is called an *exchange pair* in $\mathscr{C}_\Gamma$. As shown in [Iyama and Yoshino 2008], any exchange pair (τ_i, τ_i^*) induces the following nonsplit triangles (unique up to isomorphism) which are referred to as *exchange triangles*:

$$\tau_i \xrightarrow{f} \tau \longrightarrow \tau_i^* \longrightarrow \tau_i[1] \quad \text{and} \quad \tau_i^* \longrightarrow \tau \xrightarrow{g} \tau_i \longrightarrow \tau_i^*[1]$$
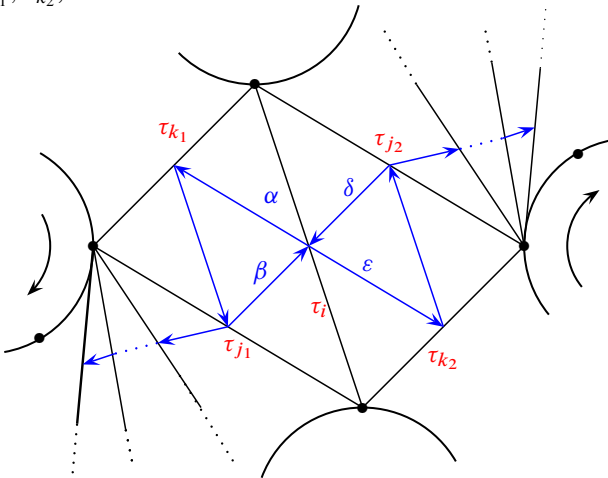
Here f is a minimal left $\mathrm{add}(T_\Gamma \setminus \{\tau_i\})$-approximation and g is a minimal right $\mathrm{add}(T_\Gamma \setminus \{\tau_i\})$-approximation. Since

$$\mathrm{End}(T_\Gamma, T_\Gamma) \cong \mathrm{End}(T_\Gamma[1], T_\Gamma[1]) \cong J(Q_\Gamma, W_\Gamma),$$

we know that the quiver of the endomorphism algebra of T_Γ in $\mathscr{C}_\Gamma$ does not contain loops at any vertex. Hence we obtain the following lemma which is a special case of Lemma 7.5 in [Keller 2010].

Lemma 4.3. *If τ_i is an internal arc of Γ, the exchange triangles are given by*

$$\tau_i \xrightarrow{f} \bigoplus_{\tau_j \to \tau_i} \tau_j \longrightarrow \tau_i^* \longrightarrow \tau_i[1] \quad and \quad \tau_i^* \longrightarrow \bigoplus_{\tau_i \to \tau_k} \tau_k \xrightarrow{g} \tau_i \longrightarrow \tau_i^*[1],$$

where f is a minimal left $\mathrm{add}(T_\Gamma \setminus \tau_i)$-approximation and g is a minimal right $\mathrm{add}(T_\Gamma \setminus \tau_i)$-approximation.

For an internal arc $\tau_i \in \Gamma$, we discussed the definition of the flip $f_\Gamma(\tau_i)$ above. On the other hand, if we view τ_i as an indecomposable rigid object in $\mathscr{C}_\Gamma$, also the mutation $\mu_\Gamma(\tau_i) := \tau_i^*$ of τ_i is defined. The following theorem shows that flip and mutation of an internal arc are compatible (viewed as objects in the cluster category $\mathscr{C}_\Gamma$):

Theorem 4.4. *If Γ is a triangulation of (S, M) and $\tau_i \in \Gamma$ is an internal arc, then*

$$\mu_\Gamma(\tau_i) = f_\Gamma(\tau_i).$$

Proof. The internal arc τ_i is a diagonal of a quadrilateral formed by the internal arcs $\tau_{j_1}, \tau_{j_2}, \tau_{k_1}, \tau_{k_2}$, as follows:

The triangles of Γ containing τ_i induce arrows $\alpha, \beta, \delta, \varepsilon$ in Q_Γ as indicated above. By Lemma 4.3 there is a nonsplit triangle in $\mathscr{C}_\Gamma$:

$$\tau_i \xrightarrow{f} \tau_{j_1} \oplus \tau_{j_2} \longrightarrow \tau_i^* \longrightarrow \tau_i[1]$$

where f is a minimal left add$(T_\Gamma \setminus \tau_i)$-approximation. We obtain the following right exact sequence in mod $J(Q_\Gamma, W_\Gamma)$ by applying $M(-) \cong \mathrm{Ext}^1_{\mathscr{C}_\Gamma}(T_\Gamma, -)$ to this triangle:

$$M(\tau_i[-1]) \longrightarrow M(\tau_{j_1}[-1]) \oplus M(\tau_{j_2}[-1]) \longrightarrow M(\tau_i^*[-1]) \longrightarrow 0.$$

This sequence is in fact a minimal projective resolution in mod $J(Q_\Gamma, W_\Gamma)$ whose projective modules can be described by strings as follows:

$$M(\tau_i[-1]) = M(V_\alpha{}^{-1}V_\varepsilon),$$
$$M(\tau_{j_1}[-1]) = M(V_\alpha{}^{-1}\beta V_\beta),$$
$$M(\tau_{j_2}[-1]) = M(V_\delta{}^{-1}\delta^{-1}V_\varepsilon).$$

Hence $M(\tau_i^*[-1]) \cong M(V_\delta^{-1}\delta^{-1}\beta V_\beta)$, which implies $w(_s(\tau_i^*)_e) = V_\delta^{-1}\delta^{-1}\beta V_\beta$ and thus $\mu_\Gamma(\tau_i) = \tau_i^* = f_\Gamma(\tau_i)$. □

For any curve γ in (S, M), the definition of the string $w(\gamma)$ depends on the Jacobian algebra $J(Q_\Gamma, W_\Gamma)$. In order to compare string modules in two Jacobian algebras arising from different triangulations of (S, M), we denote in the following by $w(\Gamma, \gamma)$ and $M(\Gamma, \gamma)$ the string and the string module in the Jacobian algebra $J(Q_\Gamma, W_\Gamma)$. Similarly, the band in $J(Q_\Gamma, W_\Gamma)$ given by a noncontractible closed curve b is denoted by $w(\Gamma, b)$. If $\gamma = \tau_i \in \Gamma$, we denote by $M(\Gamma, \tau_i)$ the associated simple decorated representation of (Q_Γ, W_Γ); see more details in [Derksen et al. 2008; Labardini-Fragoso 2009b]. It is shown in this latter reference that the flips of triangulations are compatible with the mutations of decorated representations.

On the other hand, let $\Gamma' = f_{\tau_i}(\Gamma)$ be the triangulation of (S, M) obtained by a flip along τ_i; then [Keller and Yang 2009] establishes an equivalence $\bar{\mu}_i : \mathscr{C}_\Gamma \to \mathscr{C}_{\Gamma'}$ for each $1 \leq i \leq n$. Viewing the indecomposable objects in $\mathscr{C}_\Gamma$ as noncontractible curves and noncontractible closed curves in (S, M), or as indecomposable decorated representation of (Q_Γ, W_Γ), then [Plamondon 2010] shows the compatibility between the equivalence $\bar{\mu}_i$ and the mutation of decorated representations of (Q_Γ, W_Γ). Therefore, we get following lemma.

Lemma 4.5. *Let $\Gamma' = f_{\tau_i}(\Gamma)$ be the triangulation of (S, M) obtained by a flip along τ_i and let $T_{\Gamma'}$ be the corresponding cluster-tilting object in $\mathscr{C}_{\Gamma'}$. If $\bar{\mu}_i : \mathscr{C}_\Gamma \longrightarrow \mathscr{C}_{\Gamma'}$ denotes the equivalence just discussed, then*

$$\mathrm{Ext}^1_{\mathscr{C}_{\Gamma'}}(T_{\Gamma'}, \bar{\mu}_i(\gamma)) = M(\Gamma', \gamma)$$

for any non-boundary curve γ in (S, M).

Proof. Let $\mathrm{Ind}_s\mathscr{C}_\Gamma$ be the set of all indecomposable string objects in $\mathscr{C}_\Gamma$ which is indexed by non-boundary curves in (S, M), and

$$D_\Gamma = \{M(\Gamma, \gamma)|\ \gamma \text{ curve in } (S, M)\}.$$

Then by Proposition 4.1 in [Plamondon 2010], we have the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ind}_s\,\mathscr{C}_\Gamma & \xrightarrow{\mathrm{Ext}^1_{\mathscr{C}_\Gamma}(T_\Gamma,\,-)} & D_\Gamma \\
\Big\downarrow{\mu_i} & & \Big\downarrow{\bar{\mu}_i} \\
\mathrm{Ind}_s\,\mathscr{C}_{\Gamma'} & \xrightarrow{\mathrm{Ext}^1_{\mathscr{C}_{\Gamma'}}(T_{\Gamma'},\,-)} & D_{\Gamma'}
\end{array}
$$

where μ_i is the mutation of decorated representations. Therefore, for each curve γ in (S, M) we have

$$
\mathrm{Ext}^1_{\mathscr{C}_{\Gamma'}}(T_{\Gamma'}, \bar{\mu}_i(\gamma)) = \mu_i(\mathrm{Ext}^1_{\mathscr{C}_\Gamma}(T_\Gamma, \gamma))
$$
$$
\cong \mu_i(M(\Gamma, \gamma)) \cong M(f_{\tau_i}(\Gamma), \gamma) = M(\Gamma', \gamma),
$$

where the last isomorphism is given by the main result in [Labardini-Fragoso 2009b]. This completes the proof. $\qquad\square$

Since $\bar{\mu}_i : \mathscr{C}_\Gamma \to \mathscr{C}_{\Gamma'}$ is an equivalence, the objects $\bar{\mu}_i(\gamma)$ in $\mathscr{C}_{\Gamma'}$ can also be described by curves in (S, M) and we denote $\bar{\mu}_i(\gamma)$ again by γ in $\mathscr{C}_{\Gamma'}$.

Corollary 4.6. *Each triangulation of (S, M) yields a cluster-tilting object in $\mathscr{C}_{(S,M)}$ and $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \gamma) = 0$ if γ is an internal arc in (S, M).*

Proof. Let Γ be the triangulation of (S, M) that we studied before, and Γ' be another triangulation. By Theorem 4.1, there exists a sequence of flips which transform Γ to Γ':

$$
\Gamma = \Gamma_0 \xrightarrow{f_{i_1}} \Gamma_1 \xrightarrow{f_{i_2}} \Gamma_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_n}} \Gamma_n = \Gamma',
$$

where Γ_j is a triangulation of (S, M) and f_{i_j} is a flip for $1 \le j \le n$. We know that Γ induces a cluster-tilting object T_Γ which is the direct sum of all internal arcs in Γ, and Theorem 4.4 implies that Γ_1 also induces a cluster-tilting object T_{Γ_1} given by all internal arcs in Γ_1. Lemma 4.5 allows us to keep doing this until $\Gamma_n = \Gamma'$, which completes the proof. $\qquad\square$

Let Γ be a triangulation of (S, M) and let γ and δ be two non-boundary curves in (S, M). From Theorem 2.3 we know that the morphism space $\mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ in $\mathscr{C} = \mathscr{C}_\Gamma$ can be decomposed as k-vector space as follows:

$$
\mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}\big(M(\Gamma, \gamma), M(\Gamma, \delta)\big) \oplus \big\{ f \in \mathrm{Hom}_{\mathscr{C}}(\gamma, \delta) \,|\, f \text{ factors through } T_\Gamma \big\}.
$$

Moreover, by Lemma 3.3 in [Palu 2008], the morphisms factoring through T_Γ can be replaced by the morphisms of two other related objects in $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$, so $\mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ is given by

$$
\mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \delta)) \oplus D\,\mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(\tau^{-1}M(\Gamma, \delta), \tau M(\Gamma, \gamma)).
$$

In [Crawley-Boevey 1989] a basis of the Hom-space between string modules is explicitly described using factor strings and substrings. Using Theorem 2.4, one can translate this into a description using curves and the triangulation Γ of (S, M); thus one could give a basis of $\mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ combinatorially in terms of Γ.

Another possibility is to choose a triangulation Γ' such that a given morphism in $\mathscr{C}$ factoring through T_Γ can be described by a morphism in

$$\mathrm{Hom}_{J(Q_{\Gamma'}, W_{\Gamma'})}\big(M(\Gamma', \gamma), M(\Gamma', \delta)\big).$$

This is illustrated in the following example.

Example 4.7. We consider the following two curves γ, δ from Example 2.6:
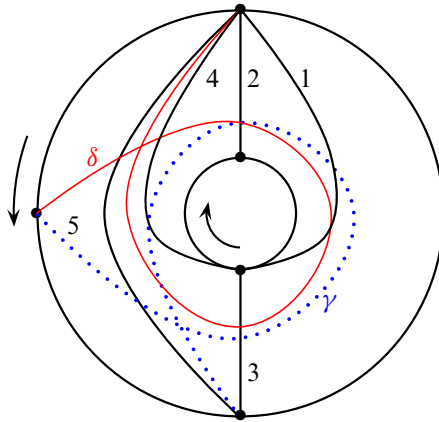
(the solid red line represents δ, and the dotted line γ). It is easy to see that

$$M(\Gamma, \delta) = \begin{matrix} & 1 & \\ 3 & & 2 \\ & 4 & \\ & 5 & \end{matrix} \quad \text{and} \quad M(\Gamma, \gamma) = \begin{matrix} 5 & & 1 \\ 3 & & 2 \\ & & 4 \end{matrix}$$

Hence $\mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \delta))$ is one-dimensional with a basis element f mapping the factor string 5 of $w(\Gamma, \gamma)$ to the substring 5 of $w(\Gamma, \delta)$. From Corollary 3.19 one obtains that γ and δ lie in a tube of rank 3 in $\mathscr{C}$ as follows:
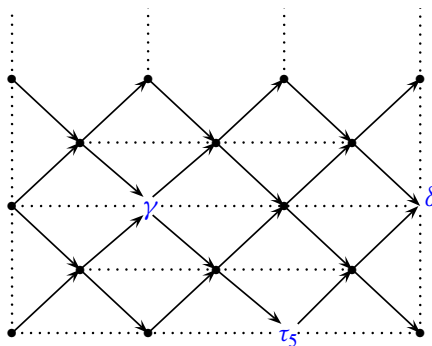
Thus there is, besides the morphism f induced from $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$, another nonzero morphism $g \in \mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ factoring through τ_5. Obviously, g cannot be described by morphisms in $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$ since $M(\Gamma, \tau_5) = 0$, but we can realize g in another module category: Applying a flip along τ_5 we obtain a new triangulation Γ' whose associated quiver with potential is $Q_{\Gamma'} = \tilde{A}_4$, $W_{\Gamma'} = 0$. Denote by $5^* = M(\Gamma', \tau_5)$ the simple $\tilde{A}_4$-module concentrated in 5; then

$$M(\Gamma', \delta) = \begin{smallmatrix} & 1 & \\ 3 & & 2 \\ 5^* & & 4 \end{smallmatrix} \qquad M(\Gamma', \gamma) = \begin{smallmatrix} & 1 & \\ 3 & 2 & 5^* \\ & 4 & \end{smallmatrix} \qquad M(\Gamma', \tau_5) = \quad 5^*$$

Hence $\mathrm{Hom}_{\tilde{A}_4}(M(\Gamma', \gamma), M(\Gamma', \delta))$ is also one-dimensional, given by a basis element g' mapping the factor string 5^* of $w(\Gamma', \gamma)$ to the substring 5^* of $w(\Gamma', \delta)$, which factors through $M(\Gamma', \tau_5)$. Therefore $g \in \mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ can be described by $g' \in \mathrm{Hom}_{\tilde{A}_4}(M(\Gamma', \gamma), M(\Gamma', \delta))$. Moreover, since $\tilde{A}_4$ is hereditary, we can conclude that

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(\gamma, \delta) &\cong \mathrm{Hom}_{\mathscr{C}(\tilde{A}_4)}(M(\Gamma', \gamma), M(\Gamma', \delta)) \\ &= \mathrm{Hom}_{\tilde{A}_4}(M(\Gamma', \gamma), M(\Gamma', \delta)) \oplus \mathrm{Ext}^1_{\tilde{A}_4}(M(\Gamma', \gamma), M(\Gamma', \gamma)) \end{aligned}$$

is two-dimensional. Hence

$$\mathrm{Hom}_{\mathscr{C}}(\gamma, \delta) \cong \mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \delta)) \oplus \mathrm{Hom}_{\tilde{A}_4}(M(\Gamma', \gamma), M(\Gamma', \delta))$$

as k-vector spaces.

Using Lemma 3.3 in [Palu 2008], we have

$$\tau M(\Gamma, \gamma) = M(\Gamma, \delta) = \begin{smallmatrix} & 1 & \\ 3 & & 2 \\ & 4 & \\ & 5 & \end{smallmatrix} \quad \text{and} \quad \tau^{-1} M(\Gamma, \delta) = M(\Gamma, \gamma) = \begin{smallmatrix} 5 & & 1 \\ 3 & & 2 \\ & 4 & \end{smallmatrix}$$

Hence, $\mathrm{Hom}_{\mathscr{C}}(\gamma, \delta)$ is given by

$$\mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \delta)) \oplus D\, \mathrm{Hom}_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \delta))$$

as k-vector space.

5. Extensions and intersections

Next we study the relation between extensions in the category $\mathscr{C} = \mathscr{C}_{(S,M)}$ and intersections of curves in (S, M). Given any curve in (S, M) with self-intersections, we explicitly construct one or two new curves, sometimes resolving the self-intersection, and sometimes increasing the winding number. These curves serve as middle term of certain nonsplit short exact sequences which allow to prove the following theorem:

Theorem 5.1. $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \gamma) = 0$ if and only if γ is an internal arc in (S, M).

Proof. Let γ be a curve in (S, M) and fix a triangulation Γ of (S, M). We denote by $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ the internal arcs of Γ that intersect γ, as indicated here:
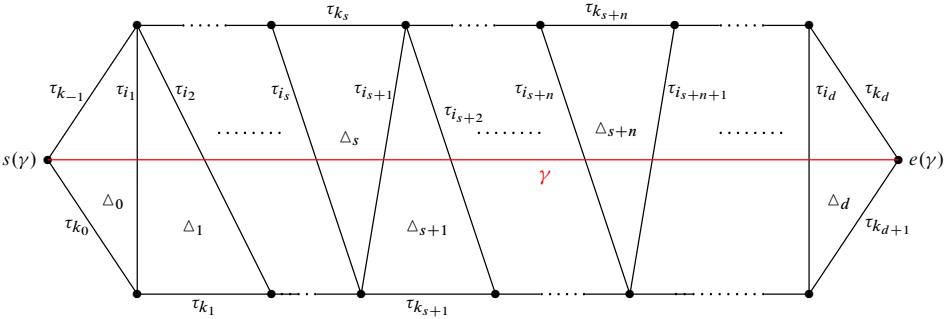
Along its way, the curve γ passes through (not necessarily distinct) triangles $\triangle_0, \triangle_1, \ldots, \triangle_d$. For $1 \leq l \leq d - 1$, the triangle $\triangle_l$ is formed by the internal arcs τ_{i_l} and $\tau_{i_{l+1}}$ and a third arc which is denoted by τ_{k_l}. In the first triangle $\triangle_0$, we denote the arcs clockwise by τ_{i_1}, τ_{k_0} and $\tau_{k_{-1}}$. Similarly, $\triangle_d$ is formed by τ_{i_d}, τ_{k_d} and $\tau_{k_{d+1}}$, as shown in the figure.

By [Corollary 4.6](#) we only need to prove one direction of the theorem, namely that $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \gamma) \neq 0$ in case the curve γ has self-intersections. To do so, it is enough to prove $\mathrm{Ext}^1_{J(Q_{\Gamma'}, W_{\Gamma'})}(M(\Gamma', \gamma), M(\Gamma', \gamma)) \neq 0$ for some triangulation Γ'. In some cases one can choose $\Gamma' = \Gamma$, but sometimes it will be necessary to change Γ to Γ' by a sequence of flips in order to realize the extension over some algebra $J(Q_{\Gamma'}, W_{\Gamma'})$.

We therefore assume that the curve γ has self-intersections. Thus there are $0 < r < r' < 1$ such that $\gamma(r) = \gamma(r')$. We choose r and r' in such a way that the restriction $b = \gamma|_{[r,r']}$ is a simple noncontractible closed curve. Hence the subword $w(\Gamma, \gamma|_{[r,r']})$ of $w(\Gamma, \gamma)$ defines a band $w(\Gamma, b)$ in $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$ as follows:
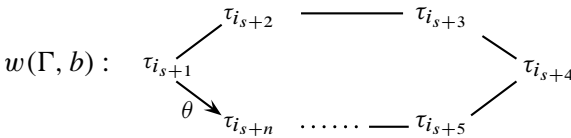
The difference between the string $w(\Gamma, \gamma|_{[r,r']})$ and the band $w(\Gamma, b)$ is the arrow between the endpoints. Up to duality, we may assume that this arrow θ (which is induced by the triangle $\triangle_{i_s}$), is oriented from $\tau_{i_{s+1}}$ towards $\tau_{i_{s+n}}$. We distinguish several cases of how the band $w(\Gamma, b)$ is embedded in the string $w(\Gamma, \gamma)$:

Case I: $w(\Gamma, \gamma)$ contains the band $w(\Gamma, b)$.

In this case, $w(\Gamma, \gamma)$ contains the string $w(\Gamma, \gamma|_{[r,r']})$ and the arrow θ and thus $m \geq 1$ consecutive copies of $w(\Gamma, b)$. We extend the subword of $w(\Gamma, \gamma)$ which is induced by b to maximal length, that is, we choose $j \leq s$ and $l \geq s+n$ such that all

arrows between $\tau_{i_{j+1}}$ and τ_{i_l} belong to $w(\Gamma, b)$, but the arrows $\tau_{i_j} \xrightarrow{\alpha} \tau_{i_{j+1}}$ induced by $\triangle_{i_j}$ and $\tau_{i_l} \xrightarrow{\beta} \tau_{i_{l+1}}$ induced by $\triangle_{i_l}$ are not induced by b.

Without loss of generality, assume α is oriented as $\tau_{i_j} \to \tau_{i_{j+1}}$. Then β has orientation $\tau_{i_l} \to \tau_{i_{l+1}}$ since γ intersects itself. Therefore the arrows $\alpha' : \tau_{i_{j+1}} \to \tau_{k_j}$ induced by $\triangle_{i_j}$ and the arrow $\beta' : \tau_{k_l} \to \tau_{i_l}$ induced by $\triangle_{i_l}$ belong to $w(\Gamma, b)$. We assume $\tau_{i_{j+t}} = \tau_{i_l}$, $\tau_{i_{j+t+1}} = \tau_{k_l}$ with $1 \le t \le n$. If $m = 1$, the situation is depicted as follows:
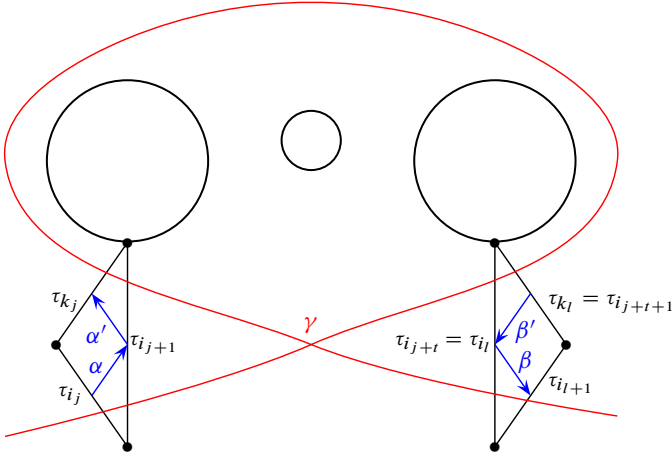
Here, the boundaries are indicated by the circles, and $\tau_{i_j} = \tau_{k_0}$ when $j = 0$ (also note that τ_{i_j} might be a boundary arc). We rewrite the figure from the preceding page accordingly:
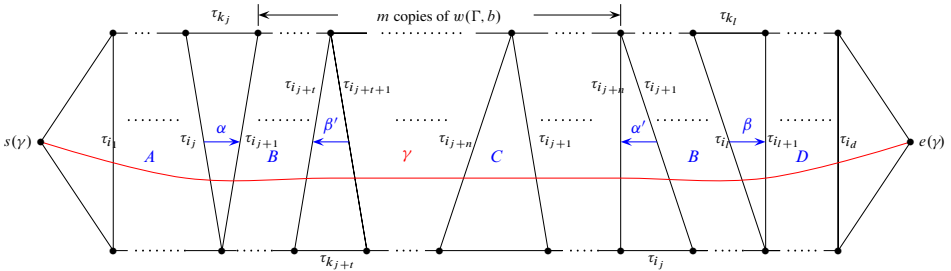
Thus $w(\Gamma, \gamma)$ is given by

$$w(\Gamma, \gamma) = A - \tau_{i_j} \xrightarrow{\alpha} \tau_{i_{j+1}} - B - \tau_{i_l} \xleftarrow{\beta'} \tau_{k_l} - C - \tau_{i_{j+n}} \xleftarrow{\alpha'} \tau_{i_{j+1}} - B - \tau_{i_l} \xrightarrow{\beta} \tau_{i_{l+1}} - D.$$

Consider the following two strings in mod $J(Q_\Gamma, Q_\Gamma)$:

$$w(\Gamma, \gamma') = A - \tau_{i_j} \xrightarrow{\alpha} \tau_{i_{j+1}} - B - \tau_{i_l} \xleftarrow{\beta'} \tau_{k_l} - C - \tau_{i_{j+n}} \xleftarrow{\alpha'} \tau_{i_{j+1}} - B - \tau_{i_l} \xleftarrow{\beta'} \tau_{k_l}$$
$$- C - \tau_{i_{j+n}} \xleftarrow{\alpha'} \tau_{i_{j+1}} - B - \tau_{i_l} \xrightarrow{\beta} \tau_{i_{l+1}} - D,$$

$$w(\Gamma, \gamma'') = A - \tau_{i_j} \xrightarrow{\alpha} \tau_{i_{j+1}} - B - \tau_{i_l} \xrightarrow{\beta} \tau_{i_{l+1}} - D,$$

where γ' and γ'' are the corresponding curves in (S, M). If $m = 1$, then γ', γ'' can be visualized as follows, where γ' is given by the full curve and γ'' is described by the dotted curve:
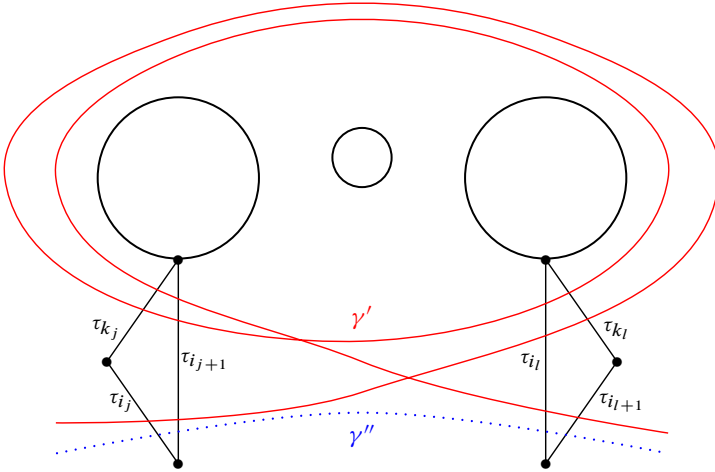

We easily construct a nonsplit exact sequence in mod $J(Q_\Gamma, W_\Gamma)$:

$$0 \longrightarrow M(\Gamma, \gamma) \xrightarrow{\begin{bmatrix} f \\ g \end{bmatrix}} M(\Gamma, \gamma') \oplus M(\Gamma, \gamma'') \longrightarrow M(\Gamma, \gamma) \longrightarrow 0,$$

where f identifies the factor string

$$A - \tau_{i_j} \xrightarrow{\alpha} \tau_{i_{j+1}} - B - \tau_{i_l} \xleftarrow{\beta'} \tau_{k_l} - C - \tau_{i_{j+n}} \xleftarrow{\alpha'} \tau_{i_{j+1}} - B - \tau_{i_l}$$

of $w(\Gamma, \gamma)$ with a substring of $w(\Gamma, \gamma')$ and g sends the factor string

$$\tau_{i_{j+1}} - B - \tau_{i_l} \xrightarrow{\beta} \tau_{i_{l+1}} - D$$

of $w(\Gamma, \gamma)$ to the same substring of $w(\Gamma, \gamma'')$.

Case II: $w(\Gamma, \gamma)$ does not contain the band $w(\Gamma, b)$.

In this case, $w(\Gamma, \gamma)$ does not contain θ but it contains $w(\Gamma, \gamma|_{[r,r']})$ as a subword. We distinguish a number of subcases:

(II.1) $\underline{0 = s < s + n = d}$. Here τ_{i_s} equals τ_{k_0} and the subword $w(\Gamma, \gamma|_{[r,r']})$ equals $w(\Gamma, \gamma)$ (see figure):
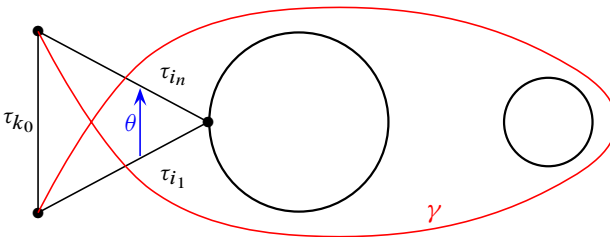
Since $w(\Gamma, b)$ is a band we know that there exists a string of the form

$$w(\Gamma, \gamma') = \tau_{i_1} \leftrightsquigarrow \tau_{i_n} \xleftarrow{\theta} \tau_{i_1} \leftrightsquigarrow \tau_{i_n}$$

in mod $J(Q_\Gamma, W_\Gamma)$, where γ' can be visualized as follows:
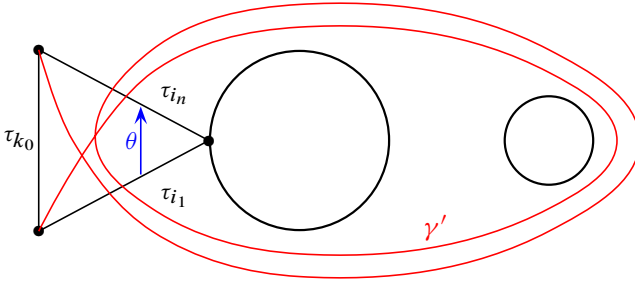
It is easy to see that there is a nonsplit exact sequence in mod $J(Q_\Gamma, W_\Gamma)$:

$$0 \longrightarrow M(\Gamma, \gamma) \xrightarrow{f} M(\Gamma, \gamma') \longrightarrow M(\Gamma, \gamma) \longrightarrow 0$$

where f is induced by the embedding of the string $w(\Gamma, \gamma)$ as a substring of $w(\Gamma, \gamma')$.

(II.2) $\underline{0 = s < s + n < d}$. Here $\tau_{i_s} = \tau_{i_0} = \tau_{k_0}$ and the subword $w(\Gamma, \gamma|_{[r,r']})$ is a proper subword of $w(\Gamma, \gamma)$ which ends before reaching the endpoint $e(\gamma)$ of γ:
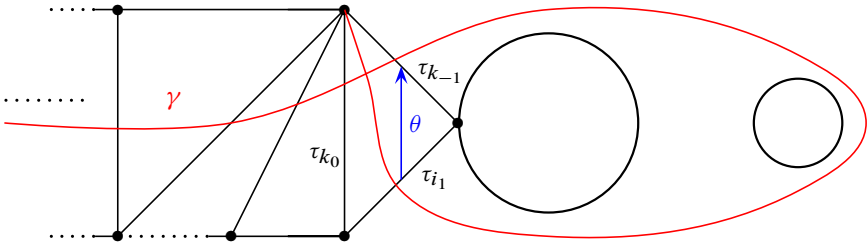
Since γ intersects itself, $\tau_{k_{-1}} = \tau_{i_n}$ and $\tau_{i_{n+1}} = \tau_{k_0}$. We rewrite the diagram from page 557 accordingly:
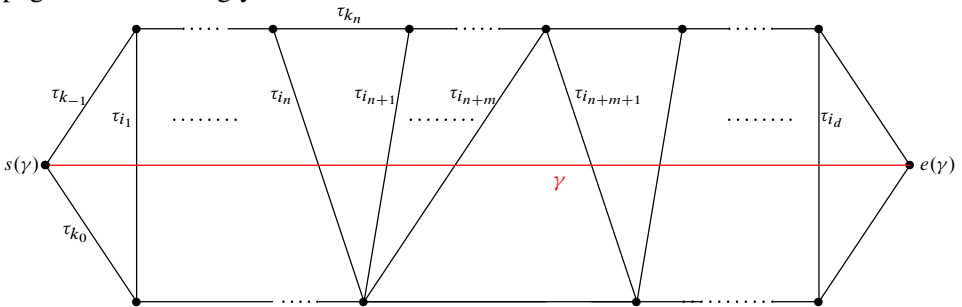
where $\tau_{i_{n+2}}, \tau_{i_{n+3}} \ldots \tau_{i_{n+m}}$ with $m \geq 1$ are all internal arcs lying clockwise before $\tau_{i_{n+1}}$. After applying flips along $\tau_{i_{n+1}}, \tau_{i_{n+2}} \ldots \tau_{i_{n+m}}$, we get a new triangulation

Γ' and a new band $w(\Gamma', b)$ in mod $J(Q_{\Gamma'}, W_{\Gamma'})$ related to same noncontractible closed curve b:

$$w(\Gamma', b): \quad \tau^*_{i_{n+m}} \quad \begin{array}{c} \tau^*_{i_{n+m-1}} \cdots\cdots \tau^*_{i_{n+1}} \\ \\ \tau_{i_n} \cdots\cdots\cdots \end{array} \quad \tau_{i_1}$$
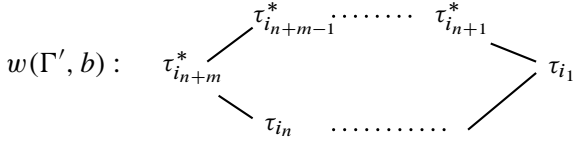
If $d = n+m$, the subword $w(\Gamma', b)$ is equal to

$$w(\Gamma', \gamma) = \tau^*_{i_{n+m}} - \tau^*_{i_{n+m-1}} \leftrightsquigarrow \tau_{i_n}$$

in mod $J(Q_{\Gamma'}, W_{\Gamma'})$, and this is the same case as in II.1 when mod $J(Q_{\Gamma'}, W_{\Gamma'})$ is considered. If $d > n+m$, then

$$w(\Gamma', \gamma) = \tau^*_{i_{n+m}} - \tau^*_{i_{n+m-1}} \leftrightsquigarrow \tau_{i_n} - \tau^*_{i_{n+m}} - \cdots ;$$

hence $w(\Gamma', \gamma)$ contains the band $w(\Gamma', b)$ as subword — the same case as in I.

(II.3) $\underline{0 < s < s+n < d}$. Here the subword $w(\Gamma, \gamma|_{[r,r']})$ is a proper subword of $w(\Gamma, \gamma)$ that starts after τ_{i_1} and ends before reaching the endpoint $e(\gamma)$ of γ. Since γ intersects itself, there must exist $1 \leq t \leq s$ such that $\tau_{i_{s+n+l}} = \tau_{i_{s+1-l}}$ for $1 \leq l \leq s-t+1$ and $\tau_{i_{s+n+t+1}} = \tau_{k_{t-1}}$. The situation is this:
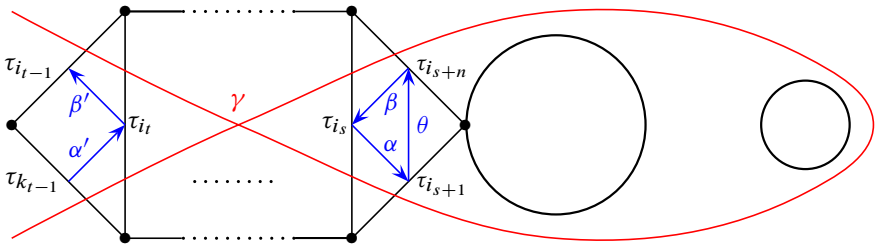
where α and β are arrows induced by the triangle $\triangle_{i_s}$, and α' and β' are induced by the triangle $\triangle_{i_{t-1}}$. We rewrite the diagram from page 557 accordingly:
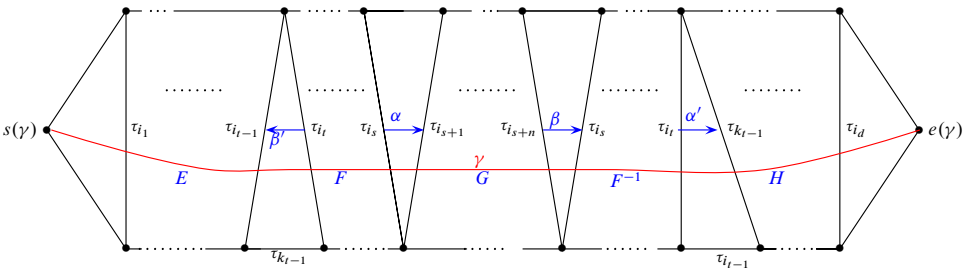
Hence $w(\Gamma, \gamma)$ can be given as

$$E - \tau_{i_{t-1}} \xleftarrow{\beta'} \tau_{i_t} - F - \tau_{i_s} \xrightarrow{\alpha} \tau_{i_{s+1}} - G - \tau_{i_{s+n}} \xrightarrow{\beta} \tau_{i_s} - F^{-1} - \tau_{i_t} \xleftarrow{\alpha'} \tau_{k_{t-1}} - H.$$

We consider the two curves in (S, M) marked γ' (solid line) and γ'' (dotted line) in the figure:
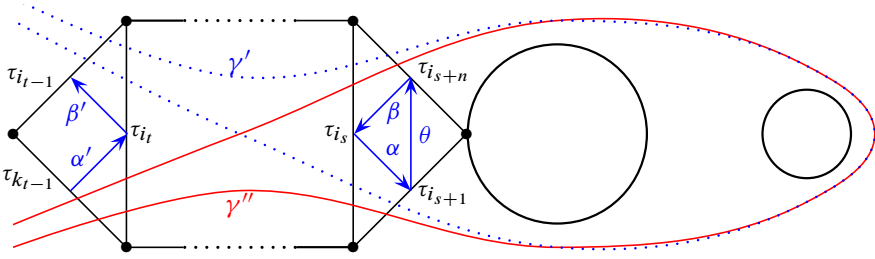
Then $w(\Gamma, \gamma')$ is given by

$$E - \tau_{i_{t-1}} \xleftarrow{\beta'} \tau_{i_t} - F - \tau_{i_s} \xrightarrow{\alpha} \tau_{i_{s+1}} - G - \tau_{i_{s+n}} \xrightarrow{\beta} \tau_{i_s} - F^{-1} - \tau_{i_t} \xrightarrow{\beta'} \tau_{i_{t-1}} - E^{-1}$$

and $w(\Gamma, \gamma'')$ by

$$H^{-1} - \tau_{k_{t-1}} \xrightarrow{\alpha'} \tau_{i_t} - F - \tau_{i_s} \xrightarrow{\alpha} \tau_{i_{s+1}} - G^{-1} - \tau_{i_{s+n}} \xrightarrow{\beta} \tau_{i_s} - F^{-1} - \tau_{i_t} \xleftarrow{\alpha'} \tau_{k_{t-1}} - H.$$

Hence there is a nonsplit exact sequence in $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$

$$0 \longrightarrow M(\Gamma, \gamma) \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} M(\Gamma, \gamma') \oplus M(\Gamma, \gamma'') \longrightarrow M(\Gamma, \gamma) \longrightarrow 0,$$

where f sends the factor string $w_0 = E - \tau_{i_{t-1}} \xleftarrow{\beta'} \tau_{i_t} - F - \tau_{i_s}$ of $w(\Gamma, \gamma)$ to w_0^{-1} as a substring of $w(\Gamma, \gamma')$ and g identifies the factor string

$$w_1 = \tau_{i_t} - F - \tau_{i_s} \xrightarrow{\alpha} \tau_{i_{s+1}} - G - \tau_{i_{s+n}} \xrightarrow{\beta} \tau_{i_s} - F^{-1} - \tau_{i_t} \xleftarrow{\alpha'} \tau_{k_{t-1}} - H$$

with w_1^{-1} as a substring of $w(\Gamma, \gamma'')$.

(II.4) $\underline{0 < s < s + n = d}$. Here the subword $w(\Gamma, \gamma|_{[r,r']})$ is a proper subword of $w(\Gamma, \gamma)$ which ends at the endpoint $e(\gamma)$ of γ. This subcase is dual to II.2. $\qquad\square$

Remark 5.2. $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \gamma) = 0$ implies $\mathrm{Ext}^1_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \gamma)) = 0$ for any triangulation Γ of (S, M). But $\mathrm{Ext}^1_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \gamma), M(\Gamma, \gamma)) = 0$ for one triangulation Γ of (S, M) does not imply $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \gamma) = 0$ in general. We reconsider δ in Example 4.7. It is easy to see that

$$M(\Gamma, \delta) = \begin{matrix} & 1 & \\ 3 & & 2 \\ & 4 & \\ & 5 & \end{matrix} \quad \text{and} \quad M(\Gamma, \delta[1]) = \begin{matrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{matrix}$$

Hence $\mathrm{Ext}^1_{J(Q_\Gamma, W_\Gamma)}(M(\Gamma, \delta), M(\Gamma, \delta)) = 0$ by the Auslander–Reiten formula in $\mathrm{mod}\, J(Q_\Gamma, W_\Gamma)$, since the nonzero morphism from $M(\Gamma, \delta)$ to $\tau(M(\Gamma, \delta)) = M(\Gamma, \delta[1])$ factors through the injective module I_5. But after applying a flip along 5, we get $\mathrm{Ext}^1_{\tilde{A}_4}(M(\Gamma', \delta), M(\Gamma', \delta)) \neq 0$, which implies that $\mathrm{Ext}^1_{\mathscr{C}}(\delta, \delta) \neq 0$.

Similarly as in Theorem 5.1, we can study the extensions of two different curves by their intersections in (S, M).

Proposition 5.3. *Let γ, δ be distinct curves in (S, M) and suppose $I(\gamma, \delta) \neq 0$. Then $\mathrm{Ext}^1_{\mathcal{C}}(\gamma, \delta) \neq 0 \neq \mathrm{Ext}^1_{\mathcal{C}}(\delta, \gamma)$.*

Proof. We only consider one of the intersections of γ and δ. After some flips (if needed), there is a triangulation Γ' such that $w(\Gamma', \gamma)$ and $w(\Gamma', \delta)$ share a common subword (associated to the intersection) $w = \tau_{j_1} \cdots \tau_{j_s}$ with $s \geq 1$, as shown:
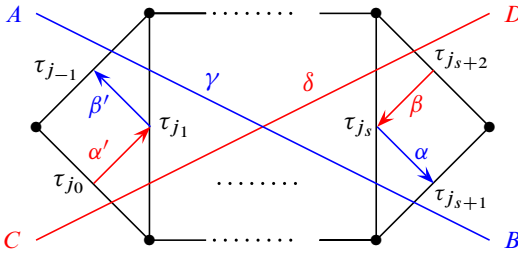
Here either τ_{j_0} or $\tau_{j_{-1}}$ might be a boundary arc, and one of $\tau_{j_{s+1}}$ and $\tau_{j_{s+2}}$ can also be a boundary arc. Hence $w(\Gamma', \gamma)$ and $w(\Gamma', \delta)$ are of the form

$$w(\Gamma', \gamma) = A - \tau_{j_{-1}} \xleftarrow{\beta'} \tau_{j_1} \leftrightsquigarrow \tau_{j_s} \xrightarrow{\alpha} \tau_{j_{s+1}} - B,$$

$$w(\Gamma', \delta) = C - \tau_{j_0} \xrightarrow{\alpha'} \tau_{j_1} \leftrightsquigarrow \tau_{j_s} \xleftarrow{\beta} \tau_{j_{s+2}} - D.$$

Consider the strings

$$(\Gamma', \gamma') = C - \tau_{j_0} \xrightarrow{\alpha'} \tau_{j_1} \leftrightsquigarrow \tau_{j_s} \xrightarrow{\alpha} \tau_{j_{s+1}} - B,$$

$$w(\Gamma', \gamma'') = A - \tau_{j_{-1}} \xleftarrow{\beta'} \tau_{j_1} \leftrightsquigarrow \tau_{j_s} \xleftarrow{\beta} \tau_{j_{s+2}} - D,$$

where γ' and γ'' are given in the picture:
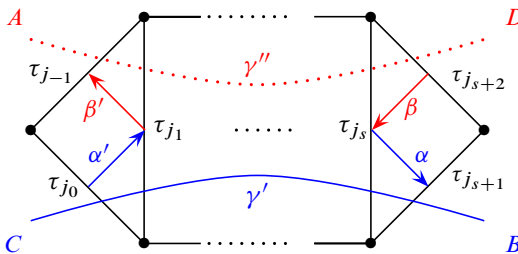
Then there is a nonsplit exact sequence in mod $J(Q_{\Gamma'}, W_{\Gamma'})$

$$0 \longrightarrow M(\Gamma', \gamma) \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} M(\Gamma', \gamma') \oplus M(\Gamma', \gamma'') \longrightarrow M(\Gamma', \delta) \longrightarrow 0,$$

where f sends the factor string

$$w_0 = \tau_{j_1} \leftrightsquigarrow \tau_{j_s} \xrightarrow{\alpha} \tau_{j_{s+1}} - B$$

of $w(\Gamma', \gamma)$ to w_0 as a substring of $w(\Gamma', \gamma')$ and g identifies the factor string $w_1 = A \!-\! \tau_{j-1} \overset{\beta'}{\longleftarrow} \tau_{j_1} \longleftrightarrow \tau_{j_s}$ as a substring of $w(\Gamma', \gamma'')$. Therefore

$$\mathrm{Ext}^1_{J(Q_{\Gamma'}, W_{\Gamma'})}(M(\Gamma', \delta), M(\Gamma', \gamma)) \neq 0,$$

which implies that $\mathrm{Ext}^1_{\mathscr{C}}(\delta, \gamma) \neq 0$. Then, since $\mathscr{C}$ is 2-Calabi–Yau, we obtain

$$\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \delta) \cong D\,\mathrm{Ext}^1_{\mathscr{C}}(\delta, \gamma) \neq 0. \qquad \square$$

Proposition 5.3 and Corollary 4.6 imply:

Corollary 5.4. *Let γ, δ be distinct internal arcs in (S, M). Then $\mathrm{Ext}^1_{\mathscr{C}}(\gamma, \delta) = 0$ if and only if $I(\gamma, \delta) = 0$.*

Corollary 5.5. *There is a bijection between triangulations of (S, M) and cluster-tilting objects of $\mathscr{C}_{(S,M)}$. In particular, each indecomposable object without self-extensions is reachable from the cluster-tilting object T_Γ (the initial cluster-tilting object).*

Proof. Corollary 4.6 implies that each triangulation of (S, M) yields a cluster-tilting object in $C_{(S,M)}$. Hence it suffices to prove that each cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$ gives a triangulation of (S, M). By Theorem 5.1, we can assume $T = \tau_1 \oplus \cdots \oplus \tau_n$ where each τ_i is an internal arc in (S, M) corresponding to T_i. The definition of a triangulation and the above corollary yield a unique triangulation $\Gamma_T = \{\tau_1, \cdots, \tau_n, \tau_{n+1}, \cdots, \tau_{n+m}\}$ where $\tau_{n+1}, \cdots, \tau_{n+m}$ are boundary arcs in (S, M). $\qquad \square$

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References

[Amiot 2009] C. Amiot, "Cluster categories for algebras of global dimension 2 and quivers with potential", *Ann. Inst. Fourier (Grenoble)* **59**:6 (2009), 2525–2590. MR 2011c:16026 Zbl 05673905

[Assem et al. 2008] I. Assem, T. Brüstle, and R. Schiffler, "Cluster-tilted algebras as trivial extensions", *Bull. Lond. Math. Soc.* **40**:1 (2008), 151–162. MR 2009c:16086

[Assem et al. 2010] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, and P.-G. Plamondon, "Gentle algebras arising from surface triangulations", *Algebra Number Theory* **4**:2 (2010), 201–229. MR 2011h:16045 Zbl 05704468

[Buan et al. 2006] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, "Tilting theory and cluster combinatorics", *Adv. Math.* **204**:2 (2006), 572–618. MR 2007f:16033 Zbl 1127.16011

[Buan et al. 2008] A. B. Buan, R. J. Marsh, and I. Reiten, "Cluster mutation via quiver representations", *Comment. Math. Helv.* **83**:1 (2008), 143–177. MR 2008k:16026 Zbl 1193.16016

[Butler and Ringel 1987] M. C. R. Butler and C. M. Ringel, "Auslander–Reiten sequences with few middle terms and applications to string algebras", *Comm. Algebra* **15**:1-2 (1987), 145–179. MR 88a:16055 Zbl 0612.16013

[Caldero and Chapoton 2006] P. Caldero and F. Chapoton, "Cluster algebras as Hall algebras of quiver representations", *Comment. Math. Helv.* **81**:3 (2006), 595–616. MR 2008b:16015 Zbl 1119.16013

[Caldero et al. 2006] P. Caldero, F. Chapoton, and R. Schiffler, "Quivers with relations arising from clusters (A_n case)", *Trans. Amer. Math. Soc.* **358**:3 (2006), 1347–1364. MR 2007a:16025 Zbl 1137.16020

[Crawley-Boevey 1989] W. W. Crawley-Boevey, "Maps between representations of zero-relation algebras", *J. Algebra* **126**:2 (1989), 259–263. MR 90k:16035 Zbl 0685.16018

[Derksen et al. 2008] H. Derksen, J. Weyman, and A. Zelevinsky, "Quivers with potentials and their representations. I. Mutations", *Selecta Math.* (*N.S.*) **14**:1 (2008), 59–119. MR 2010b:16021

[Fomin and Zelevinsky 2002] S. Fomin and A. Zelevinsky, "Cluster algebras. I. Foundations", *J. Amer. Math. Soc.* **15**:2 (2002), 497–529. MR 2003f:16050

[Fomin et al. 2008] S. Fomin, M. Shapiro, and D. Thurston, "Cluster algebras and triangulated surfaces. I. Cluster complexes", *Acta Math.* **201**:1 (2008), 83–146. MR 2010b:57032

[Geiss et al. 2005] C. Geiss, B. Leclerc, and J. Schröer, "Semicanonical bases and preprojective algebras", *Ann. Sci. École Norm. Sup.* (4) **38**:2 (2005), 193–253. MR 2007h:17018 Zbl 1131.17006

[Geiß et al. 2006] C. Geiß, B. Leclerc, and J. Schröer, "Rigid modules over preprojective algebras", *Invent. Math.* **165**:3 (2006), 589–632. MR 2007g:16023 Zbl 1167.16009

[Ginzburg 2006] V. Ginzburg, "Calabi–Yau algebras", preprint, 2006. arXiv math/0612139

[Iyama and Yoshino 2008] O. Iyama and Y. Yoshino, "Mutation in triangulated categories and rigid Cohen–Macaulay modules", *Invent. Math.* **172**:1 (2008), 117–168. MR 2008k:16028 Zbl 1140.18007

[Keller 2006] B. Keller, "On differential graded categories", pp. 151–190 in *International Congress of Mathematicians*, vol. 2, edited by M. Sanz-Solé et al., Eur. Math. Soc., Zürich, 2006. MR 2008g:18015 Zbl 1140.18008

[Keller 2010] B. Keller, "Cluster algebras, quiver representations and triangulated categories", pp. 76–160 in *Triangulated categories*, London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, Cambridge, 2010. MR 2011h:13033 Zbl 1215.16012

[Keller and Reiten 2007] B. Keller and I. Reiten, "Cluster-tilted algebras are Gorenstein and stably Calabi-Yau", *Adv. Math.* **211**:1 (2007), 123–151. MR 2008b:18018 Zbl 1128.18007

[Keller and Yang 2009] B. Keller and D. Yang, "Quiver mutation and derived equivalences", preprint, 2009. arXiv 0906.0761

[Koenig and Zhu 2008] S. Koenig and B. Zhu, "From triangulated categories to abelian categories: cluster tilting in a general framework", *Math. Z.* **258**:1 (2008), 143–160. MR 2008m:18021 Zbl 1133.18005

[Küçük 2005] A. Küçük, "The fundamental group of an oriented surface of genus n with k boundary surfaces", *Appl. Math. Comput.* **160**:1 (2005), 141–145. MR 2100181

[Labardini-Fragoso 2009a] D. Labardini-Fragoso, "Quivers with potentials associated to triangulated surfaces", *Proc. Lond. Math. Soc.* (3) **98**:3 (2009), 797–839. MR 2010b:16033 Zbl 05551833

[Labardini-Fragoso 2009b] D. Labardini-Fragoso, "Quivers with potentials associated to triangulated surfaces, II: arc representations", preprint, 2009. arXiv 0909.4100

[Palu 2008] Y. Palu, "Cluster characters for 2-Calabi-Yau triangulated categories", *Ann. Inst. Fourier* (*Grenoble*) **58**:6 (2008), 2221–2248. MR 2009k:18013 Zbl 1154.16008

[Plamondon 2010] P.-G. Plamondon, "Cluster algebras via cluster categories with infinite-dimensional morphism spaces", preprint, 2010. arXiv 1004.0830

[Reiten and Van den Bergh 2002] I. Reiten and M. Van den Bergh, "Noetherian hereditary abelian categories satisfying Serre duality", *J. Amer. Math. Soc.* **15**:2 (2002), 295–366. MR 2003a:18011 Zbl 0991.18009

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