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On the coefficients of some classes of starlike functions

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Abstract. Let Ω be the class of functions ω, $\omega(0) = 0$, $|\omega(z)| < 1$, holomorphic in the unit disc K; let a and b be arbitrary fixed numbers, $-1 \leqslant b < a \leqslant 1$, $\wp(a, b)$ — the class of functions P, $P(0) = 1$, holomorphic in K, such that $P \in \wp(a, b)$ iff $P(z) = (1+a\omega(z))(1+b\omega(z))^{-1}$ for some function $\omega \in \Omega$ and every z in K. Let $S^*(a, b)$ denote the class of functions $f, f(0) = 0, f'(0) = 1$, holomorphic in K, such that $f \in S^*(a, b)$ iff $zf'(z)(f(z))^{-1} = P(z)$ for some $P \in \wp(a, b)$; $\Sigma^*(a, b)$ — the class of meromorphic functions of the form $F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ and satisfying the condition: $zF'(z)(F(z))^{-1} = P(z)$, $P \in \wp(a, b)$.

The author obtains sharp estimates of the coefficients of functions of the families $S^*(a, b)$ and $\Sigma^*(a, b)$.

1. Introduction. Let Ω be the family of all functions of the form

$$\omega(z) = c_1 z + c_2 z^2 + \dots \tag{1.1}$$

which are holomorphic in the unit disc $K = \{z\colon |z| < 1\}$ and satisfy the condition $|\omega(z)| < 1$ for $z \in K$.

Janowski introduced (cf. [2]) a class $\wp(a, b)$ of holomorphic functions

$$P(z) = 1 + p_1 z + p_2 z^2 + \dots \tag{1.2}$$

in the unit disc; by definition, p is in $\wp(a, b)$ if and only if

$$P(z) = \frac{1+a\omega(z)}{1+b\omega(z)}, \quad z \in K, \tag{1.3}$$

for some function $\omega \in \Omega$; a, b are arbitrarily fixed numbers that satisfy the condition $-1 \leqslant b < a \leqslant 1$. It is easy to notice that $\wp(a, b) \subset \wp$ and $\wp(1, -1) \equiv \wp$, where $\wp$ is the well-known family of functions of Carathéodory type.

Let us denote by $S^*(a, b)$ (cf. [2]) the class of functions

$$f(z) = z + a_2 z^2 + \dots \tag{1.4}$$

holomorphic in K, defined by the condition: f is in $S^*(a, b)$ if and only if there exists a function $P \in \wp(a, b)$ such that

$$\frac{zf'(z)}{f(z)} = P(z), \quad z \in K. \tag{1.5}$$

Since $\wp(a, b) \subset \wp$ and $\wp(1, -1) \equiv \wp$, we have $S^*(a, b) \subset S^*$ and $S^*(1, -1) \equiv S^*$, where S^* is the well-known class of starlike functions.

Moreover, let $\Sigma^*(a, b)$ denote the family of meromorphic functions

$$F(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots \tag{1.6}$$

in the unit disc such that

$$\frac{-zF'(z)}{F(z)} = P(z), \quad z \in K, \tag{1.7}$$

for some function P in $\wp(a, b)$.

It is not difficult to see that $\Sigma^*(a, b) \subset \Sigma^*$ and $\Sigma^*(1, -1) \equiv \Sigma^*$, where Σ^* is the class of starlike meromorphic functions.

In this paper we give estimates of the coefficients of the families $S^*(a, b)$ and $\Sigma^*(a, b)$.

Our results contain those of Clunie [1], Janowski [3], Kaczmarski [4], Plaskota [5], Pomerenke [6], Robertson [7], and Wieczorek [8], as particular cases.

2. Estimates of the coefficients of functions of class $S^*(a, b)$. We shall prove the following result.

THEOREM 1. *If $f \in S^*(a, b)$ and $-1 \leqslant b < a \leqslant 1$, then*

$$|a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=1}^{n-1} |a-kb| \quad \textit{for} \quad n = 2, 3, \ldots, p \tag{2.1}$$

and

$$|a_n| \leqslant \frac{1}{(n-1)(p-2)!} \prod_{k=1}^{p-1} |a-kb| \quad \textit{for} \quad n = p+1, \ldots, \tag{2.2}$$

where $p \in \left\langle \frac{1+a}{1+b}, \frac{2+a+b}{1+b} \right)$ *is a natural number. If $b = -1$, then*

$$|a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (a+k) \quad \textit{for} \quad n = 2, 3, \ldots \tag{2.3}$$

The bounds (2.1) *and* (2.3) *are attained by the function*

$$f(z) = \begin{cases} z(1+\varepsilon bz)^{(a-b)/b} & \textit{for} \quad b \neq 0, \\ z \exp(a\varepsilon z) & \textit{for} \quad b = 0, \end{cases} \quad |\varepsilon| = 1. \tag{2.4}$$

Proof. If $f \in S^*(a, b)$, then

$$\frac{zf'(z)}{f(z)} = \frac{1+a\omega(z)}{1+b\omega(z)}, \quad z \in K, \tag{2.5}$$

for some function $\omega \in \Omega$.

From (2.5) it follows that

$$zf'(z) - f(z) = \big(af(z) - bzf'(z)\big)\omega(z);$$

hence

$$\sum_{k=1}^{\infty} (k-1) a_k z^k = \omega(z) \cdot \sum_{k=1}^{\infty} (a-kb) a_k z^k \tag{2.6}$$

and it is easy to see that $|a_2| \leqslant a-b$.

Thus estimate (2.1) holds true for $n = 2$. Now suppose that $n \geqslant 2$. Let us write equality (2.6) as follows:

$$\sum_{k=1}^{n} (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = \omega(z) \sum_{k=1}^{n-1} (a-kb) a_k z^k,$$

where the series $\sum_{k=n+1}^{\infty} d_k z^k$ is convergent in the unit disc. By using the method of Clunie's (cf. [1]) we get:

$$\Big|\sum_{k=1}^{n} (k-1) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k\Big| < \Big|\sum_{k=1}^{n-1} (a-kb) a_k z^k\Big|. \tag{2.7}$$

Hence, putting $z = re^{it}$, $0 < r < 1$, $0 \leqslant t < 2\pi$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \Big|\sum_{k=1}^{n} (k-1) a_k r^k e^{itk} + \sum_{k=n+1}^{\infty} d_k r^k e^{itk}\Big|^2 dt \leqslant \frac{1}{2\pi} \int_0^{2\pi} \Big|\sum_{k=1}^{n-1} (a-kb) a_k r^k e^{itk}\Big|^2 dt.$$

Integrating we get

$$\sum_{k=1}^{n} (k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leqslant \sum_{k=1}^{n-1} (a-kb)^2 |a_k|^2 r^{2k}. \tag{2.8}$$

In particular, from (2.8) follows

$$\sum_{k=1}^{n} (k-1) |a_k|^2 r^{2k} \leqslant \sum_{k=1}^{n-1} (a-kb)^2 |a_k|^2 r^{2k}. \tag{2.9}$$

Passing in (2.9) to the limit as $r \to 1$ we obtain

$$\sum_{k=1}^{n} (k-1)^2 |a_k|^2 \leqslant \sum_{k=1}^{n-1} (a-kb)^2 |a_k|^2.$$

Thus

$$(n-1)^2|a_n|^2 \leqslant \sum_{k=1}^{n-1} \big((a-kb)^2-(k-1)^2\big)|a_k|^2.$$

We easily observe that $\big(a-(n-1)b\big)^2-(n-2)^2 \geqslant 0$ iff $n \leqslant p$ and $-1 < b < a \leqslant 1$; moreover, if $b = -1$, then $(a+n-1)^2-(n-2)^2 \geqslant 0$ for $n = 2, 3, \ldots$ and every $a \in (-1, 1\rangle$. Estimates (2.1), (2.2) and (2.3) are obtained by induction. The function f defined by formula (2.4) belongs to the family $S^*(a, b)$. Let a_n^* denote the coefficient at z^n in the power series representing the function (2.4). Then we have

$$a_n^* = \frac{\varepsilon^{n-1}}{(n-1)!} \prod_{k=1}^{n-1} (a-kb);$$

thus the estimates (2.1) and (2.3) are sharp. Q.E.D.

Theorem 1 implies the following corollaries:

Corollary 1. *The values of the function* $f \in S^*(a, b)$ *include the disc*

$$|w| < \frac{1}{2+a-b}. \tag{2.10}$$

In fact, let $f \in S^*(a, b)$ and $f(z) \neq w_0$ for every $z \in K$. Then the function

$$g(z) = \frac{f(z)}{1-f(z)/w_0} = z+(a_2+1/w_0)z^2+\ldots$$

belongs to the family S. Thus $|a_2+1/w_0| \leqslant 2$. Since $|a_2| \leqslant a-b$, then (2.10) follows.

By specifying the values of the parameters appearing in Theorem 1 we obtain some interesting particular cases.

Corollary 2. *By putting* $a = 1-2\alpha$, $b = -1$ *in* (2.3) *we obtain the result of* [7] *for starlike functions of order* α.

Corollary 3. *For* $a = 1$ *and* $b = \frac{1}{M}-1$, $M \geqslant 1$, *we obtain the result of Janowski* (cf. [3]).

Corollary 4. *By substituting* $b = -a$, $0 < a < 1$, *in* (2.1), (2.2) *and* (2.10) *we get corresponding results obtained in* [8].

Corollary 5. *If* $a = 1-\lambda-\lambda m$, $b = -m$, *then we obtain the result of* [5].

Corollary 6. *By applying Theorem* 1 *to the case where* $a = 1$ *and* $b = -1$, *we obtain* $|a_n| \leqslant n$ *for starlike functions of order* 0.

3. Estimates for coefficients of functions of class $\Sigma^*(a, b)$. Now we shall find estimates for coefficients of functions belonging to the family $\Sigma^*(a, b)$.

THEOREM 2. *If* $F \in \Sigma^*(a, b)$, *then*

$$|a_n| \leqslant \frac{a-b}{n+1} \quad \text{for } n = 0, 1, 2, \ldots \tag{3.1}$$

The bound is attained by the function

$$F^*(z) = \begin{cases} \frac{1}{z}(1+bz^{n+1})^{(b-a)/b(n+1)} & \text{for } b \neq 0, \\ \frac{1}{z}\exp\left(\frac{-a}{n+1} z^{n+1}\right) & \text{for } b = 0. \end{cases} \tag{3.2}$$

Proof. It follows from the definitions of the families $\Sigma^*(a, b)$ and $\wp(a, b)$ that $F \in \Sigma^*(a, b)$ if and only if

$$-\frac{zF'(z)}{F(z)} = \frac{1+a\omega(z)}{1+b\omega(z)}, \quad z \in K, \tag{3.3}$$

for some function $\omega \in \Omega$.

From (3.3) and (1.6) we have

$$\sum_{k=0}^{\infty} (k+1) a_k z^{k+1} = -\Big(a-b+\sum_{k=0}^{\infty} (a+kb) a_k z^{k+1}\Big)\omega(z). \tag{3.4}$$

Hence and from (1.1) we have

$$|a_0| \leqslant a-b, \quad |a_1| \leqslant (a-b)/2. \tag{3.5}$$

Let $n \geqslant 2$. By applying the method of Clunie [1] we finally obtain the inequality

$$(n+1)^2 |a_n|^2 \leqslant (a-b)^2 + \sum_{k=0}^{n-1} \big((a+kb)^2-(k+1)^2\big)|a_k|^2. \tag{3.6}$$

Since $(a+lb)^2-(l+1)^2 \leqslant 0$ for $l = 0, 1, 2, \ldots$ and $-1 \leqslant b < a \leqslant 1$, we have

$$(n+1)^2 |a_n|^2 \leqslant (a-b)^2. \tag{3.7}$$

From inequalities (3.5) and (3.7) we conclude that the estimate (3.1) holds true for $n = 0, 1, 2, \ldots$

The function F defined by formula (3.2) belongs to the family $\Sigma^*(a, b)$, and if $F^*(z) = (1/z)+a_n^* z^n + \ldots$, then $|a_n^*| = (a-b)/(n+1)$ for $n = 0, 1, 2, \ldots$, then estimate (3.1) is sharp.

COROLLARY 1. *If* $F \in \Sigma^*(a, b)$, *then*

$$\sum_{k=0}^{\infty} \big((1-b^2)k^2+2(1-ab)k+(1-a^2)\big)|a_k|^2 \leqslant (a-b)^2. \tag{3.8}$$

In fact, by equality (3.4) and the condition $|\omega(z)| < 1$, we have

$$\sum_{k=0}^{\infty} |(k+1)\,a_k z^{k+1}|^2 < \Big|a-b+\sum_{k=0}^{\infty}(a+kb)\,a_k z^k\Big|^2;$$

thus

$$\int_0^{2\pi}\Big|\sum_{k=0}^{\infty}(k+1)\,a_k r^{k+1} e^{i(k+1)t}\Big|^2 dt \leqslant \int_0^{2\pi}\Big|a-b+\sum_{k=0}^{\infty}(a+kb)\,a_k r^k e^{i(k+1)t}\Big|^2 dt.$$

Hence, by integrating and passing to the limit as r tends to 1, we obtain (3.8).

COROLLARY 2. *By putting* $a = 1-\alpha-\alpha m$, $b = -m$ *in* (3.1) *and* (3.8) *we get the corresponding result obtained in* [4], *and by letting* $b = -a$, $0 < a \leqslant 1$, *we get the result of* [8]; *finally, we remark that if* $b = -1$ *and* $a = 1-2\alpha$, *then we get the result obtained in* [6].

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