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On the Collective Motion in Even-Even Spherical Nuclei*

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A method formally analogous to that developed by Bogoliubov, Tolmachev and Shirkov to investigate the collective excitations in superconductors is applied to even-even spherical nuclei in order to investigate the mechanism of the nuclear collective motion from the standpoint of particle excitations. Our theory leads, in principle, to the same results as those obtained by Belyaev. However, the method of description of the nuclear collective motion is quite different from that of Belyaev's paper in which the "cranking model" of Inglis is employed, and the various physical parameters used by Belyaev can be derived uniquely from the "first" principle. Thus, in so far as the vibrational motion is concerned, the physical implication underlying the nuclear collective model proposed by Bohr and Mottelson is made clear.

It is outside the scope of this paper to relate the effective inter-particle interactions used in this paper with the nuclear forces known from the two nucleon problems.

§ 1. Introduction

The main purpose of this paper is to investigate the mechanism of collective motion in even-even spherical nuclei from the standpoint of particle excitations. It is not our purpose to go into detailed quantitative calculations, but rather to develop the basic idea. The *j-j* coupling shell model is the starting point of our theory, and the pairing correlation between two nucleons is taken into account by means of the Bogoliubov transformation.[3]

Our theory leads, in principle, to the same results as those obtained by Belyaev.[4] However, the method of description of the nuclear collective motion is quite different from that of Belyaev's paper[4] in which the "cranking model" of Inglis[5] is employed. It is attempted to propose a method which is more fundamental than the "cranking model". The method is based on an extension of the theory of Sawada et al.[6] of the plasma oscillation, and is formally analogous to the method developed by Bogoliubov, Tolmachev and Shirkov[3] to investigate the collective excitations in superconductors.

To take into account the pairing correlation for which the "short range" part of the effective two-body interaction between particles plays an important role, the technique of the Bogoliubov transformation[3] used in the new theory of super-

* The basic idea which will be developed in this paper corresponds, in some sense, to the generalization of that proposed in our previous papers.[1),2)]

conductivity will be adopted, and the results of the application of the transformation to even-even spherical nuclei will be recapitulated in § 2, § 3 and § 4 in a form especially suited for our purpose.*

The fundamental idea and the physical basis of our method describing the nuclear collective vibrations will be developed in § 5, § 6 and § 7. In order to make clear that the essential idea of the theory of Sawada et al.[6] can be applied to the *finite size* nuclear system, the *explicit* relationship of this idea with that of the "time-dependent self-consistent field method" underlying the "cranking model"[5] will be established in § 5 and § 6, provided that the "long range" part of the effective two-body interaction between particles plays an important role for the collective vibration of even-even spherical nuclei. Thus it will be shown in § 7 that a method formally analogous to that developed by Bogoliubov, Tolmachev and Shirkov[3] to investigate the collective excitations in superconductors can describe the collective motion of even-even spherical nuclei.

By using this method it will be shown in § 8 that there are two kinds of quadrupole vibration-modes for even-even spherical nuclei. To display these collective modes, the "method of auxiliary variables"[7] will be employed in § 9. The relation between the collective variables thus obtained and the "deformation variables" used in the Bohr-Mottelson model[8] will be clarified in § 10. In § 11 and § 12, the normal vibrations of even-even spherical nuclei will be discussed, and the inertial parameter and the surface tension parameter for the quadrupole vibration will be uniquely determined. It will be shown that the results thus obtained are essentially equivalent to those obtained by Belyaev[4] and are in agreement with the observed trends.

It is hoped that our approach will be helpful to clarify the concept of the nuclear collective motion and to give a further insight into the various aspects of the unified model.

§ 2. The Bogoliubov transformation

An application of the Bogoliubov transformation[3] in the new theory of superconductivity to the nuclear system has been developed by Belyaev.[4] The close connection between the basic idea of this method and Racah's seniority concept in the *j-j* coupling shell model has also been discussed by many authors.[9] In this and next sections, we shall recapitulate the results in a form especially suited for our purpose.

Let us consider a system of nucleons which are moving in a spherically symmetrical self-consistent potential-well V and choose the wave functions of a nucleon in this well as the basic functions of the second quantization representation. If we adopt the *j-j* coupling shell model, such single-particle states are characterized

* The reader who is familiar with the application of the Bogoliubov transformation to the nuclear system need not read § 2 and § 3 in full but the final results (3·14) and (3·15).

by the quantum numbers (n, l, j, m). Hereafter we shall denote a set of the quantum numbers (n, l, j) by N. The Hamiltonian for the system of interacting nucleons is then

$$H=\sum_{\substack{Nm\\N'm'}}\langle Nm|T|N'm'\rangle a_{Nm}^{+}a_{N'm'}-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G|N_2'm_2'N_1'm_1'\rangle \times a_{N_1m_1}^{+}a_{N_2m_2}^{+}a_{N_2'm_2'}a_{N_1'm_1'}-\lambda\sum_{Nm}a_{Nm}^{+}a_{Nm}$$

$$=\sum_{\substack{Nm\\N'm'}}\{(\epsilon_N-\lambda)\delta_{NN'}\delta_{mm'}+\langle Nm|V|N'm'\rangle\}a_{Nm}^{+}a_{N'm'} -\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G|N_2'm_2'N_1'm_1'\rangle a_{N_1m_1}^{+}a_{N_2m_2}^{+}a_{N_2'm_2'}a_{N_1'm_1'}, \quad (2\cdot1)$$

where ϵ_N is the single-particle energy in N-state and λ is the well-known chemical potential. The sign of G and V is chosen to be positive for an attractive interaction and an attractive well respectively.

Now it is convenient for our purpose to divide the effective two-body interaction G approximately into three parts:

$$G=G^{(0)}+G^{(1)}+G^{(2)}, \quad (2\cdot2)$$

where $G^{(0)}$ the part mainly contributing to the "pairing interaction", $G^{(1)}$ the part mainly contributing to the spherically symmetrical self-consistent field, and $G^{(2)}$ is the one that is responsible for the collective vibrations. The possibility of such an approximate division of G will be discussed in § 3. Corresponding to (2·2), the Hamiltonian (2·1) may be divided as follows:

$$H=H^{(0)}+H^{(\text{res.})}, \quad (2\cdot3\text{a})$$

$$H^{(0)}=\sum_{\substack{Nm\\N'm'}}\{(\epsilon_N-\lambda)\delta_{NN'}\delta_{mm'}+\langle Nm|V|N'm'\rangle\}a_{Nm}^{+}a_{N'm'} -\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G^{(0)}+G^{(1)}|N_2'm_2'N_1'm_1'\rangle a_{N_1m_1}^{+}a_{N_2m_2}^{+}a_{N_2'm_2'}a_{N_1'm_1'}, \quad (2\cdot3\text{b})$$

$$H^{(\text{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G^{(2)}|N_2'm_2'N_1'm_1'\rangle a_{N_1m_1}^{+}a_{N_2m_2}^{+}a_{N_2'm_2'}a_{N_1'm_1'}. \quad (2\cdot3\text{c})$$

Following Belyaev's,[4] let us introduce the new Fermi operators α_{Nm} by the Bogoliubov transformation:[3]

$$a_{Nm}=u_{Nm}\alpha_{Nm}+v_{Nm}\alpha_{N-m}^{+}, \quad (2\cdot4)$$

where u_{Nm} and v_{Nm} are real numbers which obey the conditions

$$u_{Nm}=u_{N-m}=u_{N\nu}$$

$$v_{Nm}=-v_{N-m}=\begin{cases}+v_{N\nu} & m>0\\ -v_{N\nu} & m<0,\end{cases} \quad \nu\equiv|m|, \quad (2\cdot5)$$

$$u_{Nm}^2+v_{Nm}^2=1. \qquad (2\cdot6)$$

After the transformation $(2\cdot4)$, the Hamiltonian $H^{(0)}$ has the following structure:

$$H^{(0)}=U+H_{20}+H_{11}+H', \qquad (2\cdot7)$$

where

$$U=\sum_{N\nu}(\epsilon_N-\lambda)\cdot 2v_{N\nu}^2-\sum_{N\nu}\Delta_{N\nu}u_{N\nu}v_{N\nu}+\sum_{N\nu}\langle N\nu|V|N\nu\rangle v_{N\nu}^2, \qquad (2\cdot8)$$

$$H_{20}=\sum_{N\nu}\{(\epsilon_N-\lambda)\cdot 2u_{N\nu}v_{N\nu}-\Delta_{N\nu}(u_{N\nu}^2-v_{N\nu}^2)\}(\alpha_{N\nu}^+\alpha_{N-\nu}^++\alpha_{N-\nu}\alpha_{N\nu}), \qquad (2\cdot9)$$

$$H_{11}=\sum_{N\nu}\{(\epsilon_N-\lambda)(u_{N\nu}^2-v_{N\nu}^2)+\Delta_{N\nu}\cdot 2u_{N\nu}v_{N\nu}\}(\alpha_{N\nu}^+\alpha_{N\nu}+\alpha_{N-\nu}^+\alpha_{N-\nu}). \qquad (2\cdot10)$$

H' is a small perturbation term which will be neglected hereafter. $\Delta_{N\nu}$ in $(2\cdot8)$—$(2\cdot10)$ is defined by

$$\Delta_{N\nu}=\sum_{N_1\nu_1}\langle N\nu\ N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle u_{N_1\nu_1}v_{N_1\nu_1}, \qquad (2\cdot11)$$

where

$$\langle N\nu\ N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle\equiv\langle N\ +\nu,\ N\ -\nu|G^{(0)}|N_1\ -\nu_1,\ N_1\ +\nu_1\rangle$$
$$-\langle N\ +\nu,\ N\ -\nu|G^{(0)}|N_1\ +\nu_1,\ N_1\ -\nu_1\rangle. \qquad (2\cdot12)$$

The matrix elements $\langle N\nu|V|N'\nu'\rangle$ in $(2\cdot8)$ have the form

$$\langle N\nu|V|N'\nu'\rangle\equiv\langle N\ +\nu|V|N'\ +\nu'\rangle\delta_{\nu\nu'}=\langle N\ -\nu|V|N'\ -\nu'\rangle\delta_{\nu\nu'}. \qquad (2\cdot13)$$

In deriving $(2\cdot8)$—$(2\cdot10)$, we have made an approximation

$$\langle N\nu\ N'\nu'|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle=\langle N\nu\ N'\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle\delta_{\nu\nu'}$$
$$\simeq 0 \quad \text{for} \quad N\neq N', \qquad (2\cdot14)$$

and used the relation

$$\langle N\nu|V|N'\nu'\rangle=\sum_{N_1\nu_1}\{\langle N\ +\nu,\ N_1\ -\nu_1|G^{(1)}|N_1\ -\nu_1,\ N'\ +\nu'\rangle$$
$$-\langle N+\nu,\ N_1-\nu_1|G^{(1)}|N'\ +\nu',\ N_1\ -\nu_1\rangle+\langle N+\nu,\ N_1+\nu_1|G^{(1)}|N_1+\nu_1,\ N'+\nu'\rangle$$
$$-\langle N\ +\nu,\ N_1\ +\nu_1|G^{(1)}|N'\ +\nu',\ N_1\ +\nu_1\rangle\}v_{N_1\nu_1}^2, \qquad (2\cdot15)$$

which means the definition of the self-consistent field V.

§ 3. Ground- and excited-states of $H^{(0)}$

Let us determine the transformation coefficients u_{Nm} and v_{Nm} in $(2\cdot4)$ by the condition

$$\langle c_0|\alpha_{N-\nu}\alpha_{N\nu}H^{(0)}|c_0\rangle=0, \qquad (3\cdot1)$$

where $|c_0\rangle$ is the vacuum of the quasi-particles α_{Nm}, i.e.

$$\alpha_{Nm}|c_0\rangle=0, \quad \langle c_0|\alpha_{Nm}^+=0. \qquad (3\cdot2)$$

The vacuum state $|c_0\rangle$ thus determined becomes the ground state of $H^{(0)}$ and corresponds to the "lowest seniority state" of even-even spherical nuclei.

The condition (3·1) means that $H_{20}=0$, i.e.

$$(\epsilon_N-\lambda)\cdot 2u_{N\nu}v_{N\nu}-\Delta_{N\nu}(u_{N\nu}^2-v_{N\nu}^2)=0. \tag{3·3}$$

Using (3·3), (2·6) and (2·11), we find the following equation for $\Delta_{N\nu}$:

$$\Delta_{N\nu}=\frac{1}{2}\sum_{N_1\nu_1}\frac{\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle}{\sqrt{(\epsilon_{N_1}-\lambda)^2+\Delta_{N_1\nu_1}^2}}\Delta_{N_1\nu_1}. \tag{3·4}$$

A. *Solutions of the equation for* $\Delta_{N\nu}$

Eq. (3·4) has a trivial solution

$$\Delta_{N\nu}=0 \quad \text{or} \quad u_{N\nu}v_{N\nu}=0, \tag{3·5}$$

which corresponds to the sharp Fermi surface. If $G^{(0)}$ is sufficiently weak, (3·5) is the only solution of (3·4). However, if $G^{(0)}$ satisfies the inequality

$$\frac{1}{2}\sum_{N_1\nu_1}\frac{\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle}{|\epsilon_{N_1}-\lambda|}>1, \tag{3·6}$$

then there exists also a non-trivial solution.

Hereafter we shall make the following approximations:

i) Since the most essential contribution to the sum in (3·4) is given by the transitions between the states in the same shell,* we shall neglect the transitions between different shells, i.e. we shall assume

$$\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle \simeq \langle N\nu\, N\nu|G^{(0)}|N\nu_1 N\nu_1\rangle\delta_{NN_1}. \tag{3·7}$$

ii) To simplify the problem, we shall assume further that the inequality (3·6) under (3·7) may be fulfilled only for the outermost partially-filled shell (the N_0-shell),** and the trivial solution (3·5) is the only solution for the other shells.***

iii) In order that the non-trivial solution exists for the N_0-shell, the matrix elements $\langle N_0\nu_0 N_0\nu_0|G^{(0)}|N_0\nu_0' N_0\nu_0'\rangle$ ("pairing interaction") must have a same sign, because otherwise there will occur a cancellation. So we shall approximate the matrix element $\langle N_0\nu_0 N_0\nu_0|G^{(0)}|N_0\nu_0' N_0\nu_0'\rangle$ by an average over the m_0-states, i.e.

$$\langle N_0\nu_0\, N_0\nu_0|G^{(0)}|N_0\nu_0'\, N_0\nu_0'\rangle \simeq \overline{\langle N_0\nu_0\, N_0\nu_0|G^{(0)}|N_0\nu_0'\, N_0\nu_0'\rangle}=\overline{G^{(0)}}>0. \tag{3·8}$$

* As discussed by Belyaev,[4] transitions between different shells lead simply to some renormalization of $G^{(0)}$.

** Hereafter we shall denote the quantum numbers of the single-particle states in this outermost partially-filled shell by adding the suffix 0.

*** This assumption corresponds to that we are considering the spherical nuclei for which the level distances $|\epsilon_N-\epsilon_{N_0}|$ satisfy the condition

$$\overline{G_N^{(0)}}\,\Omega_N-\overline{G^{(0)}}\,\Omega(1-n/\Omega)<2(\epsilon_N-\epsilon_{N_0}) \quad \text{for} \quad \epsilon_N>\epsilon_{N_0},$$
$$\overline{G_N^{(0)}}\,\Omega_N+\overline{G^{(0)}}\,\Omega(1-n/\Omega)<2(\epsilon_{N_0}-\epsilon_N) \quad \text{for} \quad \epsilon_N<\epsilon_{N_0}.$$

Here $\overline{G_N^{(0)}}$ is an average of $\langle N\nu N\nu|G^{(0)}|N\nu' N\nu'\rangle$ over the m-states and $\Omega_N=(2j+1)/2$. $\overline{G^{(0)}}$ and Ω are defined by (3·8) and (3·11) respectively, and n is the number of particles in the N_0-shell. For the nuclei for which such a condition is not satisfied, we must take into account that the inequality (3·6) under (3·7) may also be fulfilled for the shells near the N_0-shell.

Under these approximations the non-trivial solution, which exists only for the N_0-shell, is obtained by solving the equation

$$\Delta_{N_0}=\frac{1}{2}\overline{G}^{(0)}\sum_{\nu_0}\Delta_{N_0}/\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta_{N_0}^2}, \tag{3·9}$$

and becomes

$$\Delta_{N_0}=\{(\overline{G}^{(0)2}\Omega^2/4)-(\epsilon_{N_0}-\lambda)^2\}^{1/2} \tag{3·10}$$

where

$$\Omega=(2j_0+1)/2. \tag{3·11}$$

B. *Elimination of the chemical potential* λ

The chemical potential λ is determined through the following equation:

$$\langle c_0|\sum_{m_0}a^+_{N_0m_0}a_{N_0m_0}|c_0\rangle=\sum_{\nu_0}2v^2_{N_0\nu_0}=\sum_{\nu_0}\{1-(\epsilon_{N_0}-\lambda)/\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta_{N_0}^2}\}=n, \tag{3·12}$$

where n is the number of particles in the N_0-shell, and is given by

$$\lambda=\epsilon_{N_0}-(\overline{G}^{(0)}\Omega/2)(1-n/\Omega). \tag{3·13}$$

By the use of (3·13), λ can be eliminated from all final results. The results in which λ is eliminated are summarized as follows:

$$\Delta_{N\nu}=\Delta_N=\begin{cases}(\overline{G}^{(0)}\Omega/2)\{1-(1-n/\Omega)^2\}^{1/2} & \text{for the } N_0\text{-shell.}\\ 0 & \text{for the shells other than the } N_0\text{-shell.}\end{cases} \tag{3·14}$$

$$\left.\begin{aligned}&u_{Nm}=u_{N-m}=u_N\\ &v_{Nm}=-v_{N-m}=\begin{cases}+v_N & m>0\\ -v_N & m<0,\end{cases}\\ &u_N=\begin{cases}(1-n/2\Omega)^{1/2} & \text{for the } N_0\text{-shell.}\\ 0 & \text{for the closed occupied shells.}\\ 1 & \text{for the open unoccupied shells.}\end{cases}\\ &v_N=\begin{cases}(n/2\Omega)^{1/2} & \text{for the } N_0\text{-shell.}\\ 1 & \text{for the closed occupied shells.}\\ 0 & \text{for the open unoccupied shells.}\end{cases}\end{aligned}\right\} \tag{3·15}$$

C. *Expression of* $H^{(0)}$ *and excited states*

With the aid of (3·14) and (3·15), $H^{(0)}$ becomes

$$H^{(0)}=U+\sum_{Nm}E_N\alpha^+_{Nm}\alpha_{Nm}, \tag{3·16}$$

where U is given by (2·8) and means the energy of the ground state (the vacuum state $|c_0\rangle$) and

$$\begin{aligned}E_N&=\sqrt{(\epsilon_N-\lambda)^2+\Delta_N^2}\\ &=\sqrt{\{\epsilon_N-\epsilon_{N_0}+(\overline{G}^{(0)}\Omega/2)(1-n/\Omega)\}^2+\Delta_N^2}.\end{aligned} \tag{3·17}$$

The excited states of $H^{(0)}$ are characterized by those with the quasi-particles. It is well known that the excited states of even-even nuclei contain an even number of quasi-particles, e.g., for the ground state the number of quasi-particles is zero (the vacuum state $|c_0\rangle$) and the first excited states are described in terms of two quasi-particles in the N_0-shell, etc. The number of quasi-particles in the N_0-shell is thus a generalized seniority number.

§ 4. Phenomenological analysis of the two-body interaction G

We now come to the stage to discuss the possibility of the division of G into (2·2). Recently discussions on this possibility were made by many authors.[9),10)] Here we shall follow Weisskopf's argument.[10)]

Let us expand the two-body interaction $G(x_1, x_2)$ in spherical harmonics:*

$$G(x_1, x_2)=\sum_L G_L(r_1 r_2)\, Y_{L0}(\theta_{12})=\sum_L \sqrt{\frac{4\pi}{2L+1}}\, G_L(r_1 r_2)\sum_{M=-L}^{+L} Y_{LM}(\theta_1, \varphi_1) \times Y^*_{LM}(\theta_2, \varphi_2). \qquad (4\cdot1)$$

There is a close connection between the above sum and the division in (2·2). The spherically symmetrical self-consistent field V is determined mainly by the $L=0$ term in (4·1). This means that $G^{(1)}$ corresponds to the $L=0$ term. On the other hand, the inequality (3·6) under the approximation (3·7) suggests that $G^{(0)}$ is mainly composed of the high harmonics in (4·1).[4)] We may thus conclude that the $L=0$ term corresponds to $G^{(1)}$ and the sum over $L=1$, 2 and 3 give rise to $G^{(2)}$ and the sum over $L=4, 5, \cdots$ give rise to $G^{(0)}$. We may call $G^{(1)}$ and $G^{(2)}$ the "long range" part of G and call $G^{(0)}$ the "short range" part of G, though it is not really accurate.

The $L=1$ term is not really important, because its main contribution is only a displacement of the center of mass. The interesting terms composing $G^{(2)}$, thus, are those with $L=2$ and $L=3$.

It is the main purpose of this paper to investigate how such terms bring about the quadrupole- and octupole-vibrations of spherical nuclei. In the following, our main attention will be paid especially on the $L=2$ term. Then, $H^{(\mathrm{res.})}$ in (2·3c) becomes

$$H^{(\mathrm{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1 m_1 N_2 m_2|G^{(2)}|N_2' m_2' N_1' m_1'\rangle a^+_{N_1 m_1} a^+_{N_2 m_2} a_{N_2' m_2'} a_{N_1' m_1'}$$

$$=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(Nm)}\langle N_1 N_2|g_2|N_2' N_1'\rangle(-1)^{2j_1'+2j_2'-1}(-1)^M(-1)^{-m_1'}(j_1 j_1' m_1 -m_1'|2M) \times(-1)^{-m_2'}(j_2 j_2' m_2 -m_2'|2-M)\, a^+_{N_1 m_1} a^+_{N_2 m_2} a_{N_2' m_2'} a_{N_1' m_1'}, \qquad (4\cdot2)$$

where

* For simplicity, we neglect the spin dependence of G.

$$\langle N_1 N_2|g_2|N_2' N_1'\rangle \equiv \frac{1}{2L+1}\sqrt{\frac{(2j_1+1)(2j_1'+1)(2j_2+1)(2j_2'+1)}{4\pi(2L+1)}}$$
$$\times\left(j_1 j_1' \frac{1}{2} -\frac{1}{2}|L0\right)\left(j_2 j_2' \frac{1}{2} -\frac{1}{2}|L0\right)\langle n_1 l_1 n_2 l_2|G_L|n_2' l_2' n_1' l_1'\rangle\Big|_{L=2}. \tag{4·3}$$

Here $\langle n_1 l_1 n_2 l_2|G_L|n_2' l_2' n_1' l_1'\rangle$ means the radial integral part of the L-th term in (4·1) and $(jj'm\, m'|LM)$ denotes the Clebsch-Gordan coefficient.

§ 5. Concept of nuclear collective motion and the time-dependent self-consistent field method

In order to make clear the physical meaning of our method which will be developed in the next section, we shall, in this section, start from the general consideration of the nuclear collective motion.

As is well known, the remarkable success of the nuclear shell model has suggested that the main part of the inter-nucleon interaction can be treated as a spherically symmetrical self-consistent field V. The essential point of the nuclear collective model is to assume that an additional self-consistent field may be extracted from the remaining part of the interaction. Bohr and Mottelson,[8] and Hill and Wheeler[11] have suggested that such an additional self-consistent field is non-spherical and time-dependent, and have grasped the time-variation of the additional self-consistent field as the nuclear collective motion.

Such a fundamental picture on the nuclear collective motion is directly formulated in the framework of the time-dependent self-consistent field method (the TDSCF-method). The discussions of the nuclear collective motion in the framework of the TDSCF-method were made by Nogami[12] and Ferrell[13] for the vibrational motion of closed shell nuclei and by Shono and Tanaka[14] for the rotational motion. In this section, we shall recapitulate the TDSCF-method in a form convenient for later discussions.

Let us consider the Hamiltonian (2·3a) in the case of $G^{(0)}=0$:

$$H=H^{(0)}+H^{(\mathrm{res.})}$$
$$=\sum_{Nm}\epsilon_N a_{Nm}^+ a_{N'm'} - \frac{1}{2}\sum_{(Nm)}\langle N_1 m_1 N_2 m_2|G^{(2)}|N_2' m_2' N_1' m_1'\rangle a_{N_1 m_1}^+ a_{N_2 m_2}^+ a_{N_2' m_2'} a_{N_1' N_1'} \tag{5·1}$$

where the second term is given by (4·2). Here we omit the chemical potential λ. In the coordinate representation, (5·1) is

$$H=\int\psi^+(x)\left(\frac{1}{2m}p^2-V(x)\right)\psi(x)dx$$
$$-\frac{1}{2}\int\psi^+(x_1)\psi^+(x_2)\{G_2(r_1 r_2)Y_{20}(\theta_{12})\}\psi(x_2)\psi(x_1)dx_1 dx_2,^* \tag{5·1'}$$

* The spin variable is assumed to be included in the coordinate x.

where $\psi(x)=\sum_{Nm} a_{Nm}\phi_{Nm}(x)$ and $\phi_{Nm}(x)$ are the single-particle wave functions in the spherically symmetrical self-consistent potential V.

The time derivative of $\psi(x)$ is expressed by

$$i\hbar\dot{\psi}(x)\equiv[\psi(x),\ H]=\mathsf{H}(p,\ x)\psi(x), \tag{5·2}$$

where

$$\mathsf{H}(p_1,\ x_1)=\frac{1}{2m}p_1^2-V(x_1)-\int dx_2\,\boldsymbol{\rho}(x_2,\ x_2)\,\{G_2(r_1r_2)\,Y_{20}(\theta_{12})\}$$

$$\equiv\frac{1}{2m}p_1^2-V(x_1)-\boldsymbol{U}(x_1). \tag{5·3}$$

Here the density matrix operator $\boldsymbol{\rho}(x,\ x')$ is defined by

$$\boldsymbol{\rho}(x,\ x')=\psi^+(x')\psi(x)=\sum_{\substack{Nm\\N'm'}} a^+_{Nm}a_{N'm'}\phi_{N'm'}(x)\phi^*_{Nm}(x') \tag{5·4}$$

and the time derivative of $\boldsymbol{\rho}$ is given by

$$i\hbar\dot{\boldsymbol{\rho}}\equiv[\mathsf{H},\ \boldsymbol{\rho}]. \tag{5·5}$$

The expectation value of $\boldsymbol{\rho}$ at time t is

$$\rho(t)=\langle\Psi(t)|\boldsymbol{\rho}|\Psi(t)\rangle, \tag{5·6}$$

where $|\Psi(t)\rangle$ is a state vector of H at time t, (and is not that of $H^{(0)}$). Using (5·5) we have the equation of motion for $\rho(t)$:

$$i\hbar\frac{\partial\rho(t)}{\partial t}=\langle\Psi(t)|[\mathsf{H},\ \boldsymbol{\rho}]|\Psi(t)\rangle. \tag{5·7}$$

If we make an approximation which is essential to the Hartree method :

$$\langle\Psi(t)|[\mathsf{H},\ \boldsymbol{\rho}]|\Psi(t)\rangle\simeq[\langle\Psi(t)|\mathsf{H}|\Psi(t)\rangle,\ \rho(t)], \tag{5·8}$$

we obtain the fundamental equation in the TDSCF-method :

$$i\hbar\frac{\partial\rho(t)}{\partial t}=[H(t),\ \rho(t)], \tag{5·9}$$

where

$$H(t)\equiv\langle\Psi(t)|\mathsf{H}|\Psi(t)\rangle=\frac{1}{2m}p^2-V(x)-U(x,\ t). \tag{5·10}$$

In the TDSCF-method, Eq. (5·9) is simplified through the following procedures.

i) $\rho(t)$ is expanded in the form

$$\rho(t)=\rho^{(0)}+\rho^{(1)}(t). \tag{5·11}$$

Here $\rho^{(0)}$ is the unperturbed density matrix and is equivalent to

$$\rho^{(0)}=\langle c_0|\boldsymbol{\rho}|c_0\rangle \tag{5·12}$$

where $|c_0\rangle$ is the ground state of $H^{(0)}$, and $\rho^{(1)}$ is the density matrix which should be related to the time dependent self-consistent field $U(x, t)$:

$$U(x_1, t)=\int dx_2\rho^{(1)}(x_2, x_2; t)G_2(r_1 r_2)Y_{20}(\theta_{12}). \tag{5·13}$$

ii) The first order self-consistent perturbation theory is used, which means that the products of the type $U\rho^{(1)}$ are neglected. Under this approximation, Eq. (5·9) is linearized and becomes

$$i\hbar\frac{\partial\rho^{(1)}}{\partial t}=\left[\frac{1}{2m}p^2-V, \rho^{(1)}\right]-[U, \rho^{(0)}]. \tag{5·14}$$

The main problem of the TDSCF-method is, as is well known, to solve Eq. (5·14) and to find a characteristic frequency corresponding to the collective vibration.

§ 6. Description of the nuclear collective vibration in the framework of second quantization

In this section we shall develop a method describing the nuclear collective vibration in the framework of second quantization. The essential idea of this method is based on an extension of the theory of Sawada et al.[6] of the plasma oscillation, and has been suggested by many authors.[15),16),17),1),2)]

Keeping a direct comparison with the TDSCF-method, we shall develop this idea in this section. In order to avoid unnecessary complications and to emphasize the physical implication of the method, we shall, however, illustrate here the fundamental idea by taking closed shell nuclei. The generalization of the method will be discussed in § 7.

We shall, at first, begin with rewriting Eq. (5·14) in a form more suitable for our purpose. Noting that for closed shell nuclei

$$\left.\begin{aligned}&\langle Nm|\rho^{(0)}|N'm'\rangle=\delta_{NN'}\delta_{mm'}\theta_N,\\&\langle Nm|\rho^{(1)}|N'm'\rangle=(1-\theta_N)\theta_{N'}\langle Nm|\rho^{(1)}|N'm'\rangle+(1-\theta_{N'})\theta_N\langle Nm|\rho^{(1)}|N'm'\rangle,^*\end{aligned}\right\} \tag{6·1}$$

let us introduce the following quantities with a definite angular momentum ($L=2, M$) :

$$\left.\begin{aligned}\overline{C_{2M}^{+}(NN')}&=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}\langle N'm'|\rho^{(1)}|Nm\rangle,\\\overline{C_{2M}(NN')}&=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}\langle Nm|\rho^{(1)}|N'm'\rangle.\end{aligned}\right\} \tag{6·2}$$

Here $\langle Nm|\rho|N'm'\rangle$ is given by

* In the TDSCF-method such a presupposition on the matrix elements of $\rho^{(1)}$ is usually made, in order to solve Eq. (5·14) and to find a characteristic frequency corresponding to the collective vibration.

$$\langle Nm|\rho|N'm'\rangle=\int\psi^*_{Nm}(x)\rho(x,\,x')\psi_{N'm'}(x')dx\,dx'$$

and θ_N is defined as follows:

$$\theta_N=\begin{cases}1 & \text{for the closed occupied shells.}\\ 0 & \text{for the open unoccupied shells.}\end{cases} \tag{6·3}$$

Using (6·1), (6·2) and (5·13), we find, after some calculations, that Eq. (5·14) becomes

$$i\hbar\frac{\partial}{\partial t}\overline{C^+_{2M}(NN')}=(\epsilon_{N'}-\epsilon_N)\overline{C^+_{2M}(NN')}$$
$$+\sum_{N_1N_1'}\langle NN_1|g_2|N_1'N'\rangle(1-\theta_N)\theta_{N'}\{\overline{C^+_{2M}(N_1N_1')}-(-1)^M\overline{C_{2-M}(N_1N_1')}\}, \tag{6·4}$$

where $\langle N_1N_2|g_2|N_2'N_1'\rangle$ is given by (4·3) and has the property

$$\langle N_1N_2|g_2|N_2'N_1'\rangle=\langle N_1'N_2|g_2|N_2'N_1\rangle=\langle N_1N_2'|g_2|N_2N_1'\rangle. \tag{6·5}$$

The equation equivalent to (6·4) is also obtained in the framework of second quantization in the following way.

i) We pick up the following particle-hole pair terms from $H^{(\mathrm{res.})}$ in (5·1):

$$H^{(1)}=-\frac{1}{2}\sum_{(Nm)}(N_1m_1N_2m_2|G^{(2)}|N_2'm_2'N_1'm_1')\{(1-\theta_{N_1})a^+_{N_1m_1}\cdot\theta_{N_1'}a_{N_1'm_1'}$$
$$+\theta_{N_1}a^+_{N_1m_1}\cdot(1-\theta_{N_1'})a_{N_1'm_1'}\}\{(1-\theta_{N_2})a^+_{N_2m_2}\cdot\theta_{N_2'}a_{N_2'm_2'}+\theta_{N_2}a^+_{N_2m_2}\cdot(1-\theta_{N_2'})a_{N_2'm_2'}\}$$
$$=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1N_2|g_2|N_2'N_1'\rangle\{C^+_{2M}(N_1N_1')-(-1)^MC_{2-M}(N_1N_1')\}$$
$$\times\{C_{2M}(N_2N_2')-(-1)^MC^+_{2-M}(N_2N_2')\}, \tag{6·6}$$

where

$$\left.\begin{aligned}C^+_{2M}(NN')&=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}a^+_{Nm}a_{N'm'},\\ C_{2M}(NN')&=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}a^+_{N'm'}a_{Nm}.\end{aligned}\right\} \tag{6·7}$$

ii) Commutation relations between C^+_{2M} and $C_{2M'}$ are approximated by the Boson-like commutation relations

$$\left.\begin{aligned}&[C_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]\simeq[C_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]|c_0\rangle\\ &\qquad=\delta_{MM'}\delta_{N_1N_2}\delta_{N_1'N_2'}(1-\theta_{N_1})\theta_{N_1'},\\ &[C_{2M}(N_1N_1'),\ C_{2M'}(N_2N_2')]=[C^+_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]=0.\end{aligned}\right\} \tag{6·8}$$

Using (6·8), we find that

$$i\hbar\dot{C}^+_{2M}(NN')\equiv[C^+_{2M}(NN'),\ H^{(0)}+H^{(1)}]=(\epsilon_{N'}-\epsilon_N)C^+_{2M}(NN')$$
$$+\sum_{N_1N_1'}\langle NN_1|g_2|N_1'N'\rangle(1-\theta_N)\theta_{N'}\{C^+_{2M}(N_1N_1')-(-1)^MC_{2-M}(N_1N_1')\}. \tag{6·9}$$

Now it is clear that Eq. (6·9) is equivalent to Eq. (6·4) in the TDSCF-method.* From this fact we may conclude that the approximations made in i) and ii) correspond to those of (5·8) and ii) of § 5 in the TDSCF-method.

So far we have considered the simplified system with the Hamiltonian (5·1) where $G^{(0)}=0$ is assumed. A particular advantage of the present formulation in the framework of second quantization is that the extension of the method to the case of $G^{(0)}\neq 0$ is very easy. In the following, we shall use this method and shall investigate the collective motion in the system with the Hamiltonian (2·3).

§ 7. Reduction of the Hamiltonian

Let us consider the Hamiltonian (2·3) and perform the Bogoliubov transformation (2·4). We then obtain the transformed Hamiltonian, $H=H^{(0)}+H^{(\text{res.})}$, where $H^{(0)}$ is given by (3·16) while

$$\begin{aligned} H^{(\text{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G^{(2)}|N_2'm_2'N_1'm_1'\rangle\{u_{N_1m_1}\alpha^+_{N_1m_1}+v_{N_1m_1}\alpha_{N_1-m_1}\}\\ \times\{u_{N_2m_2}\alpha^+_{N_2m_2}+v_{N_2m_2}\alpha_{N_2-m_2}\}\{u_{N_2'm_2'}\alpha_{N_2'm_2'}\\ +v_{N_2'm_2'}\alpha^+_{N_2'-m_2'}\}\{u_{N_1'm_1'}\alpha_{N_1'm_1'}+v_{N_1'm_1'}\alpha^+_{N_1'-m_1'}\}, \end{aligned} \tag{7·1}$$

u_{Nm} and v_{Nm} being given by (3·15).

In order to extend the method developed in § 6 to the present system, we now introduce the operators

$$\left.\begin{aligned} C^+_{2M}(NN')=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)u_{Nm}v_{N'm'}\alpha^+_{Nm}\alpha^+_{N'-m'}\\ C_{2M}(NN')=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)u_{Nm}v_{N'm'}\alpha_{N'-m'}\alpha_{Nm}, \end{aligned}\right\} \tag{7·2}$$

and pick up the following quasi-particle pair terms from (7·1):

$$\begin{aligned} H^{(\text{I})}=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1N_2|g_2|N_2'N_1'\rangle\{C^+_{2M}(N_1N_1')-(-1)^MC_{2-M}(N_1N_1')\}\\ \times\{C_{2M}(N_2N_2')-(-1)^MC^+_{2-M}(N_2N_2')\}. \end{aligned} \tag{7·3}$$

For closed shell nuclei in the special case of $G^{(0)}=0$, we get, with the aid of (2·4) and (3·15),

$$\begin{aligned} u_{Nm}\alpha^+_{Nm}=(1-\theta_N)a^+_{Nm},\quad v_{Nm}\alpha^+_{N-m}=\theta_Na_{Nm},\\ u_{Nm}\alpha_{Nm}=(1-\theta_N)a_{Nm},\quad v_{Nm}\alpha_{N-m}=\theta_Na^+_{Nm}, \end{aligned} \tag{7·4}$$

and the operators in (7·2) are reduced to those in (6·7). The operators in (7·2), therefore, corresponds to the generalization of those in (6·7), and the generalization of (6·6) corresponds to (7·3).

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* In the course of performing this work, it has come to the author's notice that Ehrenreich and Cohen[18] have made a similar proof for the case of the many-electron system.

Now we decompose C_{2M}^{+} and C_{2M} in the following way:

$$C_{2M}^{+}(NN')=C_{2M}^{(\mathrm{d})+}(NN')+C_{2M}^{(\mathrm{nd})+}(NN'),$$
$$C_{2M}^{(\mathrm{d})+}(NN')\equiv C_{2M}^{+}(NN')\delta_{NN'},\quad C_{2M}^{(\mathrm{nd})+}(NN')\equiv C_{2M}^{+}(NN')(1-\delta_{NN'}). \tag{7·5}$$

It follows from (3·15) that $C_{2M}^{(\mathrm{d})+}$ and $C_{2M}^{(\mathrm{d})}$ exist only for the *partially filled** N_0-shell and vanish for the other shells. It is also clear from (3·5) that, in the absence of the pairing interaction ($G^{(0)}=0$), $C_{2M}^{(\mathrm{d})+}$ and $C_{2M}^{(\mathrm{d})}$ vanish. Thus we are able to rewrite $H^{(\mathrm{I})}$ in (7·3) as

$$\begin{aligned}H^{(\mathrm{I})}=&-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1N_2|g_2|N_2'N_1'\rangle\{C_{2M}^{(\mathrm{nd})+}(N_1N_1')-(-1)^M C_{2-M}^{(\mathrm{nd})}(N_1N_1')\}\\&\qquad\times\{C_{2M}^{(\mathrm{nd})}(N_2N_2')-(-1)^M C_{2-M}^{(\mathrm{nd})+}(N_2N_2')\}\\&-\frac{1}{2}\sum_{M=-2}^{+2}\langle N_0N_0|g_2|N_0N_0\rangle\{C_{2M}^{(\mathrm{d})+}(N_0N_0)-(-1)^M C_{2-M}^{(\mathrm{d})}(N_0N_0)\}\\&\qquad\times\{C_{2M}^{(\mathrm{d})}(N_0N_0)-(-1)^M C_{2-M}^{(\mathrm{d})+}(N_0N_0)\}\\&-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{NN'}[\langle N_0N|g_2|N'N_0\rangle\{C_{2M}^{(\mathrm{d})+}(N_0N_0)-(-1)^M C_{2-M}^{(\mathrm{d})}(N_0N_0)\}\\&\qquad\times\{C_{2M}^{(\mathrm{nd})}(NN')-(-1)^M C_{2-M}^{(\mathrm{nd})+}(NN')\}+\text{herm. conj.}].\end{aligned} \tag{7·6}$$

In order to investigate the collective motion in the system with the Hamiltonian (2·3), we shall hereafter use the reduced Hamiltonian

$$H_{\mathrm{R}}=H^{(0)}+H^{(\mathrm{I})}, \tag{7·7}$$

where $H^{(0)}$ is given by (3·16) and $H^{(\mathrm{I})}$ is given by (7·6). Furthermore we shall hereafter employ the following commutation relations corresponding to the generalization of (6·8):

$$\left.\begin{aligned}&[C_{2M}^{(\mathrm{nd})}(N_1N_1'),\ C_{2M'}^{(\mathrm{nd})+}(N_2N_2')]\simeq[C_{2M}^{(\mathrm{nd})}(N_1N_1'),\ C_{2M'}^{(\mathrm{nd})+}(N_2N_2')]|c_0\rangle\\&\qquad=\delta_{MM'}\delta_{N_1N_2}\delta_{N_1'N_2'}(u_{N_1}^2v_{N_1'}^2)(1-\delta_{N_1N_1'}),\\&[C_{2M}^{(\mathrm{nd})}(N_1N_1'),\ C_{2M'}^{(\mathrm{nd})}(N_2N_2')=[C_{2M}^{(\mathrm{nd})+}(N_1N_1'),\ C_{2M'}^{(\mathrm{nd})+}(N_2N_2')]=0,\end{aligned}\right\} \tag{7·8a}$$

$$\left.\begin{aligned}&[C_{2M}^{(\mathrm{d})}(N_0N_0),\ C_{2M'}^{(\mathrm{d})+}(N_0N_0)]\simeq[C_{2M}^{(\mathrm{d})}(N_0N_0),\ C_{2M'}^{(\mathrm{d})+}(N_0N_0)]|c_0\rangle\\&\qquad=2\delta_{MM'}(u_{N_0}^2v_{N_0}^2),\\&[C_{2M}^{(\mathrm{d})}(N_0N_0),\ C_{2M'}^{(\mathrm{d})}(N_0N_0)]=[C_{2M}^{(\mathrm{d})+}(N_0N_0),\ C_{2M'}^{(\mathrm{d})+}(N_0N_0)]=0,\end{aligned}\right\} \tag{7·8b}$$

$$\left.\begin{aligned}&[C_{2M}^{(\mathrm{d})}(N_0N_0),\ C_{2M'}^{(\mathrm{nd})+}(NN')]=[C_{2M}^{(\mathrm{d})}(N_0N_0),\ C_{2M'}^{(\mathrm{nd})}(NN')]=0,\\&[C_{2M}^{(\mathrm{d})+}(N_0N_0),\ C_{2M'}^{(\mathrm{nd})}(NN')]=[C_{2M}^{(\mathrm{d})+}(N_0N_0),\ C_{2M'}^{(\mathrm{nd})+}(NN')]=0.\end{aligned}\right\} \tag{7·8c}$$

§ 8. Two modes of collective motion

The commutation relations (7·8) suggest that there exist two modes of col-

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* It is easily seen from (3·15) that $C_{2M}{}^{(\mathrm{d})+}$ and $C_{2M}{}^{(\mathrm{d})}$ vanish when the N_0-shell is closed.

lective motion: One mode is connected with $C_{2M}^{(\mathrm{nd})+}$ and $C_{2M}^{(\mathrm{nd})+}$, and another is connected with $C_{2M}^{(\mathrm{d})}$ and $C_{2M}^{(\mathrm{d})+}$. In this section we shall discuss these two modes separately. For this purpose we now rewrite the reduced Hamiltonian (7·7) as

$$H_{\mathrm{R}}=H_{\mathrm{R}}^{(0)}+H_{\mathrm{R}}' \tag{8·1}$$

where

$$H_{\mathrm{R}}'=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{NN'}[\langle N_0 N|g_2|N' N_0\rangle\{C_{2M}^{(\mathrm{d})+}(N_0 N_0)-(-1)^M C_{2-M}^{(\mathrm{d})}(N_0 N_0)\}$$
$$\times\{C_{2M}^{(\mathrm{nd})}(NN')-(-1)^M C_{2-M}^{(\mathrm{nd})+}(NN')\}+\text{herm. conj.}], \tag{8·2}$$

and consider the system with the Hamiltonian $H_{\mathrm{R}}^{(0)}$ for the moment. The term H_{R}' will be taken into account later on in § 9.

A. *Collective mode connected with* $C_{2M}^{(\mathrm{nd})}$ *and* $C_{2M}^{(\mathrm{nd})+}$

Using (7·8a) and (7·8c), we get the equation corresponding to Eq. (6·9):

$$[C_{2M}^{(\mathrm{nd})+}(NN'),\ H_{\mathrm{R}}^{(0)}]=-(E_{N'}+E_N)C_{2M}^{(\mathrm{nd})+}(NN')$$
$$+\sum_{N_1'N_1'}\langle NN_1|g_2|N_1' N'\rangle(u_N^2 v_{N'}^2)(1-\delta_{NN'})\{C_{2M}^{(\mathrm{nd})+}(N_1 N_1')$$
$$-(-1)^M C_{2-M}^{(\mathrm{nd})}(N_1 N_1')\}, \tag{8·3}$$

where E_N is given by (3·17).

Now let us introduce an operator

$$B_{2M}^{+}=\sum_{NN'}a(NN')C_{2M}^{(\mathrm{nd})+}(NN')-\sum_{NN'}b(NN')(-1)^M C_{2-M}^{(\mathrm{nd})}(NN'), \tag{8·4}$$

where $a(NN')$ and $b(NN')$ are determined by the following simultaneous equations:

$$(E-E_0)a(NN')=(E_N+E_{N'})a(NN')-\sum_{N_1N_1'}a(N_1 N_1')\langle NN_1|g_2|N_1' N'\rangle$$
$$\times(u_{N_1}^2 v_{N_1'}^2)(1-\delta_{N_1N_1'})+\sum_{N_1N_1'}b(N_1 N_1')\langle NN_1|g_2|N_1' N'\rangle(u_{N_1}^2 v_{N_1'}^2)(1-\delta_{N_1N_1'}),$$
$$(E-E_0)b(NN')=-(E_N+E_{N'})b(NN')-\sum_{N_1N_1'}a(N_1 N_1')\langle NN_1|g_2|N_1' N'\rangle$$
$$\times(u_{N_1}^2 v_{N_1'}^2)(1-\delta_{N_1N_1'})+\sum_{N_1N_1'}b(N_1 N_1')\langle NN_1|g_2|N_1' N'\rangle(u_{N_1}^2 v_{N_1'}^2)(1-\delta_{N_1N_1'}), \tag{8·5}$$

and the normalization condition:

$$[B_{2M},\ B_{2M}^{+}]=1=\sum_{NN'}a(NN')^2(u_N^2 v_{N'}^2)(1-\delta_{NN'})$$
$$-\sum_{NN'}b(NN')^2(u_N^2 v_{N'}^2)(1-\delta_{NN'}). \tag{8·6}$$

Then it is easily shown with the aid of (8·3) that B_{2M}^{+} satisfies

$$[H_{\mathrm{R}}^{(0)},\ B_{2M}^{+}]=(E-E_0)B_{2M}^{+}.$$

This means that B_{2M}^{+} is the operator which creates an eigenstate $|\Psi\rangle$ with energy E from the ground state $|\Psi_0\rangle$ of $H_{\mathrm{R}}^{(0)}$ with energy E_0:

$$|\Psi\rangle = B_{2M}^{+}|\Psi_0\rangle.$$

We do not discuss here Eq. (8·5) in detail, and note only the following. If $\langle NN_1|g_2|N_1'N'\rangle$ are separable, i.e. $\langle NN_1|g_2|N_1'N'\rangle = \langle N|g_2|N'\rangle\langle N_1|g_2|N_1'\rangle$,* we obtain the eigenvalue equation from (8·5) :[1]

$$1=\sum_{NN'}\left\{\frac{1}{(E-E_0)+(E_{N'}+E_N)}-\frac{1}{(E-E_0)-(E_{N'}+E_N)}\right\}(u_N^2 v_{N'}^2)(1-\delta_{NN'})$$
$$\times\langle NN|g_2|N'N'\rangle. \tag{8·7}$$

This equation has a root $(E-E_0)$ which is real and positive and is smaller than the smallest pair (quasi-particle) excitation energy $(E_{N'}+E_N)$ with $N \neq N'$, provided that the condition

$$1>\sum_{NN'}\frac{2\langle NN|g_2|N'N'\rangle(u_N^2 v_{N'}^2)(1-\delta_{NN'})}{E_{N'}+E_N} \tag{8·8}$$

is satisfied. The state with such a lowest excitation energy, $(E-E_0)_{\mathrm{min.}}\equiv\hbar\omega_\beta$, can be regarded as that of the collective motion connected with $C_{2M}^{(\mathrm{nd})}$ and $C_{2M}^{(\mathrm{nd})+}$. In this case, $a(NN')$ and $b(NN')$ are given by

$$a(NN')=\frac{K\langle N|g_2|N'\rangle}{\hbar\omega_\beta-(E_N+E_{N'})},\quad b(NN')=\frac{K\langle N|g_2|N'\rangle}{\hbar\omega_\beta+(E_N+E_{N'})}, \tag{8·9}$$

where the normalization constant K is determined by (8·6), i.e.

$$K=1\Bigg/\Bigg[\sum_{NN'}\frac{\langle NN|g_2|N'N'\rangle u_N^2 v_{N'}^2(1-\delta_{NN'})}{\{\hbar\omega_\beta-(E_N+E_{N'})\}^2}$$
$$-\sum_{NN'}\frac{\langle NN|g_2|N'N'\rangle u_N^2 v_{N'}^2(1-\delta_{NN'})}{\{\hbar\omega_\beta+(E_N+E_{N'})\}^2}\Bigg]^{1/2}. \tag{8·10}$$

B. *Collective mode connected with* $C_{2M}^{(\mathrm{d})}$ *and* $C_{2M}^{(\mathrm{d})+}$

In this case, we obtain the following equation corresponding to Eq. (6·9) :

$$[C_{2M}^{(\mathrm{d})+}(N_0N_0),\ H_{\mathrm{R}}^{(0)}]=-2E_{N_0}C_{2M}^{(\mathrm{d})+}(N_0N_0)$$
$$+2\langle N_0N_0|g_2|N_0N_0\rangle(u_{N_0}^2 v_{N_0}^2)\{C_{2M}^{(\mathrm{d})+}(N_0N_0)-(-1)^M C_{2-M}^{(\mathrm{d})}(N_0N_0)\}. \tag{8·11}$$

Now let us consider the matrix element, $\langle\Psi'|(E'-H_{\mathrm{R}}^{(0)})C_{2M}^{(\mathrm{d})+}(N_0N_0)|\Psi_0\rangle$, where $|\Psi'\rangle$ is an eigenstate of $H_{\mathrm{R}}^{(0)}$ with energy E' and $|\Psi_0\rangle$ is the ground state of $H_{\mathrm{R}}^{(0)}$. This vanishes, but can also be written as

$$0=(E'-E_0)\langle\Psi'|C_{2M}^{(\mathrm{d})+}(N_0N_0)|\Psi_0\rangle-\langle\Psi'|[H_{\mathrm{R}}^{(0)},\ C_{2M}^{(\mathrm{d})+}(N_0N_0)]|\Psi_0\rangle. \tag{8·12a}$$

Similarly, we find

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* The "quadrupole-quadrupole interaction" adopted by Elliott[19] and Moszkowski[20] is an example of such a type.

$$0=(E'-E_0)\langle\Psi'|(-1)^M C^{(1)}_{2-M}(N_0N_0)|\Psi_0\rangle-\langle\Psi'|[H^{(0)}_{\rm R},\ (-1)^M C^{(1)}_{2-M}(N_0N_0)]|\Psi_0\rangle. \tag{8·12b}$$

Using (8·11) and combining (8·12a) and (8·12b), we are led to the eigenvalue equation

$$1=\left\{\frac{1}{\hbar\omega_\eta+2E_{N_0}}-\frac{1}{\hbar\omega_\eta-2E_{N_0}}\right\}2\langle N_0N_0|g_2|N_0N_0\rangle u^2_{N_0}v^2_{N_0}, \tag{8·13}$$

where $\hbar\omega_\eta\equiv E'-E_0$ and $E_{N_0}=\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta^2_{N_0}}=\frac{1}{2}G^{(0)}\Omega$.

Solving Eq. (8·13), we obtain

$$\hbar\omega_\eta=G^{(0)}\Omega\sqrt{1-\theta_n/\theta'_{n_0}}, \tag{8·14}$$

where

$$\theta_n=\{1-(1-n/\Omega)^2\}, \tag{8·15a}$$

$$\theta'_{n_0}=G^{(0)}\Omega/4\langle N_0N_0|g_2|N_0N_0\rangle. \tag{8·15b}$$

θ_n is the occupation factor used by Belyaev (see ref. 4), Eq. (67)) and θ'_{n_0} represents a ratio of the strength of $G^{(0)}$ to that of $G^{(2)}$.

For the collective mode under consideration, the creation operator $B^{(\eta)+}_{2M}$, which leads to

$$|\Psi'\rangle=B^{(\eta)+}_{2M}|\Psi_0\rangle, \tag{8·16}$$

is easily obtained in the same way as that in the subsection **A.** The result is

$$B^{(\eta)+}_{2M}=a(N_0N_0)C^{(1)+}_{2M}(N_0N_0)-b(N_0N_0)(-1)^M C^{(1)}_{2-M}(N_0N_0), \tag{8·17}$$

where

$$a(N_0N_0)=\frac{\hbar\omega_\eta+G^{(0)}\Omega}{\sqrt{2\theta_nG^{(0)}\Omega\hbar\omega_\eta}},\quad b(N_0N_0)=\frac{\hbar\omega_\eta-G^{(0)}\Omega}{\sqrt{2\theta_nG^{(0)}\Omega\hbar\omega_\eta}}. \tag{8·18}$$

Here it should be noted that the collective mode under consideration vanishes in the absence of the pairing interaction ($G^{(0)}=0$).

C. *Derivation of collective variables*

Now let us introduce the following two kinds of collective coordinates $(q^{(\beta)}_M, q^{(\eta)}_M)$ and their conjugate momenta $(p^{(\beta)}_M, p^{(\eta)}_M)$:

$$\left.\begin{aligned} q^{(\beta)}_M&=-i\sqrt{\frac{\hbar}{2I_\beta\omega_\beta}}(B^{(\beta)+}_{2M}-(-1)^M B^{(\beta)}_{2-M}),\ q^{(\beta)+}_M=(-1)^M q^{(\beta)}_{-M},\\ p^{(\beta)}_M&=\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}(B^{(\beta)}_{2M}+(-1)^M B^{(\beta)+}_{2-M}),\ p^{(\beta)+}_M=(-1)^M p^{(\beta)}_{-M},\end{aligned}\right\} \tag{8·19}$$

$$\left.\begin{aligned} q^{(\eta)}_M&=-i\sqrt{\frac{\hbar}{2I_\eta\omega_\eta}}(B^{(\eta)+}_{2M}-(-1)^M B^{(\eta)}_{2-M}),\ q^{(\eta)+}_M=(-1)^M q^{(\eta)}_{-M},\\ p^{(\eta)}_M&=\sqrt{\frac{\hbar I_\eta\omega_\eta}{2}}(B^{(\eta)}_{2M}+(-1)^M B^{(\eta)+}_{2-M}),\ p^{(\eta)+}_M=(-1)^M p^{(\eta)}_{-M},\end{aligned}\right\} \tag{8·20}$$

which satisfy the commutation relations

$$[q_M^{(\beta)}, p_{M'}^{(\beta)}]=i\hbar\delta_{MM'}, \ [q_M^{(\gamma)}, p_{M'}^{(\gamma)}]=i\hbar\delta_{MM'}. \tag{8·21}$$

$B_{2M}^{(\beta)+}$ is the creation operator of the collective state with the excitation energy $\hbar\omega_\beta$ and is given by (8·4) with the corresponding solution $(a(NN'), b(NN'))$ of Eq. (8·5). I_β and I_γ are the inertial parameters of the two kinds of collective modes, which will be determined in § 10.

§ 9. Use of the "method of auxiliary variables"

In the preceding section, we have found the collective variables of the system with the Hamiltonian $H_R^{(0)}$. In this section we shall take H_R' (8·2) into account, which contains the coupling term between two modes $(q_M^{(\beta)}, q_M^{(\gamma)})$. For this purpose, we shall employ the "method of auxiliary variables",[7] which makes it possible to display the dependence of the Hamiltonian on the collective variables.

Let us consider the Hamiltonian (8·1) and start with the Schrödinger equation

$$H_R|\Phi\rangle\equiv(H_R^{(0)}+H_R')|\Phi\rangle=\tilde{E}|\Phi\rangle. \tag{9·1}$$

Here we introduce two kinds of auxiliary variables, $(\beta_M, \pi_M^{(\beta)})$ and $(\gamma_M, \pi_M^{(\gamma)})$, which satisfy the commutation relations

$$[\beta_M, \pi_{M'}^{(\beta)}]=i\hbar\delta_{MM'}, \ [\gamma_M, \pi_{M'}^{(\gamma)}]=i\hbar\delta_{MM'}. \tag{9·2}$$

Corresponding to $(q_M^{(\beta)}, p_M^{(\beta)})$ and $(q_M^{(\gamma)}, p_M^{(\gamma)})$, they have the conjugate relations

$$\left.\begin{aligned}\beta_M^+=(-1)^M\beta_{-M}, \ \pi_M^{(\beta)+}=(-1)^M\pi_{-M}^{(\beta)},\\ \gamma_M^+=(-1)^M\gamma_{-M}, \ \pi_M^{(\gamma)+}=(-1)^M\pi_{-M}^{(\gamma)}.\end{aligned}\right\} \tag{9·3}$$

To compensate the introduction of the auxiliary variables, we impose on $|\Phi\rangle$ the supplementary conditions

$$\beta_M|\Phi\rangle=0, \ \gamma_M|\Phi\rangle=0. \tag{9·4}$$

Now let us successively perform the following unitary transformations:

$$|\Phi^{(1)}\rangle=U_{(1)}^{-1}|\Phi\rangle, \ U_{(1)}=\exp\{i\sum_M(\pi_M^{(\beta)}q_M^{(\beta)}+\pi_M^{(\gamma)}q_M^{(\gamma)})/\hbar\}, \tag{9·5a}$$

$$|\Phi^{(2)}\rangle=U_{(2)}^{-1}|\Phi^{(1)}\rangle, \ U_{(2)}=\exp\{-i\sum_M(p_M^{(\beta)}\beta_M+p_M^{(\gamma)}\gamma_M)/\hbar\}, \tag{9·5b}$$

$$|\Phi_c\rangle=U_{(1)}^{-1}|\Phi^{(2)}\rangle, \ U_{(1)}=\exp\{i\sum_M(\pi_M^{(\beta)}q_M^{(\beta)}+\pi_M^{(\gamma)}q_M^{(\gamma)})/\hbar\}. \tag{9·5c}$$

Then the transformed Hamiltonian and the supplementary conditions become

$$\mathcal{H}=U_{(1)}^{-1}U_{(2)}^{-1}U_{(1)}^{-1}H_R U_{(1)}U_{(2)}U_{(1)}, \tag{9·6}$$

$$q_M^{(\beta)}|\Phi_c\rangle=0, \quad q_M^{(\gamma)}|\Phi_c\rangle=0. \tag{9·7}$$

This representation is the so-called "collective representation", in which Bohr and Mottelson[8] have developed the nuclear unified model.

It should be noted that the Hamiltonian (9·6) contains neither $(q_M^{(\eta)}, p_M^{(\eta)})$ nor $(q_M^{(\beta)}, p_M^{(\beta)})$, i.e. it satisfies

$$[\mathcal{H}, q_M^{(\beta)}]=[\mathcal{H}, p_M^{(\beta)}]=0, \quad [\mathcal{H}, q_M^{(\eta)}]=[\mathcal{H}, p_M^{(\eta)}]=0. \tag{9·8}$$

This implies that the collective variables $(q_M^{(\beta)}, p_M^{(\beta)}; q_M^{(\eta)}, p_M^{(\eta)})$ in the original Hamiltonian H_R are completely replaced by the auxiliary variables $(\beta_M, \pi_M^{(\beta)}; \eta_M \pi_M^{(\eta)})$. In this representation, therefore, the collective modes of our system can be visualized through the auxiliary variables.

Employing the operator identity

$$\exp(iS)O\exp(-iS)=O+i[S, O]-\frac{1}{2}[S, [S, O]]+\cdots,$$

and using the commutation relations (7·8) and (9·2), we obtain the explicit form of $\mathcal{H}$ of the following structure:

$$\mathcal{H}=\mathcal{H}_{\text{coll.}}+\mathcal{H}_{\text{intr.}}+\mathcal{H}_{\text{coupl}}, \tag{9·9}$$

$$\mathcal{H}_{\text{coll.}}=\sum_M\left\{\frac{1}{2I_\beta}|\pi_M^{(\beta)}|^2+\frac{1}{2}C_\beta|\beta_M|^2\right\}+\sum_M\left\{\frac{1}{2I_\eta}|\pi_M^{(\eta)}|^2+\frac{1}{2}C_\eta|\eta_M|^2\right\}$$
$$-\frac{1}{2}\kappa\sum_M\{\beta_M\eta_M^+ +\eta_M\beta_M^+\}, \tag{9·10}$$

$$\mathcal{H}_{\text{intr.}}=H_R-\sum_M\left\{\frac{1}{2I_\beta}|p_M^{(\beta)}|^2+\frac{1}{2}C_\beta|q_M^{(\beta)}|^2\right\}-\sum_M\left\{\frac{1}{2I_\eta}|p_M^{(\eta)}|^2+\frac{1}{2}C_\eta|q_M^{(\eta)}|^2\right\}$$
$$+i\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\Omega\Theta_n}\sum_M\sum_{NN'}\langle N_0 N|g_2|N' N_0\rangle\{C_{2M}^{(\text{nd})}(NN')$$
$$-(-1)^M C_{2-M}^{(\text{nd})+}(NN')\}q_M^{(\eta)}, \tag{9·11}$$

$$\mathcal{H}_{\text{coupl.}}=\sum_M\left[\kappa q_M^{(\beta)+}-i\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\Omega\Theta_n}\sum_{NN'}\langle N_0 N|g_2|N' N_0\rangle\right.$$
$$\left.\times\{C_{2M}^{(\text{nd})}(NN')-(-1)^M C_{2-M}^{(\text{nd})+}(NN')\}\right]\cdot\eta_M, \tag{9·12}$$

where

$$C_\beta\equiv I_\beta\omega_\beta^2, \quad C_\eta\equiv I_\eta\omega_\eta^2, \tag{9·13}$$

$$\kappa=\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\Omega\Theta_n}\frac{1}{\hbar}\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}\sum_{NN'}\{a(NN')-b(NN')\}(2u_N^2 v_{N'}^2)(1-\delta_{NN'})$$
$$\times\langle N_0 N|g_2|N' N_0\rangle. \tag{9·14}$$

Here $\mathcal{H}_{\text{coll.}}$ represents the energy of collective motion, and $\mathcal{H}_{\text{intr.}}$ can be interpreted as the energy of the intrinsic (particle) motion. $\mathcal{H}_{\text{coupl.}}$ represents the interaction between the collective and the intrinsic motions.

For the spherical nuclei under consideration, we can safely treat $\mathcal{H}_{\text{coupl.}}$ as a small perturbation term. Therefore, we shall neglect this term hereafter. In this case the Schrödinger equation in the "collective representation" becomes

$$(\mathcal{H}_{\text{coll.}}+\mathcal{H}_{\text{intr.}})|\Phi_c\rangle=\tilde{E}\,|\Phi_c\rangle \tag{9·15}$$

with the supplementary condition (9·7).

Let us write $|\Phi_c\rangle$ as

$$|\Phi_c\rangle=|\chi_{\text{intr.}}\rangle|\phi_{\text{coll.}}\rangle. \tag{9·16}$$

Then $|\chi_{\text{intr.}}\rangle$ satisfies the equation

$$\mathcal{H}_{\text{intr.}}|\chi_{\text{intr.}}\rangle=\tilde{E}_{\text{intr.}}|\chi_{\text{intr.}}\rangle \tag{9·17}$$

with the supplementary condition $q_M^{(\beta)}|\chi_{\text{intr.}}\rangle=q_M^{(\eta)}|\chi_{\text{intr.}}\rangle=0$, and $|\phi_{\text{coll.}}\rangle$ satisfies

$$\mathcal{H}_{\text{coll.}}|\phi_{\text{coll.}}\rangle=\tilde{E}_{\text{coll.}}|\phi_{\text{coll.}}\rangle. \tag{9·18}$$

The total energy of our system thus becoms $\tilde{E}=\tilde{E}_{\text{intr.}}+\tilde{E}_{\text{coll.}}$

The Schödinger equation (9·18) describes the collective motion of our system. In the following sections, we shall investigate the properties of Eq. (9·18).

§ 10. Mass quadrupole moments and "surface deformation variables"

Prior to discussing Eq. (9·18), we shall, in this section, clarify the connection between our variables (β_M, η_M) and the "surface deformation variables" α_M used in the Bohr-Mottelson model.[8] For this purpose, we consider the mass quadrupole moment operator.

In the Bohr-Mottelson model,[8] as is well known, the quadrupole moment operator is givan by

$$\mathcal{Q}_M=\mathcal{Q}_M^{(\text{intr.})}+\mathcal{Q}_M^{(\text{coll.})}, \tag{10·1}$$

the first part of which is associated with the intrinsic structure.* The second part is due to the surface deformation and is given by

$$\mathcal{Q}_M^{(\text{coll.})}=\frac{3}{\sqrt{5\pi}}AR_0^2\alpha_M\equiv\bar{Q}\alpha_M, \tag{10·2}$$

where A is the atomic number and R_0 is the nuclear radius.

Now let us consider the quadrupole moment operator of our system. In the original representation the quadrupole moment operator is

$$\begin{aligned}Q_M&=\sqrt{\frac{16\pi}{5}}\sum_{\substack{Nm\\N'm'}}\langle Nm|r^2Y_{2M}|N'm'\rangle a_{Nm}^+a_{N'm'}\\&=i\sum_{NN'}\langle N|q_2|N'\rangle\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)a_{Nm}^+a_{N'm'},\end{aligned} \tag{10·3}$$

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* For the even-even nuclei under consideration, $\langle\chi_{\text{intr.}}^{(0)}|\mathcal{Q}_M^{\text{intr.}}|\chi_{\text{intr.}}^{(0)}\rangle=0$, where $\chi_{\text{intr.}}^{(0)}$ is the ground state of $\mathcal{H}_{\text{intr.}}$.

where

$$\langle N|q_2|N'\rangle=\sqrt{\frac{16\pi}{5}}\sqrt{\frac{(2j+1)(2j'+1)}{20\pi}}\left(jj'\frac{1}{2}-\frac{1}{2}|20\right)\langle nl|r^2|n'l'\rangle. \quad (10\cdot4)$$

Performing the Bogoliubov transformation (2·4) with (3·15) and making use of the approximations discussed in § 6 and § 7, we have

$$Q_M=-i\sum_{NN'}\langle N|q_2|N'\rangle\{C_{2M}^{(\mathrm{nd})+}(NN')-(-1)^M C_{2-M}^{(\mathrm{nd})}(NN')\}$$
$$-i\langle N_0|q_2|N_0\rangle\{C_{2M}^{(\mathrm{d})+}(N_0N_0)-(-1)^M C_{2-M}^{(\mathrm{d})}(N_0N_0)\}. \quad (10\cdot5)$$

In the "collective representation", (10·5) changes into

$$\mathcal{Q}_M=U_{(1)}^{-1}U_{(2)}^{-1}U_{(1)}^{-1}Q_M U_{(1)}U_{(2)}U_{(1)}=\mathcal{Q}_M^{(\mathrm{intr.})}+\mathcal{Q}_M^{(\mathrm{coll})}, \quad (10\cdot6)$$

$$\mathcal{Q}_M^{(\mathrm{intr})}=Q_M-k_1q_M^{(\beta)}-k_2q_M^{(\eta)},^* \quad (10\cdot6\mathrm{a})$$

$$\mathcal{Q}_M^{(\mathrm{coll})}=k_1\beta_M+k_2\eta_M, \quad (10\cdot6\mathrm{b})$$

where

$$k_1=\frac{1}{\hbar}\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}\sum_{NN'}\langle N|q_2|N'\rangle\{a(NN')-b(NN')\}(2u_N^2v_{N'}^2)(1-\delta_{NN'}),$$

$$k_2=\frac{1}{\hbar}\sqrt{I_\eta}\langle N_0|q_2|N_0\rangle\sqrt{G^{(0)}\Omega\,\theta_n}.$$

Comparing (10·6b) with (10·2), we can find the connection between (β_M, η_M) and α_M:

$$\alpha_M=(\bar{Q}^{-1}k_1)\beta_M+(\bar{Q}^{-1}k_2)\eta_M. \quad (10\cdot7)$$

If we determine the inertial parameters (I_β, I_η) introduced in (8·19) and (8·20) as

$$I_\beta=\frac{2\hbar\bar{Q}^2}{\omega_\beta}[\sum_{NN'}\langle N|q_2|N'\rangle\{a(NN')-b(NN')\}(2u_N^2v_{N'}^2)(1-\delta_{NN'})]^{-2},$$

$$I_\eta=\hbar^2\bar{Q}^2/G^{(0)}\Omega\,\theta_n\langle N_0|q_2|N_0\rangle^2, \quad (10\cdot8)$$

Eq. (10·7) becomes

$$\alpha_M=\beta_M+\eta_M. \quad (10\cdot9)$$

Now the physical meaning of the variables β_M and η_M is clear. η_M represents the deformation associated with the configuration of the nucleons in the N_0-shell, and vanishes when the N_0-shell is closed. Thus, β_M represents the deformation of the core mainly, and itself becomes α_M when the N_0-shell is closed.

With the aid of (10·9), the polarizability of the core by the outside nucleons

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* Using the supplementary condition $q_M^{(\beta)}|\chi_{\mathrm{intr.}}^{(0)}\rangle=q_M^{(\eta)}|\chi_{\mathrm{intr.}}^{(0)}\rangle=0$, and making use of the approximation $|\chi_{\mathrm{intr.}}^{(0)}\rangle\simeq|c_0\rangle$, we find $\langle\chi_{\mathrm{intr.}}^{(0)}|\mathcal{Q}_M^{(\mathrm{intr.})}|\chi_{\mathrm{intr.}}^{(0)}\rangle=0$, where $|\chi_{\mathrm{intr.}}^{(0)}\rangle$ is the ground state of $\mathcal{H}_{\mathrm{intr.}}$.

can be obtained in the following way. Let us pick up the "potential energy" term for the collective motion from $\mathcal{H}_{\text{coll.}}$:

$$2U \equiv \sum_M \{C_\beta|\beta_M|^2 + C_\eta|\eta_M|^2 - \kappa(\beta_M\eta_M^+ + \eta_M\beta_M^+)\}$$

$$= \sum_M \{C_\beta|\alpha_M|^2 - (C_\beta+\kappa)(\eta_M\alpha_M^+ + \alpha_M\eta_M^+) + (C_\beta + C_\eta + 2\kappa)|\eta_M|^2\}. \quad (10\cdot 10)$$

For a fixed value of η_M, the equilibrium deformation α_M^0 is determined from (10·10) and is given by

$$\alpha_M^0(\eta_M) = \{1/(1-\varepsilon)\}\cdot\eta_M, \quad (10\cdot 11)$$

where

$$\varepsilon = \kappa/(C_\beta+\kappa). \quad (10\cdot 12)$$

With the aid of (10·2), (10·11) is written as

$$\mathcal{Q}_{M=0}^{(\eta)} = (1-\varepsilon)\mathcal{Q}_{M=0}^0 \quad (10\cdot 13)$$

where $\mathcal{Q}_{M=0}^{(\eta)} \equiv \bar{Q}\eta_{M=0}$ is the quadrupole moment associated with the outside nucleons. Eq. (10·13) means that the quantity ε defined by (10·12) describes the polarizability of the core by the outside nucleons.

§ 11. Normal vibrations of even-even spherical nuclei

The lower excited states of even-even spherical nuclei are described by Eq. (9·18). In order to investigate the energy spectrum, we shall transform the Hamiltonian (9·10) into that of the normal vibrations:

$$\mathcal{H}_{\text{coll.}} = \frac{1}{2}\sum_M \{|\dot{\alpha}_M^{(1)}|^2 + \omega_{(1)}^2|\alpha_M^{(1)}|^2 + |\dot{\alpha}_M^{(2)}|^2 + \omega_{(2)}^2|\alpha_M^{(2)}|^2\}. \quad (11\cdot 1)$$

Here the normal coordinates $\alpha_M^{(1)}$ and $\alpha_M^{(2)}$ are defined through

$$\left.\begin{aligned}
\beta_M &= \frac{1}{\sqrt{I_\beta}}\sqrt{\frac{(\omega_\eta^2-\omega_{(1)}^2)}{(\omega_\eta^2-\omega_{(1)}^2)+(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\alpha_M^{(1)} \\
&\quad + \frac{1}{\sqrt{I_\beta}}\sqrt{\frac{(\omega_{(2)}^2-\omega_\eta^2)}{(\omega_{(2)}^2-\omega_\eta^2)+(\omega_{(2)}^2-\omega_\beta^2)}}\cdot\alpha_M^{(2)}, \\
\eta_M &= \frac{1}{\sqrt{I_\eta}}\sqrt{\frac{(\omega_\beta^2-\omega_{(1)}^2)}{(\omega_\eta^2-\omega_{(1)}^2)+(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\alpha_M^{(1)} \\
&\quad - \frac{1}{\sqrt{I_\eta}}\sqrt{\frac{(\omega_{(2)}^2-\omega_\beta^2)}{(\omega_{(2)}^2-\omega_\eta^2)+(\omega_{(2)}^2-\omega_\beta^2)}}\cdot\alpha_M^{(2)},
\end{aligned}\right\} \quad (11\cdot 2)$$

while the corresponding eigenfrequencies are given by

$$\omega_{(1)}^2 = \frac{1}{2}(\omega_\beta^2+\omega_\eta^2) - \frac{1}{2}(\omega_\beta^2+\omega_\eta^2)$$

$$\times\left[1-\frac{4\omega_\beta^2}{(\omega_\beta^2+\omega_\eta^2)^2}\left\{\omega_\eta^2-\frac{G^{(0)}\Omega\,\theta_n}{2\hbar^2\cdot\hbar\omega_\beta}\Big[\sum_{NN'}\{a(NN')-b(NN')\}\right.\right.$$

$$\left.\left.\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0 N|g_2|N'N_0\rangle\Big]^2\right\}\right]^{1/2}, \tag{11·3a}$$

$$\omega_{(2)}^2=\frac{1}{2}(\omega_\beta^2+\omega_\eta^2)+\frac{1}{2}(\omega_\beta^2+\omega_\eta^2)$$

$$\times\left[1-\frac{4\omega_\beta^2}{(\omega_\beta^2+\omega_\eta^2)^2}\left\{\omega_\eta^2-\frac{G^{(0)}\Omega\,\theta_n}{2\hbar^2\cdot\hbar\omega_\beta}\Big[\sum_{NN'}\{a(NN')-b(NN')\}\right.\right.$$

$$\left.\left.\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0 N|g_2|N'N_0\rangle\Big]^2\right\}\right]^{1/2}. \tag{11·3b}$$

If $\hbar\omega_\beta\gg\hbar\omega_\eta$, then (11·3a) and (11·3b) become

$$\hbar\omega_{(1)}\simeq G^{(0)}\Omega\sqrt{1-\theta_n/\theta_{n_0}}, \tag{11·4a}$$

$$\hbar\omega_{(2)}\simeq\hbar\omega_\beta, \tag{11·4b}$$

where

$$\frac{1}{\theta_{n_0}}=\frac{1}{G^{(0)}\Omega}\Big[4\langle N_0N_0|g_2|N_0N_0\rangle+\frac{1}{2\hbar\omega_\beta}\{\sum_{NN'}\{a(NN')-b(NN')\}$$

$$\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0N|g_2|N'N_0\rangle\}^2\Big]$$

$$=\frac{1}{G^{(0)}\Omega}[4\langle N_0N_0|g_2|N_0N_0\rangle+\bar{Q}^{-2}\{\varepsilon/(1-\varepsilon)\}^2C_\beta\langle N_0|q_2|N_0\rangle^2]. \tag{11·5}$$

Eq. (11·4a) should be compared with Eq. (163) in Belyaev's paper,[4] which has been obtained by the use of the "cranking model" of Inglis.[5]

As has been pointed out by Belyaev,[4] the normal vibration of the first type ($\hbar\omega_{(1)}$) preserves the equilibrium relation between β_M and η_M. Indeed, the condition $\alpha_M^{(2)}=0$ can be rewritten as

$$\beta_M=\sqrt{\frac{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\eta_M\simeq\frac{\kappa}{C_\beta}\eta_M. \tag{11·6}$$

With the aid of (10·9), (11·6) becomes

$$\alpha_M=\left(1+\sqrt{\frac{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}}\right)\eta_M\simeq\{1/(1-\varepsilon)\}\eta_M \tag{11·7}$$

which is equivalent to (10·11). In the vibration of the first type the core thus adjusts itself adiabatically to the deformation of the outside nucleons.

By the use of (11·6), the normal coordinate of the first type, $\alpha_M^{(1)}$, becomes

$$\alpha_M^{(1)}=\sqrt{I_\eta}\sqrt{\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\eta_M\simeq\sqrt{I_\eta}\,\eta_M. \tag{11·8}$$

Using (11·7), we can rewrite (11·8) as

$$\alpha_M^{(1)}=\sqrt{I_\beta I_\eta}\cdot\frac{\sqrt{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}}{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}}\cdot\alpha_M\simeq(1-\varepsilon)\sqrt{I_\eta}\,\alpha_M. \qquad (11\cdot 9)$$

This indicates that the normal vibration of the first type, characterizing the lower excited states of even-even spherical nuclei, corresponds to that of α_M used in the Bohr-Mottelson model.[8] It should be noted, however, that for closed shell nuclei η_M vanishes as discussed in § 10 and the vibration of the first type disappears. In this case, therefore, the vibration of α_M becomes that of the second type ($\hbar\omega_\beta$).

§ 12. Determination of the inertial parameter and the surface tension parameter

The relation (11·9) uniquely determines the inertial parameter, B, and the surface tension parameter, C, for the quadrupole vibration of even-even nuclei. With the aid of (11·9), the Hamiltonian for the normal vibration of the first type,

$$\mathcal{H}_{\text{coll.}}^{(1)}=\frac{1}{2}\sum_M\{|\dot{\alpha}_M^{(1)}|^2+\omega_{(1)}^2|\alpha_M^{(1)}|^2\}, \qquad (12\cdot 1)$$

can be rewritten as

$$\mathcal{H}_{\text{coll.}}^{(1)}=\sum_M\left\{\frac{1}{2B}|\pi_M|^2+\frac{1}{2}C|\alpha_M|^2\right\}, \qquad (12\cdot 2)$$

where π_M is the momentum conjugate to α_M and satisfies $[\pi_M,\alpha_{M'}]=-i\hbar\delta_{MM'}$. B and C are the inertial parameter and the surface tension parameter respectively, and are uniquely determined as

$$B=I_\beta I_\eta\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{\{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}\}^2}\simeq(1-\varepsilon)^2 I_\eta, \qquad (12\cdot 3)$$

$$C=I_\beta I_\eta\cdot\omega_{(1)}^2\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{\{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}\}^2}\simeq(1-\varepsilon)^2 I_\eta\omega_{(1)}^2. \qquad (12\cdot 4)$$

Inserting (10·8) in (12·3), we get

$$\begin{aligned}B&\simeq(1-\varepsilon)^2\frac{\hbar^2\bar{Q}^2}{G^{(0)}\Omega\Theta_n\langle N_0|q_2|N_0\rangle^2}\\&=\frac{9}{8}(1-\varepsilon)^2\frac{\hbar^2A^2R_0^4(2\Omega-1)(2\Omega+1)\Omega}{G^{(0)}\pi\langle n_0 l_0|r^2|n_0 l_0\rangle^2(\Omega-1)^2(\Omega+1)}\cdot\frac{1}{n(2\Omega-n)},\end{aligned} \qquad (12\cdot 5)$$

where n is the number of nucleons in the N_0-shell and $\Omega=(2j_0+1)/2$ ((3·11)). Thus the ratio between (12·5) and $B_{\text{irrot.}}$ becomes

$$B/B_{\text{irrot.}}\simeq3(1-\varepsilon)^2\frac{\hbar^2AR_0^2}{G^{(0)}m\langle n_0 l_0|r^2|n_0 l_0\rangle^2}\cdot\frac{(2\Omega-1)(2\Omega+1)\Omega}{(\Omega-1)^2(\Omega+1)}\cdot\frac{1}{n(2\Omega-n)}, \qquad (12\cdot 6)$$

where $B_{\text{irrot.}}$ is the inertial parameter for the oscillations of an irrotational liquid drop :[8)]

$$B_{\text{irrot.}}=\frac{3}{8\pi}AmR_0^2 \quad (m: \text{ the nucleon mass}).$$

The formula (12·6) should be compared with Eq. (155) in Belyaev's paper,[4)] which has been obtained by using the "cranking formula".

Now let us remember that ε, the polarizability of the core by the outside nucleons, is expressed as

$$\begin{aligned}\varepsilon=&\frac{\kappa}{C_\beta+\kappa}\\ =&\bar{Q}\langle N_0|q_2|N_0\rangle^{-1}[\sum_{NN'}\{a(NN')-b(NN')\}(2u_N^2 v_{N'}^2)(1-\delta_{NN'})\langle N_0 N|g_2|N' N_0\rangle]\\ &\times\{\sqrt{2I_\beta\hbar\omega_\beta}\cdot\omega_\beta+\bar{Q}\langle N_0|q_2|N_0\rangle^{-1}[\sum_{NN'}\{a(NN')-b(NN')\}(2u_N^2 v_{N'}^2)\\ &\qquad\times(1-\delta_{NN'})\langle N_0 N|g_2|N' N_0\rangle]\}^{-1}.\end{aligned}$$

This indicates that ε is determined mainly by the properties of the core and does not depend appreciably on the pairing interaction ($G^{(0)}$) and on the number of nucleons in the partially-filled N_0-shell. Neglecting such a dependence of ε, we can empirically estimate the value from the data of nuclei with one particle outside closed shells. The rough estimates of (12·6) with $\langle n_0 l_0|r^2|n_0 l_0\rangle\simeq\frac{3}{5}R_0^2$, $R_0=1.4A^{1/3}\times10^{-13}$ cm, $\overline{G^{(0)}}=50$ Mev/A* and $\varepsilon=0.8$** are given in Table I together with experimental values. Though the estimation is very rough, the calculated values are in qualitative agreement with observed trends.

With the aid of (12·5) and (11·4a), the surface tension parameter (12·4) can be written as

$$C\simeq(1-\varepsilon)^2\frac{\bar{Q}^2\cdot\overline{G^{(0)}}\Omega}{\langle N_0|q_2|N_0\rangle^2\theta_n}\{1-\theta_n/\theta_{n_0}\},\tag{12·7}$$

where $\bar{Q}\equiv(3/\sqrt{5\pi})AR_0^2$, and $\langle N_0|q_2|N_0\rangle$ is given by (10·4). θ_{n_0} is given by (11·5) and represents a ratio between the strength of $G^{(0)}$ and of $G^{(2)}$. Formula (12·7) should be compared with Eq. (114) in Belyaev's paper.[4)]

As $(1-\varepsilon)^2\bar{Q}^2\cdot\overline{G^{(0)}}\Omega/\langle N_0|q_2|N_0\rangle^2\theta_n>0$, the stability of the spherical shape of even-even nuclei is determined by the sign of $\{1-\theta_n/\theta_{n_0}\}$. If $\theta_{n_0}>1$, the nucleus

* The value of $\overline{G^{(0)}}$ is determined approximately from the equation

$$\overline{G^{(0)}}\,\Omega\approx P(j_0)_{\text{exp.}},$$

where $P(j_0)_{\text{exp.}}$ is the experimental value of pairing energy. In $1g_{9/2}$- and $2d_{5/2}$-shell, $P(j_0)_{\text{exp.}}\simeq 25(2j_0+1)/A$ Mev.

** The $E2$-transition rate in $_{82}Pb^{207}$ implies that the effective charge, $e_{\text{eff.}}\equiv\varepsilon/(1-\varepsilon)\cdot Z/A\cdot e$, of Pb^{207} is $1.1e$. From this fact, we get $\varepsilon=0.73$. Note that Pb^{207} is the nucleus with one neutron hole inside the *double magic* closed shell.

Table I. Inertial parameter of even-even nuclei

a) P. H. Stelson and T. K. McGowan, Phys. Rev. **110** (1958), 489

b) K. Alder, A. Bohr, T. Huus, B. R. Mottelson and A. Winther, Rev. Mod. Phys. **28** (1956), 432.

nucleus	proton configuration	neutron configuration	B/B_{irrot} (exp.)		$B/B_{\text{irrot.}}$ (cal.)
			Ref. a)	Ref. b)	
$_{32}Ge^{74}$	$(2p_{3/2})^4$: closed	$(1g_{9/2})^2$		10	14
$_{32}Ge^{76}$	$(2p_{3/2})^4$: closed	$(1g_{9/2})^4$		9.3	9.7
$_{42}Mo^{98}$	$(1g_{9/2})^2$	$(2d_{5/2})^6$: closed	13		20
$_{44}Ru^{100}$	$(1g_{9/2})^4$	$(2d_{5/2})^6$: closed	9.6		14
$_{46}Pd^{110}$	$(1g_{9/2})^6$	$(1d_{5/2})^6\,(1g_{7/2})^8$: closed	9.9	8.5	16
$_{48}Cd^{112}$	$(1g_{9/2})^8$	$(1d_{5/2})^6\,(1g_{7/2})^8$: closed	10.5	12	23
$_{48}Cd^{114}$	$(1g_{9/2})^8$	$(3s_{1/2})^2$: closed	10.5	11	23
$_{50}Sn^{116}$	$(1g_{9/2})^{10}$: closed	$(1h_{11/2})^2$	14		19
$_{50}Sn^{118}$	〃	$(1h_{11/2})^4$	13		12
$_{50}Sn^{120}$	〃	$(1h_{11/2})^6$	14		11
$_{50}Sn^{122}$	〃	$(1h_{11/2})^8$	13		13
$_{50}Sn^{124}$	〃	$(1h_{11/2})^{10}$	16		20

remains spherical for any occupation. If $\Theta_{n_0} < \Theta_n (\leq 1)$, however, the spherical nucleus is unstable and becomes the deformed nucleus. With the aid of (8·15a), the condition $\Theta_{n_0} < \Theta_n$ can be rewritten as

$$n_0 < n < (2\Omega - n_0). \tag{12·8}$$

Here n_0 is the "characteristic number" of nucleons in the N_0-shell, which has been stressed by Weisskopf,[10] and is given by

$$n_0 = \Omega(1 - \sqrt{1 - \Theta_{n_0}}). \tag{12·9}$$

Thus a sharp transition from the spherical nuclei into the deformed nuclei occurs at the "characteristic number" n_0, provided that $\Theta_{n_0} < 1$.

§ 13. Concluding remarks

Assuming the possibility of the division, into three parts, of the effective interaction between particles as in Eq. (2·2), we have investigated the mechanism of collective motion in even-even spherical nuclei from the standpoint of particle excitations. In so far as the vibrational motion is concerned, the physical implication underlying the nuclear collective model proposed by Bohr and Mottelson[8] has been made clear.

Our method of description of the nuclear collective motion is quite different from that of the "cranking model".[5] The method is more fundamental than that of the "cranking model" in the sense that it is developed by using the effective interaction between particles explicitly. Indeed, with the aid of the method, the various physical parameters used by Belyaev[4] can be derived uniquely from the "first" principle.

In this paper, we have neglected effects of the coupling between collective and intrinsic (particle) motions. Such effects will be considered in a subsequent paper. It is outside the scope of this paper and remains to be investigated to relate the effective interaction used in this paper with the nuclear force known from the two-body problems.

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Note added in proof: Very recently R. Arvieu and M. Vénéroni made an attempt similar to ours [Comptes rendus **250** (1960), 992; preprint].

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