On the Cohomology Ring of a Sphere Bundle[1]

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1. Introduction. This paper is concerned with the general problem of determining the cohomology ring of a fibre space whose fibre is a sphere in terms of the cohomology of the base space and various invariants of the fibre space structure, such as characteristic cohomology classes (in particular, Stiefel-Whitney classes and Pontrjagin classes).

The first part of the paper is concerned with two formulas which give explicit relations between cup products in the total space of an orientable sphere bundle and triple products[2] in the base space (for which one of the arguments is the characteristic class of the bundle). In general, these formulas do not suffice to completely determine the cohomology ring of the total space. An example is given of a non-trivial case where they do suffice, however.

It should be mentioned that one of these two formulas is due originally to G. HIRSCH [7], who gave it under more restrictive hypotheses than we do. Apparently his proof of this formula is different from ours.

The second part of this paper is concerned with the problem of determining the cohomology ring of a sphere bundle whose characteristic class vanishes. This special case is simpler than the general case, and our results are correspondingly more complete. In this case the structure of the cohomology ring of the bundle may be expressed succinctly by the statement that it is a quadratic extension of the cohomology ring of the base space.[3] From this it follows that the

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[2]The triple product is a secondary cohomology operation on three variables which assigns to any elements $u \in H^p(B)$, $v \in H^q(B)$, and $w \in H^r(B)$ such that $u \cdot v = v \cdot w = 0$ a coset of a certain subgroup of $H^{p+q+r-1}(B)$. This coset is denoted by $\langle u, v, w \rangle$. For the definition and properties, see [16].

[3]This fact has apparently been known for some time, although never stated explicitly in this form; see, for example, LIAO [10], or HIRSCH [5].

The definition of a quadratic extension of a ring is the obvious generalization of the definition of a quadratic extension of a field; see N. BOURBAKI's *Algébre,* chap. II, §7. For our purposes, the definition of BOURBAKI needs to be modified slightly because we are dealing with graded, anti-commutative rings.

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complete determination of the structure of the cohomology ring of such a sphere bundle depends on two invariants. One of these invariants turns out to be essentially a Stiefel-Whitney class. The situation with regard to the other invariant is more complicated. In case the dimension of the fibre is even, it is related to one of the Pontrjagin classes.

Finally, we give some applications of the results in part II. As a first application, we are able to put the formulas for the secondary obstruction to a cross section of a 2-sphere bundle in a more convenient and explicit form. The second application is to prove the impossibility of imbedding certain manifolds differentiably in Euclidean spaces of certain dimensions. In most of the examples given, we show that certain orientable n-dimensional manifolds cannot be imbedded differentiably in Euclidean space of dimension approximately $\frac{3}{2}n$. These results depend on knowledge of the Pontrjagin classes of the tangent bundle of the manifold, and the method works even in cases where all the Stiefel-Whitney classes vanish.

Parts I and II are independent of each other to a large extent.

2. Notation and Terminology. Throughout this paper we will use the term "fibre space" to mean a "locally trivial fibre space" in accordance with the following definition:

Definition. A *fibre space* is an ordered quadruple (E, π, B, F) such that E, B, and F are topological spaces, $\pi : E \to B$ is a continuous map, and the following condition holds: Each point $x \in B$ has a neighborhood U such that there exists a homeomorphism φ of $U \times F$ onto $\pi^{-1}(U)$ having the property that $\pi[\varphi(y, z)] = y$ for any $y \in U$ and $z \in F$.

On the other hand, the term "fibre bundle" will be reserved for fibre spaces which admit a structural group. We will use the definition of fibre bundle as given in Steenrod's book [14], and use the notation (E, p, B, F, G) to denote a fibre bundle with fibre F and group G. A fibre bundle whose fibre F is an n-sphere, S^n, and whose group is the group of all $(n + 1) \times (n + 1)$ real orthogonal matrices of determinant $+1$ (denoted by $\mathrm{SO}(n + 1)$) will be called an n-sphere bundle. Analogously, a fibre space whose fibre F is an n-sphere will be called an n-sphere space.

We will assume that *all* n-sphere bundles and n-sphere spaces with which we are concerned satisfy the following orientability condition: If S^n_x denotes the fibre over the point $x \in B$, then the local system of groups defined by $H^n(S^n_x)$, $x \in B$, is a simple system. Of course, if we are using integers mod 2 as coefficients for cohomology, this condition is superfluous.

We will use a notation similar to that of R. Thom [15] for the Gysin sequence of a sphere space (E, π, B, S^{k-1}):

$$\cdots \xrightarrow{\psi} H^{q-k}(B) \xrightarrow{\mu} H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \xrightarrow{\mu} \cdots .$$

The homomorphism μ is multiplication by the characteristic class, $W_k \in H^k(B, Z)$.

Throughout this paper we will use the ring of integers, Z, the ring of integers mod m, Z_m $(m > 1)$, or the field of rational numbers, Z_0, as coefficients for cohomology. For the sake of simplicity, we will assume that the base space of any fibre space or fibre bundle that we consider is compact, although this assumption is probably not absolutely necessary. Unless otherwise indicated, we will use Cech-Alexander-Spanier cohomology with compact supports.

Part I: Relations Between Cup Products in the Total Space and Triple Products in the Base Space

3. Statement of Results. When confronted with the problem of determining the cohomology ring of a sphere space, one of the most natural questions to ask is the following: What are the properties of the homomorphisms of the Gysin sequence of a sphere space (E, π, B, S^{k-1}) with respect to cup products? The answer is obvious for the homomorphisms μ and π^*. For ψ we have the following result:

Lemma 1. *If $x \in H^p(B)$ and $y \in H^q(E)$, then*

$$\psi[(\pi^*x)\cdot y] = (-1)^p x\cdot(\psi y),$$

$$\psi[y\cdot(\pi^*x)] = (-1)^{kp}(\psi y)\cdot x$$

(coefficients in any ring).

The proof, which is quite easy, will be given in section 4. In case $x\cdot(\psi y) = 0$, it follows by exactness of the Gysin sequence that there exists an element $z \in H(B)$ such that

$$\pi^*(z) = (\pi^*x)\cdot y. \tag{3.1}$$

Of course, the element z is only determined modulo the image of μ. The question now arises, can we determine the set of all possible elements z satisfying (1) in terms of x and $\psi(y)$ by means of operations in $H^*(B)$ alone? The following theorem gives an affirmative answer to this question.

Theorem I. *If $u \in H^p(B)$, $v \in H^q(B)$, $u\cdot v = 0$, and $\mu(v) = 0$, then*

$$(\pi^*)^{-1}[(\pi^*u)(\psi^{-1}v)] = (-1)^{p+1}\langle u, v, W_k\rangle$$

(coefficients in any commutative ring).

The proof of this theorem will be given in section 5. By definition, the triple product $\langle u, v, W_k\rangle$ is a coset of the subgroup $[H^{p+q-1}(B)\cdot W_k + u\cdot H^{q+k-1}(B)]$ of $H^{p+q+k-1}(B)$. This theorem asserts that $(-1)^{p+1}(\pi^*)^{-1}[\pi^*u)(\psi^{-1}v)]$ is precisely the same coset.

Lemma 1 and theorem I may be looked on as giving information on the product of two elements of $H^*(E)$, of which one is in the image of π^* and the other is not in the image of π^*. The next question that arises is the following: Is there a similar theorem giving information about the product of two elements of $H^*(E)$, neither of which is in the image of π^*? If $x, y \in H^*(E)$, can we express $\psi(x\cdot y)$ as a function of $\psi(x)$ and $\psi(y)$? This last question can be reworded as follows: If $u, v \in H^*(B)$, and $\mu(u) = \mu(v) = 0$, let $(\psi^{-1}u)(\psi^{-1}v)$ denote the set of all elements $x\cdot y$ for $x \in \psi^{-1}u$ and $y \in \psi^{-1}v$. Then can we express $\psi[(\psi^{-1}u)\cdot(\psi^{-1}v)]$ as a function of u and v? An easy calculation using lemma 1 shows that $\psi[(\psi^{-1}u)\cdot(\psi^{-1}v)]$ is a coset of the subgroup $H^{p+k-1}(B)\cdot v + u\cdot H^{q+k-1}(B)$ where p and q are the degrees of u and v respectively.

Theorem*[4] *II. *Under the hypotheses above, if* $u \in H^p(B)$, $v \in H^q(B)$, $\mu(u) = \mu(v) = 0$, *then*

$$\psi[(\psi^{-1}u)\cdot(\psi^{-1}v)] = (-1)^{p+k+1}\langle u, W_k, v\rangle.$$

The proof of this theorem will be given in section 6.

Although theorems I and II do not suffice in general to completely determine the cohomology ring $H^*(E)$, we will give an example later where they do suffice.

4. Preliminaries to the Proofs of Theorems I and II. Let A denote the mapping cylinder of the projection $\pi : E \to B$. According to R. Thom [15], the cohomology sequence of the pair (A, E) is isomorphic to the Gysin sequence of the fibre space (E, π, B, S^{k-1}). To be precise, there exist isomorphisms

$$\phi^* : H^{q-k}(B) \to H^q(A - E)$$

$$j : H^q(B) \to H^q(A)$$

such that the following diagram is commutative:

$$\begin{tikzcd}
 & H^{q-k}(B) \arrow[r, "\mu"] \arrow[d, "\phi^*"] & H^q(B) \arrow[d, "j"] \arrow[dr, "\pi^*"] & \\
H^{q-1}(E) \arrow[ur, "\psi"] \arrow[dr, "\delta^*"'] & & & H^q(E) \\
 & H^q(A - E) \arrow[r, "h"'] & H^q(A) \arrow[ur, "i"'] &
\end{tikzcd}$$

In this diagram the top line is the Gysin sequence, while the bottom line is the cohomology sequence of the pair (A, E).

In addition, Thom proves the following important property of the isomorphisms ϕ^* and j: For any element $x \in H^{q-k}(B)$,

$$\phi^*(x) = j(x)\cdot U \tag{4.1}$$

where $U \in H^k(A - E)$ is the image of $1 \in H^0(B)$ under the isomorphism ϕ^* (recall that we have assumed that B is compact!). The isomorphism j is induced by the natural projection $A \to B$ of the mapping cylinder onto the base space,

[4]This theorem is due originally to G. Hirsch [7]. However, Hirsch states the theorem under more restrictive hypotheses, and in his formulation there is an additional term on the right.

while the definition of ϕ^* is more complicated. The reader can, if he wishes, take equation (4.1) as the definition of ϕ^*.

We are now in a position to prove lemma 1. To prove the second equation of lemma 1, it suffices to prove that

$$\phi^*\psi[y(\pi^*x)] = (-1)^{kp}\phi^*[(\psi y)\cdot x],$$

since ϕ^* is an isomorphism. We now compute as follows:

$$\phi^*\psi[y(\pi^*x)] = \delta^*[y(\pi^*x)] = \delta^*[y(ijx)] = (\delta^*y)(jx),$$

$$\begin{aligned}\phi^*[(\psi y)x] &= j[(\psi y)x]\cdot U = (j\psi y)(jx)U \\ &= (-1)^{kp}[(j\psi y)U](jx) = (-1)^{kp}(\phi^*\psi y)(jx) = (-1)^{kp}(\delta^*y)(jx).\end{aligned}$$

The proof of the first formula of lemma 1 is similar.

We conclude this section by recalling Thom's definition of the generalized characteristic classes. First of all, the characteristic class W_k is defined by

$$j(W_k) = h(U).$$

Second, the generalized Stiefel-Whitney classes are defined by

$$\phi^*(W_i) = Sq^i(U), \qquad 0 \leqq i \leqq k-1,$$

where W_i is an integral or mod 2 cohomology class according as i is odd or even. If i is odd, then $2W_i = 0$. One may also consider W_i reduced mod 2 for i odd.

Actually, we will be mainly concerned with the classes W_k and W_{k-1} in what follows.

5. Proof of Theorem I. First of all, an easy computation shows that (under the hypotheses of theorem I) $(\pi^*)^{-1}[(\pi^*u)(\psi^{-1}v)]$ is a coset of the subgroup $\mu[H^{p+q-1}(B)] + u\cdot H^{q+k-1}(B)$. By definition, the triple product $\langle u, v, W_k\rangle$ is a coset of the subgroup $H^{p+q-1}(B)\cdot W_k + u\cdot H^{q+k-1}(B)$. Thus $(\pi^{*-1})[\pi^*u)(\psi^{-1}v)]$ and $(-1)^{p+1}\langle u, v, W_k\rangle$ are cosets of the same subgroup. Therefore, to prove theorem I, it obviously suffices to prove that

$$(5.1) \qquad (\pi^*u)(\psi^{-1}v) = (-1)^{p+1}\pi^*\langle u, v, W_k\rangle.$$

Rather than prove (5.1) directly, we will reduce it to an equivalent statement involving only the cohomology sequence of the pair (A, E). To this end, note that $\pi^*(u) = ij(u)$, and $\psi^{-1}v = \delta^{*-1}\phi^*v = \delta^{*-1}[(jv)U]$, therefore

$$(5.2) \qquad (\pi^*u)(\psi^{-1}v) = (iju)\cdot\delta^{*-1}[(jv)\cdot U].$$

Also, $\pi^*\langle u, v, W_k\rangle = ij\langle u, v, W_k\rangle$, and $j\langle u, v, W_k\rangle = \langle ju, jv, jW_k\rangle = \langle ju, jv, hU\rangle$, since j is an isomorphism onto. Thus we have

$$(5.3) \qquad \pi^*\langle u, v, W_k\rangle = i\langle ju, jv, hU\rangle.$$

It follows that (5.1) is equivalent to the following equation:

$$(5.4) \qquad (iju)\cdot\delta^{*-1}[(jv)\cdot U] = (-1)^{p+1}i\langle ju, jv, hU\rangle.$$

The hypothesis $u \cdot v = 0$ is equivalent to $(ju) \cdot (jv) = 0$, and the hypothesis $\mu(v) = 0$ is equivalent to the statement that $0 = j(vW_k)$, or, alternatively, that $(jv)(hU) = 0$. Therefore, if we write $u_0 = ju$ and $v_0 = jv$, theorem I is equivalent to the following:

Theorem I′. *Suppose $u_0 \in H^p(A)$, $v_0 \in H^q(A)$, and $u_0 v_0 = 0$, $v_0(hU) = 0$. Then*

$$(iu_0)\,\delta^{*^{-1}}(v_0 U) = (-1)^{p+1} i\langle u_0, v_0, hU\rangle.$$

We will now prove theorem I′. Let $C^* = \sum_{p \geqq 0} C^p$ denote the ring of Alexander-Spanier cochains of A and $I^* = \sum_{p \geqq 0} I^p$ denote the ring of Alexander-Spanier cochains with compact support of $A - E$. Then I^* is a 2-sided ideal in C^* which is stable under the coboundary operator (*i.e.*, $\delta(I^*) \subset I^*$). Moreover, the quotient ring C^*/I^* may be used for the cochain ring of E, *i.e.*, $H^q(C^*/I^*) = H^q(E)$.

Choose representative cocycles $u' \in C^p$, $v' \in C^q$, $U' \in I^k$ for u_0, v_0, and U respectively. Since $u_0 v_0 = 0$, there exists a cochain $a \in C^{p+q-1}$ such that $\delta(a) = u'v'$. Similarly, there exists a cochain $b \in C^{q+k-1}$ such that $\delta b = v'U'$. Then the cocycle $aU' - (-1)^p u'b$ is, by definition, a representative of the triple product $\langle u_0, v_0, hU\rangle$. On the other hand, since $U' \in I^*$, $v'U' \in I^*$, and b is a cocycle mod I^*. Therefore the cohomology class of b (mod I^*) is obviously a representative of $\delta^{*^{-1}}(v_0 U)$. Therefore the cohomology class of $u'b$ (mod I^*) is a representative of $(iu_0)\delta^{*^{-1}}(v_0 U)$. Since $U' \in I^*$, $aU' \in I^*$, and

$$(-1)^{p+1} u'b \equiv aU' - (-1)^p u'b \pmod{I^*}.$$

From this theorem I' follows.

This completes the proof of theorem I.

6. Proof of Theorem II. First of all, by applying the isomorphism ϕ^*, we will reduce theorem II to an equivalent theorem about the cohomology sequence of the pair (A, E). For this purpose we make the following two computations:

$$\begin{aligned} \phi^*\psi[(\psi^{-1}u)(\psi^{-1}v)] &= \delta^*[(\psi^{-1}u)(\psi^{-1}v)] \\ &= \delta^*[\delta^{*^{-1}}(\phi^* u) \cdot \delta^{*^{-1}}(\phi^* v)] \qquad (6.1) \\ &= \delta^*[\delta^{*^{-1}}(ju \cdot U) \cdot \delta^{*^{-1}}(jv \cdot U)]. \end{aligned}$$

$$\phi^*\langle u, W_k, v\rangle = (j\langle u, W_k, v\rangle) \cdot U = \langle ju, jW_k, jv\rangle \cdot U = \langle ju, hU, jv\rangle \cdot U. \qquad (6.2)$$

Also $\mu(u) = 0$ if and only if $j\mu(u) = 0$. Now $j\mu(u) = h\phi^*(u) = h[(ju) \cdot U]$. Therefore $\mu(u) = 0$ if and only if $h[(ju) \cdot U] = 0$. A similar equation holds for v.

From these computations, it follows that if we set $j(u) = u_0$, $j(v) = v_0$, then theorem II is equivalent to the following:

Theorem II′. *Let $u_0 \in H^p(A)$ and $v_0 \in H^q(A)$ be elements such that $h[u_0 U] = 0$, $h[v_0 U] = 0$. Then*

$$\delta^*[\delta^{*^{-1}}(u_0 U) \cdot \delta^{*^{-1}}(v_0 U)] = (-1)^{p+k+1}\langle u_0, hU, v_0\rangle \cdot U.$$

We will now prove theorem II′. As in the proof of theorem I′, let $C^* = \sum C^p$ and $I^* = \sum I^p$ denote the ring of Alexander-Spanier cochains with compact support of A and $A - E$ respectively. We may assume that STEENROD's cup-i products are defined in C^*, and that they have the usual properties (see [3], *exposé* 14, or [13]). Choose representative cochains $u' \varepsilon C^p$, $v' \varepsilon C^q$, and $U' \varepsilon I^k$ for u_0, v_0, and U respectively. Since $h(u_0 U) = 0$, there exists a cochain a such that $\delta(a) = u'U'$; similarly, there exists a cochain b such that $\delta(b) = v'U'$. Then a and b are cocycles mod I^*, and cohomology classes mod I^* are elements of $\delta^{*-1}(u_0 U)$ and $\delta^{*-1}(v_0 U)$ respectively. Therefore the cocycle

$$\delta(ab) = u'U'b + (-1)^{p+k-1} av'U' \tag{6.3}$$

represents $\delta^*[\delta^{*-1}(u_0 U) \cdot \delta^{*-1}(v_0 U)]$.

We will now transform the expression on the right side of (6.3) as follows. First of all,

$$\delta(b \cup_1 U') = (-1)^q bU' - (-1)^{kq+q} U'b + (v'U') \cup_1 U';$$

therefore

$$U'b = (-1)^{kq} bU' + (-1)^{kq+q}(v'U') \cup_1 U' - (-1)^{kq+q} \delta(b \cup_1 U'). \tag{6.4}$$

By a theorem of G. HIRSCH ([6], theorem 1),

$$(v'U') \cup_1 U' = v'(U' \cup_1 U') + (v' \cup_1 U')U'. \tag{6.5}$$

Substituting (6.5) in (6.4), we obtain

$$\begin{aligned} U'b = (-1)^{kq} bU' &+ (-1)^{kq+q}(v' \cup_1 U')U' \\ &+ (-1)^{kq+q} v'(U' \cup_1 U') - (-1)^{kq+q} \delta(b \cup_1 U'). \end{aligned} \tag{6.6}$$

If we substitute (6.6) into (6.3), there results the following:

$$\begin{aligned} \delta(ab) = [(-1)^{p+k-1} av' &+ (-1)^{kq} u'b + (-1)^{kq+q} u'(v' \cup_1 U')]U' \\ &+ (-1)^{kq+q} u'v'(U' \cup_1 U') - (-1)^{kq+q+p} \delta[u'(b \cup_1 U')]. \end{aligned} \tag{6.7}$$

Next, we will compute a representative for $\langle u_0, hU, v_0 \rangle$. We have

$$\delta(v' \cup_1 U') = (-1)^{q+k-1} v'U' + (-1)^{qk+q+k} U'v';$$

therefore

$$\begin{aligned} U'v' &= (-1)^{qk+q+k} \delta(v' \cup_1 U') + (-1)^{qk} v'U' \\ &= \delta[(-1)^{qk+q+k} v' \cup_1 U' + (-1)^{qk} b]. \end{aligned}$$

From the definition of the triple product (see [16]), it follows that the cocycle

$$\begin{aligned} av' - (-1)^p u'[(-1)^{qk+q+k} & v' \cup_1 U' + (-1)^{qk} b] \\ &= av' - (-1)^{qk+q+k+p} u'(v' \cup_1 U') - (-1)^{p+qk} u'b \end{aligned} \tag{6.8}$$

is a representative of $\langle u_0, hU, v_0\rangle$. In order to utilize this expression for a representative of $\langle u_0, hU, v_0\rangle$, we will rewrite (6.7) in the following form:

$$\begin{aligned}\delta(ab) = {}& [(-1)^{p+k-1}av' + (-1)^{kq+k}u'b + (-1)^{kq+q}u'(v' \cup_1 U')]\cdot U' \\ & + (-1)^{kq}u'z - (-1)^{kq+q+p}\delta[u'(b \cup_1 U')]\end{aligned} \tag{6.9}$$

where

$$z = [1 - (-1)^k]bU' + (-1)^q v'(U' \cup_1 U').$$

Now clearly $z \in I^{q+2k-1}$, and an easy computation shows that $\delta(z) = 0$, *i.e.*, z = a cocycle. Thus the term $(-1)^{kq}u'z$ on the right side of equation (6.9) represents an element of the subgroup $u_0 \cdot H^{q+2k-1}(A - E)$ of $H^{p+q+2k-1}(A - E)$. Since $\langle u_0, hU, v_0\rangle \cdot U$ is a coset of $u_0H^{q+2k-1}(A - E) + H^{p+2k-1}(A - E)\cdot v_0$, comparison of (6.8) and (6.9) shows that $\delta(ab)$ is a representative of $(-1)^{p+k-1}$ $\langle u_0, hU, v_0\rangle \cdot U$, as required. This completes the proof.

7. An Example. We will consider in this section sphere bundles $(E, \pi, B, S^1, \mathrm{SO}(2))$ such that the fibre is a 1-sphere (*i.e.*, $k = 2$), and the base space B is a finite, connected 5-dimensional CW-complex which satisfies the following conditions:

(a) B has only one vertex.

(b) B has no cells of dimension 1, 3, or 4.

Such a cell complex may be constructed by taking a cluster of 2-spheres having but a single common point and adjoining 5-cells by arbitrary continuous maps. It is clear that $H^1(B) = H^3(B) = H^4(B) = 0$, while $H^2(B)$ and $H^5(B)$ are free abelian groups. As is well known, given any element of $H^2(B, Z)$, there exists an orientable 1-sphere bundle over B having that element as characteristic class. We will consider only bundles over B such that the characteristic class W_2 is a member of some basis of $H^2(B)$ (*i.e.*, the subgroup generated by W_2 is a direct summand of $H^2(B)$).

The Gysin sequence of such a sphere bundle is readily seen to split into the following pieces:

$$0 \to H^0(B) \xrightarrow{\pi^*} H^0(E) \to 0,$$

$$0 \to H^0(B) \xrightarrow{\mu} H^2(B) \xrightarrow{\pi^*} H^2(E) \to 0,$$

$$0 \to H^3(B) \xrightarrow{\psi} H^2(E) \to 0,$$

$$0 \to H^5(B) \xrightarrow{\pi^*} H^5(E) \to 0,$$

$$0 \to H^6(E) \xrightarrow{\psi} H^5(B) \to 0.$$

Thus the only non-zero cohomology groups of E are in dimensions 0, 2, 3, 5, and 6. Moreover, there is no torsion. The product of an element of degree two and an element of degree three can be computed by means of theorem I, and the product of two elements of degree three can be computed by use of theorem

II. In each case the products in E depend on the triple products in B. Moreover, if the 5-cells of B were attached to the 2-skeleton of B by maps representing non-trivial triple Whitehead products, there will be non-zero triple products in $H^*(B)$; for the proof see sections 2, 3, and 4 of [16].

Part II: The Cohomology Ring of a Sphere Bundle Whose Characteristic Class Vanishes

8. Statement of Results. Let (E, π, B, S^{k-1}) be a sphere space whose characteristic class, W_k, is zero. The Gysin sequence of such a fibre space breaks up into pieces of length 3, as follows:

$$0 \to H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\psi} H^{q-k+1}(B) \to 0.$$

Moreover, the group $H^q(E)$ is a trivial extension of $H^q(B)$ by $H^{q-k+1}(B)$. This may be proved as follows: Choose an element $a \in H^{k-1}(E)$ such that $\psi(a) = 1$, where $1 \in H^0(B)$ is the unit of $H^*(B)$. Then if $x \in H^{q-k+1}(B)$,

$$\psi[a \cdot \pi^*(x)] = (-1)^{kq}(\psi a) \cdot x = (-1)^{kq} x.$$

Therefore, if we define

$$\theta : H^{q-k+1}(B) \to H^q(E)$$

by

$$\theta(x) = (-1)^{kq} a \cdot \pi^*(x),$$

it follows that

$$\psi\theta(x) = x,$$

which proves that the extension is trivial. Thus given any element $u \in H^q(E)$, there exist unique elements $u_1 \in H^q(B)$ and $u_2 \in H^{q-k+1}(B)$ such that

$$u = \pi^*(u_1) + a \cdot \pi^*(u_2).$$

In particular, there exist unique elements $\alpha \in H^{2k-2}(B)$ and $\beta \in H^{k-1}(B)$ such that

$$a^2 = \pi^*(\alpha) + a \cdot \pi^*(\beta). \tag{8.1}$$

Moreover, in view of the anti-commutativity of the cohomology ring of E and the fact that π^* is an isomorphism which preserves products, if α and β are known, the multiplicative structure of $H^*(E)$ is completely determined.

Of course a is not uniquely determined. If a' is another element of $H^{k-1}(E)$ such that $\psi(a') = 1$, then by exactness of the Gysin sequence there exists an element $b \in H^{k-1}(B)$ such that $\pi^*(b) = a' - a$. By analogy with (8.1), there exist unique elements $\alpha' \in H^{2k-2}(B)$ and $\beta' \in H^{k-1}(B)$ such that

$$a'^2 = \pi^*(\alpha') + a' \cdot \pi^*(\beta'). \tag{8.1'}$$

An easy calculation shows that α' and β' are related to α and β as follows:[5]

(8.2) $\beta' = \beta + 2b$ if k is odd,

(8.3) $\beta' = \beta$ if k is even,

(8.4) $\alpha' = \alpha - b\beta - b^2$ for any k, odd or even.

Thus, in case k is even, β is an invariant of the given sphere space; in case k is odd, the coset of β mod $2H^{k-1}(B)$ is an invariant of the given fibre space (note that in the above formulas, b may be an arbitrary element of $H^{k-1}(B)$).

Theorem[6] ***III.*** *If k is even, then $\beta = W_{k-1}$; if k is odd, then $\beta \equiv W_{k-1}$* mod 2, i.e., $W_{k-1} = \beta$ *reduced* mod 2.

The proof of this theorem will be given in section 9. In either case, β is identified with a standard invariant of the given sphere space.

As formula (8.4) indicates, the situation with regard to α is more complicated. If k is odd, an easy computation using formulas (8.2) and (8.4) shows that

$$4\alpha' + \beta'^2 = 4\alpha + \beta^2,$$

i.e., $4\alpha + \beta^2$ is an invariant of the given fibre space.

Theorem IV. *If k is odd, and $(E, \pi, B, S^{k-1}, \mathrm{SO}(k))$ is a $(k-1)$-sphere bundle with $W_k = 0$ and B a polyhedron, then $4\alpha + \beta^2 = P_{2k-2}$, the Pontrjagin class in dimension $2k-2$.*

The proof of this theorem will be given in sections 10–12. It does not seem possible to drop the hypothesis that the given fibre space admit the rotation group as structural group, because at present it is not known whether or not it is possible to define Pontrjagin classes for the most general sphere spaces as we have defined them in section 2.

In case $H^{2k-2}(B, Z)$ has no elements of order 2 (and hence no elements of order 4), theorems III and IV together give the complete determination of the cohomology ring of a k-sphere bundle (k odd) admitting the rotation group as structural group and such that $W_k = 0$. Examples seem to indicate that no such simple result is possible in case k is even. It should be noted that when k is even, $2\alpha = 0$ and $2\beta = 0$; this follows from the anti-commutativity of multiplication. Hence if $H^{2k-2}(B)$ contains no elements of order 2, it follows that $\alpha = 0$.

On the other hand, the problem of determining the cohomology ring with mod 2 coefficients of a sphere bundle with vanishing characteristic class seems like an interesting and difficult problem which is not even touched on here.

9. Proof of Theorem III. It is readily seen that β in equation (8.1) is given by

(9.1) $$\beta = \pm\psi(a^2)$$

[5]Formulas (8.2) and (8.4) are given by Hirsch for the case $k = 3$ in [5], section 2.1.

[6]This theorem is closely related to theorem 22.1 of Liao [10].

(the sign is immaterial). Therefore, to prove theorem III, it suffices to prove that $\psi(a^2) = W_{k-1}$ if k is even, and $\psi(a^2) \equiv W_{k-1} \bmod 2$ if k is odd (W_{k-1} is of order 2, hence $W_{k-1} = -W_{k-1}$). Applying the isomorphism ϕ^* to these relations, and using the facts that $\phi^* \circ \psi = \delta^*$ and $\phi^*(W_{k-1}) = Sq^{k-1}U$ (see section 4), we see that theorem III is equivalent to the following:

$$\delta^*(a^2) = Sq^{k-1}U \qquad \text{if } k \text{ is even,} \tag{9.2}$$

$$\delta^*(a^2) \equiv Sq^{k-1}(U) \mod 2 \qquad \text{if } k \text{ is odd.} \tag{9.3}$$

Now (9.2) and (9.3) are easy consequences of the fact that $\delta^*(a) = U$ and the fact that δ^* and Sq^{k-1} commute (see theorem (9.6) of STEENROD, [13]) plus the fact that $a^2 = Sq^{k-1}a$ if k is even, and $a^2 \equiv Sq^{k-1}a \bmod 2$ if k is odd.

10. Preliminaries to the Proof of Theorem IV. We assume that the reader is familiar with the classification theorem for principal fibre bundles:[7] Given any topological group G, there exists a universal principal G-bundle (E_G, p, B_G, G, G) such that the principal G-bundles over a polyhedron B are in 1–1 correspondence with the homotopy classes of maps $B \rightarrow B_G$. The correspondence is established by assigning to each map $f: B \rightarrow B_G$ the induced bundle $f^{-1}(E_G, p, B_G, G, G)$ (for details see STEENROD, [14], H. CARTAN, [3], and J. MILNOR, [11]).

In this section we shall use the term "characteristic class" in the most general sense: If (E, π, B, G, G) is a principal G-bundle corresponding to a continuous map $f: B \rightarrow B_G$ and $w \in H^n(B_G, \Pi)$, then $f^*(w) \in H^n(B, \Pi)$ will be called the *characteristic class of (E, π, B, G, G) which corresponds to w*. Our main objective in this section is to define a modified type of "universal" bundle which will be universal only for those principal G-bundles for which a given charadteristic class vanishes.

First of all, it is necessary to recall the following facts:

(10.1) If Π is a countable abelian group, then it is possible to realize the Eilenberg-MacLane space $K(\Pi, n)$ as an abelian topological group (MILNOR, [11]).

(10.2) If P is a polyhedron and Y is a space of type $K(\Pi, n)$, the homotopy classes of maps $P \rightarrow Y$ are in 1–1 correspondence with the elements of the cohomology group $H^n(P, \Pi)$; the correspondence is established by assigning to the map $f: P \rightarrow Y$ the cohomology class $f^*(e) \in H^n(P, \Pi)$, where $e \in H^n(Y, \Pi)$ is the fundamental cohomology class of Y (this result can be proved by the standard methods of obstruction theory).

From these facts we can deduce the following lemma:

Lemma 2. *Let A be an abelian topological group which is a space of type $K(\Pi, n-1)$ and let P be a polyhedron. Then the equivalence classes of principal*

[7]In the course of the proof of theorem IV, we will allow for consideration fibre bundles whose base space is not compact. Instead of using Alexander-Spanier cohomology with compact supports, we will use ordinary singular cohomology.

A-bundles with base space P are in 1–1 correspondence with the elements of the cohomology group $H^n(P, \Pi)$. The correspondence is defined by assigning to each such bundle its characteristic class.

Proof: The classifying space B_A is readily seen to be a space of type $K(\Pi, n)$. The result now follows from (10.2) and the classification theorem for fibre bundles.

We will call the principal A-bundle over P with $w \in H^n(P, \Pi)$ as characteristic class the bundle which "kills off the cohomology class w". Let (E, p, P, A, A) be such a bundle; it has the following important property:

(10.3) Given any polyhedron Q and continuous map $f : Q \to P$, there exists a continuous map $g : Q \to E$ such that $f = p \circ g$ if and only if $f^*(w) = 0$.

This result[8] may be easily proved by the methods of obstruction theory as outlined, for example, in part III of Steenrod [14].

Definition. Let B_G be the classifying space for the topological group G and let $w \in H^n(B_G, \Pi)$. Denote the space obtained from B_G by killing off the cohomology class w by (B_G, w). We will call (B_G, w) the "classifying space for G-bundles with vanishing characteristic class w". Note that there exists a map $\phi : (B_G, w) \to B_G$ which exhibits (B_G, w) as a principal $K(\Pi, n-1)$-bundle over B_G. Let $((E_G, w), p, (B_G, w), G, G)$ denote the principal G-bundle over (B_G, w) induced by ϕ. Then this bundle has the following characteristic property:

Lemma 3. *Let (E, π, B, G, G) be any principal G-bundle over the polyhedron B such that its characteristic class corresponding to $w \in H^n(B_G, \Pi)$ vanishes. Then there exists a continuous map $f : B \to (B_G, w)$ such that (E, π, B, G, G) is isomorphic to the G-bundle over B induced by f. (Note that the homotopy class of f need not be unique.)*

So far, all the results of this section have referred to principal G-bundles. By using the following lemma, it is possible to extend them to bundles with arbitrary fibre which admit G as structural group.

Lemma 4. *Let (E, p, B, F, G) be a fibre bundle, and let (E', p', B, G, G) be the associated principal bundle. If $f : X \to B$ is any continuous map, then the induced bundles $f^{-1}(E, p, B, F, G)$ and $f^{-1}(E', p', B, G, G)$ are associated bundles.*

This lemma can be paraphrased as follows: The operations of taking induced bundles and of taking associated bundles commute. The proof is obvious provided one uses the definitions of "induced bundle" and "associated bundle" in terms of coordinate transformations in B; see Steenrod [14], §8.1 and §10.1.

For the statement of the next two lemmas, let G_0 be a closed subgroup of the topological group G. Then G operates on the coset space G/G_0, and associated with any principal G-bundle is a fibre bundle with fibre G/G_0 and group G.

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[8] The facts we have just reviewed are rather well known; for a brief discussion of them, see also [4], *exposé* 17 (by R. Thom).

Lemma 5. *If (E, p, B, G, G) is a principal G-bundle, and $\pi : E/G_0 \to E/G = B$ denotes the natural projection, then $(E/G_0, \pi, B, G/G_0, G)$ is a fibre bundle which is associated with (E, p, B, G, G).*

For the proof, see H. Cartan [3], *exposé* 7, theorem 2.

Lemma 6. *If (E_G, p, B_G, G, G) is a universal principal G-bundle and $(E, \pi, B_G, G/G_0, G)$ is the associated bundle with fibre G/G_0, then E has the same homotopy type as B_{G_0}, the classifying space for G_0.*

Proof: By Lemma 5, $(E_G/G_0, \pi, B_G, G/G_0, G)$ is an associated bundle with fibre G/G_0. The natural projection $q : E_G \to E_G/G_0$ defines a principal G_0-bundle $(E_G, q, E_G/G_0, G_0, G_0)$. Since E_G is contractible, this principal G_0-bundle is a universal G_0-bundle, and E_G/G_0 is the classifying space for G_0.

We will use this lemma in the case where $G = \mathrm{SO}(k)$, and $G_0 = \mathrm{SO}(k-1)$. Then $G/G_0 = S^{k-1}$. Thus if $(E_{\mathrm{SO}(k)}, p, B_{\mathrm{SO}(k)}, \mathrm{SO}(k), \mathrm{SO}(k))$ is a universal SO(k)-bundle, and $(E, \pi, B_{\mathrm{SO}(k)}, S^{k-1}, \mathrm{SO}(k))$ is the associated $(k-1)$-sphere bundle, then $E = B_{\mathrm{SO}(k-1)}$.

11. The Universal Gysin Sequence. Let $(E_{\mathrm{SO}(n)}, p_n, B_{\mathrm{SO}(n)}, \mathrm{SO}(n), \mathrm{SO}(n))$ be a universal SO(n)-bundle, and let $(B_{\mathrm{SO}(n-1)}, \pi_n, B_{\mathrm{SO}(n)}, S^{n-1}, \mathrm{SO}(n))$ denote the associated $(n-1)$-sphere bundle. We will call the Gysin sequence of the latter bundle the "universal Gysin sequence". We will denote the homomorphisms of the universal Gysin sequence by $\mu_n : H^q(B_{\mathrm{SO}(n)}) \to H^{q+n}(B_{\mathrm{SO}(n)})$, $\pi_n^* : H^q(B_{\mathrm{SO}(n)}) \to H^q(B_{\mathrm{SO}(n-1)})$, and $\psi_n : H^q(B_{\mathrm{SO}(n-1)}) \to H^{q-n+1}(B_{\mathrm{SO}(n)})$, respectively.

The cohomology ring of the classifying space $B_{\mathrm{SO}(n)}$ has been studied rather thoroughly (see the following articles by A. Borel and the references given there: [1], [2]). We will use the following notation for certain cohomology classes of $B_{\mathrm{SO}(n)}$:

(a) Universal Stiefel-Whitney classes mod 2:

$$w_i(n) \in H^i(B_{\mathrm{SO}(n)}, Z_2), \qquad 2 \leqq i \leqq n.$$

(b) Universal integral Stiefel-Whitney classes:

$$W_i(n) \in H^i(B_{\mathrm{SO}(n)}, Z), \qquad 2 < i \leqq n, \qquad i \ \text{odd}.$$

(c) Universal Euler-Poincaré class:

$$W_n(n) \in H^n(B_{\mathrm{SO}(n)}, Z), \qquad n \ \text{even}.$$

(d) Universal Pontrjagin classes:

$$P_{4i}(n) \in H^{4i}(B_{\mathrm{SO}(n)}, Z), \qquad 1 \leqq i \leqq \tfrac{1}{2}n.$$

The integral Stiefel-Whitney classes are of order 2, while the Euler-Poincaré class and the Pontrjagin classes are of infinite order. The class $w_i(n)$ is the reduction mod 2 of $W_i(n)$, i odd or $i = n$, n even. It is known that all torsion elements of $H^*(B_{SO(n)}, Z)$ are of order 2.

Our main objective in this section is to prove the following two lemmas which will be used in the next section to prove theorem IV.

Lemma 7. *For n odd, $\psi_n[W_{n-1}(n-1)]$ is twice a generator of $H^0(B_{SO(n)}, Z)$.*

Lemma 8. *For n odd, $\pi_n^*[P_{2n-2}(n)] = [W_{n-1}(n-1)]^2$.*

Proof of lemma 7: Consider the following portion of the universal Gysin sequence (n odd, integer coefficients):

$$0 \to H^{n-1}(B_{SO(n)}) \xrightarrow{\pi_n^*} H^{n-1}(B_{SO(n-1)}) \xrightarrow{\psi_n} H^0(B_{SO(n)}) \xrightarrow{\mu_n} H^n(B_{SO(n)}).$$

Now μ_n is defined as follows: $\mu_n(x) = x \cdot W_n(n)$. Since $2W_n(n) = 0$, the kernel of $\mu_n : H^0(B_{SO(n)}) \to H^n(B_{SO(n)})$ is the subgroup of $H^0(B_{SO(n)})$ which is of index 2. By exactness, it follows that the cokernel of π_n^* is an infinite cyclic group; and to prove the lemma, it suffices to prove that $W_{n-1}(n-1)$ is a generator of this cokernel. Note that it follows from exactness that π_n^* maps the torsion subgroup of $H^{n-1}(B_{SO(n)})$ isomorphically onto the torsion subgroup of $H^{n-1}(B_{SO(n-1)})$.

Let $T(m)$ be a maximal toral subgroup of SO$(n-1)$ (here $m = \frac{1}{2}(n-1)$). Then $T(m)$ is also a maximal torus in SO(n) (we are considering that SO$(n-1)$ is imbedded in SO(n) as a subgroup). This leads to the following commutative diagram involving the integral cohomology rings of the corresponding classifying spaces:

$$\begin{tikzcd}
H^*(B_{SO(n)}) \arrow[rr, "\pi_n^*"] \arrow[dr, "\rho_1"'] & & H^*(B_{SO(n-1)}) \arrow[dl, "\rho_2"] \\
& H^*(B_{T(m)}) &
\end{tikzcd}$$

Now $H^*(B_{T(m)})$ is a polynomial algebra $Z[x_1, \cdots, x_m]$, where the generators x_i are of degree 2. The following facts about the homomorphisms ρ_1 and ρ_2 follow easily from the work of BOREL (*op. cit.*):

(a) The kernels of ρ_1 and ρ_2 consist of the torsion elements of $H^*(B_{SO(n)})$ and $H^*(B_{SO(n-1)})$, respectively.

(b) The image of ρ_1 is the ring of symmetric polynomials in the variables $x_1^2, x_2^2, \cdots, x_m^2$.

(c) The image of ρ_2 is the subring of $H^*(B_{T(m)})$ generated by the symmetric polynomials in $x_1^2, x_2^2, \cdots, x_m^2$ and the monomial $x_1x_2 \cdots x_m$.

(d) $\rho_1[P_{4i}(n)] = \rho_2[P_{4i}(n-1)] =$ the i^{th} elementary symmetric function in the variables $x_1^2, x_2^2, \cdots, x_m^2$ $(0 \leqq i \leqq m)$.

(e) $\rho_1[W_{n-1}(n-1)] = x_1x_2 \cdots x_m$.

From these facts it readily follows that $W_{n-1}(n-1)$ generates the cokernel of $\pi_n^* : H^{n-1}(B_{SO(n)}) \to H^{n-1}(B_{SO(n-1)})$ as desired.

Proof of lemma 8: Let $r_1 : H^*(B_{SO(n)}, Z) \to H^*(B_{SO(n)}, Z_2)$ and $r_2 : H^*(B_{SO(n-1)}, Z) \to H^*(B_{SO(n-1)}, Z_2)$ denote the homomorphisms which are reduction modulo 2. Since all torsion elements of $H^*(B_{SO(n-1)}, Z)$ are of order 2, in order to prove lemma 8, it suffices to prove the following two statements:

$$(\alpha) \qquad \rho_2\pi_n^*[P_{2n-2}(n)] = \rho_2[W_{n-1}(n-1)]^2.$$

$$(\beta) \qquad r_2\pi_n^*[P_{2n-2}(n)] = r_2[W_{n-1}(n-1)]^2.$$

Now (α) follows easily from statements (d) and (e) listed in the proof of lemma 7. To prove (β), one uses the following three facts:

$$(\mathrm{f}) \qquad r_1(P_{4i}(n)) = [w_{2i}(n)]^2.$$

$$(\mathrm{g}) \qquad r_2(W_i(n-1)) = w_i(n-1).$$

(h) In the universal Gysin sequence with coefficients mod $2, \pi_n^*[w_i(n)] = w_i(n-1)$, $0 \leqq i \leqq n-1$, and $\pi_n^*[w_n(n)] = 0$. These results are all proved in the work of Borel referred to above.

12. Proof of Theorem IV. Let $\phi_k : B_k' \to B_{SO(k)}$ denote the space and map obtained by "killing off" the integral cohomology class $W_k(k) \in H^k(B_{SO(k)}, Z)$ as described in section 8, and let $(E_k', \pi_k', B_k', S^{k-1}, SO(k))$ denote the induced bundle, $\phi^{-1}(B_{SO(k-1)}, \pi_k, B_{SO(k)}, S^{k-1}, SO(k))$. Then $(E_k', \pi_k', B_k', S^{k-1}, SO(k))$ is the universal bundle for $(k-1)$-sphere bundles with vanishing characteristic class W_k. Note also that there exists a continuous map $\Phi_k : E_k' \to B_{SO(k-1)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} E_k' & \xrightarrow{\Phi_k} & B_{SO(k-1)} \\ \downarrow \pi_k' & & \downarrow \pi_k \\ B_k' & \xrightarrow{\phi_k} & B_{SO(k)} \end{array} \tag{12.1}$$

Now let $(E, \pi, B, S^{k-1}, SO(k))$ be any $(k-1)$-sphere bundle with vanishing characteristic class. There exists a map $\eta : B \to B_{SO(k)}$ such that $(E, \pi, B, S^{k-1}, SO(k))$ is isomorphic to $\eta^{-1}(B_{SO(k-1)}, \pi_k, B_{SO(k)}, S^{k-1}, SO(k))$. Since the characteristic class W_k of (E, π, B, S^{k-1}) vanishes, $\eta^*(W_k(k)) = 0$, and hence there exists a map $f : B \to B_k'$ such that $\phi \circ f = \eta$. It follows that $(E, \pi, B, S^{k-1}, SO(k))$ is isomorphic to $f^{-1}(E_k', \pi_k', B_k', S^{k-1}, SO(k))$.

Lemma 9. *If k is odd and theorem IV is true for $(E_k', \pi_k', B_k', S^{k-1}, SO(k))$, then it is also true for $(E, \pi, B, S^{k-1}, SO(k))$,*

The proof of this lemma is straightforward and is left to the reader.

We will now prove that theorem IV is true for the bundle $(E_k', \pi_k', B_k', S^{k-1}, SO(k))$. The maps ϕ_k and Φ_k in diagram (12.1) induce maps of the universal Gysin sequence into that of (E_k', π_k', B_k') as indicated in the following diagram:

$$\begin{array}{ccccccccc} \cdots \xrightarrow{\mu_k} & H^i(B_{SO(k)}) & \xrightarrow{\pi_k^*} & H^i(B_{SO(k-1)}) & \xrightarrow{\psi_k} & H^{i-k+1}(B_{SO(k)}) & \xrightarrow{\mu_k} \cdots \\ & \downarrow \phi_k^* & & \downarrow \Phi_k^* & & \downarrow \phi_k^* & \\ 0 \to & H^i(B_k') & \xrightarrow{\pi_k'^*} & H^i(E_k') & \xrightarrow{\psi_k'} & H^{i-k+1}(B_k') & \to 0 \end{array}$$

Let $\omega \in H^0(B_{SO(k)}, Z)$ and $\omega' \in H^0(B'_k, Z)$ denote the unit elements. Assume that $a \in H^{k-1}(E'_k)$, $\alpha \in H^{2k-2}(B'_k)$, and $\beta \in H^{k-1}(B'_k)$ are chosen so that

$$(12.2) \qquad \begin{aligned} \psi'_k(a) &= \omega', \\ a^2 &= \pi'^*_k(\alpha) + \pi'^*_k(\beta)\cdot a. \end{aligned}$$

Choose $\epsilon = \pm 1$ so that

$$\psi_k[\epsilon W_{k-1}(k-1)] = 2\omega,$$

which is possible by lemma 7. One now computes easily that (writing $W_{k-1}(k-1) = W$)

$$\psi'_k[2a - \epsilon\Phi^*_k(W)] = 0,$$

hence by exactness there exists an element $c \in H^{k-1}(B)'_k$ such that

$$\pi'^*_k(c) = 2a - \epsilon\Phi^*_k(W)$$

or

$$(12.3) \qquad 2a = \epsilon\Phi^*_k(W) + \pi'^*_k(c).$$

Squaring both sides of this equation, we obtain

$$(12.4) \qquad 4a^2 = \Phi^*_k(W^2) + 2\epsilon\Phi^*_k(W)\pi'^*_k(c) + \pi'^*_k(c^2).$$

Now by lemma 8,

$$(12.5) \qquad \Phi^*_k(W^2) = \Phi^*_k[\pi^*_k(P_{2k-2}(k))] = \pi'^*_k\phi^*_k[P_{2k-2}(k)] = \pi'^*_k(P_{2k-2}),$$

and by (12.3),

$$(12.6) \qquad \epsilon\Phi^*_k(W) = 2a - \pi'^*_k(c).$$

Substituting (12.5) and (12.6) in (12.4), we obtain

$$(12.7) \qquad 4a^2 = \pi'^*_k(P_{2k-2} - c^2) + 4a\cdot\pi'^*_k(c).$$

If now we multiply equation (12.2) by four and compare with (12.7), we have

$$\begin{aligned} 4\alpha &= P_{2k-2} - c^2, \\ 4\beta &= 4c. \end{aligned}$$

To complete the proof, we will show that $\beta = c$. On the one hand, if we reduce both sides of equation (12.3) modulo two, we see that

$$c \equiv W_{k-1} \bmod 2.$$

But since $\beta \equiv W_{k-1} \bmod 2$ by theorem III, it follows that $\beta \equiv c \bmod 2$, *i.e.*, there exists an element x such that

$$\beta - c = 2x.$$

On the other hand, $4(\beta - c) = 0$, hence $8x = 0$. We will complete the proof by showing that the torsion subgroup of $H^{k-1}(B_k')$ contains only elements of order 2; it then follows that $8x = 0$ implies $2x = 0$, or $\beta = c$.

Lemma 10. *The torsion subgroup of $H^{k-1}(B_k')$ contains only elements of order 2.*

Proof: The map $\phi_k : B_k' \to B_{SO(k)}$ is a fibre map with fibre a space of type $K(Z, k-1)$. Consider the following portion of the cohomology sequence of this fibre space, which is exact by a theorem of SERRE ([12], proposition 5, chap. III):

$$0 \to H^{k-1}(B_{SO(k)}) \xrightarrow{\phi_k^*} H^{k-1}(B_k') \xrightarrow{i^*} H^{k-1}(Z, k-1) \xrightarrow{\tau^*} H^k(B_{SO(k)}).$$

By the definition of the space B_k' and the map ϕ_k, the transgression τ^* maps a generator of the infinite cyclic group $H^{k-1}(Z, k-1)$ onto $W_k(k)$, which is a cohomology class of order 2. Therefore the kernel of τ^*, which is also the image of i^*, is the subgroup of $H^{k-1}(Z, k-1)$ of index two. It follows that $H^{k-1}(B_k')$ is isomorphic to the direct sum of $H^{k-1}(B_{SO(k)})$ and an infinite cyclic group. Since the torsion elements of $H^*(B_{SO(k)})$ are all of order 2, the result follows.

13. Application to the Determination of the Secondary Obstruction to the Cross Section of a 2-sphere Bundle. Let $(E, p, B, S^2, SO(3))$ be an orientable 2-sphere bundle with B a polyhedron and vanishing characteristic class $W_3 \in H^3(B, Z)$. Since $W_3 = 0$, there exists a cross section over the 3-skeleton of B and the secondary obstruction to the extension of the cross section to the 4-skeleton is defined. A comparison of our formulas in section 8 with the work of LIAO [10] shows that the set of all possible elements α in formula (8.1) is precisely this secondary obstruction. Using theorem IV, we immediately obtain the following result:

Theorem*[9] *V. *Let $(E, p, B, S^2, SO(3))$ be a 2-sphere bundle with B a polyhedron, $W_3 = 0$, and no 2-torsion in $H^4(B, Z)$. Then there exists a cross section over the 4-skeleton if and only if there exists an integral cohomology class $\beta \in H^2(B, Z)$ such that*

(a) $\beta \equiv W_2 \bmod 2$,

(b) $\beta^2 = P_4$,

where W_2 and P_4 denote the Stiefel-Whitney class and Pontrjagin class respectively.

[9] It has been pointed out to me that a recent paper by W. T. WU entitled *Proof of a Certain Conjecture of Hopf* (Scientia Sinica, vol. 4 (1955) pp. 491–500) also indicates a connection between the Pontrjagin class P_4 and the second obstruction in a 2-sphere bundle. However, WU does not indicate how the Stiefel-Whitney class W_2 enters into the picture. WU bases his proofs on papers of HOPF (*Sur une formule de la théorie des espaces fibrés*, Colloque de Topologie (Espaces Fibrés), Brussels, 1950, pp. 117–121) and V. BOLTYANSKI (*Vector Fields in Manifolds*, Doklady Akad. Nauk URSS, 80 (1950) pp. 305–307), but apparently he did not know of LIAO's thesis.

14. Application to Prove Non-imbeddability of Certain Differentiable Manifolds in Euclidean Space. In this section we show how Theorems III and IV can be applied to prove the non-imbeddability of certain manifolds in Euclidean space of certain dimensions. Our method requires knowledge of certain Pontrjagin classes, but has the advantage that it works in some cases where all the Stiefel-Whitney classes of the tangent bundle to the manifold vanish.

We will use the following notation:

R^n = Euclidean n-space,

$P_n(C)$ = n-dimensional complex projective space ($2n$ real dimensions),

$P_n(Q)$ = n-dimensional quaternionic projective space ($4n$ real dimensions).

To illustrate our methods, we will prove the following results:

Theorem*[10] *VI. (a) *For $m > 1$, $P_{2m}(C)$ cannot be imbedded differentiably in R^{6m+1}.*

(b) *$P_n(Q)$ cannot be imbedded differentiably in R^{6n+1} for $n > 2$.*

First, we state a series of lemmas on which our method depends.

Lemma 11. *Let M^n be a compact orientable manifold imbedded differentiably in R^m, $m > n$. Then the characteristic class of the normal bundle to M^n vanishes.*

This is an old theorem of Seifert and Whitney. A recent proof may be found in Thom [15], corollary III, 15.

For the statements of the next two lemmas, let M^n be a compact, connected, orientable n-dimensional manifold imbedded differentiably in the sphere S^{n+1}, and let A and B denote the components of $S^{n+1} - M^n$. Then $\bar{A} = A \cup M^n$ and $\bar{B} = B \cup M^n$, where the bar denotes closure. Let $i : H^q(\bar{A}) \to H^q(M^n)$ and $j : H^q(\bar{B}) \to H^q(M^n)$ denote homomorphisms induced by inclusion maps.

Lemma 12. *For $q \geqq n$, $H^q(\bar{A}) = H^q(\bar{B}) = 0$.*

This follows from the Alexander duality theorem.

Lemma 13. *For $0 < q < n$, the injections $i : H^q(\bar{A}) \to H^q(M^n)$ and $j : H^q(\bar{B}) \to H^q(M^n)$ are isomorphisms into, and $H^q(M^n)$ is the direct sum of the image subgroups.*

This follows from the exactness of the Mayer-Vietoris sequence of the triad $(S^{n+1}; \bar{A}, \bar{B})$.

For any differentiable manifold M^n, we will denote certain characteristic classes as follows:

Stiefel-Whitney class (mod 2) of the tangent bundle,

$$w_i(M^n) \in H^i(M^n, Z_2).$$

[10] In a recent paper entitled *On Curvature and Characteristic Classes of a Riemannian Manifold* (Abh. Math. Sem. Univ. Hamburg, **20** (1955) pp. 117–126), S. S. Chern proves the following weaker result: $P_{2m}(C)$ cannot be imbedded differentiably in R^{6m-1}. He also proves that $P_{2m+1}(C)$ cannot be imbedded differentiably in R^{6m+1}. In his proofs, he uses Pontrjagin classes with real coefficients.

Pontrjagin class of the tangent bundle,

$$P_{4i}(M^n) \in H^{4i}(M^n, Z).$$

Stiefel-Whitney class (mod 2) of the normal bundle for any imbedding in Euclidean space,

$$\bar{w}_i(M^n) \in H^i(M^n, Z_2).$$

Pontrjagin class of the normal bundle for any imbedding in Euclidean space,

$$\bar{P}_{4i}(M^n) \in H^{4i}(M^n).$$

Total Stiefel-Whitney classes:

$$w(M^n) = \sum_{i \geq 0} w_i(M^n),$$

$$\bar{w}(M^n) = \sum_{i \geq 0} \bar{w}_i(M^n).$$

Total Pontrjagin classes:

$$P(M^n) = \sum_{i \geq 0} P_{4i}(M^n),$$

$$\bar{P}(M^n) = \sum_{i \geq 0} \bar{P}_{4i}(M^n).$$

According to the Whitney duality theorem, $w(M^n) \cdot \bar{w}(M^n) = 1$. Similarly, if $H^*(M^n, Z)$ has no 2-torsion, then $P(M^n) \cdot \bar{P}(M^n) = 1$ (see A. Borel [1]).

Lemma 14. *If y is a generator of $H^2(P_n(C), Z_2)$, then $w(P_n(C)) = (1 + y)^{n+1}$; similarly, if y denotes a generator of $H^4(P_n(Q), Z_2)$, then $w(P_n(Q)) = (1 + y)^{n+1}$.*

This lemma may be proved by the method of Wu [18]. The first statement is due originally to E. Stiefel.

Corollary. *$\bar{w}(P_n(C)) = (1 + y)^{-n-1}$ and $\bar{w}(P_n(Q)) = (1 + y)^{-n-1}$, where y denotes a generator of $H^2(P_n(C), Z_2)$ or $H^4(P_n(Q), Z_2)$ as is appropriate.*

To prove the corollary, one uses the Whitney duality theorem.

Lemma 15. *If x denotes a generator of $H^2(P_n(C), Z)$, then $P(P_n(C)) = (1 + x^2)^{n+1}$ and $\bar{P}(P_n(C)) = (1 + x^2)^{-n-1}$.*

For the proof, see Hirzebruch [9], *Satz* 4.10.2.

Lemma 16. *For proper choice of a generator x of $H^4(P_n(Q), Z)$, $P(P_n(Q)) = (1 + x)^{2n+2}(1 + 4x)^{-1}$ and $\bar{P}(P_n(Q)) = (1 + x)^{-2n-2}(1 + 4x)$.*

The proof is given in a paper by Hirzebruch [8].

Lemma 17. *For every integer $m > 1$ there exists a prime p such that*

$$4m + 2 < p < 6m.$$

Proof: According to a theorem of I. SCHUR ([19], *Hilfsatz* I), for any integer $x \geqq 29$, there exists a prime p such that $x < p \leqq \frac{5}{4}x$. Therefore, if we take $x = 4m + 2$, we see that the lemma is true for $m \geqq 7$. It is readily checked directly that the lemma is true for $m = 2, 3, 4, 5$, and 6.

Alternatively, one could use the main theorem of a paper by R. BREUSCH [20] to prove this lemma.[11]

Proof of theorem VI (a): Suppose the theorem is false and that $P_{2m}(C)$ can be imbedded differentiably in R^{6m+1}. We may add the point at infinity and thus assume that $P_{2m}(C)$ is imbedded differentiably in S^{6m+1}. We assume that S^{6m+1} is given a definite Riemannian metric. Choose a positive number ϵ so small that given any point $a \in S^{6m+1}$ of distance $\leqq \epsilon$ from $P_{2m}(C)$, there exists a unique geodesic segment through a of length $\leqq \epsilon$ normal to $P_{2m}(C)$. Let N denote the set of all points $a \in S^{6m+1}$ whose distance from $P_{2m}(C)$ is $< \epsilon$. We will call N an *open tubular neighborhood* of $P_{2m}(C)$ in S^{6m+1}. Let E denote the boundary of N and $\pi : E \to P_{2m}(C)$ the projection defined by assigning to any point $a \in E$ the point $\pi(a) \in P_{2m}(C)$ where the unique geodesic segment through a of length ϵ normal to $P_{2m}(C)$ meets $P_{2m}(C)$. Then $(E, \pi, P_{2m}(C), S^{2m}, \mathrm{SO}(2m + 1))$ is a sphere bundle which is associated with the bundle of all normal vectors to $P_{2m}(C)$ in S^{6m+1}.

Since E is a differentiable hypersurface in S^{6m+1}, we can apply lemmas 12 and 13 to it. We will denote the complement of $\bar{N}$ in S^{6m+1} by R. Since the injections $H^q(\bar{N}) \to H^q(E)$ and $H^q(\bar{R}) \to H^q(E)$ are isomorphisms into for *all* q, we may identify $H^*(\bar{N})$ and $H^*(\bar{R})$ with certain subrings of $H^*(E)$; this we will do. With this convention,

$$\tag{14.1} \begin{cases} H^0(\bar{N}) = H^0(\bar{R}) = H^0(E), \\ H^q(E) = H^q(\bar{R}) + H^q(\bar{N}) \qquad (0 < q < 6m, \quad \text{direct sum}), \\ H^{6m}(\bar{R}) = H^{6m}(\bar{N}) = 0. \end{cases}$$

Moreover, the subring $H^*(\bar{N})$ is obviously the same as the image of the projection $\pi^* : H^*(P_{2m}(C)) \to H^*(E)$.

On the other hand, the characteristic class of the normal bundle vanishes by lemma 11; hence we can use theorems III and IV to compute the cohomology ring of E. According to these theorems, we may choose an element $z \in H^{2m}(E, Z)$ such that every element $u \in H^q(E)$ can be expressed uniquely in the form

$$u = \pi^*(u_1) + z \cdot \pi^*(u_2)$$

and

$$z^2 = \pi^*(\alpha) + z \cdot \pi^*(\beta)$$

where $\beta \in H^{2m}(E, Z)$, $\alpha \in H^{4m}(E, Z)$ are any elements such that

[11]The author is indebted to N. C. ANKENY for pointing out the existence of references [19] and [20].

$$\beta \equiv \bar{w}_{2m}(P_{2m}(C)) \qquad \bmod 2,$$

$$4\alpha + \beta^2 = \bar{P}_{4m}(P_{2m}(C)).$$

By the corollary to lemma 14, $\bar{w}_{2m}(P_{2m}(C)) = \binom{3m}{m}y^m$; hence we may choose $\beta = \binom{3m}{m}x^m$. By lemma 15, $\bar{P}_{4m}(P_{2m}(C)) = (-1)^m\binom{3m}{m}x^{2m}$; hence

$$\alpha = \tfrac{1}{4}[\bar{P}_{4m} - \beta^2] = \frac{1}{4}\left[(-1)^m\binom{3m}{m} - \binom{3m}{m}^2\right]x^{2m}.$$

If we identify the generator $x \in H^2(P_{2m}(C))$ with $\pi^*(x)$, then the structure of the integral cohomology ring of E may be described as follows: It is a commutative ring generated by the elements x and z of degrees 2 and $2m$ respectively, and subject to the following relations:

$$\text{(14.2)} \qquad \begin{cases} x^{2m+1} = 0, \\ z^2 = \dfrac{1}{4}\left[(-1)^m\dbinom{3m}{m} - \dbinom{3m}{m}^2\right]x^{2m} + \dbinom{3m}{m}x^m z. \end{cases}$$

With this notation, $H^*(\bar{N})$ is the subring generated by x, since $H^*(\bar{N}) = \pi^*H^*(P_{2m}(C))$.

We will now apply all this data to obtain a contradiction. $H^{2m}(E)$ and $H^{4m}(E)$ are free abelian groups of rank 2 with bases $\{x^m, z\}$ and $\{x^{2m}, x^m z\}$ respectively, while $H^{2m}(\bar{N})$ and $H^{4m}(\bar{N})$ are infinite cyclic groups generated by x^m and x^{2m} respectively. If we apply (14.1) with $q = 2m$ and $q = 4m$, we see that $H^{2m}(\bar{R})$ and $H^{4m}(\bar{R})$ must also be infinite cyclic groups with generators of the form

$$u = \lambda x^m + z,$$

$$v = \nu x^{2m} + x^m z$$

respectively for appropriate integers λ and ν. Since $H^{6m}(\bar{R}) = 0$, we must have $uv = 0$. Using (14.2), we obtain

$$uv = (\lambda x^m + z)(\nu x^{2m} + x^m z) = \left[\lambda + \nu + \binom{3m}{m}\right]x^{2m}z.$$

Since $x^{2m}z$ is a generator of the infinite cyclic group $H^{6m}(E)$, this implies

$$\text{(14.3)} \qquad \lambda + \nu + \binom{3m}{m} = 0.$$

Next, since $H^*(\bar{R})$ is a subring, u^2 must be an integral multiple of v. Again computing using (14.2), we obtain

$$u^2 = (\lambda x^m + z)^2 = \left\{\lambda^2 + \frac{1}{4}\left[(-1)^m\binom{3m}{m} - \binom{3m}{m}^2\right]\right\}x^{2m} + \left[2\lambda + \binom{3m}{m}\right]x^m z.$$

Therefore if u^2 is a multiple of $v = \nu x^{2m} + x^m z$, we must have

$$\lambda^2 + \frac{1}{4}\left[(-1)^m\binom{3m}{m} - \binom{3m}{m}^2\right] = \nu\left[2\lambda + \binom{3m}{m}\right]. \tag{14.4}$$

Now eliminate ν between (14.3) and (14.4). The result is the following quadratic equation for λ:

$$3\lambda^2 + 3\binom{3m}{m}\lambda + \frac{1}{4}\left[(-1)^m\binom{3m}{m} + 3\binom{3m}{m}^2\right] = 0. \tag{14.5}$$

The discriminant of this quadratic equation is $(-1)^{m+1}3\binom{3m}{m}$. As is well known, if a quadratic equation has integral solutions, then the discriminant is a perfect square. Now if m is even, the discriminant is negative, and hence not a perfect square. If m is odd, $m = 2n + 1$, the discriminant is $3\binom{6n+3}{2n+1}$; since there is always a prime p such that $4n + 2 < p < 6n + 3$ if $n \geqq 1$ (see lemma 17), it follows that $3\binom{6n+3}{2n+1}$ cannot be a perfect square for $n > 1$. Thus, for $m > 1$, $(-1)^{m+1}3\binom{3m}{m}$ is never a perfect square. This is a contradiction. Hence theorem VI(a) is proved.

Remarks: 1. Theorem VI(a) is false for $m = 1$. WHITNEY [17] has given a differentiable imbedding of $P_2(C)$ in R^7.

2. This theorem does not give the best possible result for all values of m. For example, for $m = 2$, it can be proved using Stiefel-Whitney classes of the normal bundle that $P_4(C)$ cannot be imbedded differentiably in R^{14}, or more generally that $P_k(C)$ cannot be imbedded differentiably in R^{4k-2} for $k = 2^n$. I do not know whether or not theorem VI(a) gives the best possible results for those values of m for which the Stiefel-Whitney classes are useless.

3. For most values of m, it does not appear possible to prove by this method the analogue of theorem VI(a) for $P_{2m+1}(C)$ (namely, that $P_{2m+1}(C)$ cannot be imbedded differentiably in R^{6m+3}). An exception is the case $m = 3$: One can prove that $P_7(C)$ cannot be imbedded differentiably in R^{21}. The details of the proof are left to the reader. This example is especially interesting because $\bar{w}(P_7(C)) = 1$, *i.e.*, all Stiefel-Whitney classes of the normal bundle vanish.

Proof of theorem VI (b) in case $n = 2m + 1$, $m > 0$: Here again one assumes the theorem is false, and that $P_{2m+1}(Q)$ can be imbedded in S^{12m+7}. By exactly the same method as in the proof of theorem VI (a), one chooses a positive number ϵ and constructs the spaces N, E, and R, and the projection $\pi : E \to P_{2m+1}(Q)$. Then $(E, \pi, P_{2m+1}(Q), S^{4m+2}, \mathrm{SO}(4m + 3))$ is a sphere bundle which is an associate of the bundle of normal vectors, and E is a hypersurface in S^{12m+7}. One may consider that $H^*(\bar{N})$ and $H^*(\bar{R})$ are subalgebras of $H^*(E)$, and we have the analogue of (14.1):

$$\begin{cases} H^q(E) = H^q(\bar{R}) + H^q(\bar{N}), & 0 < q < 12m + 6, \\ H^{12m+6}(\bar{R}) = H^{12m+6}(\bar{N}) = 0. \end{cases} \tag{14.6}$$

By lemma 16,

$$\bar{P}_{8m+4}(P_{2m+1}(Q)) = 3\binom{6m+2}{2m-1}x^{2m+1}$$

where $x \in H^4(P_{2m+1}(Q))$ is an appropriately chosen generator. For simplicity, we will let $q = \frac{3}{4}\binom{6m+2}{2m-1}$; then $\bar{P}_{8m+4} = 4qx^{2m+1}$ and $q \neq 0$. Making use of theorem IV, one sees that the structure of $H^*(E)$ may be described as follows: It is a commutative ring generated by $x \in H^4(E)$ and $z \in H^{4m+2}(E)$ subject to the relations

$$\begin{cases} x^{2m+2} = 0, \\ z^2 = qx^{2m+1}. \end{cases} \tag{14.7}$$

Furthermore, the subring $H^*(\bar{N})$ is the subring generated by x. From (14.6) it follows that $H^{4k+2}(\bar{R})$ is an infinite cyclic group generated by zx^i, where $i = k - m, m \leqq k \leqq 3m$, while $H^j(\bar{R}) = 0$ for all other values of $j > 0$. However, this leads to an immediate contradiction, since $H^*(\bar{R})$ is not closed under multiplication. For example, $(zx)^2 = z^2x^2 = qx^{2m+3}$ by (14.7), and $qx^{2m+3} \in H^*(\bar{N})$, $qx^{2m+2} \neq 0$.

Proof of theorem VI (b) in case $n = 2m$, $m > 1$: The basic idea of the proof in this case is the same as in the previous case. The details of the computations are much the same as in the proof of theorem VI (a). As before, we assume the theorem false and that $P_{2m}(Q)$ can be imbedded differentiably in S^{12m+1}, and then construct the spaces N, E, and R, and the projection $\pi : E \to P_{2m}(Q)$ as before. $(E, \pi, P_{2m}(Q), S^{4m}, \mathrm{SO}(4m+1))$ is the normal sphere bundle. We have the direct sum decomposition

$$H^q(E) = H^q(\bar{N}) + H^q(\bar{R}), \qquad 0 < q < 12m, \tag{14.8}$$

and

$$H^{12m}(\bar{N}) = H^{12m}(\bar{R}) = 0. \tag{14.9}$$

By the corollary to lemma 14, $\bar{w}_{4m}(P_{2m}(Q)) = \binom{3m}{m}y^m$, where y is a generator of $H^4(P_{2m}(Q))$. By lemma 16, $\bar{P}_{8m}(P_{2m}(Q)) = \left[\binom{6m+1}{2m} - 4\binom{6m}{2m-1}\right]x^{2m}$, where x is an appropriately chosen generator of $H^4(P_{2m}(Q), Z)$. If we identify the generator $x \in H^4(P_{2m}(Q))$ and $\pi^*(x) \in H^4(E)$, then, by using theorems III and IV, we see that the structure of $H^*(E)$ can be described as follows: $H^*(E)$ is a commutative ring generated by $x \in H^4(E)$ and $z \in H^{4m}(E)$ subject to the following relations:

$$\begin{cases} x^{2m+1} = 0, \\ z^2 = \frac{1}{4}\left[\binom{6m+1}{2m} - 4\binom{6m}{2m-1} - \binom{3m}{m}^2\right]x^{2m} + \binom{3m}{m}x^m z. \end{cases} \tag{14.10}$$

Also, $H^*(\bar{N})$ is the subring generated by x. We will now apply (14.8) for $q = 4m$ and $q = 8m$ to determine $H^{4m}(\bar{R})$ and $H^{8m}(\bar{R})$. We see that $H^{4m}(E)$ and $H^{8m}(E)$ are free abelian groups of rank 2 generated by $\{x^m, z\}$ and $\{x^{2m}, x^m z\}$ respectively, while $H^{4m}(\bar{N})$ and $H^{8m}(\bar{N})$ are infinite cyclic groups generated by x^m and x^{2m} respectively. It follows from (14.8) that $H^{4m}(\bar{R})$ and $H^{8m}(\bar{R})$ are also infinite cyclic groups, and they must be generated by

$$u = \lambda x^m + z,$$
$$v = \nu x^{2m} + x^m z,$$

respectively, for appropriate choices of the integers λ and ν. By (14.9), $H^{12m}(\bar{R}) = 0$; therefore $uv = 0$. If we compute using (14.10), we find that

$$uv = \left[\lambda + \nu + \binom{3m}{m}\right] x^{2m} z,$$

and since $x^{2m}z$ is a generator of $H^{12m}(E)$, we must have

$$\lambda + \nu + \binom{3m}{m} = 0. \tag{14.11}$$

Since $H^*(\bar{R})$ is a subring of $H^*(E)$, u^2 must be an integral multiple of v. Now it follows from (14.10) that

$$u^2 = \left\{\lambda^2 + \frac{1}{4}\left[\binom{6m+1}{2m} - 4\binom{6m}{2m-1} - \binom{3m}{m}^2\right]\right\} x^{2m} + \left[2\lambda + \binom{3m}{m}\right] x^m z.$$

If u^2 is to be an integral multiple of $v = \nu x^{2m} + x^m z$, we must have

$$\lambda^2 + \frac{1}{4}\left[\binom{6m+1}{2m} - 4\binom{6m}{2m-1} - \binom{3m}{m}^2\right] = \nu\left[2\lambda + \binom{3m}{m}\right]. \tag{14.12}$$

Elimination of ν between (14.11) and (14.12) gives the following quadratic equation for λ:

$$3\lambda^2 + 3\binom{3m}{m}\lambda + \frac{1}{4}\left[\binom{6m+1}{2m} - 4\binom{6m}{2m-1} + 3\binom{3m}{m}^2\right] = 0. \tag{14.13}$$

The discriminant of this quadratic equation is $9\binom{6m-1}{2m-2}$; since there is always a prime between $4m + 2$ and $6m$ if $m > 1$ (see lemma 17), it follows that, for $m > 1$, $9\binom{6m-1}{2m-2}$ cannot be a perfect square. Therefore (14.13) does not have any integral solutions for $m > 1$, which is a contradiction. This completes the proof of theorem VI (b).

Remarks. 1. Apparently the restriction that $n > 2$ is necessary in theorem VI (b). Although there is apparently nowhere in the literature a description of an explicit differentiable imbedding of $P_2(Q)$ in R^{13}, Dr. F. Peterson has communicated to the author a method for a topological imbedding of $P_2(Q)$ in R^{13}. Presumably this imbedding could be "smoothed out" so as to obtain a differentiable imbedding.

2. This theorem does not always give the best possible result. For example, if $n = 2^k$, then it can be proved that $P_n(Q)$ cannot be imbedded in R^{8n-4}. The proof uses only Stiefel-Whitney classes mod 2.

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